Linear perturbations of low angular momentum accretion flow in the Kerr metric and the corresponding emergent gravity phenomena

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(Received 6 April 2018; published 28 December 2018)

For certain geometric configurations of matter falling onto a rotating black hole, we develop a novel linear perturbation analysis scheme to perform the stability analysis of stationary integral accretion solutions corresponding to the steady state low angular momentum, inviscid, barotropic, irrotational, general relativistic accretion of hydrodynamic fluid. We demonstrate that such steady states remain stable under linear perturbation, and hence, the stationary solutions are reliable to probe the black hole spacetime using the accretion phenomena. We report that a relativistic acoustic geometry emerges out as a consequence of such a stability analysis procedure. We study various properties of that sonic geometry in detail. We construct the causal structures to establish the one to one correspondences of the sonic points with the acoustic black hole horizons and the shock location with an acoustic white hole horizon. The influence of the spin of the rotating black holes on the emergence of such acoustic spacetime has been discussed.

DOI: 10.1103/PhysRevD.98.123022

I. INTRODUCTION

Understanding the accretion process is important to study the observational signature of astrophysical black holes [1,2]. One studies the dynamical and the radiative properties of black hole accretion to construct the characteristic black hole spectra, and such spectra is analyzed observationally to probe the spacetime at close proximity of the horizon of astrophysical black holes. Because of the inner boundary conditions posed by the presence of the event horizon, black hole accretion is necessarily transonic [3], except for the possible cases of wind fed accretion of supersonic stellar winds [4].

For low angular momentum accretion, flow can manifest multitransonicity, i.e., one may observe the transition from the subsonic to supersonic flow at more than one place during the course of the motion of the matter falling towards the horizon, originating out from infinity (from a reasonably large distance). Accretion solutions passing through more than one sonic point may be connected through a discontinuous shock wave [3,5–32]. Study of the aforementioned shocked multitransonic accretion is, usually, accomplished for steady state matter flow, and the stationary integral accretion solutions are considered to probe the shock formation phenomena as well as to construct the corresponding black hole spectra ([15] and the references therein), although full time-dependent numerical simulation works are also performed to understand several characteristic features of the black hole accretion in general. For low angular momentum stationary integral accretion solutions, it is usually assumed that the flow in inviscid, barotropic, and irrotational.

It is relevant to note in this aspect that nonsteady features (time variability) and various kind of local as well as global fluctuations may be present for large-scale astrophysical fluid flows, which may jeopardize the steady state, and the stationary integral flow solutions may not be used to construct the black hole spectra in such circumstances. To ensure that one can use the stationary integral flow solutions to study the black hole accretion in a reliable way, it is thus imperative to establish that the steady state accretion model under consideration remains stable under perturbation.

In our present work, we would like to investigate the effect of the linear perturbation on stationary accretion solutions obtained for steady state general relativistic accretion onto rotating astrophysical black holes, i.e., accretion flow studied in the Kerr metric. For low angular momentum inviscid accretion, conical wedge-shaped flow is ideal to simulate the geometric configuration of matter accreting onto the black hole. It is to be mentioned here that by ‘low’ angular momentum, we essentially mean accretion flow with sub-Keplerian angular momentum distribution. For such flow, the stable Keplerian orbit may not form and hence it is not necessary to introduce the viscous dissipation to make the stable circular orbit collapse and to let the matter accrete onto the black hole. The inviscid assumption

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is considered to be justified for such flow profile. It is believed that such flow structure is a common feature for accretion onto the supermassive black hole at our Galactic centre (see [33] and references therein). Such sub-Keplerian weakly rotating flows may be observed in various astrophysical systems, for detached binary systems fed by accretion from OB stellar winds [34,35], for instance. Also for semi-detached low-mass nonmagnetic binaries [36], and for super-massive black holes fed by accretion from slowly rotating central stellar clusters ([37,38] and references therein) such flows are common. Even for a standard Keplerian accretion disc, turbulence may produce such low angular momentum flow (see, e.g., [39] and references therein). In supersonic astrophysical flows, perturbation of various kinds may produce discontinuities. The type of low angular momentum flow which will be discussed in the present work is somewhat different from thick accretion disc models in the sense that considerable amount of radial advective velocity is included as the initial boundary condition for our flow model. Such advection may be a consequence of high-velocity stellar wind fed accretion. Such advective accretion flows in the Kerr metric with complete general relativistic treatment for shock formation in conical flow has not been treated in the literature before.

The equations governing such flow will be derived from the first principle and the stationary integral flow solutions corresponding to the steady state will be obtained. It will be demonstrated that for certain values of initial conditions governing the flow, such integral solutions may pass through two sonic points, and flows passing through two sonic points will be connected through a stationary shock wave. Such stationary integral solutions will then be linearly perturbed to demonstrate that the perturbation does not diverge, which ensures that the stationary integral solutions are reliable because the corresponding steady state remains stable under linear perturbation. While performing the aforementioned procedure of linear stability analysis, one observes, as will be demonstrated in the consequent sections, the linear perturbation (the corresponding "sound wave") that propagates within the accreting fluid with a certain speed of propagation. It is also observed that the propagation of such linear acoustic perturbation can be described using a particular kind of acoustic spacetime (conformal to a certain form of black hole metric), that spacetime is further described by a metric. One can write down the specific form of such an acoustic metric embedded within the background stationary accreting flow.

Such findings lead to very interesting consequences. In the field of analogue gravity phenomena, it has been suggested that a black holelike spacetime can be generated within a transonic fluid by linearly perturbing such a flow. The propagation of linear perturbation within such a fluid can be described using a spacetime metric, conformal to the Painleve-Gullstrand-Lemaitre [40–42] presentation of the Schwarzschild metric [43–49]. The acoustic metric can possess corresponding acoustic horizons, depending on certain criteria, and such horizons may be of the black hole or white hole types [50]. Acoustic black holes are formed where the background fluid makes a transition from the subsonic to the supersonic state, and acoustic white holes are formed where the background fluid may make a transition from the supersonic state to the subsonic state.

Since the Hawking effect, as well as its counterpart in analogue gravity, is a kinematical effect, one can define the acoustic surface gravity in a more general way, i.e., acoustic surface gravity can be evaluated on any kind of acoustic horizon. Following such an approach, it has also been stated in the literature that depending on certain initial conditions, there is a theoretical probability that a classical analogue system can have an infinitely large (or, at least, extremely large) value of acoustic surface gravity [51].

The aforementioned information leads us to the conclusion that not only can an accreting black hole system be considered as a classical analogue model, but the acoustic geometry is a natural consequence of the process of the linear perturbation of the stationary integral accretion solutions of steady, inviscid, barotropic, irrotational flows of matter onto astrophysical black holes. In the present work, we precisely demonstrate that. We discuss that the emergent gravity phenomena are the natural consequence of the linear stability analysis of steady-state accretion, and hence, we make the crucial connection between two apparently disjoint fields of research; namely, the astrophysical accretion process and the emergent gravity phenomenon. We make a formal correspondence between the sonic surfaces in accretion astrophysics with acoustic black hole horizons in analogue gravity, between the discontinuous stationary shock in multitransonic black hole accretion with an acoustic white hole in analogue spacetime, through the construction of a corresponding causal structure at and around the sonic points and shock locations, respectively.

In the subsequent section, we show that acoustic surface gravity corresponding to an accreting black hole system can be calculated in terms of the gradient of the background steady state fluid velocity, as well as that of the sound speed, with both evaluated at the acoustic horizon. At sonic points, the transition from the subsonic to the supersonic state is continuous, and hence, such gradients, as well as the acoustic surface gravity, have a finite value. At the location of the formation of a stationary shock, there is a discontinuous transition from the supersonic state to the subsonic state corresponding to the stationary integral solutions. The gradient of the fluid, as well as the sound velocity, diverge at the shock location. As a result, the corresponding value of the acoustic surface gravity evaluated at the shock location (acoustic white hole) is infinite. This is an important finding since it manifests that the theoretical results obtained in [51] are actually relevant for a realistic physical system as well.

In accretion astrophysics, it is usually believed that the majority of the black holes contain nonzero spin angular momentum, i.e., most of the astrophysical black holes are of the Kerr type [52–71]. It is thus important to understand how the black hole spin (the Kerr parameter) influences the overall features of the emergent gravity phenomena as observed, while linearly perturbing the background solutions in the Kerr metric.

In the present work, we will demonstrate the black hole spin dependence of the corresponding analogue gravity phenomena. In this way, we also try to understand how the properties of the background black hole metric influence the characteristic feature of the sonic metric embedded within the accreting fluids.

In what follows, we demonstrate how one obtains the governing equations corresponding to the low angular momentum, inviscid, polytropic, irrotational, axially symmetric, non-self-gravitating general relativistic accretion of hydrodynamic fluid onto a rotating black hole in the background Kerr metric. We shall set $G = c = M_{\mathrm{BH}} = 1$ where G is the universal gravitational constant, c is the velocity of light, and M_{BH} is the mass of the black hole. The radial distance will be scaled by GM_{BH}/c^2, and any velocity will be scaled by c. We shall use the negative-time-positive-space metric convention.

II. BASIC EQUATIONS GOVERNING THE FLOW

We consider the following metric for a stationary rotating spacetime

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + 2g_{\phi t}d\phi dt + g_{\phi\phi}d\phi^2, \tag{1}$$

where the metric elements are functions of r, θ, and ϕ. The metric elements in the Boyer-Lindquist coordinates are given by [72]

$$g_{tt} = \left(1 - \frac{2}{\mu r}\right), \qquad g_{rr} = \frac{\mu r^2}{\Delta}, \qquad g_{\theta\theta} = \mu r^2,$$
$$g_{\phi t} = g_{t\phi} = -\frac{2a\sin^2\theta}{\mu r}, \qquad g_{\phi\phi} = \frac{\Sigma}{\mu r^2}\sin^2\theta, \tag{2}$$

where

$$\mu = 1 + \frac{a^2}{r^2}\cos^2\theta, \qquad \Delta = r^2 - 2r + a^2,$$
$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \tag{3}$$

The event horizon of the Kerr black hole is located at $r_+ = 1 + \sqrt{1 - a^2}$.

We assume the hydrodynamic fluid accreting onto the Kerr black hole to be perfect, irrotational, and described by an adiabatic equation of state. The energy momentum tensor for such fluids is given by

$$T^{\mu\nu} = (p + \varepsilon)v^\mu v^\nu + pg^{\mu\nu}, \tag{4}$$

where p and ε are the pressure and the energy density of the fluid, respectively. v^μ is the four velocity of the fluid that satisfies the normalization condition $v^\mu v_\mu = -1$. The adiabatic equation of state is given by the relation $p = k\rho^\gamma$, where ρ is the rest-mass energy density and $\gamma = c_p/c_v$ is the adiabatic index (c_p and c_v are specific heats at constant pressure and at constant volume, respectively). The total energy density ε is the sum of the rest-mass energy density and the internal energy density (due to the thermal energy), i.e., $\varepsilon = \rho + \varepsilon_{\mathrm{thermal}}$. The continuity equation, which ensures the conservation of mass, is given by

$$\nabla_\mu(\rho v^\mu) = 0, \tag{5}$$

where the covariant divergence is defined as $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\lambda}v^\lambda$ with the Christoffel symbols $\Gamma^\nu_{\mu\lambda} = \frac{1}{2}g^{\nu\sigma}[\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\sigma\lambda} - \partial_\sigma g_{\mu\lambda}]$. The energy-momentum conservation equation is given by

$$\nabla_\mu T^{\mu\nu} = 0. \tag{6}$$

A substitution of Eq. (4) in Eq. (6) provides the general relativistic Euler equation for a barotropic ideal fluid as

$$(p + \varepsilon)v^\mu\nabla_\mu v^\nu + (g^{\mu\nu} + v^\mu v^\nu)\nabla_\mu p = 0. \tag{7}$$

The specific enthalpy of the flow is defined as $h = (p + \varepsilon)/\rho$. We assume the flow to be isentropic; i.e., the specific entropy of the flow s/ρ is constant where s is the entropy density. Therefore, for an isentropic flow, the following thermodynamical identity where T is the temperature of the fluid,

$$dh = Td\left(\frac{s}{\rho}\right) + \frac{1}{\rho}dp, \tag{8}$$

gives $dp = \rho dh$, which, when used in $h = (p + \varepsilon)/\rho$, also gives $d\varepsilon = \rho dh$. Thus, the adiabatic sound speed is given by

$$c_s^2 = \frac{\partial p}{\partial \varepsilon}\bigg|_{\frac{s}{\rho}=\mathrm{constant}} = \frac{\rho}{h}\frac{\partial h}{\partial \rho}. \tag{9}$$

The relativistic Euler equation for isentropic flow can thus be written as

$$v^\mu\nabla_\mu v^\nu + \frac{c_s^2}{\rho}(v^\mu v^\nu + g^{\mu\nu})\partial_\mu\rho = 0. \tag{10}$$

For general relativistic irrotational isentropic fluid, the irrotationality condition is given by [45]

$$\partial_\mu(hv_\nu) - \partial_\nu(hv_\mu) = 0. \tag{11}$$

III. ACCRETION FLOW GEOMETRY

We consider an axially symmetric accretion flow in the Kerr background. The flow is assumed to be symmetric about the equatorial plane. The four velocity components are written as (v^t, v^r, v^θ, v^ϕ). We assume that the velocity component along the vertical direction is negligible compared to the radial component v^r; i.e., $v^\theta \ll v^r$. Also, due to axial symmetry, the ∂_ϕ term in the continuity equation given by Eq. (5) would vanish. Thus, the continuity equation for such flow can be written as

$$\partial_t(\rho v^t \sqrt{-g}) + \partial_r(\rho v^r \sqrt{-g}) = 0, \tag{12}$$

where g is the determinant of the metric $g_{\mu\nu}$. For the Kerr metric, $g = -\sin^2\theta r^4 \mu^2$. The accretion flow variables, i.e., the velocity components and the density, are, in general, functions of the t, r, θ coordinates. However, assuming that the flow thickness is small compared to the radial size of the accretion disc, one can do an averaging of any flow variable $f(t, r, \theta)$ along the θ direction by the following approximation [73]

$$\int f(t, r, \theta) d\theta \approx H_\theta f\left(t, r, \theta = \frac{\pi}{2}\right), \tag{13}$$

where H_θ is the characteristic angular scale of the flow. Such an averaging is very common in accretion disc literature, and it is usually known as a vertical averaging of the flow. Thus, the continuity equation for vertically averaged axially symmetric accretion can be written as [74,75]

$$\partial_t(\rho v^t \sqrt{-\tilde{g}} H_\theta) + \partial_r(\rho v^r \sqrt{-\tilde{g}} H_\theta) = 0, \tag{14}$$

where $\tilde{g}$ is the value of g on the equatorial plane; i.e., $\tilde{g} = -r^4$. The advantage of vertical averaging is that all of the variables are defined by their values measured on the equatorial plane, and any information about the geometry of the flow along the vertical direction is contained in the term H_θ. H_θ is a function of the local flow thickness $H(r)$. The angular scale H_θ of the flow thickness, i.e., the angle made by the flow thickness at the center of the black hole at any radial distances r from the center of the black hole along the equatorial plane, is given by $H_\theta = H(r)/r$, assuming the flow thickness to be small at all r.

There are mainly three models of flow geometry in the literature. The simplest one is called the constant height model (CH). For such flow geometry, the flow thickness remains constant for all r; i.e., for such flow, $H(r) =$ constant or $H_\theta \propto 1/r$. In the second model, the flow geometry is considered to be quasispherical or wedge-shaped conical. Such flow is known as conical flow (CF). For conical flow geometry, the angular scale H_θ remains constant at all r. In other words, the flow thickness is proportional to the radial distance or $H(r) \propto r$. The third model is the most complicated one. In this model, the flow is considered to be in hydrostatic equilibrium in the vertical direction, and it is known as vertical equilibrium model (VE). Further details on such classification are available in [76]. In the VE model, the local flow thickness will also depend on the flow variables. The general relativistic calculation by [77] for a stationary case gives the flow thickness for the VE model through the following relation

$$-\frac{2p}{\rho} + \frac{H_{\mathrm{VE}}^2(r)}{r^4}(v_\phi^2 - a^2(v_t^2 - 1)) = 0. \tag{15}$$

As evident from the above relation, the flow thickness in the VE model is a complicated function of the flow variables p, ρ, v_ϕ, v_t as well as the black hole spin a.

In the present work, we consider the conical flow model, where the accretion flow is assumed to maintain a wedge-shaped conical geometry. As mentioned earlier, in such flow the local flow thickness is proportional to the radial distance measured along the equatorial plane; i.e., $\frac{H}{r} =$ constant or H_θ being the characteristic angular scale of local flow is constant for such conical flow geometry. Thus, H_θ does not depend on the accretion flow variables like velocity or density. Therefore, linear perturbation of these quantities (discussed in the next section) will have no effect on it. For simplicity, therefore, we will write H_θ simply as H_0. The same is true for the CH model also. However, due to the complicated dependence of $H(r)$ on the flow variables in the VE model, the flow thickness will also be perturbed when the flow variables are perturbed. This will make the analysis too complicated to be presented here, and it may be reported elsewhere. Therefore, as mentioned earlier, we do not consider the CH and VE models and work only with the CF model. From now on, all of the equations will be derived by assuming that the flow variables are vertically averaged and their values are computed on the equatorial plane.

IV. LINEAR PERTURBATION ANALYSIS AND THE ACOUSTIC GEOMETRY

The scheme of the linear perturbation analysis would be the following: We shall write the accretion variables, e.g., four velocity components and density about their stationary background values up to first order in perturbation. These expressions are then used in the basic governing equations, such as the continuity equation, normalization condition, and the irrotationality condition. Keeping only the terms that are linear in the perturbed quantities gives equations relating different perturbed quantities up to first order in perturbations. Further manipulations of these equations gives a wave equation that describes the propagation of the perturbation of the mass accretion rate, which is defined later in this section. Such wave equations mimic the wave equation for a massless scalar field in curved spacetime.

Finally, comparing theses two wave equations, one obtains the acoustic metric.

Below we derive some useful relations using the irrotationality condition [Eq. (11)], the normalization condition $v^\mu v_\mu = -1$, and the axial symmetry, which will later be used to derive the wave equation for linear perturbation. From the irrotationality condition given by Eq. (11), with $\mu = t$ and $\nu = \phi$, and with axial symmetry, we have

$$\partial_t(hv_\phi) = 0. \tag{16}$$

Again, with $\mu = r$ and $\nu = \phi$ and the axial symmetry, the irrotationality condition gives

$$\partial_r(hv_\phi) = 0, \tag{17}$$

so we get that hv_ϕ is a constant of the motion. Equation (16) gives

$$\partial_t v_\phi = -\frac{v_\phi c_s^2}{\rho}\partial_t\rho. \tag{18}$$

Substituting $v_\phi = g_{\phi\phi}v^\phi + g_{\phi t}v^t$ in the above equation provides

$$\partial_t v^\phi = -\frac{g_{\phi t}}{g_{\phi\phi}}\partial_t v^t - \frac{v_\phi c_s^2}{g_{\phi\phi}\rho}\partial_t\rho. \tag{19}$$

The normalization condition $v^\mu v_\mu = -1$ provides

$$g_{tt}(v^t)^2 = 1 + g_{rr}(v^r)^2 + g_{\phi\phi}(v^\phi)^2 + 2g_{\phi t}v^\phi v^t, \tag{20}$$

which, after differentiating with respect to t, gives

$$\partial_t v^t = \alpha_1\partial_t v^r + \alpha_2\partial_t v^\phi, \tag{21}$$

where $\alpha_1 = -\frac{v_r}{v_t}$, $\alpha_2 = -\frac{v_\phi}{v_t}$, and $v_t = -g_{tt}v^t + g_{\phi t}v^\phi$. Substituting $\partial_t v^\phi$ in Eq. (21) using Eq. (19) gives

$$\partial_t v^t = \left(\frac{-\alpha_2 v_\phi c_s^2/(\rho g_{\phi\phi})}{1+\alpha_2 g_{\phi t}/g_{\phi\phi}}\right)\partial_t\rho + \left(\frac{\alpha_1}{1+\alpha_2 g_{\phi t}/g_{\phi\phi}}\right)\partial_t v^r. \tag{22}$$

We perturb the velocities and density around their steady background values as following

$$v^\mu(r,t) = v_0^\mu(r) + v_1^\mu(r,t) \tag{23}$$

$$\rho(r,t) = \rho_0(r) + \rho_1(r,t), \tag{24}$$

where $\mu = t, r, \phi$ and the subscript "0" denotes the stationary background part and the subscript "1" denotes the linear perturbations. Using Eqs. (23)–(24) in Eq. (22), and retaining only the terms of first order in perturbed quantities, we obtain

$$\partial_t v_1^t = \eta_1\partial_t\rho_1 + \eta_2\partial_t v_1^r \tag{25}$$

where

$$\eta_1 = -\frac{c_{s0}^2}{\Lambda v_0^t\rho_0}[\Lambda(v_0^t)^2 - 1 - g_{rr}(v_0^r)^2],$$

$$\eta_2 = \frac{g_{rr}v_0^r}{\Lambda v_0^t} \quad \text{and} \quad \Lambda = g_{tt} + \frac{g_{\phi t}^2}{g_{\phi\phi}}. \tag{26}$$

A. Linear perturbation of mass accretion rate

For stationary background flow, the ∂_t part of the equation of continuity, i.e., Eq. (14), vanishes, and integration over spatial coordinate provides $\sqrt{-\tilde{g}}H_0\rho_0 v_0^r =$ constant. Multiplying the quantity $\sqrt{-\tilde{g}}H_0\rho_0 v_0^r$ by the azimuthal component of volume element $d\phi$, and integrating the final expression, gives the mass accretion rate, $\Psi_0 = \tilde{\Omega}\sqrt{-\tilde{g}}H_0\rho_0 v_0^r$. Ψ_0 gives the rate of infall of mass through a particular surface. $\tilde{\Omega}$ arises due to the integral over ϕ and is just a geometrical factor, and therefore, we can redefine the mass accretion rate by setting it to unity without any loss of generality. Thus, we define

$$\Psi_0 \equiv \sqrt{-\tilde{g}}H_0\rho_0 v_0^r. \tag{27}$$

Now, let us define a quantity $\Psi \equiv \sqrt{-\tilde{g}}H\rho(r,t)v^r(r,t)$ which has the stationary value equal to Ψ_0. Using the perturbed quantities given by Eqs. (23) and (24), we have

$$\Psi(r,t) = \Psi_0 + \Psi_1(r,t), \tag{28}$$

where

$$\Psi_1(r,t) = \sqrt{-\tilde{g}}H_0(\rho_0 v_1^r + v_0^r\rho_1). \tag{29}$$

Using Eqs. (23)–(25) and (28) in the continuity of Eq. (14), and differentiating Eq. (29) with respect to t, gives, respectively,

$$\rho_0\eta_2\partial_t v_1^r + (v_0^t + \rho_0\eta_1)\partial_t\rho_1 = -\frac{1}{\sqrt{-\tilde{g}}H_0}\partial_r\Psi_1, \tag{30}$$

and

$$\rho_0\partial_t v_1^r + v_0^r\partial_t\rho_1 = \frac{1}{\sqrt{-\tilde{g}}H_0}\partial_t\Psi_1. \tag{31}$$

In deriving Eq. (30), we have used Eq. (25). With these two equations given by Eqs. (30) and (31), we can write $\partial_t v_1^r$ and $\partial_t\rho_1$ solely in terms of derivatives of Ψ_1 as

$$\partial_t v_1^r = \frac{1}{\sqrt{-\tilde{g}}H_0\rho_0\tilde{\Lambda}}[-(v_0^t + \rho_0\eta_1)\partial_t\Psi_1 - v_0^r\partial_r\Psi_1]$$

$$\partial_t\rho_1 = \frac{1}{\sqrt{-\tilde{g}}H_0\rho_0\tilde{\Lambda}}[\rho_0\eta_2\partial_t\Psi_1 + \rho_0\partial_r\Psi_1], \tag{32}$$

where $\tilde{\Lambda}$ is given by

$$\tilde{\Lambda} = \frac{g_{rr}(v_0^r)^2}{\Lambda v_0^t} - v_0^t + \frac{c_{s0}^2}{\Lambda v_0^t}[\Lambda(v_0^t)^2 - 1 - g_{rr}(v_0^r)^2]. \tag{33}$$

Now, let us go back to the irrotationality condition given by the Eq. (11). Using $\mu = t$ and $\nu = r$ gives the following equation

$$\partial_t(hg_{rr}v^r) - \partial_r(hv_t) = 0. \tag{34}$$

For stationary flow, this provides $\xi_0 = -h_0 v_{t0} =$ constant, which is the specific energy of the system. We substitute the density and velocities in Eq. (34) using Eq. (23), (24), and

$$v_t(r,t) = v_{t0}(r) + v_{t1}(r,t). \tag{35}$$

Keeping only the terms that are linear in the perturbed quantities, and differentiating with respect to time t, gives

$$\partial_t(h_0 g_{rr}\partial_t v_1^r) + \partial_t\left(\frac{h_0 g_{rr} c_{s0}^2 v_0^r}{\rho_0}\partial_t\rho_1\right) - \partial_r(h_0\partial_t v_{t1}) - \partial_r\left(\frac{h_0 v_{t0} c_{s0}^2}{\rho_0}\partial_t\rho_1\right) = 0. \tag{36}$$

We can also write

$$\partial_t v_{t1} = \tilde{\eta}_1\partial_t\rho_1 + \tilde{\eta}_2\partial_t v_1^r, \tag{37}$$

with

$$\tilde{\eta}_1 = -\left(\Lambda\eta_1 + \frac{g_{\phi t} v_{\phi 0} c_{s0}^2}{g_{\phi\phi}\rho_0}\right), \qquad \tilde{\eta}_2 = -\Lambda\eta_2. \tag{38}$$

Using Eq. (37) in Eq. (36), and dividing the resultant equation by $h_0 v_{t0}$, provides

$$\partial_t\left(\frac{g_{rr}}{v_{t0}}\partial_t v_1^r\right) + \partial_t\left(\frac{g_{rr}c_{s0}^2 v_0^r}{\rho_0 v_{t0}}\partial_t\rho_1\right) - \partial_r\left(\frac{\tilde{\eta}_2}{v_{t0}}\partial_t v_1^r\right) - \partial_r\left(\left(\frac{\tilde{\eta}_1}{v_{t0}} + \frac{c_{s0}^2}{\rho_0}\right)\partial_t\rho_1\right) = 0, \tag{39}$$

where we have used $h_0 v_{t0} =$ constant. Finally, substituting $\partial_t v_1^r$ and $\partial_t\rho_1$ in Eq. (39) using Eq. (32), we get

$$\begin{aligned}&\partial_t\left[k(r)\left(-g^{tt} + (v_0^t)^2\left(1 - \frac{1}{c_{s0}^2}\right)\right)\right] \\ &+ \partial_t\left[k(r)\left(v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right)\right)\right] \\ &+ \partial_r\left[k(r)\left(v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right)\right)\right] \\ &+ \partial_r\left[k(r)\left(g^{rr} + (v_0^r)^2\left(1 - \frac{1}{c_{s0}^2}\right)\right)\right] = 0,\end{aligned} \tag{40}$$

where

$$k(r) = \frac{g_{rr} v_0^r c_{s0}^2}{v_0^t v_{t0}\tilde{\Lambda}} \quad \text{and} \quad g^{tt} = \frac{1}{\Lambda} = \frac{1}{g_{tt} + g_{\phi t}^2/g_{\phi\phi}}. \tag{41}$$

Equation (40) can be written as $\partial_\mu(f^{\mu\nu}\partial_\nu\Psi_1) = 0$, where $f^{\mu\nu}$ is given by the symmetric matrix

$$f^{\mu\nu} = \frac{g_{rr} v_0^r c_{s0}^2}{v_0^t v_{t0}\tilde{\Lambda}} \times \begin{bmatrix} -g^{tt} + (v_0^t)^2\left(1 - \frac{1}{c_{s0}^2}\right) & v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right) \\ v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right) & g^{rr} + (v_0^r)^2\left(1 - \frac{1}{c_{s0}^2}\right)\end{bmatrix}. \tag{42}$$

This is the main result of this section, and it will be used in the next section to obtain the acoustic metric and in Sec. VIII for linear stability analysis of the stationary accretion solutions in the Kerr metric. In the Schwarzschild limit $(a = 0)$, we have $v_{t0}\tilde{\Lambda} = 1 + (1 - c_{s0}^2)g_{\phi\phi}(v_0^\phi)^2$. Thus, the $f^{\mu\nu}$ in Eq. (42) matches the result obtained by [74] in the Schwarzschild limit.

B. The acoustic metric

The linear perturbation analysis performed in the previous section provides the equation describing the propagation of the linear perturbation of the mass accretion rate $\Psi_1(r,t)$, and it is given by the following equation

$$\partial_\mu(f^{\mu\nu}\partial_\nu\Psi_1) = 0, \tag{43}$$

where μ, ν each run over r, t. This equation could be compared to the wave equation of a massless scalar field φ propagating in a curved spacetime given by [78]

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = 0. \tag{44}$$

Thus, comparing these two equations, one obtains the acoustic metric $G^{\mu\nu}$, which is related to $f^{\mu\nu}$ in the following way

$$\sqrt{-G}G^{\mu\nu} = f^{\mu\nu}, \tag{45}$$

where G is the determinant of the acoustic metric $G_{\mu\nu}$. $f^{\mu\nu}$ could be written as $f^{\mu\nu} = k(r)\tilde{f}^{\mu\nu}$, where $k(r)$ is the overall multiplicative factor, and $\tilde{f}^{\mu\nu}$ is the matrix part as given in Eq. (42). Thus, $G^{\mu\nu} = (k(r)/\sqrt{-G})\tilde{f}^{\mu\nu}$, and therefore, $G^{\mu\nu}$ is related to $\tilde{f}^{\mu\nu}$ by a conformal factor given by $k(r)/\sqrt{-G}$. One of our main goals of the present work is to show that the acoustic horizon is the transonic surface of the accretion flow and to demonstrate that by studying the causal structure of the acoustic spacetime. However, the location of the event horizon, or the causal structure of the spacetime, does not depend on the conformal factor of the spacetime metric. Thus, in order to investigate these properties of the acoustic spacetime, we can take $G^{\mu\nu}$ to be the same as $\tilde{f}^{\mu\nu}$ by ignoring the conformal factor. Thus, the acoustic metric $G^{\mu\nu}$ and $G_{\mu\nu}$, apart from the conformal factor, are given by

$$G^{\mu\nu} = \begin{bmatrix} -g^{tt} + (v_0^t)^2\left(1 - \frac{1}{c_{s0}^2}\right) & v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right) \\ v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right) & g^{rr} + (v_0^r)^2\left(1 - \frac{1}{c_{s0}^2}\right) \end{bmatrix} \tag{46}$$

and

$$G_{\mu\nu} = \begin{bmatrix} -g^{rr} - (v_0^r)^2\left(1 - \frac{1}{c_{s0}^2}\right) & v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right) \\ v_0^r v_0^t\left(1 - \frac{1}{c_{s0}^2}\right) & g^{tt} - (v_0^t)^2\left(1 - \frac{1}{c_{s0}^2}\right) \end{bmatrix}. \tag{47}$$

V. LOCATION OF THE ACOUSTIC EVENT HORIZON

The metric corresponding to the acoustic spacetime is given by Eq. (47). The metric elements of $G_{\mu\nu}$ are independent of time, and thus, the metric is stationary. In general relativity, the event horizon for such stationary spacetime is defined as a time like a hypersurface $r =$ constant whose normal $n_\mu = \partial_\mu r = \delta_\mu^r$ is null with respect to the spacetime metric. In a similar way, we can define the event horizon of the acoustic spacetime as a null timelike hypersurface. Thus, the location of the acoustic horizon is given by the condition [25,79–81]

$$G^{\mu\nu} n_\mu n_\nu = G^{\mu\nu}\delta_\mu^r \delta_\nu^r = G^{rr} = 0. \tag{48}$$

Therefore, on the event horizon, we have the following condition

$$c_{s0}^2 = \frac{g_{rr}(v_0^r)^2}{1 + g_{rr}(v_0^r)^2}. \tag{49}$$

Now it is convenient to move to the corotating frame as defined in [73]. Let u be the radial velocity of the fluid in the corotating frame, which is referred as the "advective velocity," and let $\lambda = -v_\phi/v_t$ be the specific angular momentum. For stationary flow, the advective velocity and the specific angular momentum will be denoted with a subscript "0" as earlier. In this corotating frame, we can write v^r, v^t, and v_t in terms of u, λ as

$$v^r = \frac{u}{\sqrt{g_{rr}(1 - u^2)}} \tag{50}$$

$$v^t = \sqrt{\frac{(g_{\phi\phi} + \lambda g_{\phi t})^2}{(g_{\phi\phi} + 2\lambda g_{\phi t} - \lambda^2 g_{tt})(g_{\phi\phi} g_{tt} + g_{\phi t}^2)(1 - u^2)}} \tag{51}$$

and

$$v_t = -\sqrt{\frac{g_{tt} g_{\phi\phi} + g_{\phi t}^2}{(g_{\phi\phi} + 2\lambda g_{\phi t} - \lambda^2 g_{tt})(1 - u^2)}}. \tag{52}$$

In the corotating frame, Eq. (49) becomes

$$u_0^2|_{\rm h} = c_{s0}^2|_{\rm h}, \tag{53}$$

where the subscript "h" implies that the quantity is to be evaluated at the horizon and would imply the same hereafter. Thus, we see that the acoustic horizon is located at a radius where the advective velocity u_0 becomes equal to the speed of sound c_{s0}, which is exactly the surface known as the transonic surface. Thus, the transonic surface of the accretion flow and the acoustic horizon coincide.

VI. CAUSAL STRUCTURE OF THE ACOUSTIC SPACETIME

An acoustic null geodesic corresponding to the radially traveling ($d\phi = 0$, $d\theta = 0$) acoustic phonons is given by $ds^2 = 0$. Thus,

$$\left(\frac{dr}{dt}\right)_\pm \equiv b_\pm = \frac{-G_{rt} \pm \sqrt{G_{rt}^2 - G_{rr}G_{tt}}}{G_{rr}}, \tag{54}$$

where the acoustic metric elements $G_{tt}, G_{rt} = G_{tr}, G_{rr}$ are given by Eq. (47). These are expressed in terms of the background metric elements, the sound speed, and the velocity variables $u_0(r)$ and $\lambda_0 = -v_{\phi 0}/v_{t0}$ using Eqs. (50) and (51)

$$\begin{aligned} G_{tt} &= -\frac{1}{g_{rr}(1 - u_0^2)}\left(1 - \frac{u_0^2}{c_{s0}^2}\right) \\ G_{tr} = G_{rt} &= \frac{u_0}{(1 - u_0^2)}\left(1 - \frac{1}{c_{s0}^2}\right) \\ &\quad \times \sqrt{\frac{(g_{\phi\phi} + \lambda_0 g_{\phi t})^2}{g_{rr}(g_{\phi\phi} + 2\lambda_0 g_{\phi t} - \lambda_0^2 g_{tt})(g_{\phi\phi} g_{tt} + g_{\phi t}^2)}} \\ G_{rr} &= \frac{1}{g_{tt} g_{\phi\phi} + g_{\phi t}^2} \\ &\quad \times \left(g_{\phi\phi} - \frac{(g_{\phi\phi} + \lambda_0 g_{\phi t})^2}{(g_{\phi\phi} + 2\lambda_0 g_{\phi t} - \lambda_0^2 g_{tt})} \frac{(1 - \frac{1}{c_{s0}^2})}{(1 - u_0^2)}\right). \end{aligned} \tag{55}$$

$t(r)$ can be obtained as

$$t(r)_\pm = t_0 + \int_{r_0}^{r} \frac{dr}{b_\pm}. \tag{56}$$

We can introduce a new set of coordinates as following

$$dz = dt - \frac{1}{b_+}dr, \quad \text{and} \quad dw = dt - \frac{1}{b_-}dr. \tag{57}$$

In terms of these new coordinates, the acoustic line element can be written as

$$ds^2|_{\phi=\theta=\text{const}} = \mathcal{D}dzdw, \tag{58}$$

where $\mathcal{D}$ is found to be equal to G_{tt}.

$b_\pm(r)$ is a function of the stationary solution $u_0(r)$ and the sound speed $c_{s0}(r)$. Therefore, we have to first obtain $u_0(r)$ and $c_{s0}(r)$ for the stationary accretion flow. This is done by simultaneously numerically integrating the equations describing the gradient of the advective velocity du_0/dr and the gradient of the sound speed dc_{s0}/dr, which are derived in Appendix A. We use the fourth order Runge-Kutta method to integrate these equations. The solutions are characterized by the parameters [ξ_0, γ, λ_0, a]. Remember that $\xi_0 = -h_0 v_{t0}$ is the specific energy of the flow that is a conserved quantity for the flow under consideration. Thus, given a particular set of [ξ_0, γ, λ_0, a], we get $u_0(r)$ and $c_{s0}(r)$ by numerically solving Eqs. (A8) and (A7) simultaneously, and then, using these solutions of $u_0(r)$ and $c_{s0}(r)$, we get $b_\pm(r)$. The integration in Eq. (56) is then performed by applying the Euler method. Finally, we plot $t(r)_\pm$ as a function of r to see the causal structure of the acoustic spacetime.

A. Monotransonic case

Let us first consider the case where the accretion flow is monotransonic. For such accretion flow, there exists only one transonic surface. In other words, the flow starts its journey from large radial distance subsonically, i.e., $|u_0| < |c_{s0}|$ or $\mathcal{M} = |u_0|/|c_{s0}| < 1$, where $\mathcal{M}$ is the Mach number of the flow, and at some certain radial distance r, the advective velocity becomes equal to the speed of sound, or $\mathcal{M} = 1$. The radius r at which $\mathcal{M}$ becomes equal to 1 is called the transonic point. For the flow under consideration, the transonic points are the critical points of the flow where the denominator in the expression of du_0/dr becomes 0 (see Appendix A). Thus, the transonic points are given by $r = r_{\rm crit}$, which in turn are obtained by solving Eq. (A10) for given values of the parameters $[\xi_0, \gamma, \lambda_0, a]$. For $r < r_{\rm crit}$, the flow is supersonic, i.e., $\mathcal{M} > 1$, and it remains supersonic all the way up to the event horizon r_+.

We would like to choose the parameters $[\xi_0, \gamma, \lambda_0]$ in a way such that Eq. (A10) has exactly one solution outside the event horizon (i.e., for $r > r_+$) for all values of a and see how the radius of the transonic surface, or equivalently $r_{\rm crit}$, varies with the black hole spin a. Then, with the same $[\xi_0, \gamma, \lambda_0]$, we pick a few values of the black hole spin a and draw the causal structure of the acoustic space time, and we show that the location of the acoustic horizon matches $r_{\rm crit}$ for that value of a. In Fig. 1, we plot the critical points $r_{\rm crit}$ of monotransonic flow as a function of the black hole spin.

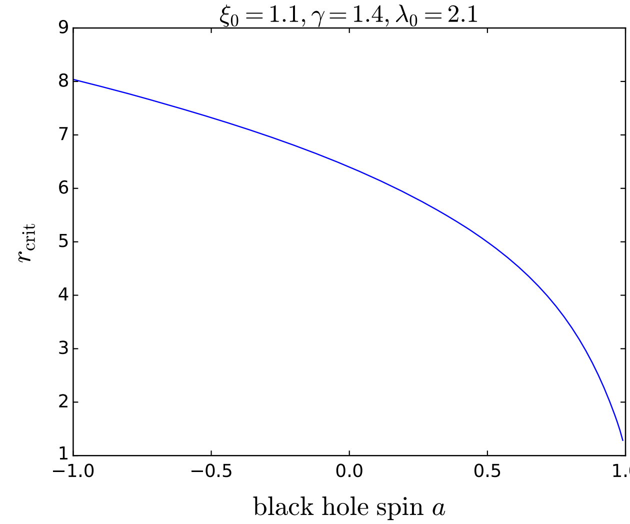

FIG. 1. The critical points $r_{\rm crit}$ (which are transonic points of the monotransonic accretion flow) are plotted as a function of the black hole spin a for the set of values $[\xi_0, \gamma, \lambda_0] = [1.1, 1.4, 2.1]$.

In Fig. 2, we show the causal structure of the acoustic spacetime for monotransonic accretion flow. The parameters $[\xi_0, \gamma, \lambda_0] = [1.1, 1.4, 2.1]$ are the same for all of the plots, while the black hole spins are $a = -0.9, -0.5, 0, 0.5, 0.9$ by row, from top to bottom. Solid lines represent $t_+(r)$ vs r, i.e., $z =$ constant lines, and the dotted lines represent the $t_-(r)$ vs r, i.e., $w =$ constant lines. It is illustrated from the causal structures that the radius of the acoustic horizon, where $t_+(r)$ diverges, is same as the critical points $r_{\rm crit}$ for the given value of $[\xi_0, \gamma, \lambda_0, a]$.

B. Multitransonic case

For a given set of values of the parameters $[\xi_0, \gamma, \lambda_0, a]$, Eq. (A10) can have more than one, or more specifically three, solutions for $r > r_+$. The corresponding flow in such a case is said to be a multicritical flow as it allows multiple critical points $r_{\rm in}, r_{\rm mid}, r_{\rm out}$, such that $r_{\rm in} < r_{\rm mid} < r_{\rm out}$. These critical points can be characterized by performing a critical point analysis. Such analysis shows that the inner and outer critical points $r_{\rm in}$ and $r_{\rm mid}$, respectively, are of the saddle type, whereas the middle critical point $r_{\rm mid}$ is the center type. Thus, the accretion flow can only pass through the outer or inner critical points. When the accretion flow passes through both the outer and inner critical points, the accretion flow is called a multitransonic flow. Multicritical flows are not necessarily multitransonic flows. This could be understood as the following: suppose the flow starts its journey from large radial distance subsonically, and at $r = r_{\rm out}$, it makes a transition from the subsonic state to the supersonic state. Thus, $r_{\rm out}$ is basically the outer acoustic horizon. After the flow becomes supersonic, it may encounter a shock formation that makes the flow subsonic from supersonic discontinuously, i.e., the dynamical variables, such as the

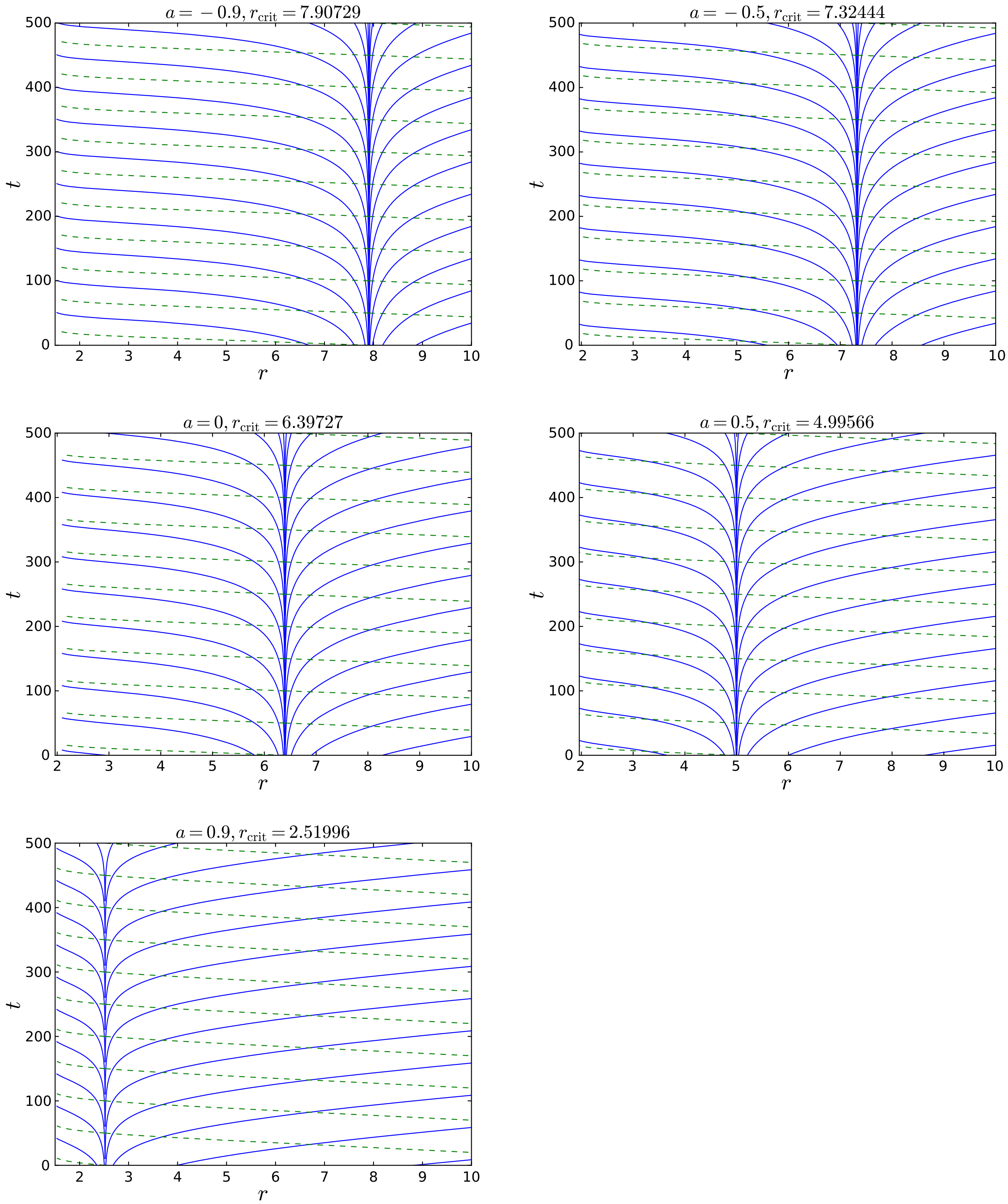

FIG. 2. Causal structure of the acoustic spacetime for monotransonic accretion. $t_+(r)$ vs r, i.e., $z =$ constant lines are represented by the solid lines, and $t_-(r)$ vs r, i.e., $w =$ constant lines are represented by the dashed lines. $t_\pm(r)$ are given by Eq. (56). The causal structures are plotted with $[\xi_0, \gamma, \lambda_0] = [1.1, 1.4, 2.1]$ for $a = -0.9, -0.5, 0, 0.5, 0.9$, by rows, from top to bottom. It could be noticed that the acoustic horizon where the $t_+(r)$ lines diverges, it coincides with the critical point $r_{\rm crit}$. This further illustrates that the transonic surface of the accretion flow is indeed the acoustic horizon of the embedded acoustic spacetime.

velocity, sound speed, density, and pressure, make a discontinuous jump. After it becomes subsonic due the shock formation, it again passes through the inner critical point and becomes supersonic from subsonic. Therefore, in the presence of shock formation, the flow can pass through both the outer and inner critical points, and hence, the flow is multitransonic. However, all of the set parameters $[\xi_0, \gamma, \lambda_0, a]$ that allow multiple critical

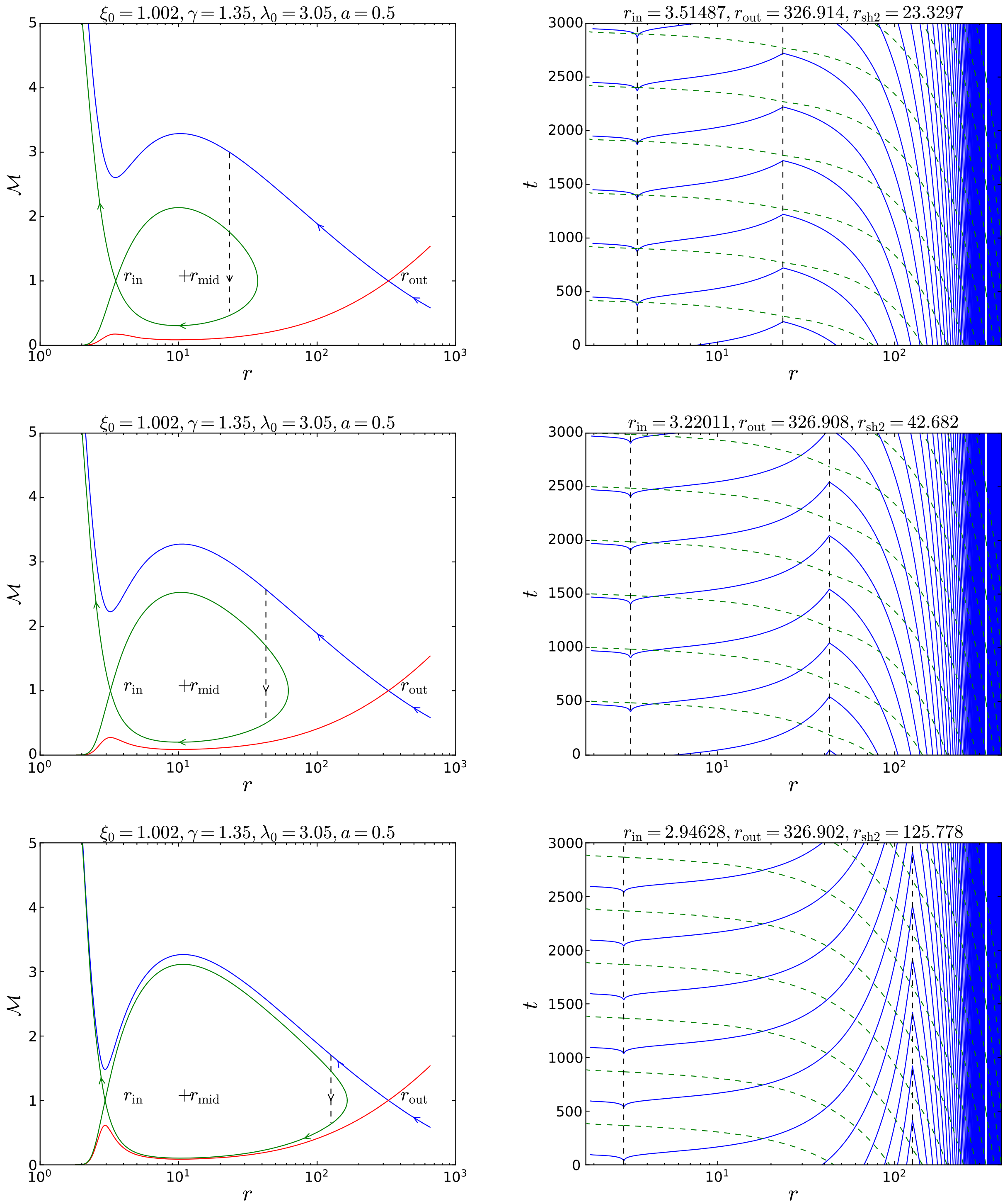

FIG. 3. Mach number $\mathcal{M}$ vs r plot (on the left) and the corresponding causal structure (on the right). The parameters [$\xi_0 = 1.002$, $\gamma = 1.35$, $\lambda_0 = 3.05$] are the same as where the black hole spin is $a = 0.5$ (top panel), $a = 0.55$ (middle panel), and $a = 0.6$ (bottom panel). The solid lines represents $t_+(r)$ vs r lines, and the dashed lines represents the $t_-(r)$ vs r lines.

points do not allow shock formation. In other words, only a subset of parameters allowing multiple critical points allows shock formation. This is best shown by plotting the parameter space.

We have assumed a nondissipative inviscid accretion flow. Therefore, the flow has conserved specific energy and mass accretion rate. Thus, the shock produced in such a flow is assumed to be of the energy preserving Rankine

Hugonoit type, which satisfies the general relativistic Rankine Hugonoit conditions [25,29,82–87]

$$\begin{aligned} [\![\rho v^\mu \eta_\mu]\!] &= [\![\rho v^r]\!] = 0 \\ [\![T_{t\mu}\eta^\mu]\!] &= [\![(p+\varepsilon)v_t v^r]\!] = 0 \\ [\![T_{\mu\nu}\eta^\mu\eta^\nu]\!] &= [\![(p+\varepsilon)(v^r)^2 + pg^{rr}]\!] = 0, \end{aligned} \tag{59}$$

where $\eta_\mu = \delta^r_\mu$ is the normal to the surface of the shock formation. $[\![f]\!]$ is defined as $[\![f]\!] = f_+ - f_-$, where f_+ and f_- are values of f after and before the shock, respectively. The first condition comes from the conservation of the mass accretion rate, and the last two conditions come from the energy-momentum conservation. These conditions are to be satisfied at the location of the shock formation. In order to find out the location of the shock formation, it is convenient to construct a shock-invariant quantity, which depends only on u_0, c_{s0} and γ, using the conditions above. The first and second conditions are trivially satisfied, owing to the constancy of the mass accretion rate and the specific energy. The first condition is basically $(\Psi_0)_+ = (\Psi_0)_-$, and the third condition is $(T^{rr})_+ = (T^{rr})_-$. Thus, we can define a shock-invariant quantity $S_{\rm sh} = T^{rr}/\Psi_0$ that also satisfies $[\![S_{\rm sh}]\!] = 0$ and is given by (see Appendix B)

$$S_{\rm sh} = \frac{(u_0^2(\gamma - c_{s0}^2) + c_{s0}^2)}{u_0\sqrt{1 - u_0^2(\gamma - 1 - c_{s0}^2)}}. \tag{60}$$

The procedure to find the location of the shock formation is the following. Let us denote the values of $S_{\rm sh}$ along the flow passing through the outer critical point as $S_{\rm sh}^{\rm out}$ and the same for the flow passing through inner critical point as $S_{\rm sh}^{\rm in}$. At the location of the shock formation $r_{\rm sh}$, we have $S_{\rm sh}^{\rm out} = S_{\rm sh}^{\rm in}$. Thus, evaluating the $S_{\rm sh}^{\rm out}$ and $S_{\rm sh}^{\rm in}$ we find $r_{\rm sh}$ by noting the value of r, for which $S_{\rm sh}^{\rm out} = S_{\rm sh}^{\rm in}$. In general, there are two such values of $r_{\rm sh}$ such that one is between the inner and middle critical points $r_{\rm in} < r_{\rm sh1} < r_{\rm mid}$ and the other one is between the middle and outer critical points $r_{\rm mid} < r_{\rm sh2} < r_{\rm out}$. However, the literature shows that the shock formation at $r_{\rm sh1}$ is unstable, and the formation at $r_{\rm sh2}$ is stable. Therefore, only $r_{\rm sh2}$ is the allowed location of the shock formation, and hence, we shall hereafter refer to only this location as the location of the shock formation.

In the left column of Fig. 3, we show the phase portraits of the flow, i.e., the Mach number vs radial distance plots for three different values of the Kerr parameter $a = 0.5$, 0.55, 0.6, keeping $[\xi_0, \gamma, \lambda_0]$ to be the same as $[\xi_0 = 1.002$, $\gamma = 1.35$, $\lambda_0 = 3.05]$. These chosen values of the parameters $[\xi_0, \gamma, \lambda_0, a]$ allow the flow to be multicritical, as well as multitransonic, by allowing shock formation. The shock transition of the flow has been denoted by a vertical dashed line in the phase portrait, which implies that the shock formation at that location makes the flow to jump from the supersonic state in the branch passing through the outer critical point to the subsonic state in the branch passing through the inner critical point.

In the right column of Fig. 3, we show the causal structure corresponding the flow shown by the phase portrait in the left column in the particular row. In the causal structure plots, the vertical dashed line in the left is the location of the inner critical point, and the vertical dashed line on the right is the location of the shock formation. The outer critical point is located at the white line separating densely populated diverging $t_+(r)$ lines. It is obvious from the causal structure that the inner and outer critical points are the inner and outer acoustic horizons of the acoustic spacetime. Also, it could be noticed that for an observer in the region $r_{\rm in} < r < r_{\rm sh2}$, the surface of the shock formation would resemble a white hole horizon. Thus, the shock formation can be regraded as the presence of an acoustic white hole.

VII. ACOUSTIC SURFACE GRAVITY

The Hawking temperature of an astrophysical black hole is given in terms of the surface gravity, which can be derived by using the Killing vector that is null on the event horizon. Similarly, the analogue Hawking temperature $T_{\rm AH}$ may be given in terms of the acoustic surface gravity κ as $T_{\rm AH} = \hbar\kappa/(2\pi K_B)$ in the units we are working with. K_B is the Boltzmann constant and $\hbar = h/2\pi$, where h is the Planck constant. Suppose χ^μ is the Killing vector of the acoustic spacetime that is null on the acoustic horizon, i.e., $\chi^\mu\chi_\mu|_{\rm h} = G_{\mu\nu}\chi^\mu\chi^\nu|_{\rm h} = 0$. Then the acoustic surface gravity is obtained by using the following relation [81,88]

$$\nabla_\alpha(-\chi^\mu\chi_\mu) = 2\kappa\chi_\alpha. \tag{61}$$

The acoustic metric given by Eq. (47) is independent of time t. Therefore, we have the stationary Killing vector $\chi^\mu = \delta^\mu_t$, which is null on the horizon, i.e., $G_{\mu\nu}\chi^\mu\chi^\nu|_{\rm h} = G_{tt}|_{\rm h} = 0$. Now, $\chi_\mu = G_{\mu\nu}\chi^\nu = G_{\mu\nu}\delta^\nu_t = G_{\mu t}$. Therefore, from the $\alpha = r$ component of Eq. (61), the acoustic surface gravity is obtained to be

$$\kappa = \frac{1}{2G_{rt}}\partial_r(-G_{tt})|_{u_0^2 = c_s^2}. \tag{62}$$

Using the expressions of G_{tt} and G_{rt} from Eq. (55) provides

$$\kappa = \left|\kappa_0\left(\frac{du_0}{dr} - \frac{dc_{s0}}{dr}\right)\right|_{\rm h}, \tag{63}$$

where

$$\kappa_0 = \frac{\sqrt{(g_{tt}g_{\phi\phi} + g_{\phi t}^2)(g_{\phi\phi} + 2\lambda_0 g_{\phi t} - \lambda_0^2 g_{tt})}}{(1 - c_{s0}^2)(g_{\phi\phi} + \lambda_0 g_{\phi t})\sqrt{g_{rr}}} \tag{64}$$

and the subscript "h," as mentioned earlier, denotes that the quantities have been evaluated at the acoustic horizon.

On the equatorial plane ($\theta = \frac{\pi}{2}$), the metric elements are given by

$$g_{tt} = 1 - \frac{2}{r}, \qquad g_{\phi t} = -\frac{2a}{r},$$
$$g_{\phi\phi} = \frac{r^3 + a^2 r + 2a^2}{r} \tag{65}$$

Thus, κ_0 can be further written as

$$\kappa_0 = \frac{r\sqrt{(r^2 - 2r + a^2)(g_{\phi\phi} + 2\lambda_0 g_{\phi t} - \lambda_0^2 g_{tt})}}{(1 - c_{s0}^2)(r^3 + a^2 r + 2a^2 - 2a\lambda_0)\sqrt{g_{rr}}}. \tag{66}$$

The acoustic surface gravity is thus obtained as a function of the background metric elements and the stationary values of the accretion variables. The surface gravity depends explicitly on the black hole spin a.

VIII. STABILITY ANALYSIS

The wave equation describing the propagation of the mass accretion rate, as given by Eq. (40), could be used to check whether the steady state accretion flow solutions are stable under linear perturbations. We discuss two different kind of solutions of the wave equation given by Eq. (40). We follow the technique introduced by [89] for this purpose. Let us take the trial solution as

$$\Psi_1(r,t) = P_\omega(r)\exp[i\omega t]. \tag{67}$$

Using this trial solution in the wave equation $\partial_\mu(f^{\mu\nu}\partial_\nu\Psi_1) = 0$, where $f^{\mu\nu}$ is Eq. (42), provides

$$-\omega^2 f^{tt} P_\omega + i\omega[f^{tr}\partial_r P_\omega + \partial_r(f^{rt}P_\omega)] + \partial_r(f^{rr}\partial_r P_\omega) = 0. \tag{68}$$

A. Standing wave analysis

In order to form a standing wave, the amplitude of the wave $P_\omega(r)$ must vanish at two different radii r_1 and r_2 for all times, i.e., $P_\omega(r_1) = 0 = P_\omega(r_2)$. The outer point r_2 could be located at the source, at a large distance from which accreting materials are coming. However, in order for the inner condition $P_\omega(r_1) = 0$ to be satisfied, the accretor must have a physical surface. Also, the solution must be continuous in the range $r_1 \le r \le r_2$. If the accretor is a black hole, then the accretion flow is necessarily supersonic at the event horizon [1,3]. Also, there is no physical surface or mechanism to make the wave amplitude vanish at the horizon, and hence, in the case of a black hole, standing waves are not formed. If the accretion flow has a supersonic region, then it is also possible to develop the shock at some radius, and this would make the solution discontinuous. Therefore, in order for the standing wave to be formed, the flow must be subsonic in the region $r_1 \le r \le r_2$. For this reason, we consider subsonic flow in the following.

Multiplying Eq. (68) by $P_\omega(r)$ and integrating the resulting equation between r_1 and r_2 gives

$$\omega^2 \int_{r_1}^{r_2} P_\omega^2 f^{tt} dr - i\omega \int_{r_1}^{r_2} \partial_r[f^{tr}P_\omega^2] dr - \int_{r_1}^{r_2} [P_\omega \partial_r(f^{rr}\partial_r P_\omega)] dr = 0. \tag{69}$$

Boundary conditions at r_1 and r_2 make the middle term vanish, and integrating the last term by parts, Eq. (69) can be written as

$$\omega^2 \int_{r_1}^{r_2} P_\omega^2 f^{tt} dr + \int_{r_1}^{r_2} f^{rr}(\partial_r P_\omega)^2 dr = 0, \tag{70}$$

which provides

$$\omega^2 = -\frac{\int_{r_1}^{r_2} f^{rr}(\partial_r P_\omega)^2 dr}{\int_{r_1}^{r_2} f^{tt} P_\omega^2 dr}. \tag{71}$$

From Eq. (42), f^{tt} is given by

$$f^{tt} = \frac{g_{rr} v_0^r c_{s0}^2}{v_0^t v_{t0}\tilde{\Lambda}}\left[-g^{tt} + (v_0^t)^2\left(1 - \frac{1}{c_{s0}^2}\right)\right], \tag{72}$$

and f^{rr} is given by

$$f^{rr} = \frac{g_{rr} v_0^r c_{s0}^2}{v_0^t v_{t0}\tilde{\Lambda}}\left[g^{rr} + (v_0^r)^2\left(1 - \frac{1}{c_{s0}^2}\right)\right]. \tag{73}$$

As $g^{tt} > 0$ and $c_{s0}^2 < 1$

$$-g^{tt} + (v_0^t)^2\left(1 - \frac{1}{c_{s0}^2}\right) < 0, \tag{74}$$

and using Eq. (50), we have

$$g^{rr} + \frac{u_0^2}{g_{rr}(1 - u_0^2)}\left(1 - \frac{1}{c_{s0}^2}\right) = \frac{(1 - u_0^2) + u_0^2(1 - \frac{1}{c_{s0}^2})}{g_{rr}(1 - u_0^2)} = \frac{(1 - \frac{u_0^2}{c_{s0}^2})}{g_{rr}(1 - u_0^2)} > 0, \tag{75}$$

where we have used the fact that the accretion flow is subsonic $u_0^2 < c_{s0}^2$. Hence, $\omega^2 > 0$. Therefore, ω has two real roots, the trial solution is oscillatory, and the stationary accretion solution is stable.

B. Traveling wave analysis

We consider the traveling wave solution with the wavelength, which is small compared to the characteristic radius of the accretor, which, for the case of black hole, could be taken as the radius of the event horizon. For such solutions, the frequency will be very large, and hence, the solution could be written as a power series of the following form

$$P_\omega(r) = \exp\left[\sum_{n=-1}^{\infty} \frac{k_n(r)}{\omega^n}\right]. \tag{76}$$

Substituting the trail solution in Eq. (68) enables us to find out leading order terms by equating the coefficients of individual power of ω to zero. Thus, we get

$$\text{coefficient of } \omega^2\text{: } f^{rr}(\partial_r k_{-1})^2 + 2if^{tr}\partial_r k_{-1} - f^{tt} = 0 \tag{77}$$

$$\text{coefficient of } \omega\text{: } f^{rr}[\partial_r^2 k_{-1} + 2\partial_r k_{-1} k_0] + i[2f^{tr}\partial_r k_0 + \partial_r f^{tr}] + \partial_r f^{rr}\partial_r k_{-1} = 0 \tag{78}$$

$$\text{coefficient of } \omega^0\text{: } f^{rr}[\partial_r^2 k_0 + 2\partial_r k_{-1}\partial_r k_1 + (\partial_r k_0)^2] + \partial_r f^{rr}\partial_r k_0 + 2if^{tr}\partial_r k_1 = 0. \tag{79}$$

Equation (77) gives

$$k_{-1}(r) = i\int \frac{-f^{tr} \pm \sqrt{(f^{tr})^2 - f^{tt}f^{rr}}}{f^{rr}} dr. \tag{80}$$

Using $k_{-1}(r)$ from Eq. (80) in Eq. (78) gives

$$k_0(r) = -\frac{1}{2}\ln[\sqrt{(f^{tr})^2 - f^{tt}f^{rr}}] + \text{constant} \tag{81}$$

and using Eq. (80) and (81) in Eq. (79) gives

$$k_1(r) = \pm\frac{i}{2}\int \frac{\partial_r(f^{rr}\partial_r k_0) + f^{rr}(\partial_r k_0)^2}{\sqrt{(f^{tr})^2 - f^{tt}f^{rr}}} dr. \tag{82}$$

Now,

$$\det f^{\mu\nu} = f^{tt}f^{rr} - (f^{rt})^2 = \left(\frac{g_{rr}v_0^r c_s^2}{v_0^t v_{t0}\tilde{\Lambda}}\right)^2 \mathcal{F}, \tag{83}$$

where v_{t0} is the stationary value of v_t given by Eq. (52), and v_0^r and v_0^t are the stationary values of v^r and v^t given by Eq. (50) and (51), respectively, and

$$\mathcal{F} = \left[-g^{tt}g^{rr} + \left(1 - \frac{1}{c_s^2}\right)(-g^{tt}(v_0^r)^2 + g^{rr}(v_0^t)^2)\right]. \tag{84}$$

In terms of λ_0, u_0 and the background metric elements $\mathcal{F}$ can be written as

$$\mathcal{F} = -\frac{g_{\phi\phi}}{g_{rr}(g_{\phi\phi}g_{tt} + g_{\phi t}^2)} \times \left[1 + \frac{(1-c_s^2)}{c_s^2(1-u_0^2)}\left(\frac{(1+\lambda_0\frac{g_{\phi t}}{g_{\phi\phi}})^2}{(1+2\lambda_0\frac{g_{\phi t}}{g_{\phi\phi}} - \lambda_0^2\frac{g_{tt}}{g_{\phi\phi}})} - u_0^2\right)\right] < 0. \tag{85}$$

Thus, $k_{-1}(r)$ and $k_1(r)$ are purely imaginary, and the leading contribution to the amplitude of the wave comes from $k_0(r)$.

So that the trial solution does not diverge and is stable, the power series in Eq. (76) must converge, i.e., we have to show $|k_n/\omega_n| \gg |k_{n+1}/\omega_{n+1}|$. As the frequency is very large $\omega \gg 1$, the contributions from higher order terms are very small. Thus, it should suffice to show that $|\omega k_{-1}| \gg |k_0| \gg |k_1/\omega|$. k_{-1}, k_0, k_1 are complicated functions of the accretion variables, and thus, it is not possible to have an analytic form. However, we can find the spatial dependence at a large distance $r \to \infty$, where the spacetime is effectively Newtonian. From the constancy of the mass accretion rate, we have $v^r \propto 1/(\rho r^2)$. At the asymptotic limit, ρ approaches its constant ambient value ρ_∞, and hence, at $r \to \infty$, $v_\infty^r \propto 1/r^2$. Similarly, the sound speed has its ambient value $c_{s0\infty}$. $v_0^t \sim 1$ and $v_{t0} \sim -1$. Also, $\tilde{\Lambda}_\infty \propto (v_0^r)^2$. Thus, in this asymptotic limit, we have

$$f^{tt} \sim r^2, \qquad f^{rr} \sim r^2, \qquad f^{tr} \sim r^0, \\ (f^{rt})^2 - f^{tt}f^{rr} \sim r^4, \tag{86}$$

which gives $k_{-1} \sim r$, $k_0 \sim \ln r$, and $k_1 \sim 1/r$. Therefore, the sequence converges in the leading order even at large r. Considering the first three terms in the expansion in Eq. (76), the amplitude of the wave can be approximated as

$$|\Psi_1| \approx \left[\left(\frac{v_0^t v_{t0}\tilde{\Lambda}}{g_{rr}v_0^r c_s^2}\right)^2 \frac{1}{-\mathcal{F}}\right]^{\frac{1}{4}}. \tag{87}$$

IX. CONCLUDING REMARKS

In this work, we demonstrate that the emergence of acoustic spacetime as an analogue system is a natural outcome of the linear stability analysis of the relativistic black hole accretion. It is interesting to investigate whether, in general, the emergence of a gravity like phenomena is a consequence of linear perturbation analysis only, or any complex nonlinear perturbation (of any order) of fluid may lead to the emergent gravity phenomena. In other words, it is important to know how universal the analogue gravity phenomena is—whether black hole like spacetime can be generated by only one means (linear perturbation) or any kind of perturbation of a general nature would lead to the construction of an analogue system. In another work [90], we have started explaining this for standard Newtonian fluid flow. In our future work, we would like to explore the possibility of obtaining (or not) an acoustic spacetime through the process of higher order perturbation analysis of relativistic astrophysical accretion. It is to be noted that the correspondence between the analogue system and the accretion astrophysics can be established through the process of linear stability analysis of stationary integral accretion solutions. That means that only the steady state accretion has been considered. The body of literature in accretion astrophysics is huge and diverse, and hence, there are several excellent works that exist in literature where complete time-dependent numerical simulation has been performed to study the nonsteady flow of hydrodynamic fluid, including various kinds of time variabilities [31,32,91–120]. Also, in our present work, we limited our stability analysis procedure within a purely analytical framework and did not opt for any numerical studies in this aspect. There are, however, a number of works existing in the literature (for some recent works, see [121–124]) which study, fully numerically, the stability analysis of spherically or axially symmetric black hole accretion in two or three dimensions. We, however, did not concentrate on such an approach since our main motivation was to explore how the emergent gravity phenomena can be observed through the stability analysis of steady-state solutions of hydrodynamic accretion.

In the present work, we have explicitly performed the perturbation analysis to make a correspondence between the analogue gravity and the accretion astrophysics around black holes. Various properties of the corresponding analogue spacetime, however, can be studied by examining the stationary solutions as well, both for matter flow in spherically symmetric as well as for axially symmetric accretion [25,26,76,125–129].

In theoretical physics, one of the main objectives of studying the analogue gravity phenomenon is to understand the Hawking-like effects—the emission of phonons from the close vicinity of the acoustic horizon, which is considered to be analogous to the usual Hawking radiation emanating out from standard gravitational black holes.

Even though the detailed analysis of quantum Hawking-like effects may not be possible in a purely classical analogue system, the study of the acoustic surface gravity may have significant importance in such systems. The acoustic surface gravity itself is an important entity to study, as it may help to understand the flow structure as well as the acoustic spacetime. Therefore, the acoustic surface gravity may be studied independently without studying the analogue Hawking radiationlike phenomena, characterized by the analogue Hawking temperature, which may be too small to be detected experimentally in such a system. The acoustic surface gravity plays an important role in studying the non-negligible effects associated with the analogue Hawking effects, which could be examined through modified dispersion relations. Such studies have been performed in purely analytical work as well as in experimental setups [130–135].

The deviation of the Hawking-like effect in the dispersive medium depends very sensitively on the gradient of dynamical velocity as well as that of the sound speed. In most of the above-mentioned studies, the sound speed is taken to be constant, or, in other words, the flow is taken to be isothermal. Also, the velocity gradient is estimated by prescribing a particular velocity profile using certain assumptions. On the other hand, in our current work, the values of the space gradient of both the dynamical flow velocity and the speed of sound have been computed very accurately. Thus it is obvious that the non-universal feature of the Hawking like effect could be further modified by studying the black hole accretion system as an analogue gravity system. Therefore, it is obvious that though the accreting black hole system may not provide any direct signature of the Hawking-like effect, it can still be considered as a very important, as well as a unique theoretical construct, to study analogue gravity phenomena.

Lastly, one may argue that the analogue Hawking temperature may be significant in case of accretion around a primordial black hole. However, the accretion process of primordial black holes itself is an area that is not completely understood. To study accretion in such a system, one has to first construct a self-consistent model of accretion onto such primordial black holes. Such a study is clearly beyond the scope of the present work, and hence, we concentrated only on large astrophysical black holes.

ACKNOWLEDGMENTS

M. A. S. sincerely expresses his deep gratitude to Sourav Bhattacharaya for his very useful help and guidance. T. K. D. acknowledges the support from the Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata, India, in the form of a long-term visiting scientist (a one-year sabbatical visitor). The authors thank the anonymous referees for useful comments and suggestions.

APPENDIX A: ACCRETION FLOW EQUATIONS

To derive the expression for the gradient of advective velocity du_0/dr and the gradient of the sound speed dc_{s0}/dr, we use the expressions for the two conserved quantities of the flow. The mass accretion rate Ψ_0 in terms of u_0 is given by

$$\Psi_0 = 4\pi H_0 r^2 \rho_0 \frac{u_0}{\sqrt{g_{rr}(1-u_0^2)}}, \tag{A1}$$

and the relativistic Bernoulli's constant is given by

$$\xi_0 = h_0 \sqrt{\frac{g_{tt}g_{\phi\phi} + g_{\phi t}^2}{(g_{\phi\phi} + 2\lambda_0 g_{\phi t} - \lambda_0^2 g_{tt})(1-u^2)}}. \tag{A2}$$

For adiabatic flow with conserved specific entropy, in other words an isentropic flow, the enthalpy is given by $dh = dp/\rho$, which, when used in the definition of enthalpy given $h = (p+\varepsilon)/\rho$, gives $h = d\varepsilon/d\rho$. The energy density ε includes rest-mass energy ρ and an internal energy equal to $p/(\gamma-1)$. Thus, $\varepsilon = \rho + p/(\gamma-1)$. For a polytropic equation of state $p = k\rho^\gamma$, the enthalpy is therefore given by

$$h_0 = \frac{\gamma - 1}{\gamma - (1 + c_{s0}^2)}. \quad \text{(A3)}$$

To obtain an equation for the gradient of the sound speed, one defines a new quantity $\dot{\Xi}$ via the following transformation

$$\dot{\Xi} = \Psi_0 (\gamma k)^{\frac{1}{\gamma - 1}}. \quad \text{(A4)}$$

k is a measure of the specific entropy of the accreting matter, as the entropy per particle σ is related to k as

$$\sigma = \frac{1}{\gamma - 1} \log k + \frac{\gamma}{\gamma - 1} + \text{constant}. \quad \text{(A5)}$$

Thus, $\dot{\Xi}$ represents the total inward entropy flux and could be labeled as the stationary entropy accretion rate. Expressing ρ in terms of k, γ, h, c_{s0}^2, the entropy accretion rate could be written as

$$\dot{\Xi} = 4\pi H_0 \frac{u_0}{\sqrt{g_{rr}(1 - u_0^2)}} r^2 \left(\frac{(\gamma - 1) c_{s0}^2}{\gamma - (1 + c_{s0}^2)} \right)^{\frac{1}{\gamma - 1}}. \quad \text{(A6)}$$

Taking the logarithmic derivative of the above equation with respect to r, the gradient of the sound speed could be written as

$$\frac{dc_{s0}}{dr} = -\frac{c_{s0}(\gamma - (1 + c_{s0}^2))}{2} \left[\frac{1}{u_0(1 - u_0^2)} \frac{du_0}{dr} + \frac{1}{r} + \frac{1}{2} \frac{\Delta'}{\Delta} \right], \quad \text{(A7)}$$

where $\Delta = r^2 - 2r + a^2$, as given by Eq. (3), and the Δ' denotes the first derivative of Δ with respect to r. The gradient of the advective velocity could be found by taking logarithmic derivative of Eq. (A1) and Eq. (A2) (substituting $dh/h_0 = c_{s0}^2 d\rho/\rho_0$) and eliminating $d\rho/\rho_0$, which gives

$$\frac{du_0}{dr} = \frac{u_0(1 - u_0^2)[c_{s0}^2(\frac{2}{r} + \frac{\Delta'}{\Delta}) - \frac{\Delta'}{\Delta} + \frac{B'}{B}]}{2(u_0^2 - c_{s0}^2)} = \frac{N}{D}, \quad \text{(A8)}$$

where $B = (g_{\phi\phi} + 2\lambda_0 g_{\phi t} - \lambda_0^2 g_{tt})$, and B' is the first derivative of B with respect to r. The critical points of the flow are obtained by equating $D = N = 0$. $D = 0$ gives the location of critical points at $u_0^2|_{\rm crit} = c_{s0}^2|_{\rm crit}$, and $N = 0$ gives

$$u_0^2|_{r = r_{\rm crit}} = c_{s0}^2|_{r = r_{\rm crit}} = \left. \frac{\frac{\Delta'}{\Delta} - \frac{B'}{B}}{\frac{2}{r} + \frac{\Delta'}{\Delta}} \right|_{r = r_{\rm crit}}. \quad \text{(A9)}$$

Using the above condition, we can substitute u_0^2 and c_{s0}^2 in Eq. (A2) at the critical points, which provides

$$\xi_0 = \frac{1}{1 - \frac{1}{\gamma - 1} \frac{\frac{\Delta'}{\Delta} - \frac{B'}{B}}{\frac{2}{r} + \frac{\Delta'}{\Delta}}} \left. \sqrt{\frac{2\Delta + r\Delta'}{2B + rB'}} \right|_{r = r_{\rm crit}}. \quad \text{(A10)}$$

Thus, for a given value of ξ_0, which is a constant along the flow, and that of γ, λ_0 and a, the above equation could be solved for $r_{\rm crit}$ numerically, and the critical points could be found. To find the value of the gradient of the advective velocity at the critical points, we use L'Hospital rule, which gives

$$\left. \frac{du_0}{dr} \right|_{\rm crit} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\Gamma}}{2\alpha}, \quad \text{(A11)}$$

where

$$\begin{aligned}
\alpha &= 1 + \gamma - 3c_{s0}^2|_{r = r_{\rm crit}} \\
\beta &= 2c_{s0}(1 - c_{s0}^2)(\gamma - (1 + c_{s0}^2)) \left. \left(\frac{1}{r} + \frac{\Delta'}{2\Delta} \right) \right|_{r = r_{\rm crit}} \\
\Gamma &= c_{s0}^2 (1 - c_{s0}^2)^2 \left[(\gamma - (1 + c_{s0}^2)) \left(\frac{1}{r} + \frac{\Delta}{2\Delta'} \right)^2 - \Gamma^1 \right]_{r = r_{\rm crit}} \\
\Gamma^1 &= \frac{1 - c_{s0}^2}{2c_{s0}^2} \left(\frac{\Delta'^2}{\Delta^2} - \frac{\Delta''}{\Delta} \right) - \frac{1}{2c_{s0}^2} \left(\frac{B'^2}{B^2} - \frac{B''}{B} \right) - \frac{1}{r^2}.
\end{aligned} \quad \text{(A12)}$$

Δ'' and B'' are the second derivatives of Δ and B with respect to r, respectively. For a given set of parameters $[\xi_0, \gamma, \lambda_0, a]$, we can now solve Eqs. (A8) and (A7) simultaneously to obtain the Mach number as a function of the radial coordinate r. Depending on the values of the parameters $[\xi_0, \gamma, \lambda_0, a]$, the phase portrait may contain one or more critical points.

APPENDIX B: SHOCK-INVARIANT QUANTITY

h_0 is given by Eq. (A3). $c_{s0}^2 = (1/h_0) dp/d\rho = (1/h_0) k\gamma\rho^{\gamma - 1}$, which gives ρ_0 (and hence also p, and ε) in terms of k, γ, and c_{s0}. Thus,

$$\begin{aligned}
\rho &= k^{-\frac{1}{\gamma - 1}} \left[\frac{(\gamma - 1)c_{s0}^2}{\gamma(\gamma - 1 - c_{s0}^2)} \right]^{\frac{1}{\gamma - 1}} \\
p &= k^{-\frac{1}{\gamma - 1}} \left[\frac{(\gamma - 1)c_{s0}^2}{\gamma(\gamma - 1 - c_{s0}^2)} \right]^{\frac{\gamma}{\gamma - 1}} \\
\varepsilon &= k^{-\frac{1}{\gamma - 1}} \left[\frac{(\gamma - 1)c_{s0}^2}{\gamma(\gamma - 1 - c_{s0}^2)} \right]^{\frac{1}{\gamma - 1}} \left(1 + \frac{c_{s0}^2}{\gamma(\gamma - 1 - c_{s0}^2)} \right).
\end{aligned} \quad \text{(B1)}$$

Now, $\Psi_0 = \text{constant} \times r^2 \rho v_0^r$ and $T^{rr} = (p + \varepsilon)(v_0^r)^2 + pg^{rr}$, where $v_0^r = u_0/\sqrt{g_{rr}(1 - u_0^2)}$. Therefore, the shock-invariant quantity $S_{\rm sh} = T^{rr}/\Psi_0$ becomes

$$S_{\rm sh} = \frac{(u_0^2(\gamma - c_{s0}^2) + c_{s0}^2)}{u_0 \sqrt{1 - u_0^2}(\gamma - 1 - c_{s0}^2)}, \quad \text{(B2)}$$

where we have removed any overall factor of r as a shock-invariant quantity that is to be evaluated at constant $r = r_{\rm sh}$.

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