Second-order cosmological perturbations. III. Produced by scalar-scalar coupling during radiation-dominated stage

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We study the 2nd-order scalar, vector, and tensor metric perturbations in Robertson-Walker (RW) spacetime in synchronous coordinates during the radiation dominated (RD) stage. The dominant radiation is modeled by a relativistic fluid described by a stress tensor $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + g_{\mu\nu} p$ with $p = c_s^2 \rho$, and the 1st-order velocity is assumed to be curlless. We analyze the solutions of 1st-order perturbations, upon which the solutions of 2nd-order perturbation are based. We show that the 1st-order tensor modes propagate at the speed of light and are truly radiative, but the scalar and vector modes do not. The 2nd-order perturbed Einstein equation contains various couplings of 1st-order metric perturbations, and the scalar-scalar coupling is considered in this paper. We decompose the 2nd-order Einstein equation into the evolution equations of 2nd-order scalar, vector, and tensor perturbations, and the energy and momentum constraints. The coupling terms and the stress tensor of the fluid together serve as the effective source for the 2nd-order metric perturbations. The equation of covariant conservation of stress tensor is also needed to determine ρ and U^μ. By solving this set of equations up to 2nd order analytically, we obtain the 2nd-order integral solutions of all the metric perturbations, density contrast, and velocity. To use these solutions in applications, one needs to carry out seven types of the numerical integrals. We perform the residual gauge transformations between synchronous coordinates up to 2nd order and identify the gauge-invariant modes of 2nd-order solutions.

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I. INTRODUCTION

As the theoretical foundation of cosmology, cosmological perturbation has been extensively studied to linear order [1–6], and successfully applied to large-scale structures [7], cosmic microwave background radiation (CMB) [8–15] and relic gravitational wave (RGW) [16–20], etc. Nonlinear effects beyond the linear perturbation become interesting in the era of precision cosmology [21–29]. The cosmic evolution in Big Bang cosmology is very long, from inflation to the present accelerating stage, and the effects of nonlinear perturbation during expansion might accumulate substantially and be significant for cosmology. The metric perturbations in general relativity can be decomposed into three irreducible parts: scalar, vector, and tensor, giving rise to six types of couplings of metric perturbations: scalar-scalar, scalar-vector, etc., at the 2nd-order level. In a Robertson-Walker (RW) spacetime, the 2nd-order cosmological perturbation is the lowest order of nonlinear perturbations. In the literature, Tomita studied the 2nd-order perturbations in synchronous coordinates and analyzed the 2nd-order density contrast for some special cases [30]. Gravitational instability was studied in the 2nd-order perturbations in association with large-scale structure [31–33]. Ref. [34] discussed the cosmological effects of small-scale nonlinearity. In Lambda cold dark matter (ΛCDM) framework, Ref. [35] calculated 2nd-order scalar and vector perturbations in the Poisson gauge. Gauge-invariant 2nd-order perturbations were studied in Refs. [36,37], and in Ref. [38]. The 2nd-order density perturbation was studied with the squared RGW as the source in the Arnowitt-Deser-Misner (ADM) framework [39–41].

Matarrese *et al.* studied the equations of 2nd-order scalar and tensor perturbations for the scalar-scalar coupling in Einstein–de Sitter model filled with a pressureless dust [42,43], the 2nd-order vector due to scalar-scalar coupling was explored in Poisson gauge [44,45]. In Refs. [46,47] we have performed a systematical study of 2nd-order perturbation in the matter-dominated (MD) stage, and for the scalar-scalar, scalar-tensor, and tensor-tensor couplings, we have derived all the analytical solutions of 2nd-order scalar, vector, tensor metric perturbations, and of the 2nd-order density contrast.

In the RD stage, the scale factor $a(\tau)$ can increase up to $\sim 10^{24}$ times, much more than in MD stage. The 2nd-order perturbation during RD has not been analytically explored

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in literature, which motivates our study in this paper. The dominant cosmic radiation driving the RD expansion can be modeled as a relativistic fluid with a stress tensor $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + g_{\mu\nu} p$ with the pressure $p = c_s^2 \rho$ and $c_s^2 = \frac{1}{3}$, which is more complicated than the dust model [42,43,46,47], as the velocity U^μ is involved. For this, we have to solve the equation of covariant conservation $T^{\mu\nu}{}_{;\nu} = 0$ and determine U^μ, as well as ρ, up to the 2nd order. The 2nd-order perturbed Einstein equation will be decomposed into the equations of scalar, vector, and tensor, by the procedures similar to those for MD stage [46,47]; and for the case of scalar-scalar coupling, we derive the analytical solutions of 2nd-order scalar, vector, and tensor perturbations under general initial conditions. Since the solutions contain residual gauge modes in synchronous coordinates, we shall also calculate residual gauge transformations from synchronous to synchronous up to the 2nd order, and identify the gauge-invariant modes of the 2nd-order metric perturbations, the density contrast, and the velocity.

In Sec. II, we give some basic setups, including notations of metric perturbations and the relativistic fluid model. In Sec. III, we derive the analytical solution of 1st-order perturbations and the 1st-order residual gauge transformations. We analyze how the tensor modes as dynamic degrees of freedom (d.o.f.) differ from the scalar and vector modes.

In Sec. IV, for the case of scalar-scalar coupling, we decompose the 2nd-order perturbed Einstein equation into the equations of 2nd-order scalar, vector, tensor metric perturbations, and also derive the equations of the 2nd-order density contrast and velocity from the covariant conservation of $T_{\mu\nu}$.

In Sec. V, we derive the solutions of the 2nd-order metric perturbations, density contrast, and velocity, which involve time and momentum integrals.

In Sec. VI, we perform the synchronous-to-synchronous transformation, and identify the residual gauge modes in these 2nd-order solutions.

Section VII is the conclusions and discussions.

Appendix A lists the expressions of perturbed quantities used in Einstein equations, Appendix B gives the perturbed Einstein equations up to 2nd order for a general RW spacetime, and Appendix C gives the synchronous-to-synchronous gauge transformations of the metric perturbations, the density contrast, and the velocity up to 2nd order. We use a unit with the speed of light $c = 1$.

II. BASIC SETUPS

A flat Robertson-Walker metric in synchronous coordinates is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\tau)[-d\tau^2 + \gamma_{ij} dx^i dx^j], \tag{2.1}$$

where τ is the conformal time, the indices $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$, and

$$\gamma_{ij} = \delta_{ij} + \gamma_{ij}^{(1)} + \frac{1}{2}\gamma_{ij}^{(2)} \tag{2.2}$$

with $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$ being the 1st- and 2nd-order metric perturbation, respectively. Writing $g^{ij} = a^{-2}\gamma^{ij}$, one has

$$\gamma^{ij} = \delta^{ij} - \gamma^{(1)ij} - \frac{1}{2}\gamma^{(2)ij} + \gamma^{(1)ik}\gamma_k^{(1)j}. \tag{2.3}$$

Raising and lowering the 3-dim spatial indices, such as $\gamma^{(1)ik}$ and $\gamma^{(2)ik}$, will be done by δ^{ij}. We use the same notations for metric perturbations as in Refs. [43,46,47]. The 1st- and 2nd-order metric perturbation can be further written as

$$\gamma_{ij}^{(A)} = -2\phi^{(A)}\delta_{ij} + \chi_{ij}^{(A)}, \quad \text{with} \quad A = 1, 2, \tag{2.4}$$

where $\phi^{(A)}$ is the trace part of scalar perturbation, and $\chi_{ij}^{(A)}$ is traceless and can be further decomposed into the following:

$$\chi_{ij}^{(A)} = D_{ij}\chi^{\|(A)} + \chi_{ij}^{\perp(A)} + \chi_{ij}^{\top(A)}, \text{ with } A = 1, 2, \tag{2.5}$$

where $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$ and $\chi^{\|(A)}$ is a scalar function, and $D_{ij}\chi^{\|(A)}$ is the traceless part of the scalar perturbation. There are two scalar modes of metric perturbations, $\phi^{(A)}$ and $D_{ij}\chi^{\|(A)}$ of each order. The vector metric perturbation $\chi_{ij}^{\perp(A)}$ satisfies a condition $\partial^i\partial^j\chi_{ij}^{\perp(A)} = 0$ and can be written as

$$\chi_{ij}^{\perp(A)} = \partial_i B_j^{(A)} + \partial_j B_i^{(A)}, \quad \text{with} \quad \partial^i B_i^{(A)} = 0, \quad A = 1, 2, \tag{2.6}$$

where $B_i^{(A)}$ is a curl vector, and the vector metric perturbation also has two independent modes. The tensor metric perturbation $\chi_{ij}^{\top(A)}$ satisfies the traceless and transverse condition: $\chi^{\top(A)i}{}_i = 0$, $\partial^i\chi_{ij}^{\top(A)} = 0$, having two independent modes.

The RD stage of expansion is driven by a relativistic fluid (without a shear stress), whose stress tensor is $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + g_{\mu\nu}p$, where ρ and p are the energy density and pressure measured by a comoving observer in the locally inertial frame, respectively, and $U^\mu = \frac{dx^\mu}{d\lambda}$ with $d\lambda^2 = -ds^2$ is the fluid 4-velocity with a normalization condition $g_{\mu\nu}U^\mu U^\nu = -1$. The energy density ρ is expanded up to 2nd order as the following:

$$\rho = \rho^{(0)} + \rho^{(1)} + \frac{1}{2}\rho^{(2)}, \tag{2.7}$$

where $\rho^{(0)}$ is the background density, $\rho^{(1)}$ is the 1st-order density perturbation, etc. We introduce the density contrast

$$\delta^{(A)} \equiv \frac{\rho^{(A)}}{\rho^{(0)}}, \qquad A = 1, 2. \tag{2.8}$$

The pressure is also expanded up to 2nd order

$$p = p^{(0)} + p^{(1)} + \frac{1}{2} p^{(2)}. \tag{2.9}$$

For the fluid, we introduce the 0th-, 1st-, and 2nd-order sound speeds as the following:

$$c_s^2 \equiv \frac{p^{(0)\prime}}{\rho^{(0)\prime}} = \frac{p^{(0)}}{\rho^{(0)}}, \quad c_L^2 \equiv \frac{p^{(1)}}{\rho^{(1)}}, \quad c_N^2 \equiv \frac{p^{(2)}}{\rho^{(2)}}, \tag{2.10}$$

which can be taken to be $c_s^2 = \frac{1}{3}$ and $c_L^2 = \frac{1}{3}$ in applications [48], and we assume $c_N^2 = \frac{1}{3}$ for computation convenience, its actual value should be determined by future experiments. The expansion of U^μ is expanded up to 2nd order

$$U^\mu \equiv U^{(0)\mu} + U^{(1)\mu} + \frac{1}{2} U^{(2)\mu}. \tag{2.11}$$

The coordinate 3-velocity [11] $v^i \equiv \frac{dx^i}{d\tau} = \frac{U^i}{U^0}$ is expanded up to 2nd-order

$$v^i = v^{(1)i} + \frac{1}{2} v^{(2)i}. \tag{2.12}$$

By $g_{\mu\nu} U^\mu U^\nu = -1$, the component U^0 is given up to 2nd order

$$U^{(0)0} = a^{-1}, \quad U^{(1)0} = 0, \quad U^{(2)0} = a^{-1} v^{(1)k} v^{(1)}_k. \tag{2.13}$$

Here, mU^0 is the total energy of a particle, and $\frac{1}{2} m v^{(1)k} v^{(1)}_k$ is the nonrelativistic kinetic energy in the comoving coordinate in Newtonian mechanics. By $U^i = v^i U^0$, one has

$$U^{(0)i} = 0, \quad U^{(1)i} = a^{-1} v^{(1)i}, \quad U^{(2)i} = a^{-1} v^{(2)i}. \tag{2.14}$$

One then writes the covariant velocity $U_\mu = g_{\mu\nu} U^\nu$ as the following:

$$U_0 = -a\left(1 + \frac{1}{2} v^{(1)m} v^{(1)}_m\right), \tag{2.15}$$

$$U_i = a\left(v^{(1)}_i + \gamma^{(1)}_{ij} v^{(1)j} + \frac{1}{2} v^{(2)}_i\right). \tag{2.16}$$

Note that the lowest order of U_i is 1st order.

The Einstein equation is expanded up to 2nd order of perturbations

$$G^{(A)}_{\mu\nu} \equiv R^{(A)}_{\mu\nu} - \frac{1}{2}[g_{\mu\nu} R]^{(A)} = 8\pi G T^{(A)}_{\mu\nu}, \quad \text{with} \quad A = 0, 1, 2. \tag{2.17}$$

For each order of (2.17), the (00) component is the energy constraint, $(0i)$ components are the momentum constraints, and (ij) components are the evolution equations. The 0th-order Einstein equation gives the Friedman equations

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3} a^2 \rho^{(0)}, \tag{2.18}$$

$$-2\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 = 8\pi G a^2 p^{(0)}. \tag{2.19}$$

By $c_s^2 = \frac{1}{3}$ of RD stage, Eqs. (2.18) and (2.19) have a solution $a(\tau) \propto \tau$ and

$$\rho^{(0)}(\tau) = \frac{3}{8\pi G} \frac{a'^2(\tau)}{a^4(\tau)} \propto \tau^{-4}. \tag{2.20}$$

The covariant conservation of the stress tensor is

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.21}$$

The dynamics of gravitational systems is determined by (2.17) and (2.21). The component $\mu = 0$ of (2.21) gives the energy conservation

$$\begin{aligned} &g^{00} p_{,0} + \partial_0[(\rho + p) U^0 U^0] + \partial_i[(\rho + p) U^0 U^i] \\ &\quad + \Gamma^0_{00}(\rho + p) U^0 U^0 + \Gamma^0_{ij}(\rho + p) U^i U^j \\ &\quad + \Gamma^0_{00}(\rho + p) U^0 U^0 + \Gamma^k_{k0}(\rho + p) U^0 U^0 \\ &\quad + \Gamma^k_{km}(\rho + p) U^0 U^m = 0, \end{aligned} \tag{2.22}$$

and the component $\mu = i$ gives the momentum conservation

$$\begin{aligned} &g^{ik} p_{,k} + \partial_0[(\rho + p) U^i U^0] + \partial_m[(\rho + p) U^i U^m] \\ &\quad + 2\Gamma^i_{m0}(\rho + p) U^m U^0 + \Gamma^i_{ml}(\rho + p) U^m U^l \\ &\quad + \Gamma^0_{00}(\rho + p) U^i U^0 + \Gamma^k_{k0}(\rho + p) U^i U^0 \\ &\quad + \Gamma^k_{km}(\rho + p) U^i U^m = 0, \end{aligned} \tag{2.23}$$

where the nonvanishing Christopher symbols are listed in Eqs. (A1)–(A4). To each order of perturbation, (2.22) and (2.23) will determine ρ and U^μ.

III. 1ST-ORDER PERTURBATIONS

Although 1st-order perturbation in RD stage is known in the literature, here we give a detailed analysis of the equations, solutions, and gauge modes, because the 2nd-order solutions depend upon them. In addition, the 2nd-order equations and gauge transformations have similar structures to the 1st-order ones. Moreover, the 1st-order solutions will give an insight on how the tensor perturbations as dynamic d.o.f. differ from the scalar and vector perturbations.

The (00) component of 1st-order perturbed Einstein equation is

$$G^{(1)}_{00} \equiv R^{(1)}_{00} - \frac{1}{2} g^{(0)}_{00} R^{(1)} = 8\pi G T^{(1)}_{00}. \tag{3.1}$$

The expressions of the 1st-order perturbed Einstein tensors for a general RW spacetime are listed in Appendix A. By (A9) and (A19) for the RD stage, the above gives the 1st-order energy constraint

$$-\frac{6}{\tau}\phi^{(1)\prime} + 2\nabla^2\phi^{(1)} + \frac{1}{3}\nabla^2\nabla^2\chi^{\|(1)} = \frac{3}{\tau^2}\delta^{(1)}, \tag{3.2}$$

which involves only the scalar modes $\phi^{(1)}, \chi^{\|(1)}$ on the left-hand side (lhs), and $\delta^{(1)}$ on the right-hand side (rhs). The $(0i)$ component of 1st-order perturbed Einstein equation is

$$G^{(1)}_{0i} = R^{(1)}_{0i} = 8\pi G T^{(1)}_{0i}. \tag{3.3}$$

By (A6) and (A20), the above for RD stage gives the 1st-order momentum constraint

$$2\phi^{(1)\prime}_{,i} + \frac{1}{2} D_{ij}\chi^{\|(1)\prime,j} + \frac{1}{2}\chi^{\perp(1)\prime,j}_{ij} = -\frac{4}{\tau^2} v^{(1)}_i, \tag{3.4}$$

which involves both vector and scalar modes, and the velocity $v^{(1)}_i$ on the rhs. Both constraints equations (3.2) and (3.4) do not involve the tensor modes and contain only 1st-order time derivatives, indicating that they are not dynamical equations. Now decompose the 3-velocity as $v^{(1)}_i = v^{\perp(1)}_i + v^{\|(1)}_{,i}$ with $\partial^i v^{\perp(1)}_i = 0$. Applying $\nabla^{-2}\partial^i$ to (3.4) leads to the longitudinal momentum constraint

$$2\phi^{(1)\prime} + \frac{1}{3}\nabla^2\chi^{\|(1)\prime} = -\frac{4}{\tau^2} v^{\|(1)}, \tag{3.5}$$

which tells that the longitudinal velocity $v^{\|(1)}$ is related to the scalar modes. A combination [(3.4)—∂_i (3.5)] gives the transverse momentum constraint

$$\frac{1}{2}\chi^{\perp(1)\prime,j}_{ij} = -\frac{4}{\tau^2} v^{\perp(1)}_i; \tag{3.6}$$

i.e., the curl velocity $v^{\perp(1)}_i$ is only related to the 1st-order vector mode. The (ij) component of 1st-order perturbed Einstein equation is

$$G^{(1)}_{ij} \equiv R^{(1)}_{ij} - \frac{1}{2}\delta_{ij} a^2 R^{(1)} - \frac{1}{2} a^2 \gamma^{(1)}_{ij} R^{(0)} = 8\pi G T^{(1)}_{ij}, \tag{3.7}$$

which, by (A11) and (A21), gives the 1st-order evolution equation

$$\begin{aligned} &2\phi^{(1)\prime\prime}\delta_{ij} + \frac{4}{\tau}\phi^{(1)\prime}\delta_{ij} + \phi^{(1)}_{,ij} - \nabla^2\phi^{(1)}\delta_{ij} + \frac{1}{2}D_{ij}\chi^{\|(1)\prime\prime} \\ &\quad + \frac{1}{\tau}D_{ij}\chi^{\|(1)\prime} + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(1)} - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(1)} \\ &\quad + \frac{1}{2}\chi^{\perp(1)\prime\prime}_{ij} + \frac{1}{\tau}\chi^{\perp(1)\prime}_{ij} + \frac{1}{2}\chi^{\top(1)\prime\prime}_{ij} + \frac{1}{\tau}\chi^{\top(1)\prime}_{ij} \\ &\quad - \frac{1}{2}\nabla^2\chi^{\top(1)}_{ij} = \frac{3c_L^2}{\tau^2}\delta^{(1)}\delta_{ij}, \end{aligned} \tag{3.8}$$

where $\frac{1}{2}D^k_j\chi^{\|(1)}_{,ik} + \frac{1}{2}D^k_i\chi^{\|(1)}_{,jk} - \frac{1}{2}\delta_{ij}D_{kl}\chi^{\|(1),kl} = \frac{2}{3}\nabla^2 D_{ij}\chi^{\|(1)} - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(1)}$ and $\chi^{\perp(1),k}_{ki,j} + \chi^{\perp(1),k}_{kj,i} - \nabla^2\chi^{\perp(1)}_{ij} = 0$ have been used. The evolution equation (3.8) involves 2nd-order time derivatives of all three types of the metric perturbations, and gives the dynamical equations. We decompose it into the trace and traceless parts

$$\phi^{(1)\prime\prime} + \frac{2}{\tau}\phi^{(1)\prime} - \frac{1}{3}\nabla^2\phi^{(1)} - \frac{1}{18}\nabla^2\nabla^2\chi^{\|(1)} = \frac{3c_L^2}{2\tau^2}\delta^{(1)}, \tag{3.9}$$

$$\begin{aligned} &D_{ij}\phi^{(1)} + \frac{1}{2}D_{ij}\chi^{\|(1)\prime\prime} + \frac{1}{\tau}D_{ij}\chi^{\|(1)\prime} + \frac{1}{6}\nabla^2 D_{ij}\chi^{\|(1)} \\ &\quad + \frac{1}{2}\chi^{\perp(1)\prime\prime}_{ij} + \frac{1}{\tau}\chi^{\perp(1)\prime}_{ij} + \frac{1}{2}\chi^{\top(1)\prime\prime}_{ij} + \frac{1}{\tau}\chi^{\top(1)\prime}_{ij} \\ &\quad - \frac{1}{2}\nabla^2\chi^{\top(1)}_{ij} = 0. \end{aligned} \tag{3.10}$$

Applying $3\nabla^{-2}\nabla^{-2}\partial^i\partial^j$ to Eq. (3.10) gives the scalar mode equation

$$\chi^{\|(1)\prime\prime} + \frac{2}{\tau}\chi^{\|(1)\prime} + \frac{1}{3}\nabla^2\chi^{\|(1)} + 2\phi^{(1)} = 0. \tag{3.11}$$

Note that the term $\frac{1}{3}\nabla^2\chi^{\|(1)}$ in the above has a plus sign, which is in contrast to the minus sign $-\frac{1}{3}\nabla^2\phi^{(1)}$ of the trace part equation (3.9). The scalar modes described by (3.9) and (3.11) behave as a wave in RD stage, as will be revealed by their solutions. A combination ∂^i[(3.10)$-\frac{1}{2}D_{ij}$ (3.11)] yields the vector mode equation

$$\chi^{\perp(1)\prime\prime}_{ij} + \frac{2}{\tau}\chi^{\perp(1)\prime}_{ij} = 0, \tag{3.12}$$

where an irrelevant $\mathbf{x}$-independent constant has been dropped. Observe that (3.12) is not a hyperbolic type of partial differential equation, implying that the vector modes do not propagate in space and are not a wave. Besides, (3.12) tells that the 1st-order vector modes is independent of the scalar and tensor modes, but depends on the curl velocity by (3.6). A combination [(3.10) $-\frac{1}{2}D_{ij}$ (3.11)–(3.12)], yields the tensor mode equation

$$\chi_{ij}^{\top(1)''}+\frac{2}{\tau}\chi_{ij}^{\top(1)'}-\nabla^2\chi_{ij}^{\top(1)}=0, \tag{3.13}$$

which is a hyperbolic equation, indicating that the tensor modes propagate at the speed of light and are gravitational waves. Also observe that (3.13) is not related to the scalar nor the vector modes, furthermore, it is homogenous and does not depend on the perturbed stress tensor $T_{\mu\nu}^{(1)}$ of relativistic fluid. We do not consider the neutrino free-streaming during RD stage [49,50] that can give an anisotropic stress as a source of the tensor modes. Therefore, at 1st-order, RGW is a free wave propagating in the RW spacetime background. So far, the 1st-order evolution equation (3.8) has been decomposed into three equations, (3.11), (3.12), and (3.13), for the three types of metric perturbations, which are independent of each other.

To solve the equations of the scalars, we need to know $\delta^{(1)}$ and $v_i^{(1)}$, which serve as the source for the 1st-order metric perturbations. For this, we resort to the covariant conservation of stress tensor. The 1st-order part of (2.22) gives the 1st-order equation of energy conservation (see also (4.24) in Ref. [11])

$$\rho^{(1)'}+\frac{3}{\tau}(\rho^{(1)}+p^{(1)})-3\phi^{(1)'}(\rho^{(0)}+p^{(0)}) \\ +\partial_i[(\rho^{(0)}+p^{(0)})v^{(1)i}]=0, \tag{3.14}$$

i.e., the continuity equation for a fluid. By (2.8), the above is written in terms of the 1st-order density contrast

$$\delta^{(1)'}+\frac{3}{\tau}\left(c_L^2-\frac{1}{3}\right)\delta^{(1)}=4\phi^{(1)'}-\frac{4}{3}\nabla^2 v^{\parallel(1)}, \tag{3.15}$$

which contains the scalar $\phi^{(1)}$ and the longitudinal velocity $v^{\parallel(1)}$. The sound speed c_L appears in the above 1st-order equation. Similarly, the 1st order of (2.23) gives the 1st-order momentum conservation equation (see also (4.25) in Ref. [11])

$$c_L^2\rho^{(0)}\delta_{,i}^{(1)}+\frac{4}{3}\rho^{(0)'}v_i^{(1)}+\frac{4}{3}\rho^{(0)}v_i^{(1)'}+\frac{16}{3\tau}\rho^{(0)}v_i^{(1)}=0, \tag{3.16}$$

which is also written as

$$c_L^2\delta_{,i}^{(1)}+\frac{4}{3}v_i^{(1)'}=0, \tag{3.17}$$

i.e., the Euler equation for a fluid. The vector equation (3.17) can be decomposed into longitudinal and transverse parts

$$c_L^2\delta^{(1)}+\frac{4}{3}v^{\parallel(1)'}=0, \tag{3.18}$$

$$v_i^{\perp(1)'}=0. \tag{3.19}$$

We find that the trace of evolution equation (3.9) can be given as the following combination

$$(3.9)=-\frac{1}{6}(3.2)-\frac{\tau}{6}\frac{d}{d\tau}(3.2)+\frac{\tau}{6}\nabla^2(3.5)-\frac{1}{2\tau}(3.15), \tag{3.20}$$

and so is the scalar part of the traceless evolution equation (3.11) as the following:

$$(3.11)=\nabla^{-2}\bigg[(3.2)+\tau\frac{d}{d\tau}(3.2)-\tau\nabla^2(3.5)+3\frac{d}{d\tau}(3.5) \\ +\frac{6}{\tau}(3.5)+\frac{3}{\tau}(3.15)-\frac{9}{\tau^2}(3.18)\bigg]. \tag{3.21}$$

(This also occurs for the 2nd-order scalar perturbations.) This means that we can use the equations of constraints and conservations to solve the scalars, and the solutions will satisfy the evolution equations automatically.

Now we solve for the 1st-order perturbations. First, the 1st-order tensor $\chi_{ij}^{\top(1)}$ in (3.13) is written in terms of Fourier modes

$$\chi_{ij}^{\top(1)}(\mathbf{x},\tau)=\frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{s=+,\times}\overset{s}{\epsilon}_{ij}(k)\overset{s}{h}_k(\tau), \\ \mathbf{k}=k\hat{k}, \tag{3.22}$$

with two polarization tensors satisfying

$$\overset{s}{\epsilon}_{ij}(k)\delta^{ij}=0,\qquad \overset{s}{\epsilon}_{ij}(k)k^i=0,\qquad \overset{s}{\epsilon}_{ij}(k)\overset{s'}{\epsilon}{}^{ij}(k)=2\delta_{ss'}.$$

For RGW generated during inflation [16,18,19], the two polarization modes $\overset{s}{h}_k(\tau)$ with $s=+,\times$ are usually assumed to be statistically equivalent and the superscript s can be dropped. During RD stage the mode is given by

$$h_k(\tau)=\frac{1}{a(\tau)}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\tau}{2}}\left[b_1H_{\frac{1}{2}}^{(1)}(k\tau)+b_2H_{\frac{1}{2}}^{(2)}(k\tau)\right] \\ =\frac{1}{a(\tau)}\frac{i}{\sqrt{2k}}[-b_1e^{ik\tau}+b_2e^{-ik\tau}], \tag{3.23}$$

where b_1, b_2 are $\mathbf{k}$-dependent coefficients, to be determine by the initial condition during inflation, or a possible subsequent reheating stage [16,19]. There are cosmic processes occurred during RD stage, such QCD transition, and e^+e^- annihilation [51], which modify only slightly the amplitude of RGW and will be neglected in this study.

Next, the vector mode equation (3.12) has a general solution

$$\chi_{ij}^{\perp(1)} = \frac{C_{1ij}(\mathbf{x})}{\tau} + C_{2ij}(\mathbf{x}), \tag{3.24}$$

where C_{1ij} and C_{2ij} are arbitrary, time-independent functions, and can be written in a form $C_{1ij} = \partial_j C_{1i}^{\perp} + \partial_i C_{1j}^{\perp}$ with $C_{1i}^{\perp}$ being a curl vector satisfying $\partial^i C_{1i}^{\perp} = 0$, and similar for C_{2ij}. Substituting (3.24) into the transverse momentum constraint (3.6) yields the solution of transverse velocity

$$v_i^{\perp(1)} = \frac{1}{8} C_{1ij}^{,j}(\mathbf{x}), \tag{3.25}$$

which is time independent, consistent with the transverse momentum conservation (3.19). Neither vector nor tensor mode involves the 1st-order sound speed c_L.

Then we solve for the scalars. By the longitudinal momentum conservation (3.18),

$$\delta^{(1)} = -\frac{4}{3c_L^2} v^{\parallel(1)\prime}, \tag{3.26}$$

plugging it into the energy momentum conservation (3.15) gives the following:

$$\phi^{(1)\prime} = -\frac{1}{3c_L^2} v^{\parallel(1)\prime\prime} - \frac{1}{\tau}\frac{(c_L^2 - \frac{1}{3})}{c_L^2} v^{\parallel(1)\prime} + \frac{1}{3}\nabla^2 v^{\parallel(1)}. \tag{3.27}$$

Taking $\frac{d}{d\tau}$ on the energy constraint (3.2) and combing with the momentum constraint (3.5) and canceling the scalar $\chi^{\parallel(1)}$ lead to

$$-6\frac{d}{d\tau}\left[\frac{\phi^{(1)\prime}}{\tau}\right] + \nabla^2\left[-\frac{4}{\tau^2} v^{\parallel(1)}\right] = 3\frac{d}{d\tau}\left[\frac{1}{\tau^2}\delta^{(1)}\right]. \tag{3.28}$$

Plugging $\delta^{(1)}$ of (3.26) and $\phi^{(1)\prime}$ of (3.27) into (3.28), one arrives at a 3rd-order differential equation of longitudinal velocity

$$v^{\parallel(1)\prime\prime\prime} + \frac{3c_L^2}{\tau} v^{\parallel(1)\prime\prime} - \frac{6c_L^2+2}{\tau^2} v^{\parallel(1)\prime} - c_L^2 \nabla^2 v^{\parallel(1)\prime} - \frac{c_L^2}{\tau}\nabla^2 v^{\parallel(1)} = 0, \tag{3.29}$$

which can be solved directly. By the Fourier transformation

$$v^{\parallel(1)}(\mathbf{x},\tau) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} v_{\mathbf{k}}^{\parallel(1)}, \tag{3.30}$$

(3.29) written in the $\mathbf{k}$-space is

$$v_{\mathbf{k}}^{\parallel(1)\prime\prime\prime} + \frac{3c_L^2}{\tau} v_{\mathbf{k}}^{\parallel(1)\prime\prime} + \left(c_L^2 k^2 - \frac{6c_L^2+2}{\tau^2}\right) v_{\mathbf{k}}^{\parallel(1)\prime} + \frac{c_L^2 k^2}{\tau} v_{\mathbf{k}}^{\parallel(1)} = 0, \tag{3.31}$$

which has a general solution

$$v_{\mathbf{k}}^{\parallel(1)}(\tau) = d_1\left(\frac{c_L k\tau}{2}\right)^{-\frac{3c_L^2}{2}} \Gamma\left(\frac{3c_L^2}{2}+1\right)\left(J_{\frac{3c_L^2}{2}}(c_L k\tau) + (c_L k\tau) J_{\frac{3c_L^2}{2}+1}(c_L k\tau)\right) + d_2\left(\frac{c_L k\tau}{2}\right)^3 {}_1F_2\left(2;\frac{5}{2},\frac{3c_L^2}{2}+\frac{5}{2};-\left(\frac{c_L k\tau}{2}\right)^2\right) + d_3\left(\frac{c_L k\tau}{2}\right)^{-3c_L^2} {}_1F_2\left(\frac{1}{2}-\frac{3c_L^2}{2};-\frac{3c_L^2}{2}-\frac{1}{2},1-\frac{3c_L^2}{2};-\left(\frac{c_L k\tau}{2}\right)^2\right), \tag{3.32}$$

where d_1, d_2, d_3 are $\mathbf{k}$-dependent integration constants, $J_q(x)$ is the Bessel function, and $\Gamma(x)$ is the Gamma function, and ${}_pF_q$ is the generalized hypergeometric function. For the case of the sound speed $c_L^2 = \frac{1}{3}$ [6], the solution (3.32) reduces to

$$v_{\mathbf{k}}^{\parallel(1)} = \frac{D_1}{k\tau} + D_2\left(\frac{2}{k\tau} + \frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + D_3\left(\frac{2}{k\tau} - \frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}, \tag{3.33}$$

with D_1, D_2, and D_3 being $\mathbf{k}$-dependent constants with $D_1 = 3\sqrt{3}d_2 + 2\sqrt{3}d_3$, $D_2 = -\frac{3\sqrt{3}}{4}d_2 + \frac{i}{2}\sqrt{3}d_1$, and $D_3 = -\frac{3\sqrt{3}}{4}d_2 + \frac{i}{2}\sqrt{3}d_1$. Given the solution $v^{\parallel(1)}$, other 1st-order scalar perturbations follow straight forwardly. The Fourier transform of (3.26) gives the 1st-order density contrast

$$\begin{aligned}\delta^{(1)}_{\mathbf{k}} &= -4v^{\parallel(1)\prime}_{\mathbf{k}}\\ &= \frac{4D_1}{k\tau^2} + D_2\left(\frac{8}{k\tau^2} + \frac{8i}{\sqrt{3}\tau} - \frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}\\ &\quad + D_3\left(\frac{8}{k\tau^2} - \frac{8i}{\sqrt{3}\tau} - \frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}. \end{aligned} \tag{3.34}$$

Similarly, time integration of the k-mode of (3.27) yields

$$\begin{aligned}\phi^{(1)}_{\mathbf{k}} &= -v^{\parallel(1)\prime}_{\mathbf{k}} - \frac{1}{3}k^2\int^{\tau} v^{\parallel(1)}_{\mathbf{k}} d\tau + D_4\\ &= D_1\left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right) + D_2\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}}\\ &\quad + D_3\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}}\\ &\quad - \frac{2k}{3}\int^{\tau}\left[D_2 e^{-ik\tau'/\sqrt{3}} + D_3 e^{ik\tau'/\sqrt{3}}\right]\frac{d\tau'}{\tau'} + D_4, \end{aligned} \tag{3.35}$$

where D_4 is a $\mathbf{k}$-dependent constant. Equation (3.2) in the $\mathbf{k}$-space gives

$$\begin{aligned}\chi^{\parallel(1)}_{\mathbf{k}} &= \frac{9}{k^4\tau^2}\delta^{(1)}_{\mathbf{k}} + \frac{18}{k^4\tau}\phi^{(1)\prime}_{\mathbf{k}} + \frac{6}{k^2}\phi^{(1)}_{\mathbf{k}}\\ &= -D_1\frac{2\ln\tau}{k} + D_2\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - D_3\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}}\\ &\quad + \frac{6D_4}{k^2} - \frac{4}{k}\int^{\tau}\left[D_2 e^{-ik\tau'/\sqrt{3}} + D_3 e^{ik\tau'/\sqrt{3}}\right]\frac{d\tau'}{\tau'}. \end{aligned} \tag{3.36}$$

We have checked that these solutions satisfy the scalar parts of the evolution equation, (3.9) and (3.11). The constants D_1, D_2, D_3, D_4 should be determined by the initial conditions, say, at the end of inflation.

The 1st-order solutions of (3.24) and (3.33)–(3.36) contain coordinate-dependent gauge terms, which need to be identified. [See (C16)–(C19), (C21), and (C24) in Appendix C and also Ref. [46] for the synchronous-synchronous transformation in an RW spacetime.] For RD stage with $a(\tau)\propto\tau$, one has the residual gauge transform of the metric perturbations as the following:

$$\bar{\phi}^{(1)} = \phi^{(1)} + \frac{1}{\tau^2}A^{(1)} + \frac{\nabla^2 A^{(1)}}{3}\ln\tau + \frac{1}{3}\nabla^2 C^{\parallel(1)}, \tag{3.37}$$

$$\bar{\chi}^{\parallel(1)} = \chi^{\parallel(1)} - 2A^{(1)}\ln\tau - 2C^{\parallel(1)}, \tag{3.38}$$

$$\bar{\chi}^{\perp(1)}_{ij} = \chi^{\perp(1)}_{ij} - C^{\perp(1)}_{i,j} - C^{\perp(1)}_{j,i}, \tag{3.39}$$

$$\bar{\chi}^{\top(1)}_{ij} = \chi^{\top(1)}_{ij}, \tag{3.40}$$

where $A^{(1)}$, $C^{\parallel(1),i}$, and $C^{\perp(1)i}$ with $\partial_i C^{\perp(1)i} = 0$ are small $\mathbf{x}$-dependent functions that describe the transformation. The transformation of 1st-order density contrast and 3-velocity are

$$\bar{\delta}^{(1)} = \delta^{(1)} + \frac{4}{\tau^2}A^{(1)}, \tag{3.41}$$

$$\bar{v}^{\parallel(1)} = v^{\parallel(1)} + \frac{A^{(1)}}{\tau}, \tag{3.42}$$

$$\bar{v}^{\perp(1)i} = v^{\perp(1)i}. \tag{3.43}$$

Equation (3.43) states that the transverse velocity is gauge invariant. In the $\mathbf{k}$-space these are

$$\bar{\phi}^{(1)}_{\mathbf{k}} = \phi^{(1)}_{\mathbf{k}} + A^{(1)}_{\mathbf{k}}\left(\frac{1}{\tau^2} - \frac{k^2}{3}\ln\tau\right) - \frac{k^2}{3}C^{\parallel(1)}_{\mathbf{k}}, \tag{3.44}$$

$$\bar{\chi}^{\parallel(1)}_{\mathbf{k}} = \chi^{\parallel(1)}_{\mathbf{k}} - 2A^{(1)}_{\mathbf{k}}\ln\tau - 2C^{\parallel(1)}_{\mathbf{k}}, \tag{3.45}$$

$$\bar{\chi}^{\perp(1)}_{\mathbf{k}ij} = \chi^{\perp(1)}_{\mathbf{k}ij} - C^{\perp(1)}_{\mathbf{k}i,j} - C^{\perp(1)}_{\mathbf{k}j,i}, \tag{3.46}$$

$$\bar{\chi}^{\top(1)}_{\mathbf{k}ij} = \chi^{\top(1)}_{\mathbf{k}ij}, \tag{3.47}$$

$$\bar{\delta}^{(1)}_{\mathbf{k}} = \delta^{(1)}_{\mathbf{k}} + \frac{4}{\tau^2}A^{(1)}_{\mathbf{k}}, \tag{3.48}$$

$$\bar{v}^{\parallel(1)}_{\mathbf{k}} = v^{\parallel(1)}_{\mathbf{k}} + \frac{A^{(1)}_{\mathbf{k}}}{\tau}, \tag{3.49}$$

$$\bar{v}^{\perp(1)i}_{\mathbf{k}} = v^{\perp(1)i}_{\mathbf{k}}. \tag{3.50}$$

By these we can identify the gauge modes in the 1st-order solutions. First, the 1st-order tensor modes are gauge invariant by (3.40). Next, (3.39) tells that the constant term C_{2ij} in the vector solution (3.24) is a gauge term, which can be removed by choosing the gauge transformation function $C^{\perp(1)i}$ to satisfy $C^{\perp(1)}_{i,j} + C^{\perp(1)}_{j,i} = C_{2ij}$, so that the gauge-invariant modes in the vector solution (3.24) are

$$\chi^{\perp(1)}_{ij} = \frac{C_{1ij}(\mathbf{x})}{\tau}. \tag{3.51}$$

By the relation (2.14) and the solution (3.25), the transverse part of 1st-order velocity U^i is gauge invariant, given by

$$a^{-1}v^{\perp(1)i} = \frac{C^{ij}_{1,j}(\mathbf{x})}{8\tau}. \tag{3.52}$$

Both (3.51) and (3.52) decay as τ^{-1} and can be neglected as an approximation; i.e., we assume that the relativistic fluid is irrotational during RD stage, and set

$$v_i^{\perp(1)} = \chi_{ij}^{\perp(1)} = C_{1ij} = 0. \tag{3.53}$$

This approximation will simplify calculation of 2nd-order perturbations.

The transformations (3.44) and (3.45) state that D_1 and D_4 terms in the scalar solutions (3.35) and (3.36) are gauge terms and can be removed by choosing

$$A_{\mathbf{k}}^{(1)} = -\frac{D_1}{k}, \tag{3.54}$$

$$C_{\mathbf{k}}^{\parallel(1)} = \frac{3D_4}{k^2}, \tag{3.55}$$

so that the gauge-invariant modes of two scalars are

$$\phi_{\mathbf{k}}^{(1)} = D_2\left(\frac{2}{k\tau^2}+\frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}} + D_3\left(\frac{2}{k\tau^2}-\frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}} - \frac{2k}{3}\int^\tau \left[D_2 e^{-ik\tau'/\sqrt{3}} + D_3 e^{ik\tau'/\sqrt{3}}\right]\frac{d\tau'}{\tau'}, \tag{3.56}$$

$$\chi_{\mathbf{k}}^{\parallel(1)} = D_2\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - D_3\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} - \frac{4}{k}\int^\tau \left[D_2 e^{-ik\tau'/\sqrt{3}} + D_3 e^{ik\tau'/\sqrt{3}}\right]\frac{d\tau'}{\tau'}. \tag{3.57}$$

By the transformation (3.48), the D_1 term in the density contrast solution (3.34) is a gauge term and is removed by choosing the same $A_{\mathbf{k}}^{(1)}$ as (3.54), so that the gauge-invariant mode of density contrast is

$$\delta_{\mathbf{k}}^{(1)} = D_2\left(\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}} + D_3\left(\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}. \tag{3.58}$$

Finally, (3.49) shows that the D_1 term in the velocity (3.33) is a gauge term, and the gauge-invariant mode of velocity is

$$v_{\mathbf{k}}^{\parallel(1)} = D_2\left(\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + D_3\left(\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}. \tag{3.59}$$

It is revealing to compare the behaviors of the 1st-order perturbations in RD stage. The scalar modes (3.56) and (3.57), the density contrast (3.58), and the longitudinal velocity (3.59) all contain a factor $e^{\pm ik\tau/\sqrt{3}}$, so that they are waves propagating at the sound speed $\frac{1}{\sqrt{3}}$ of the relativistic fluid. On the other hand, as mentioned below (3.12), the vector modes (3.51) and the transverse velocity (3.52) are not a wave and do not propagate, they simply decrease with time. In contrast, the tensor modes (3.23) are waves and propagate at the speed of light. It is also interesting to compare with the MD stage [43,46,47], in which the scalar, vector, and density contrast are not a wave and do not propagate, only the tensor modes propagate at the speed of light. Therefore, the tensor modes always propagate at the speed of light, regardless of the spacetime background, in contrast to the scalar and vector modes.

So far, we have obtained the gauge-invariant 1st-order solutions. As we shall see in the next section, the couplings (products) of the 1st-order solutions will appear in the 2nd-order perturbed Einstein equation, and constitute a part of effective source for the 2nd-order metric perturbations.

IV. 2ND-ORDER PERTURBED EQUATIONS BY SCALAR-SCALAR COUPLINGS

To study the 2nd-order cosmological perturbations, we need the 2nd-order perturbed Einstein equation, which are listed in Appendix B for a general RW spacetime. Since the vector mode and the curl velocity are neglected in (3.53) as an approximation, there remain only three types of products of 1st-order perturbations: scalar-scalar, scalar-tensor, and tensor-tensor. In this paper we consider the scalar-scalar coupling.

The (00) component of 2nd-order perturbed Einstein equation is given by Eq. (B8), which for RD stage is written as

$$-\frac{6}{\tau}\phi_S^{(2)\prime} + 2\nabla^2\phi_S^{(2)} + \frac{1}{3}\nabla^2\nabla^2\chi_S^{\parallel(2)} = \frac{3}{\tau^2}\delta_S^{(2)} + E_S, \tag{4.1}$$

where a subscript "S" in $\phi_S^{(2)}$, etc., indicates the scalar-scalar coupling; $\delta_S^{(2)}$ is the 2nd-order density contrast; and

$$\begin{aligned}E_S = {}& \frac{8}{\tau^2}v^{\parallel(1),k}v^{\parallel(1)}_{,k} + \frac{24}{\tau}\phi^{(1)\prime}\phi^{(1)} - 6\phi^{(1)\prime}\phi^{(1)\prime} - 6\phi^{(1)}_{,k}\phi^{(1),k} - 16\phi^{(1)}\nabla^2\phi^{(1)} - \frac{8}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)} + 2\phi^{(1),kl}\chi^{\parallel(1)}_{,kl} \\ & - \frac{2}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\parallel(1)} + \frac{1}{4}\chi^{\parallel(1)\prime,kl}\chi^{\parallel(1)\prime}_{,kl} - \frac{1}{12}\nabla^2\chi^{\parallel(1)\prime}\nabla^2\chi^{\parallel(1)\prime} + \frac{2}{\tau}\chi^{\parallel(1),kl}\chi^{\parallel(1)\prime}_{,kl} - \frac{2}{3\tau}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime} + \frac{1}{3}\chi^{\parallel(1),kl}\nabla^2\chi^{\parallel(1)}_{,kl} \\ & - \frac{1}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)} - \frac{1}{4}\chi^{\parallel(1),klm}\chi^{\parallel(1)}_{,klm} + \frac{5}{12}\nabla^2\chi^{\parallel(1)}_{,k}\nabla^2\chi^{\parallel(1),k}\end{aligned} \tag{4.2}$$

which consists the coupling terms of 1st-order scalar metric perturbations, as well as the longitudinal velocity $v^{\parallel(1)}$ which is absent in the dust model as for the MD stage [43,46]. Equation (4.1) is formally similar to the 1st-order equation (3.2), except for E_S as a part of the effective source on the rhs.

The $(0i)$ component of the 2nd-order perturbed Einstein equation (B12) for RD stage is given by

$$2\phi^{(2)\prime}_{S,i} + \frac{1}{2} D_{ij}\chi_S^{\parallel(2)\prime,j} + \frac{1}{2}\chi^{\perp(2)\prime,j}_{Sij} = -\frac{4}{\tau^2} v^{(2)}_{Si} + M_{Si}, \tag{4.3}$$

where

$$\begin{aligned} M_{Si} \equiv & \frac{16}{\tau^2}\phi^{(1)} v^{\parallel(1)}_{,i} - \frac{8}{\tau^2} v^{\parallel(1),k}\chi^{\parallel(1)}_{,ki} + \frac{8}{3\tau^2} v^{\parallel(1)}_{,i}\nabla^2\chi^{\parallel(1)} - \frac{6}{\tau^2}(1+c_L^2)\delta^{(1)} v^{\parallel(1)}_{,i} - 8\phi^{(1)\prime}\phi^{(1)}_{,i} - 8\phi^{(1)}\phi^{(1)\prime}_{,i} - \frac{4}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)\prime}_{,i} \\ & - \frac{4}{3}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)}_{,i} + \phi^{(1),k}\chi^{\parallel(1)\prime}_{,ki} - \frac{1}{3}\phi^{(1)}_{,i}\nabla^2\chi^{\parallel(1)\prime} - 2\phi^{(1)\prime,k}\chi^{\parallel(1)}_{,ki} + \frac{2}{3}\phi^{(1)\prime}_{,i}\nabla^2\chi^{\parallel(1)} + \frac{2}{3}\chi^{\parallel(1)\prime}_{,ki}\nabla^2\chi^{\parallel(1),k} \\ & - \frac{1}{18}\nabla^2\chi^{\parallel(1)}_{,i}\nabla^2\chi^{\parallel(1)\prime} - \frac{1}{3}\chi^{\parallel(1)}_{,ki}\nabla^2\chi^{\parallel(1)\prime,k} + \frac{1}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime}_{,i} - \frac{1}{2}\chi^{\parallel(1)\prime,kl}\chi^{\parallel(1)}_{,kli}. \end{aligned} \tag{4.4}$$

Equation (4.3) is similar to the 1st-order equation (3.4), except for M_{Si} in the effective source, which contains the 1st-order pressure perturbation $c_L^2\delta^{(1)} \propto p^{(1)}$ and the velocity $v^{\parallel(1)}$ from $T^{(2)}_{0i}$, as well as the scalar-scalar couplings. Equation (4.3) can be further decomposed into two parts, similar to the 1st-order case in Section III. Writing the 3-velocity as $v^{(2)}_i = v^{\perp(2)}_i + v^{\parallel(2)}_{,i}$ with $\partial^i v^{\perp(2)}_i = 0$ and applying $\nabla^{-2}\partial^i$ on (4.3) lead to the longitudinal part of momentum constraint

$$2\phi^{(2)\prime}_S + \frac{1}{3}\nabla^2\chi^{\parallel(2)\prime}_S = -\frac{4}{\tau^2} v^{\parallel(2)}_S + \nabla^{-2}M^{,l}_{Sl}, \tag{4.5}$$

where

$$\begin{aligned} \nabla^{-2}M^{,l}_{Sl} = & -8\phi^{(1)\prime}\phi^{(1)} - \frac{1}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)\prime} - \frac{4}{3}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)} - \frac{1}{18}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime} - \frac{1}{6}\chi^{\parallel(1)}_{,k}\nabla^2\chi^{\parallel(1)\prime,k} \\ & + \nabla^{-2}\Big[\frac{16}{\tau^2}\phi^{(1),k} v^{\parallel(1)}_{,k} + \frac{16}{\tau^2}\phi^{(1)}\nabla^2 v^{\parallel(1)} - \frac{8}{\tau^2} v^{\parallel(1),kl}\chi^{\parallel(1)}_{,kl} - \frac{16}{3\tau^2} v^{\parallel(1),k}\nabla^2\chi^{\parallel(1)}_{,k} + \frac{8}{3\tau^2}\nabla^2 v^{\parallel(1)}\nabla^2\chi^{\parallel(1)} \\ & - \frac{6}{\tau^2}(1+c_L^2)\delta^{(1)}\nabla^2 v^{\parallel(1)} - \frac{6}{\tau^2}(1+c_L^2)\delta^{(1)}_{,k} v^{\parallel(1),k} + \phi^{(1),kl}\chi^{\parallel(1)\prime}_{,kl} - \phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)\prime} + 2\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)} \\ & - 2\phi^{(1)\prime,kl}\chi^{\parallel(1)}_{,kl} + \frac{1}{6}\chi^{\parallel(1)}_{,k}\nabla^2\nabla^2\chi^{\parallel(1)\prime,k} + \frac{1}{6}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)\prime} \\ & + \frac{1}{6}\chi^{\parallel(1)\prime}_{,kl}\nabla^2\chi^{\parallel(1),kl} + \frac{2}{3}\nabla^2\chi^{\parallel(1)\prime}_{,k}\nabla^2\chi^{\parallel(1),k} - \frac{1}{2}\chi^{\parallel(1)\prime,klm}\chi^{\parallel(1)}_{,klm}\Big]. \end{aligned} \tag{4.6}$$

A combination [(4.3)-∂_i (4.5)] gives the transverse part of momentum constraint

$$\frac{1}{2}\chi^{\perp(2)\prime,j}_{Sij} = -\frac{4}{\tau^2} v^{\perp(2)}_{Si} + (M_{Si} - \partial_i\nabla^{-2}M^{,l}_{Sl}), \tag{4.7}$$

where

$$
\begin{aligned}
(M_{Si}-\partial_i\nabla^{-2}M^{,l}_{Sl}) ={}& \frac{16}{\tau^2}\phi^{(1)}v^{\parallel(1)}_{,i}-\frac{8}{\tau^2}v^{\parallel(1),k}\chi^{\parallel(1)}_{,ki}+\frac{8}{3\tau^2}v^{\parallel(1)}_{,i}\nabla^2\chi^{\parallel(1)}-\frac{6}{\tau^2}(1+c_L^2)\delta^{(1)}v^{\parallel(1)}_{,i}-\phi^{(1)}\nabla^2\chi^{\parallel(1)\prime}_{,i}+2\phi^{(1)\prime}_{,i}\nabla^2\chi^{\parallel(1)}\\
&+\phi^{(1),k}\chi^{\parallel(1)\prime}_{,ki}-2\phi^{(1)\prime,k}\chi^{\parallel(1)}_{,ki}+\frac{2}{3}\chi^{\parallel(1)\prime}_{,ki}\nabla^2\chi^{\parallel(1),k}-\frac{1}{6}\chi^{\parallel(1)}_{,ki}\nabla^2\chi^{\parallel(1)\prime,k}+\frac{1}{6}\chi^{\parallel(1)}_{,k}\nabla^2\chi^{\parallel(1)\prime,k}_{,i}\\
&+\frac{1}{6}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime}_{,i}-\frac{1}{2}\chi^{\parallel(1)\prime,kl}\chi^{\parallel(1)}_{,kli}+\partial_i\nabla^{-2}\bigg[-\frac{16}{\tau^2}\phi^{(1),k}v^{\parallel(1)}_{,k}-\frac{16}{\tau^2}\phi^{(1)}\nabla^2v^{\parallel(1)}+\frac{8}{\tau^2}v^{\parallel(1),kl}\chi^{\parallel(1)}_{,kl}\\
&+\frac{16}{3\tau^2}v^{\parallel(1),k}\nabla^2\chi^{\parallel(1)}_{,k}-\frac{8}{3\tau^2}\nabla^2v^{\parallel(1)}\nabla^2\chi^{\parallel(1)}+\frac{6}{\tau^2}(1+c_L^2)\delta^{(1)}\nabla^2v^{\parallel(1)}+\frac{6}{\tau^2}(1+c_L^2)\delta^{(1)}_{,k}v^{\parallel(1),k}\\
&-\phi^{(1),kl}\chi^{\parallel(1)\prime}_{,kl}+\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)\prime}-2\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)}+2\phi^{(1)\prime,kl}\chi^{\parallel(1)}_{,kl}-\frac{1}{6}\chi^{\parallel(1)}_{,k}\nabla^2\nabla^2\chi^{\parallel(1)\prime,k}\\
&-\frac{1}{6}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)\prime}-\frac{1}{6}\chi^{\parallel(1)\prime}_{,kl}\nabla^2\chi^{\parallel(1),kl}-\frac{2}{3}\nabla^2\chi^{\parallel(1)\prime}_{,k}\nabla^2\chi^{\parallel(1),k}+\frac{1}{2}\chi^{\parallel(1)\prime,klm}\chi^{\parallel(1)}_{,klm}\bigg].
\end{aligned}
\tag{4.8}
$$

Equations (4.5) and (4.7) are similar to the 1st-order equations (3.5) and (3.6), respectively.

The (ij) component of 2nd-order perturbed Einstein equation (B20) for the RD stage is given by

$$
\begin{aligned}
&2\phi^{(2)\prime\prime}_S\delta_{ij}+\frac{4}{\tau}\phi^{(2)\prime}_S\delta_{ij}+\phi^{(2)}_{S,ij}-\nabla^2\phi^{(2)}_S\delta_{ij}+\frac{1}{2}D_{ij}\chi^{\parallel(2)\prime\prime}_S+\frac{1}{\tau}D_{ij}\chi^{\parallel(2)\prime}_S+\frac{1}{6}\nabla^2D_{ij}\chi^{\parallel(2)}_S-\frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\parallel(2)}_S\\
&+\frac{1}{2}\chi^{\perp(2)\prime\prime}_{Sij}+\frac{1}{\tau}\chi^{\perp(2)\prime}_{Sij}+\frac{1}{2}\chi^{\top(2)\prime\prime}_{Sij}+\frac{1}{\tau}\chi^{\top(2)\prime}_{Sij}-\frac{1}{2}\nabla^2\chi^{\top(2)}_{Sij}=\frac{3c_N^2}{\tau^2}\delta^{(2)}_S\delta_{ij}+S_{Sij}.
\end{aligned}
\tag{4.9}
$$

The 2nd-order pressure perturbation $c_N^2\delta_S^{(2)}\propto p^{(2)}$ appears on the rhs. This 2nd-order evolution equation is similar to the 1st-order (3.8), except for the extra part in the effective source

$$
\begin{aligned}
S_{Sij} ={}& \frac{8}{\tau^2}v^{\parallel(1)}_{,i}v^{\parallel(1)}_{,j}-\frac{12}{\tau^2}c_L^2\delta^{(1)}\phi^{(1)}\delta_{ij}+\frac{6}{\tau^2}c_L^2\delta^{(1)}\chi^{\parallel(1)}_{,ij}-\frac{2}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij}-6\phi^{(1)}_{,i}\phi^{(1)}_{,j}\\
&+4\phi^{(1),k}\phi^{(1)}_{,k}\delta_{ij}+4\phi^{(1)}\nabla^2\phi^{(1)}\delta_{ij}-4\phi^{(1)}\phi^{(1)}_{,ij}-2\phi^{(1)\prime}\phi^{(1)\prime}\delta_{ij}-\frac{12}{\tau}\phi^{(1)\prime}\chi^{\parallel(1)}_{,ij}\\
&+\frac{4}{\tau}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)}\delta_{ij}-\phi^{(1)\prime}\chi^{\parallel(1)\prime}_{,ij}+\frac{1}{3}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)\prime}\delta_{ij}-6\phi^{(1)\prime\prime}\chi^{\parallel(1)}_{,ij}+2\phi^{(1)\prime\prime}\nabla^2\chi^{\parallel(1)}\delta_{ij}\\
&-\phi^{(1)}_{,j}\nabla^2\chi^{\parallel(1)}_{,i}-\phi^{(1)}_{,i}\nabla^2\chi^{\parallel(1)}_{,j}+\phi^{(1)}_{,k}\chi^{\parallel(1),k}_{,ij}-\frac{2}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)}_{,ij}+\frac{1}{3}\phi^{(1)}_{,k}\nabla^2\chi^{\parallel(1),k}\delta_{ij}\\
&+\frac{2}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)}\delta_{ij}+4\chi^{\parallel(1)}_{,ij}\nabla^2\phi^{(1)}-\frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij}-2\phi^{(1)}_{,ki}\chi^{\parallel(1),k}_{,j}\\
&-2\phi^{(1)}_{,kj}\chi^{\parallel(1),k}_{,i}+\frac{4}{3}\phi^{(1)}_{,ij}\nabla^2\chi^{\parallel(1)}+\frac{1}{4}\chi^{\parallel(1),klm}\chi^{\parallel(1)}_{,klm}\delta_{ij}-\frac{11}{36}\nabla^2\chi^{\parallel(1),k}\nabla^2\chi^{\parallel(1)}_{,k}\delta_{ij}\\
&-\frac{2}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)}\delta_{ij}-\frac{1}{6}\nabla^2\chi^{\parallel(1)}_{,i}\nabla^2\chi^{\parallel(1)}_{,j}-\frac{1}{3}\chi^{\parallel(1),k}_{,j}\nabla^2\chi^{\parallel(1)}_{,ik}-\frac{1}{3}\chi^{\parallel(1),k}_{,i}\nabla^2\chi^{\parallel(1)}_{,jk}\\
&+\frac{2}{3}\chi^{\parallel(1)}_{,ij}\nabla^2\nabla^2\chi^{\parallel(1)}+\frac{2}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)}_{,ij}-\frac{1}{2}\chi^{\parallel(1),kl}_{,i}\chi^{\parallel(1)}_{,klj}+\frac{2}{3}\chi^{\parallel(1)}_{,kij}\nabla^2\chi^{\parallel(1),k}\\
&-\frac{2}{\tau}\chi^{\parallel(1),kl}\chi^{\parallel(1)\prime}_{,kl}\delta_{ij}+\frac{2}{3\tau}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime}\delta_{ij}-\frac{3}{4}\chi^{\parallel(1)\prime,kl}\chi^{\parallel(1)\prime}_{,kl}\delta_{ij}-\chi^{\parallel(1),kl}\chi^{\parallel(1)\prime\prime}_{,kl}\delta_{ij}\\
&+\frac{1}{3}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime\prime}\delta_{ij}+\frac{13}{36}\nabla^2\chi^{\parallel(1)\prime}\nabla^2\chi^{\parallel(1)\prime}\delta_{ij}+\chi^{\parallel(1)\prime,k}_{,i}\chi^{\parallel(1)\prime}_{,kj}-\frac{2}{3}\chi^{\parallel(1)\prime}_{,ij}\nabla^2\chi^{\parallel(1)\prime},
\end{aligned}
\tag{4.10}
$$

which contains the scalar-scalar metric products, as well as the terms of $v^{\parallel(1)}$ and $c_L^2\delta^{(1)}\propto p^{(1)}$ that are absent in the dust model.

We also decompose the evolution equation (4.9), by similar procedures to the 1st-order case. First, the trace part of (4.9) is the following:

$$2\phi_S^{(2)''}+\frac{4}{\tau}\phi_S^{(2)'}-\frac{2}{3}\nabla^2\phi_S^{(2)}-\frac{1}{9}\nabla^2\nabla^2\chi_S^{\|(2)}=\frac{3c_N^2}{\tau^2}\delta_S^{(2)}+\frac{1}{3}S_{Sk}^{k}, \tag{4.11}$$

where

$$\begin{aligned}S_{Sk}^{k}=&\frac{8}{\tau^2}v_{,k}^{\|(1)}v^{\|(1),k}-\frac{36}{\tau^2}c_L^2\delta^{(1)}\phi^{(1)}+6\phi_{,k}^{(1)}\phi^{(1),k}+8\phi^{(1)}\nabla^2\phi^{(1)}-6\phi^{(1)'}\phi^{(1)'}-4\phi_{,kl}^{(1)}\chi^{\|(1),kl}+\frac{4}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}\\&+\frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}+\frac{2}{9}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}-\frac{2}{3}\chi^{\|(1),kl}\nabla^2\chi_{,kl}^{\|(1)}-\frac{5}{12}\nabla^2\chi^{\|(1),k}\nabla^2\chi_{,k}^{\|(1)}+\frac{1}{4}\chi^{\|(1),klm}\chi_{,klm}^{\|(1)}\\&-\frac{6}{\tau}\chi^{\|(1),kl}\chi_{,kl}^{\|(1)'}+\frac{2}{\tau}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)'}-\frac{5}{4}\chi^{\|(1)',kl}\chi_{,kl}^{\|(1)'}-3\chi^{\|(1),kl}\chi_{,kl}^{\|(1)''}+\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)''}+\frac{5}{12}\nabla^2\chi^{\|(1)'}\nabla^2\chi^{\|(1)'}.\end{aligned} \tag{4.12}$$

The traceless part of (4.9) is

$$D_{ij}\phi_S^{(2)}+\frac{1}{2}D_{ij}\chi_S^{\|(2)''}+\frac{1}{\tau}D_{ij}\chi_S^{\|(2)'}+\frac{1}{6}\nabla^2D_{ij}\chi_S^{\|(2)}+\frac{1}{2}\chi_{Sij}^{\perp(2)''}+\frac{1}{\tau}\chi_{Sij}^{\perp(2)'}+\frac{1}{2}\chi_{Sij}^{\top(2)''}+\frac{1}{\tau}\chi_{Sij}^{\top(2)'}-\frac{1}{2}\nabla^2\chi_{Sij}^{\top(2)}=\bar{S}_{Sij}, \tag{4.13}$$

where

$$\begin{aligned}\bar{S}_{Sij}\equiv&\,S_{Sij}-\frac{1}{3}S_{Sk}^{k}\delta_{ij}\\=&\frac{8}{\tau^2}v_{,i}^{\|(1)}v_{,j}^{\|(1)}-\frac{8}{3\tau^2}v_{,k}^{\|(1)}v^{\|(1),k}\delta_{ij}+\frac{6}{\tau^2}c_L^2\delta^{(1)}\chi_{,ij}^{\|(1)}-\frac{2}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij}-6\phi_{,i}^{(1)}\phi_{,j}^{(1)}\\&+2\phi^{(1),k}\phi_{,k}^{(1)}\delta_{ij}-4\phi^{(1)}\phi_{,ij}^{(1)}+\frac{4}{3}\phi^{(1)}\nabla^2\phi^{(1)}\delta_{ij}-\frac{12}{\tau}\phi^{(1)'}\chi_{,ij}^{\|(1)}+\frac{4}{\tau}\phi^{(1)'}\nabla^2\chi^{\|(1)}\delta_{ij}\\&-\phi^{(1)'}\chi_{,ij}^{\|(1)'}+\frac{1}{3}\phi^{(1)'}\nabla^2\chi^{\|(1)'}\delta_{ij}-6\phi^{(1)''}\chi_{,ij}^{\|(1)}+2\phi^{(1)''}\nabla^2\chi^{\|(1)}\delta_{ij}-\phi_{,j}^{(1)}\nabla^2\chi_{,i}^{\|(1)}\\&-\phi_{,i}^{(1)}\nabla^2\chi_{,j}^{\|(1)}+\phi_{,k}^{(1)}\chi_{,ij}^{\|(1),k}+\frac{1}{3}\phi_{,k}^{(1)}\nabla^2\chi^{\|(1),k}\delta_{ij}-\frac{2}{3}\phi^{(1)}\nabla^2\chi_{,ij}^{\|(1)}\\&+\frac{2}{9}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}\delta_{ij}+4\chi_{,ij}^{\|(1)}\nabla^2\phi^{(1)}-\frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij}-2\phi_{,ki}^{(1)}\chi_{,j}^{\|(1),k}\\&-2\phi_{,kj}^{(1)}\chi_{,i}^{\|(1),k}+\frac{4}{3}\phi_{,kl}^{(1)}\chi^{\|(1),kl}\delta_{ij}+\frac{4}{3}\phi_{,ij}^{(1)}\nabla^2\chi^{\|(1)}-\frac{4}{9}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij}\\&+\frac{2}{3}\chi_{,kij}^{\|(1)}\nabla^2\chi^{\|(1),k}-\frac{1}{6}\nabla^2\chi_{,i}^{\|(1)}\nabla^2\chi_{,j}^{\|(1)}-\frac{1}{6}\nabla^2\chi^{\|(1),k}\nabla^2\chi_{,k}^{\|(1)}\delta_{ij}-\frac{1}{3}\chi_{,j}^{\|(1),k}\nabla^2\chi_{,ik}^{\|(1)}\\&-\frac{1}{3}\chi_{,i}^{\|(1),k}\nabla^2\chi_{,jk}^{\|(1)}+\frac{2}{9}\chi^{\|(1),kl}\nabla^2\chi_{,kl}^{\|(1)}\delta_{ij}+\frac{2}{3}\chi_{,ij}^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}+\frac{2}{9}\nabla^2\chi^{\|(1)}\nabla^2\chi_{,ij}^{\|(1)}\\&-\frac{8}{27}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}\delta_{ij}-\frac{1}{2}\chi_{,i}^{\|(1),kl}\chi_{,klj}^{\|(1)}+\frac{1}{6}\chi^{\|(1),klm}\chi_{,klm}^{\|(1)}\delta_{ij}+\chi_{,i}^{\|(1)',k}\chi_{,kj}^{\|(1)'}\\&-\frac{1}{3}\chi^{\|(1)',kl}\chi_{,kl}^{\|(1)'}\delta_{ij}-\frac{2}{3}\chi_{,ij}^{\|(1)'}\nabla^2\chi^{\|(1)'}+\frac{2}{9}\nabla^2\chi^{\|(1)'}\nabla^2\chi^{\|(1)'}\delta_{ij}.\end{aligned} \tag{4.14}$$

Applying $3\nabla^{-2}\nabla^{-2}\partial_i\partial_j$ to (4.13) gives the equation for the scalar $\chi_S^{\|(2)}$ as the following:

$$\chi_S^{\|(2)''}+\frac{2}{\tau}\chi_S^{\|(2)'}+\frac{1}{3}\nabla^2\chi_S^{\|(2)}+2\phi_S^{(2)}=3\nabla^{-2}\nabla^{-2}\bar{S}_{Skl}^{,kl}, \tag{4.15}$$

where

$$
\begin{aligned}
\bar{S}^{,kl}_{Skl} = {}& \frac{2}{3}\nabla^2\nabla^2\left[-\frac{9}{4}\phi^{(1)}\phi^{(1)} - \frac{2}{3}\phi^{(1)}\nabla^2\chi^{\|(1)} - \frac{1}{18}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)} - \frac{1}{16}\chi^{\|(1)}_{,kl}\chi^{\|(1),kl}\right] \\
& + \nabla^2\left[\frac{4}{3\tau^2}v^{\|(1)}_{,k}v^{\|(1),k} + \frac{4}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\|(1)} - \frac{5}{3}\phi^{(1)}\nabla^2\phi^{(1)} - \frac{8}{\tau}\phi^{(1)\prime}\nabla^2\chi^{\|(1)} - \frac{2}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime} - 4\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}\right. \\
& \left. - \frac{4}{9}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,k} + \frac{4}{3}\phi^{(1)}_{,kl}\chi^{\|(1),kl} + \frac{1}{54}\nabla^2\chi^{\|(1),k}\nabla^2\chi^{\|(1)}_{,k} + \frac{11}{36}\chi^{\|(1)}_{,kl}\nabla^2\chi^{\|(1),kl} + \frac{1}{6}\chi^{\|(1)\prime,kl}\chi^{\|(1)\prime}_{,kl} - \frac{1}{9}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}\right] \\
& + \frac{8}{\tau^2}\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)} + \frac{8}{\tau^2}v^{\|(1)}_{,k}\nabla^2 v^{\|(1),k} - \frac{6}{\tau^2}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)} + \frac{6}{\tau^2}c_L^2\delta^{(1),kl}\chi^{\|(1)}_{,kl} + 2\phi^{(1),k}\nabla^2\phi^{(1)}_{,k} + 2\phi^{(1)}\nabla^2\nabla^2\phi^{(1)} \\
& - \frac{12}{\tau}\phi^{(1)\prime,kl}\chi^{\|(1)}_{,kl} + \frac{12}{\tau}\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)} - \phi^{(1)\prime,kl}\chi^{\|(1)\prime}_{,kl} + \nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime} - 6\phi^{(1)\prime\prime,kl}\chi^{\|(1)}_{,kl} + 6\nabla^2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)} \\
& + \frac{4}{3}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\phi^{(1)} + \frac{11}{3}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\phi^{(1),k} + \frac{2}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k} - 3\phi^{(1)}_{,klm}\chi^{\|(1),klm} + \frac{7}{9}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\nabla^2\chi^{\|(1),k} \\
& + \frac{5}{18}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)} - \frac{1}{2}\chi^{\|(1)}_{,klm}\nabla^2\chi^{\|(1),klm} + \frac{1}{3}\chi^{\|(1)\prime}_{,kl}\nabla^2\chi^{\|(1)\prime,kl} + \frac{1}{3}\nabla^2\chi^{\|(1)\prime}_{,k}\nabla^2\chi^{\|(1)\prime,k}.
\end{aligned} \tag{4.16}
$$

By a combination ∂^i [(4.13)$-\frac{1}{2}D_{ij}$ (4.15)], one has

$$
\frac{1}{2}\chi^{\perp(2)\prime\prime,k}_{Skj} + \frac{1}{\tau}\chi^{\perp(2)\prime,k}_{Skj} = \bar{S}^{,k}_{Skj} - \partial_j\nabla^{-2}\bar{S}^{,kl}_{Skl}. \tag{4.17}
$$

By ∇^{-2} [∂_i (4.17)$+(i\leftrightarrow j)$] and by (2.6), one gets the vector mode equation

$$
\frac{1}{2}\chi^{\perp(2)\prime\prime}_{Sij} + \frac{1}{\tau}\chi^{\perp(2)\prime}_{Sij} = V_{Sij}, \tag{4.18}
$$

where the rhs is the effective source [see (B29)]

$$
\begin{aligned}
V_{Sij} \equiv {}& \nabla^{-2}\bar{S}^{,k}_{Skj,i} + \nabla^{-2}\bar{S}^{,k}_{Ski,j} - 2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl,ij} \\
= {}& \partial_i\left[\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}_{,j} + 2\phi^{(1),k}\chi^{\|(1)}_{,kj} + \frac{1}{3}\chi^{\|(1)}_{,kj}\nabla^2\chi^{\|(1),k}\right] \\
& - \partial_i\partial_j\nabla^{-2}\left[\frac{6}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\|(1)} - \frac{12}{\tau}\phi^{(1)\prime}\nabla^2\chi^{\|(1)} - \phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime} - 6\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)} + \frac{1}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)} + 2\phi^{(1),kl}\chi^{\|(1)}_{,kl}\right. \\
& \left. + \frac{1}{12}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\chi^{\|(1),k} + \frac{1}{3}\chi^{\|(1)}_{,kl}\nabla^2\chi^{\|(1),kl}\right] + \partial_i\nabla^{-2}\left[\frac{8}{\tau^2}v^{\|(1)}_{,j}\nabla^2 v^{\|(1)} + \frac{6}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\|(1)}_{,j} + \frac{6}{\tau^2}c_L^2\delta^{(1),k}\chi^{\|(1)}_{,kj}\right. \\
& + 2\phi^{(1)}\nabla^2\phi^{(1)}_{,j} - \frac{12}{\tau}\phi^{(1)\prime,k}\chi^{\|(1)}_{,kj} - \frac{12}{\tau}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}_{,j} - \phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}_{,j} - \phi^{(1)\prime,k}\chi^{\|(1)\prime}_{,kj} - 6\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}_{,j} - 6\phi^{(1)\prime\prime,k}\chi^{\|(1)}_{,kj} \\
& \left. + \frac{4}{3}\nabla^2\chi^{\|(1)}_{,j}\nabla^2\phi^{(1)} - \frac{5}{3}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,kj} - 3\phi^{(1)}_{,kl}\chi^{\|(1),kl}_{,j} + \frac{5}{18}\nabla^2\chi^{\|(1)}_{,j}\nabla^2\nabla^2\chi^{\|(1)} - \frac{1}{2}\chi^{\|(1)}_{,klj}\nabla^2\chi^{\|(1),kl} + \frac{1}{3}\chi^{\|(1)\prime}_{,kj}\nabla^2\chi^{\|(1)\prime,k}\right] \\
& - \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[\frac{8}{\tau^2}\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)} + \frac{8}{\tau^2}v^{\|(1)}_{,k}\nabla^2 v^{\|(1),k} - \frac{6}{\tau^2}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)} + \frac{6}{\tau^2}c_L^2\delta^{(1),kl}\chi^{\|(1)}_{,kl} + 2\phi^{(1),k}\nabla^2\phi^{(1)}_{,k}\right. \\
& + 2\phi^{(1)}\nabla^2\nabla^2\phi^{(1)} - \frac{12}{\tau}\phi^{(1)\prime,kl}\chi^{\|(1)}_{,kl} + \frac{12}{\tau}\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)} - \phi^{(1)\prime,kl}\chi^{\|(1)\prime}_{,kl} + \nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime} - 6\phi^{(1)\prime\prime,kl}\chi^{\|(1)}_{,kl} \\
& + 6\nabla^2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)} + \frac{4}{3}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\phi^{(1)} + \frac{11}{3}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\phi^{(1),k} + \frac{2}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k} - 3\phi^{(1)}_{,klm}\chi^{\|(1),klm} \\
& \left. + \frac{7}{9}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\nabla^2\chi^{\|(1),k} - \frac{1}{2}\chi^{\|(1)}_{,klm}\nabla^2\chi^{\|(1),klm} + \frac{5}{18}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)} + \frac{1}{3}\chi^{\|(1)\prime}_{,kl}\nabla^2\chi^{\|(1)\prime,kl} + \frac{1}{3}\nabla^2\chi^{\|(1)\prime}_{,k}\nabla^2\chi^{\|(1)\prime,k}\right] \\
& + (i\leftrightarrow j),
\end{aligned} \tag{4.19}
$$

which consists of the scalar-scalar couplings, as well as $v^{\parallel(1)}$ and $c_L^2\delta^{(1)} \propto p^{(1)}$. Equation (4.18) is similar to the 1st-order equation (3.12) aside from the source V_{Sij}. For consistency, we have checked that V_{Sij} satisfies the condition (2.6). Although the 1st-order vector is vanishing by assumption, the 2nd-order vector is generated by the coupling of 1st-order scalar perturbations.

Finally, [(4.13) $-\frac{1}{2}D_{ij}$ (4.15)–(4.18)] gives the 2nd-order tensor mode equation

$$\frac{1}{2}\chi^{\top(2)''}_{Sij} + \frac{1}{\tau}\chi^{\top(2)'}_{Sij} - \frac{1}{2}\nabla^2\chi^{\top(2)}_{Sij} = J_{Sij}. \tag{4.20}$$

Unlike the 1st-order equation (3.13), it is inhomogeneous with the rhs being the effective source [see (B31)]

$$\begin{aligned}
J_{Sij} \equiv{}& \bar{S}_{Sij} - \frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl} - \nabla^{-2}\bar{S}^{,k}_{Skj,i} - \nabla^{-2}\bar{S}^{,k}_{Ski,j} + 2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl,ij} \\
={}& D_{ij}\Big[-\frac{3}{4}\phi^{(1)}\phi^{(1)} - \frac{1}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)} - 2\phi^{(1)}_{,k}\chi^{\parallel(1),k} + \frac{1}{6}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)} + \frac{1}{16}\chi^{\parallel(1)}_{,kl}\chi^{\parallel(1),kl} - \frac{1}{3}\chi^{\parallel(1)}_{,k}\nabla^2\chi^{\parallel(1),k}\Big] \\
&+ \frac{8}{\tau^2}v^{\parallel(1)}_{,i}v^{\parallel(1)}_{,j} - \frac{8}{3\tau^2}v^{\parallel(1)}_{,k}v^{\parallel(1),k}\delta_{ij} + \frac{6}{\tau^2}c_L^2\delta^{(1)}\chi^{\parallel(1)}_{,ij} - \frac{2}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij} + 2\phi^{(1)}\phi^{(1)}_{,ij} - \frac{2}{3}\phi^{(1)}\nabla^2\phi^{(1)}\delta_{ij} \\
&- \frac{12}{\tau}\phi^{(1)'}\chi^{\parallel(1)}_{,ij} + \frac{4}{\tau}\phi^{(1)'}\nabla^2\chi^{\parallel(1)}\delta_{ij} - \phi^{(1)'}\chi^{\parallel(1)'}_{,ij} + \frac{1}{3}\phi^{(1)'}\nabla^2\chi^{\parallel(1)'}\delta_{ij} - 6\phi^{(1)''}\chi^{\parallel(1)}_{,ij} + 2\phi^{(1)''}\nabla^2\chi^{\parallel(1)}\delta_{ij} + \frac{7}{3}\phi^{(1)}_{,ij}\nabla^2\chi^{\parallel(1)} \\
&- \frac{7}{9}\nabla^2\phi^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij} + \frac{1}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)}_{,ij} - \frac{1}{9}\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)}\delta_{ij} + 3\phi^{(1)}_{,k}\chi^{\parallel(1),k}_{,ij} - \phi^{(1)}_{,k}\nabla^2\chi^{\parallel(1),k}\delta_{ij} + 2\chi^{\parallel(1),k}\phi^{(1)}_{,kij} \\
&- \frac{2}{3}\chi^{\parallel(1),k}\nabla^2\phi^{(1)}_{,k}\delta_{ij} + 4\chi^{\parallel(1)}_{,ij}\nabla^2\phi^{(1)} - \frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij} + \frac{2}{3}\chi^{\parallel(1)}_{,ij}\nabla^2\nabla^2\chi^{\parallel(1)} - \frac{2}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)}\delta_{ij} \\
&- \frac{7}{18}\nabla^2\chi^{\parallel(1)}_{,i}\nabla^2\chi^{\parallel(1)}_{,j} + \chi^{\parallel(1)}_{,kij}\nabla^2\chi^{\parallel(1),k} - \frac{11}{54}\nabla^2\chi^{\parallel(1),k}\nabla^2\chi^{\parallel(1)}_{,k}\delta_{ij} - \frac{1}{9}\chi^{\parallel(1),k}\nabla^2\nabla^2\chi^{\parallel(1)}_{,k}\delta_{ij} + \frac{1}{3}\chi^{\parallel(1)}_{,k}\nabla^2\chi^{\parallel(1),k}_{,ij} \\
&- \frac{1}{2}\chi^{\parallel(1),kl}_{,i}\chi^{\parallel(1)}_{,klj} + \frac{1}{6}\chi^{\parallel(1),klm}\chi^{\parallel(1)}_{,klm}\delta_{ij} + \chi^{\parallel(1)',k}_{,i}\chi^{\parallel(1)'}_{,kj} - \frac{1}{3}\chi^{\parallel(1)',kl}\chi^{\parallel(1)'}_{,kl}\delta_{ij} - \frac{2}{3}\chi^{\parallel(1)'}_{,ij}\nabla^2\chi^{\parallel(1)'} + \frac{2}{9}\nabla^2\chi^{\parallel(1)'}\nabla^2\chi^{\parallel(1)'}\delta_{ij} \\
&- \frac{3}{2}D_{ij}\nabla^{-2}\Big[\frac{4}{3\tau^2}v^{\parallel(1)}_{,k}v^{\parallel(1),k} + \frac{4}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\parallel(1)} - \frac{5}{3}\phi^{(1)}\nabla^2\phi^{(1)} - \frac{8}{\tau}\phi^{(1)'}\nabla^2\chi^{\parallel(1)} - \frac{2}{3}\phi^{(1)'}\nabla^2\chi^{\parallel(1)'} - 4\phi^{(1)''}\nabla^2\chi^{\parallel(1)} \\
&- \frac{4}{9}\phi^{(1),k}\nabla^2\chi^{\parallel(1)}_{,k} + \frac{4}{3}\phi^{(1)}_{,kl}\chi^{\parallel(1),kl} + \frac{1}{54}\nabla^2\chi^{\parallel(1),k}\nabla^2\chi^{\parallel(1)}_{,k} + \frac{11}{36}\chi^{\parallel(1)}_{,kl}\nabla^2\chi^{\parallel(1),kl} + \frac{1}{6}\chi^{\parallel(1)',kl}\chi^{\parallel(1)'}_{,kl} - \frac{1}{9}\nabla^2\chi^{\parallel(1)'}\nabla^2\chi^{\parallel(1)'}\Big] \\
&- \frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\Big[\frac{8}{\tau^2}\nabla^2 v^{\parallel(1)}\nabla^2 v^{\parallel(1)} + \frac{8}{\tau^2}v^{\parallel(1)}_{,k}\nabla^2 v^{\parallel(1),k} - \frac{6}{\tau^2}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\parallel(1)} + \frac{6}{\tau^2}c_L^2\delta^{(1),kl}\chi^{\parallel(1)}_{,kl} + 2\phi^{(1),k}\nabla^2\phi^{(1)}_{,k} \\
&+ 2\phi^{(1)}\nabla^2\nabla^2\phi^{(1)} - \frac{12}{\tau}\phi^{(1)',kl}\chi^{\parallel(1)}_{,kl} + \frac{12}{\tau}\nabla^2\phi^{(1)'}\nabla^2\chi^{\parallel(1)} - \phi^{(1)',kl}\chi^{\parallel(1)'}_{,kl} + \nabla^2\phi^{(1)'}\nabla^2\chi^{\parallel(1)'} - 6\phi^{(1)'',kl}\chi^{\parallel(1)}_{,kl} \\
&+ 6\nabla^2\phi^{(1)''}\nabla^2\chi^{\parallel(1)} + \frac{4}{3}\nabla^2\nabla^2\chi^{\parallel(1)}\nabla^2\phi^{(1)} + \frac{11}{3}\nabla^2\chi^{\parallel(1)}_{,k}\nabla^2\phi^{(1),k} + \frac{2}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\parallel(1)}_{,k} - 3\phi^{(1)}_{,klm}\chi^{\parallel(1),klm} \\
&+ \frac{7}{9}\nabla^2\chi^{\parallel(1)}_{,k}\nabla^2\nabla^2\chi^{\parallel(1),k} + \frac{5}{18}\nabla^2\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)} - \frac{1}{2}\chi^{\parallel(1)}_{,klm}\nabla^2\chi^{\parallel(1),klm} + \frac{1}{3}\chi^{\parallel(1)'}_{,kl}\nabla^2\chi^{\parallel(1)',kl} + \frac{1}{3}\nabla^2\chi^{\parallel(1)'}_{,k}\nabla^2\chi^{\parallel(1)',k}\Big] \\
&- \partial_i\Big[\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)}_{,j} + 2\phi^{(1),k}\chi^{\parallel(1)}_{,kj} + \frac{1}{3}\chi^{\parallel(1)}_{,kj}\nabla^2\chi^{\parallel(1),k}\Big] - \partial_j\Big[\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\parallel(1)}_{,i} + 2\phi^{(1),k}\chi^{\parallel(1)}_{,ki} + \frac{1}{3}\chi^{\parallel(1)}_{,ki}\nabla^2\chi^{\parallel(1),k}\Big] \\
&+ \partial_i\partial_j\nabla^{-2}\Big[\frac{12}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\parallel(1)} - \frac{24}{\tau}\phi^{(1)'}\nabla^2\chi^{\parallel(1)} - 2\phi^{(1)'}\nabla^2\chi^{\parallel(1)'} - 12\phi^{(1)''}\nabla^2\chi^{\parallel(1)} + \frac{2}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)} + 4\phi^{(1),kl}\chi^{\parallel(1)}_{,kl} \\
&+ \frac{1}{6}\nabla^2\chi^{\parallel(1)}_{,k}\nabla^2\chi^{\parallel(1),k} + \frac{2}{3}\chi^{\parallel(1)}_{,kl}\nabla^2\chi^{\parallel(1),kl}\Big] - \partial_i\nabla^{-2}\Big[\frac{8}{\tau^2}v^{\parallel(1)}_{,j}\nabla^2 v^{\parallel(1)} + \frac{6}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\parallel(1)}_{,j} + \frac{6}{\tau^2}c_L^2\delta^{(1),k}\chi^{\parallel(1)}_{,kj} \\
&+ 2\phi^{(1)}\nabla^2\phi^{(1)}_{,j} - \frac{12}{\tau}\phi^{(1)',k}\chi^{\parallel(1)}_{,kj} - \frac{12}{\tau}\phi^{(1)'}\nabla^2\chi^{\parallel(1)}_{,j} - \phi^{(1)'}\nabla^2\chi^{\parallel(1)'}_{,j} - \phi^{(1)',k}\chi^{\parallel(1)'}_{,kj} - 6\phi^{(1)''}\nabla^2\chi^{\parallel(1)}_{,j} - 6\phi^{(1)'',k}\chi^{\parallel(1)}_{,kj} \\
&+ \frac{4}{3}\nabla^2\chi^{\parallel(1)}_{,j}\nabla^2\phi^{(1)} - \frac{5}{3}\phi^{(1),k}\nabla^2\chi^{\parallel(1)}_{,kj} - 3\phi^{(1)}_{,kl}\chi^{\parallel(1),kl}_{,j} + \frac{5}{18}\nabla^2\chi^{\parallel(1)}_{,j}\nabla^2\nabla^2\chi^{\parallel(1)} - \frac{1}{2}\chi^{\parallel(1)}_{,klj}\nabla^2\chi^{\parallel(1),kl} + \frac{1}{3}\chi^{\parallel(1)'}_{,kj}\nabla^2\chi^{\parallel(1)',k}\Big]
\end{aligned}$$

$$
\begin{aligned}
&-\partial_j\nabla^{-2}\bigg[\frac{8}{\tau^2}v^{\parallel(1)}_{,i}\nabla^2 v^{\parallel(1)}+\frac{6}{\tau^2}c_L^2\delta^{(1)}\nabla^2\chi^{\parallel(1)}_{,i}+\frac{6}{\tau^2}c_L^2\delta^{(1),k}\chi^{\parallel(1)}_{,ki}+2\phi^{(1)}\nabla^2\phi^{(1)}_{,i}-\frac{12}{\tau}\phi^{(1)\prime,k}\chi^{\parallel(1)}_{,ki}-\frac{12}{\tau}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)}_{,i}\\
&-\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)\prime}_{,i}-\phi^{(1)\prime,k}\chi^{\parallel(1)\prime}_{,ki}-6\phi^{(1)\prime\prime}\nabla^2\chi^{\parallel(1)}_{,i}-6\phi^{(1)\prime\prime,k}\chi^{\parallel(1)}_{,ki}+\frac{4}{3}\nabla^2\chi^{\parallel(1)}_{,i}\nabla^2\phi^{(1)}-\frac{5}{3}\phi^{(1),k}\nabla^2\chi^{\parallel(1)}_{,ki}-3\phi^{(1)}_{,kl}\chi^{\parallel(1),kl}_{,i}\\
&+\frac{5}{18}\nabla^2\chi^{\parallel(1)}_{,i}\nabla^2\nabla^2\chi^{\parallel(1)}-\frac{1}{2}\chi^{\parallel(1)}_{,kli}\nabla^2\chi^{\parallel(1),kl}+\frac{1}{3}\chi^{\parallel(1)\prime}_{,ki}\nabla^2\chi^{\parallel(1)\prime,k}\bigg]+\partial_i\partial_j\nabla^{-2}\nabla^{-2}\bigg[\frac{16}{\tau^2}\nabla^2 v^{\parallel(1)}\nabla^2 v^{\parallel(1)}+\frac{16}{\tau^2}v^{\parallel(1)}_{,k}\nabla^2 v^{\parallel(1),k}\\
&-\frac{12}{\tau^2}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\parallel(1)}+\frac{12}{\tau^2}c_L^2\delta^{(1),kl}\chi^{\parallel(1)}_{,kl}+4\phi^{(1),k}\nabla^2\phi^{(1)}_{,k}+4\phi^{(1)}\nabla^2\nabla^2\phi^{(1)}-\frac{24}{\tau}\phi^{(1)\prime,kl}\chi^{\parallel(1)}_{,kl}+\frac{24}{\tau}\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)}\\
&-2\phi^{(1)\prime,kl}\chi^{\parallel(1)\prime}_{,kl}+2\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)\prime}-12\phi^{(1)\prime\prime,kl}\chi^{\parallel(1)}_{,kl}+12\nabla^2\phi^{(1)\prime\prime}\nabla^2\chi^{\parallel(1)}+\frac{8}{3}\nabla^2\nabla^2\chi^{\parallel(1)}\nabla^2\phi^{(1)}+\frac{22}{3}\nabla^2\chi^{\parallel(1)}_{,k}\nabla^2\phi^{(1),k}\\
&+\frac{4}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\parallel(1)}_{,k}-6\phi^{(1)}_{,klm}\chi^{\parallel(1),klm}+\frac{14}{9}\nabla^2\chi^{\parallel(1)}_{,k}\nabla^2\nabla^2\chi^{\parallel(1),k}-\chi^{\parallel(1)}_{,klm}\nabla^2\chi^{\parallel(1),klm}\\
&+\frac{5}{9}\nabla^2\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)}+\frac{2}{3}\chi^{\parallel(1)\prime}_{,kl}\nabla^2\chi^{\parallel(1)\prime,kl}+\frac{2}{3}\nabla^2\chi^{\parallel(1)\prime}_{,k}\nabla^2\chi^{\parallel(1)\prime,k}\bigg], \qquad (4.21)
\end{aligned}
$$

which consists of the scalar-scalar couplings only, and gets no contribution from $T^{(2)}_{ij}$, since the relativistic fluid has been assumed to have no anisotropic stress. We have checked that the source J_{Sij} satisfies the traceless and transverse condition.

So far the 2nd-order perturbed Einstein equation has been decomposed into the equations for the 2nd-order scalar, vector, and tensor, respectively. To solve them, we need to specify the 2nd-order density contrast and velocity, and resort to the 2nd-order energy-momentum conservation (B32) and (B33). For RD stage (B32) for the scalar-scalar coupling gives the 2nd-order energy conservation

$$
\begin{aligned}
&\delta^{(2)\prime}_S+\frac{3}{\tau}\left(c_N^2-\frac{1}{3}\right)\delta^{(2)}_S+\frac{4}{3}\nabla^2 v^{\parallel(2)}_S-4\phi^{(2)\prime}_S+\frac{16}{3}v^{\parallel(1)\prime,k}v^{\parallel(1)}_{,k}+2(1+c_L^2)\delta^{(1)}_{,k}v^{\parallel(1),k}+2(1+c_L^2)\delta^{(1)}\nabla^2 v^{\parallel(1)}\\
&-6(1+c_L^2)\delta^{(1)}\phi^{(1)\prime}-16\phi^{(1)\prime}\phi^{(1)}-\frac{4}{3}D_{kl}\chi^{\parallel(1)\prime}D^{kl}\chi^{\parallel(1)}-8\phi^{(1)}_{,k}v^{\parallel(1),k}=0. \qquad (4.22)
\end{aligned}
$$

Moving the coupling terms to the rhs, this is also written as

$$
\delta^{(2)\prime}_S+\frac{3}{\tau}\left(c_N^2-\frac{1}{3}\right)\delta^{(2)}_S=-\frac{4}{3}\nabla^2 v^{\parallel(2)}_S+4\phi^{(2)\prime}_S+A_S, \qquad (4.23)
$$

which is similar to the 1st-order (3.15), where

$$
\begin{aligned}
A_S\equiv&-\frac{16}{3}v^{\parallel(1)\prime,k}v^{\parallel(1)}_{,k}-2(1+c_L^2)\delta^{(1)}_{,k}v^{\parallel(1),k}-2(1+c_L^2)\delta^{(1)}\nabla^2 v^{\parallel(1)}+6(1+c_L^2)\delta^{(1)}\phi^{(1)\prime}\\
&+16\phi^{(1)\prime}\phi^{(1)}+\frac{4}{3}\chi^{\parallel(1)\prime}_{,kl}\chi^{\parallel(1),kl}-\frac{4}{9}\nabla^2\chi^{\parallel(1)\prime}\nabla^2\chi^{\parallel(1)}+8\phi^{(1)}_{,k}v^{\parallel(1),k} \qquad (4.24)
\end{aligned}
$$

is part of the effective source. The 2nd-order pressure perturbation $c_N^2\delta^{(2)}_S\propto p^{(2)}$ appears in Eq. (4.23). From (B33) follows the 2nd-order momentum conservation

$$
c_N^2\delta^{(2)}_{S,i}+\frac{4}{3}v^{(2)\prime}_{Si}=F_{Si}, \qquad (4.25)
$$

which is similar to the 1st-order (3.17), where

$$
\begin{aligned}
F_{Si}\equiv&-2(1+c_L^2)\delta^{(1)\prime}v^{\parallel(1)}_{,i}-2(1+c_L^2)\delta^{(1)}v^{\parallel(1)\prime}_{,i}-\frac{8}{3}v^{\parallel(1)}_{,ik}v^{\parallel(1),k}-\frac{8}{3}v^{\parallel(1)}_{,i}\nabla^2 v^{\parallel(1)}-4c_L^2\delta^{(1)}_{,i}\phi^{(1)}+\frac{40}{3}v^{\parallel(1)}_{,i}\phi^{(1)\prime}\\
&+2c_L^2\delta^{(1),k}\chi^{\parallel(1)}_{,ki}-\frac{2}{3}c_L^2\delta^{(1)}_{,i}\nabla^2\chi^{\parallel(1)}-\frac{8}{3}v^{\parallel(1),k}\chi^{\parallel(1)\prime}_{,ki}+\frac{8}{9}v^{\parallel(1)}_{,i}\nabla^2\chi^{\parallel(1)\prime}. \qquad (4.26)
\end{aligned}
$$

To proceed, by $[\nabla^{-2}\partial^i$ (4.25)], (4.25) is decomposed into a longitudinal part

$$c_N^2\delta_S^{(2)} + \frac{4}{3}v_S^{\|(2)\prime} = F_S^{\|} \tag{4.27}$$

with

$$\begin{aligned}
F_S^{\|} \equiv{}& \nabla^{-2}\partial^i F_{Si}\\
={}& -\frac{4}{3}v^{\|(1)}_{,k}v^{\|(1),k} + \nabla^{-2}\Big[-2(1+c_L^2)\delta^{(1)\prime}_{,k}v^{\|(1),k} - 2(1+c_L^2)\delta^{(1)\prime}\nabla^2 v^{\|(1)} - 2(1+c_L^2)\delta^{(1),k}v^{\|(1)\prime}_{,k} - 2(1+c_L^2)\delta^{(1)}\nabla^2 v^{\|(1)\prime}\\
&-\frac{8}{3}v^{\|(1)}_{,k}\nabla^2 v^{\|(1),k} - \frac{8}{3}\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)} - 4c_L^2\delta^{(1)}_{,k}\phi^{(1),k} - 4c_L^2\phi^{(1)}\nabla^2\delta^{(1)} + \frac{40}{3}v^{\|(1),k}\phi^{(1)\prime}_{,k} + \frac{40}{3}\phi^{(1)\prime}\nabla^2 v^{\|(1)}\\
&+2c_L^2\delta^{(1),kl}\chi^{\|(1)}_{,kl} + \frac{4}{3}c_L^2\delta^{(1),k}\nabla^2\chi^{\|(1)}_{,k} - \frac{2}{3}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)} - \frac{8}{3}v^{\|(1),kl}\chi^{\|(1)\prime}_{,kl} - \frac{16}{9}v^{\|(1),k}\nabla^2\chi^{\|(1)\prime}_{,k} + \frac{8}{9}\nabla^2 v^{\|(1)}\nabla^2\chi^{\|(1)\prime}\Big].
\end{aligned} \tag{4.28}$$

By [(4.25)-∂_i (4.27)], one gets the transverse part

$$\frac{4}{3}v_{Si}^{\perp(2)\prime} = F_{Si}^{\perp} \tag{4.29}$$

with the effective source

$$\begin{aligned}
F_{Si}^{\perp} \equiv{}& F_{Si} - \partial_i F_S^{\|}\\
={}& -2(1+c_L^2)\delta^{(1)\prime}v^{\|(1)}_{,i} - 2(1+c_L^2)\delta^{(1)}v^{\|(1)\prime}_{,i} - \frac{8}{3}v^{\|(1)}_{,i}\nabla^2 v^{\|(1)} - 4c_L^2\delta^{(1)}_{,i}\phi^{(1)} + \frac{40}{3}v^{\|(1)}_{,i}\phi^{(1)\prime} + 2c_L^2\delta^{(1),k}\chi^{\|(1)}_{,ki}\\
&-\frac{2}{3}c_L^2\delta^{(1)}_{,i}\nabla^2\chi^{\|(1)} - \frac{8}{3}v^{\|(1),k}\chi^{\|(1)\prime}_{,ki} + \frac{8}{9}v^{\|(1)}_{,i}\nabla^2\chi^{\|(1)\prime} + \partial_i\nabla^{-2}\Big[2(1+c_L^2)\delta^{(1)\prime}_{,k}v^{\|(1),k} + 2(1+c_L^2)\delta^{(1)\prime}\nabla^2 v^{\|(1)}\\
&+2(1+c_L^2)\delta^{(1),k}v^{\|(1)\prime}_{,k} + 2(1+c_L^2)\delta^{(1)}\nabla^2 v^{\|(1)\prime} + \frac{8}{3}v^{\|(1)}_{,k}\nabla^2 v^{\|(1),k} + \frac{8}{3}\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)} + 4c_L^2\delta^{(1)}_{,k}\phi^{(1),k}\\
&+4c_L^2\phi^{(1)}\nabla^2\delta^{(1)} - \frac{40}{3}v^{\|(1),k}\phi^{(1)\prime}_{,k} - \frac{40}{3}\phi^{(1)\prime}\nabla^2 v^{\|(1)} - 2c_L^2\delta^{(1),kl}\chi^{\|(1)}_{,kl} - \frac{4}{3}c_L^2\delta^{(1),k}\nabla^2\chi^{\|(1)}_{,k} + \frac{2}{3}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)}\\
&+\frac{8}{3}v^{\|(1),kl}\chi^{\|(1)\prime}_{,kl} + \frac{16}{9}v^{\|(1),k}\nabla^2\chi^{\|(1)\prime}_{,k} - \frac{8}{9}\nabla^2 v^{\|(1)}\nabla^2\chi^{\|(1)\prime}\Big]
\end{aligned} \tag{4.30}$$

being a transverse vector function.

Similar to the 1st-order relations (3.20) and (3.21), we find that the trace of the 2nd-order evolution equation (4.11) can be formed by a combination

$$(4.11) = -\frac{1}{3}(4.1) - \frac{\tau}{3}\frac{d}{d\tau}(4.1) + \frac{\tau}{3}\nabla^2(4.5) - \frac{1}{\tau}(4.23), \tag{4.31}$$

and the scalar part of the 2nd-order traceless evolution equation (4.15) can be formed by a combination

$$(4.15) = \nabla^{-2}\left[(4.1) + \tau\frac{d}{d\tau}(4.1) - \tau\nabla^2(4.5) + 3\frac{d}{d\tau}(4.5) + \frac{6}{\tau}(4.5) + \frac{3}{\tau}(4.23) - \frac{9}{\tau^2}(4.27)\right]. \tag{4.32}$$

Thus, we can use the equations of constraints and conservations to solve the scalars, and the solutions will satisfy the evolution equations automatically.

V. SOLUTIONS TO THE 2ND-ORDER PERTURBATIONS

Now we solve the 2nd-order equations given in the last section. First, the 2nd-order tensor equation (4.20) has the general solution

$$\chi^{\top(2)}_{Sij}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \Big(\bar{I}_{Sij}(\mathbf{k},\tau) + \sum_{s=+,\times} \overset{s}{\epsilon}_{ij}(\mathbf{k}) \Big[-b^s_1 \sqrt{\frac{2}{\pi}} \frac{ie^{ik\tau}}{k\tau} + b^s_2 \sqrt{\frac{2}{\pi}} \frac{ie^{-ik\tau}}{k\tau} \Big] \Big), \tag{5.1}$$

where b^s_1 and b^s_2 are polarization-dependent and $\mathbf{k}$-dependent coefficients, to be determined by initial conditions, their associated term is the homogeneous solution of (4.20) and has the same form as (3.23). The integrand of the inhomogenous solution in (5.1) is given by

$$\bar{I}_{Sij}(\mathbf{k},\tau) \equiv \frac{ie^{-ik\tau}}{k\tau} \int^\tau \tau' e^{ik\tau'} \bar{J}_{Sij}(\mathbf{k},\tau') d\tau' - \frac{ie^{ik\tau}}{k\tau} \int^\tau \tau' e^{-ik\tau'} \bar{J}_{Sij}(\mathbf{k},\tau') d\tau', \tag{5.2}$$

with $\bar{J}_{Sij}$ being the Fourier transform of the source J_{Sij} in (4.21) consisting of products of 1st-order solutions.

Next, the vector mode equation (4.18) has the general solution

$$\chi^{\perp(2)}_{Sij}(\mathbf{x},\tau) = c_{1ij}(\mathbf{x}) + \frac{c_{2ij}(\mathbf{x})}{\tau} + 2\int^\tau \frac{d\tau'}{\tau'^2} \int^{\tau'} \tau''^2 V_{Sij}(\mathbf{x},\tau'') d\tau'', \tag{5.3}$$

with V_{Sij} given by (4.19), and c_{1ij} and c_{2ij} are two time-independent functions and correspond to the homogenous solution to be determined by initial values. (Actually c_{1ij} is a gauge mode as shall be seen in the next section.) Plugging the solution (5.3) into (4.7) gives the solution of transverse 2nd-order velocity

$$v^{\perp(2)}_{Si} = \frac{c^{,j}_{2ij}(\mathbf{x})}{8} + \frac{\tau^2}{4}(M_{Si} - \partial_i \nabla^{-2} M^{,k}_{Sk}) - \frac{1}{4}\int^\tau \tau'^2 V^{,j}_{Sij}(\mathbf{x},\tau') d\tau', \tag{5.4}$$

where $(M_{Si} - \partial_i \nabla^{-2} M^{,k}_{Sk})$ has been defined in (4.8). This solution can also be derived from the integration of the transverse part of the momentum conservation (4.29), as we have checked. Although the 1st-order curl vector $v^{\perp(1)}_i$ is vanishing by assumption, nevertheless, the 2nd-order $v^{\perp(2)}_i$ is generated according to (5.4).

Next, we solve the 2nd-order scalars by similar procedures to the 1st-order case. From the longitudinal part of 2nd-order momentum conservation (4.27),

$$\delta^{(2)}_S = -\frac{4}{3c^2_N} v^{\|(2)'}_S + \frac{1}{c^2_N} F^{\|}_S. \tag{5.5}$$

Plugging this $\delta^{(2)}_S$ into the energy conservation (4.23) gives $\phi^{(2)'}_S$ in terms of $v^{\|(2)}_S$, as the following:

$$\phi^{(2)'}_S = -\frac{1}{3c^2_N} v^{\|(2)''}_S - \frac{1}{\tau}\frac{c^2_N - \frac{1}{3}}{c^2_N} v^{\|(2)'}_S + \frac{1}{3}\nabla^2 v^{\|(2)}_S + \frac{1}{4c^2_N} F^{\|'}_S + \frac{3}{4\tau}\frac{c^2_N - \frac{1}{3}}{c^2_N} F^{\|}_S - \frac{1}{4} A_S. \tag{5.6}$$

Taking $[\frac{d}{d\tau}$ (4.1)] gives

$$-\frac{6}{\tau}\phi^{(2)''}_S + \frac{6}{\tau^2}\phi^{(2)'}_S + \nabla^2\Big[2\phi^{(2)'}_S + \frac{1}{3}\nabla^2 \chi^{\|(2)'}_S\Big] = \frac{3}{\tau^2}\delta^{(2)'}_S - \frac{6}{\tau^3}\delta^{(2)}_S + E'_S. \tag{5.7}$$

Plugging the longitudinal momentum constraint (4.5) into the above gives

$$-\frac{6}{\tau}\phi^{(2)''}_S + \frac{6}{\tau^2}\phi^{(2)'}_S - \frac{4}{\tau^2}\nabla^2 v^{\|(2)}_S + M^{,l}_{Sl} = \frac{3}{\tau^2}\delta^{(2)'}_S - \frac{6}{\tau^3}\delta^{(2)}_S + E'_S. \tag{5.8}$$

Then, plugging $\delta^{(2)}_S$ of (5.5) and $\phi^{(2)'}_S$ of (5.6) into (5.8) yields a 3rd-order differential equation of $v^{\|(2)}_S$ as

$$v^{\|(2)'''}_S + \frac{3c^2_N}{\tau} v^{\|(2)''}_S - \frac{6c^2_N + 2}{\tau^2} v^{\|(2)'}_S - \frac{c^2_N}{\tau}\nabla^2 v^{\|(2)}_S - c^2_N \nabla^2 v^{\|(2)'}_S = Z_S, \tag{5.9}$$

where the effective source is

$$Z_S \equiv \frac{3}{4}F^{\|''}_S + \frac{9c^2_N}{4\tau}F^{\|'}_S - \frac{9c^2_N + 3}{2\tau^2}F^{\|}_S - \frac{3c^2_N}{4}A'_S + \frac{3c^2_N}{4\tau}A_S - \frac{\tau}{2}c^2_N M^{,l}_{Sl} + \frac{\tau}{2}c^2_N E'_S. \tag{5.10}$$

Written in the $\mathbf{k}$-space, (5.9) and (5.10) are

$$v^{\|(2)'''}_{S\mathbf{k}} + \frac{3c^2_N}{\tau} v^{\|(2)''}_{S\mathbf{k}} + \Big(c^2_N k^2 - \frac{6c^2_N + 2}{\tau^2}\Big) v^{\|(2)'}_{S\mathbf{k}} + \frac{c^2_N}{\tau} k^2 v^{\|(2)}_{S\mathbf{k}} = Z_{S\mathbf{k}}, \tag{5.11}$$

$$Z_{S\mathbf{k}} \equiv \frac{3}{4}F^{\|''}_{S\mathbf{k}} + \frac{9c_N^2}{4\tau}F^{\|'}_{S\mathbf{k}} - \frac{9c_N^2+3}{2\tau^2}F^{\|}_{S\mathbf{k}} - \frac{3c_N^2}{4}A'_{S\mathbf{k}} + \frac{3c_N^2}{4\tau}A_{S\mathbf{k}} - \frac{\tau}{2}c_N^2 M^{\ l}_{S\mathbf{k}l} + \frac{\tau}{2}c_N^2 E'_{S\mathbf{k}}. \tag{5.12}$$

For a general 2nd-order sound speed c_N, the homogeneous solution of (5.11) is similar to the 1st-order solution (3.32) with a replacement of $c_L \to c_N$, the inhomogeneous solution of (5.11) is complicated. For the case $c_N^2 = \frac{1}{3}$ and a general c_L, (5.11) becomes

$$v^{\|(2)'''}_{S\mathbf{k}} + \frac{1}{\tau}v^{\|(2)''}_{S\mathbf{k}} + \left(\frac{k^2}{3} - \frac{4}{\tau^2}\right)v^{\|(2)'}_{S\mathbf{k}} + \frac{k^2}{3\tau}v^{\|(2)}_{S\mathbf{k}} = Z_{S\mathbf{k}}(\tau). \tag{5.13}$$

The explicit expression of Z_S in (5.10) is

$$\begin{aligned}
Z_S(\mathbf{x},\tau) = {}& \nabla^2\left[\frac{4\tau}{3}\phi^{(1)'}\phi^{(1)} + \frac{\tau}{18}\phi^{(1)}\nabla^2\chi^{\|(1)'} + \frac{2\tau}{9}\phi^{(1)'}\nabla^2\chi^{\|(1)} + \frac{\tau}{108}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)'} + \frac{\tau}{36}\chi^{\|(1)}_{,l}\nabla^2\chi^{\|(1)',l}\right] \\
& + \frac{4}{3\tau^2}v^{\|(1)}_{,l}v^{\|(1),l} - \frac{8}{3\tau}v^{\|(1)',l}v^{\|(1)}_{,l} + \frac{1}{2\tau}(1+c_L^2)\delta^{(1)}_{,l}v^{\|(1),l} + \frac{1}{2\tau}(1+c_L^2)\delta^{(1)}\nabla^2 v^{\|(1)} - \frac{2}{3\tau}\phi^{(1),l}v^{\|(1)}_{,l} \\
& - \frac{8}{3\tau}\phi^{(1)}\nabla^2 v^{\|(1)} + \frac{4}{3\tau}v^{\|(1),lm}\chi^{\|(1)}_{,lm} + \frac{8}{9\tau}v^{\|(1),l}\nabla^2\chi^{\|(1)}_{,l} - \frac{4}{9\tau}\nabla^2 v^{\|(1)}\nabla^2\chi^{\|(1)} + \frac{3}{2\tau}(1+c_L^2)\delta^{(1)}\phi^{(1)'} - \frac{8}{3}v^{\|(1)',l}v^{\|(1)'}_{,l} \\
& - \frac{8}{3}v^{\|(1)'',l}v^{\|(1)}_{,l} + \frac{1}{2}(1+c_L^2)\delta^{(1)'}_{,l}v^{\|(1),l} + \frac{1}{2}(1+c_L^2)\delta^{(1)}_{,l}v^{\|(1)',l} + \frac{1}{2}(1+c_L^2)\delta^{(1)'}\nabla^2 v^{\|(1)} + \frac{1}{2}(1+c_L^2)\delta^{(1)}\nabla^2 v^{\|(1)'} \\
& - \frac{3}{2}(1+c_L^2)\delta^{(1)'}\phi^{(1)'} - \frac{3}{2}(1+c_L^2)\delta^{(1)}\phi^{(1)''} - 2\phi^{(1)'}_{,l}v^{\|(1),l} - 2\phi^{(1)}_{,l}v^{\|(1)',l} - 2\tau\phi^{(1)'}\phi^{(1)''} - 2\tau\phi^{(1)}_{,l}\phi^{(1)',l} \\
& - \frac{8\tau}{3}\phi^{(1)'}\nabla^2\phi^{(1)} - \frac{8\tau}{3}\phi^{(1)}\nabla^2\phi^{(1)'} - \frac{4\tau}{9}\phi^{(1)'}\nabla^2\nabla^2\chi^{\|(1)} - \frac{5\tau}{18}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)'} + \frac{2\tau}{3}\phi^{(1)',lm}\chi^{\|(1)}_{,lm} + \frac{\tau}{6}\phi^{(1),lm}\chi^{\|(1)'}_{,lm} \\
& - \frac{4\tau}{9}\nabla^2\phi^{(1)'}\nabla^2\chi^{\|(1)} - \frac{\tau}{9}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)'} + \frac{\tau}{12}\chi^{\|(1)',lm}\chi^{\|(1)''}_{,lm} - \frac{\tau}{36}\nabla^2\chi^{\|(1)'}\nabla^2\chi^{\|(1)''} + \frac{\tau}{36}\chi^{\|(1)',lm}\nabla^2\chi^{\|(1)}_{,lm} \\
& + \frac{\tau}{18}\chi^{\|(1),lm}\nabla^2\chi^{\|(1)'}_{,lm} - \frac{\tau}{54}\nabla^2\chi^{\|(1)'}\nabla^2\nabla^2\chi^{\|(1)} - \frac{5\tau}{108}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)'} + \frac{\tau}{36}\nabla^2\chi^{\|(1)}_{,l}\nabla^2\chi^{\|(1)',l} \\
& - \frac{\tau}{36}\chi^{\|(1)}_{,l}\nabla^2\nabla^2\chi^{\|(1)',l} + \nabla^{-2}\Big[-\frac{3}{2}(1+c_L^2)\delta^{(1)'''}_{,l}v^{\|(1),l} - \frac{3}{2}(1+c_L^2)\delta^{(1)'''}\nabla^2 v^{\|(1)} - \frac{9}{2}(1+c_L^2)\delta^{(1)'',l}v^{\|(1)'}_{,l} \\
& - \frac{9}{2}(1+c_L^2)\delta^{(1)''}\nabla^2 v^{\|(1)'} - \frac{9}{2}(1+c_L^2)\delta^{(1)',l}v^{\|(1)''}_{,l} - \frac{9}{2}(1+c_L^2)\delta^{(1)'}\nabla^2 v^{\|(1)''} - \frac{3}{2}(1+c_L^2)\delta^{(1),l}v^{\|(1)'''}_{,l} \\
& - \frac{3}{2}(1+c_L^2)\delta^{(1)}\nabla^2 v^{\|(1)'''} + 4v^{\|(1)'}_{,lm}v^{\|(1)',lm} + 4v^{\|(1)}_{,lm}v^{\|(1)'',lm} - 4\nabla^2 v^{\|(1)'}\nabla^2 v^{\|(1)'} - 4\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)''} \\
& - 3c_L^2\delta^{(1)''}_{,l}\phi^{(1),l} - 3c_L^2\phi^{(1)}\nabla^2\delta^{(1)''} - 6c_L^2\delta^{(1)'}_{,l}\phi^{(1)',l} - 6c_L^2\phi^{(1)'}\nabla^2\delta^{(1)'} - 3c_L^2\phi^{(1)''}\nabla^2\delta^{(1)} - 3c_L^2\delta^{(1)}_{,l}\phi^{(1)'',l} \\
& + 10v^{\|(1)'',l}\phi^{(1)'}_{,l} + 10\phi^{(1)'}\nabla^2 v^{\|(1)''} + 20v^{\|(1)',l}\phi^{(1)''}_{,l} + 20\phi^{(1)''}\nabla^2 v^{\|(1)'} + 10v^{\|(1),l}\phi^{(1)'''}_{,l} + 10\phi^{(1)'''}\nabla^2 v^{\|(1)} \\
& + \frac{3}{2}c_L^2\delta^{(1)'',lm}\chi^{\|(1)}_{,lm} + c_L^2\delta^{(1)'',l}\nabla^2\chi^{\|(1)}_{,l} - \frac{1}{2}c_L^2\nabla^2\delta^{(1)''}\nabla^2\chi^{\|(1)} + 3c_L^2\delta^{(1)',lm}\chi^{\|(1)'}_{,lm} + 2c_L^2\delta^{(1)',l}\nabla^2\chi^{\|(1)'}_{,l} \\
& - c_L^2\nabla^2\delta^{(1)'}\nabla^2\chi^{\|(1)'} + \frac{3}{2}c_L^2\delta^{(1),lm}\chi^{\|(1)''}_{,lm} + c_L^2\delta^{(1),l}\nabla^2\chi^{\|(1)''}_{,l} - \frac{1}{2}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)''} - 2v^{\|(1)'',lm}\chi^{\|(1)'}_{,lm} \\
& - \frac{4}{3}v^{\|(1)'',l}\nabla^2\chi^{\|(1)'}_{,l} + \frac{2}{3}\nabla^2 v^{\|(1)''}\nabla^2\chi^{\|(1)'} - 4v^{\|(1)',lm}\chi^{\|(1)''}_{,lm} - \frac{8}{3}v^{\|(1)',l}\nabla^2\chi^{\|(1)''}_{,l} + \frac{4}{3}\nabla^2 v^{\|(1)'}\nabla^2\chi^{\|(1)''} \\
& - 2v^{\|(1),lm}\chi^{\|(1)'''}_{,lm} - \frac{4}{3}v^{\|(1),l}\nabla^2\chi^{\|(1)'''}_{,l} + \frac{2}{3}\nabla^2 v^{\|(1)}\nabla^2\chi^{\|(1)'''} - \frac{3}{2\tau}(1+c_L^2)\delta^{(1)''}_{,l}v^{\|(1),l} - \frac{3}{2\tau}(1+c_L^2)\delta^{(1)''}\nabla^2 v^{\|(1)} \\
& - \frac{3}{2\tau}(1+c_L^2)\delta^{(1),l}v^{\|(1)''}_{,l} - \frac{3}{2\tau}(1+c_L^2)\delta^{(1)}\nabla^2 v^{\|(1)''} - \frac{3}{\tau}(1+c_L^2)\delta^{(1)',l}v^{\|(1)'}_{,l} - \frac{3}{\tau}(1+c_L^2)\delta^{(1)'}\nabla^2 v^{\|(1)'} \\
& + \frac{4}{\tau}v^{\|(1)}_{,lm}v^{\|(1)',lm} - \frac{4}{\tau}\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)'} - \frac{3}{\tau}c_L^2\delta^{(1)'}_{,l}\phi^{(1),l} - \frac{3}{\tau}c_L^2\delta^{(1)}_{,l}\phi^{(1)',l} - \frac{3}{\tau}c_L^2\phi^{(1)'}\nabla^2\delta^{(1)} - \frac{3}{\tau}c_L^2\phi^{(1)}\nabla^2\delta^{(1)'}
\end{aligned}$$

$$+\frac{10}{\tau}v^{\|(1)',l}\phi^{(1)'}_{,l}+\frac{10}{\tau}v^{\|(1),l}\phi^{(1)''}_{,l}+\frac{10}{\tau}\phi^{(1)''}\nabla^2v^{\|(1)}+\frac{10}{\tau}\phi^{(1)'}\nabla^2v^{\|(1)'}+\frac{3}{2\tau}c_L^2\delta^{(1)',lm}\chi^{\|(1)}_{,lm}+\frac{3}{2\tau}c_L^2\delta^{(1),lm}\chi^{\|(1)'}_{,lm}$$
$$+\frac{1}{\tau}c_L^2\delta^{(1)',l}\nabla^2\chi^{\|(1)}_{,l}+\frac{1}{\tau}c_L^2\delta^{(1),l}\nabla^2\chi^{\|(1)'}_{,l}-\frac{1}{2\tau}c_L^2\nabla^2\delta^{(1)'}\nabla^2\chi^{\|(1)}-\frac{1}{2\tau}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)'}-\frac{2}{\tau}v^{\|(1)',lm}\chi^{\|(1)'}_{,lm}-\frac{2}{\tau}v^{\|(1),lm}\chi^{\|(1)''}_{,lm}$$
$$-\frac{4}{3\tau}v^{\|(1)',l}\nabla^2\chi^{\|(1)'}_{,l}-\frac{4}{3\tau}v^{\|(1),l}\nabla^2\chi^{\|(1)''}_{,l}+\frac{2}{3\tau}\nabla^2v^{\|(1)'}\nabla^2\chi^{\|(1)'}+\frac{2}{3\tau}\nabla^2v^{\|(1)}\nabla^2\chi^{\|(1)''}+\frac{6}{\tau^2}(1+c_L^2)\delta^{(1)'}_{,l}v^{\|(1),l}$$
$$+\frac{6}{\tau^2}(1+c_L^2)\delta^{(1)'}\nabla^2v^{\|(1)}+\frac{6}{\tau^2}(1+c_L^2)\delta^{(1),l}v^{\|(1)'}_{,l}+\frac{6}{\tau^2}(1+c_L^2)\delta^{(1)}\nabla^2v^{\|(1)'}+\frac{8}{\tau^2}v^{\|(1)}_{,l}\nabla^2v^{\|(1),l}+\frac{8}{\tau^2}\nabla^2v^{\|(1)}\nabla^2v^{\|(1)}$$
$$+\frac{12}{\tau^2}c_L^2\delta^{(1)}_{,l}\phi^{(1),l}+\frac{12}{\tau^2}c_L^2\phi^{(1)}\nabla^2\delta^{(1)}-\frac{40}{\tau^2}v^{\|(1),l}\phi^{(1)'}_{,l}-\frac{40}{\tau^2}\phi^{(1)'}\nabla^2v^{\|(1)}-\frac{6}{\tau^2}c_L^2\delta^{(1),lm}\chi^{\|(1)}_{,lm}-\frac{4}{\tau^2}c_L^2\delta^{(1),l}\nabla^2\chi^{\|(1)}_{,l}$$
$$+\frac{2}{\tau^2}c_L^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)}+\frac{8}{\tau^2}v^{\|(1),lm}\chi^{\|(1)'}_{,lm}+\frac{16}{3\tau^2}v^{\|(1),l}\nabla^2\chi^{\|(1)'}_{,l}-\frac{8}{3\tau^2}\nabla^2v^{\|(1)}\nabla^2\chi^{\|(1)'}\Bigg], \tag{5.14}$$

and its Fourier transform is $Z_{S\mathbf{k}}(\tau)$ can be given, which is a lengthy expression. Here, as an illustration, one typical term of (5.14) has the Fourier transformation as the following:

$$\begin{aligned}\frac{1}{(2\pi)^{3/2}}\int d^3x\left[\frac{12}{\tau^2}c_L^2\phi^{(1)}\nabla^2\delta^{(1)}\right]e^{-i\mathbf{k}\cdot\mathbf{x}}&=\frac{12c_L^2}{(2\pi)^{9/2}\tau^2}\int d^3x\left(\int d^3k_1\phi^{(1)}_{\mathbf{k_1}}e^{i\mathbf{k_1}\cdot\mathbf{x}}\right)\left(\int d^3k_2\delta^{(1)}_{\mathbf{k_2}}e^{i\mathbf{k_2}\cdot\mathbf{x}}(-|\mathbf{k_2}|^2)\right)e^{-i\mathbf{k}\cdot\mathbf{x}}\\&=-\frac{12c_L^2}{(2\pi)^3\tau^2}\int d^3k_1\int d^3k_2|\mathbf{k_2}|^2\phi^{(1)}_{\mathbf{k_1}}\delta^{(1)}_{\mathbf{k_2}}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}-\mathbf{k})\\&=-\frac{12c_L^2}{(2\pi)^3\tau^2}\int d^3k_1|\mathbf{k}-\mathbf{k_1}|^2\phi^{(1)}_{\mathbf{k_1}}\delta^{(1)}_{(\mathbf{k}-\mathbf{k_1})},\end{aligned} \tag{5.15}$$

where the 1st-order $\phi^{(1)}_{\mathbf{k}}$ and $\delta^{(1)}_{\mathbf{k}}$ are known in (3.56) and (3.58) for $c_L^2=\frac{1}{3}$. [For a general c_L, $\delta^{(1)}_{\mathbf{k}}$ and $\phi^{(1)}_{\mathbf{k}}$ are given by (3.26) and (3.27), using the solution $v^{\|(1)}$ (3.32).] Other terms in (5.14) can be calculated similarly to (5.15). Once $Z_{S\mathbf{k}}$ is obtained, the general solution of (5.13) follows:

$$\begin{aligned}v^{\|(2)}_{S\mathbf{k}}=&\frac{G_1}{k\tau}+G_2\left(\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}}+G_3\left(\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}}-\left(\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\Bigg(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\\&+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\Bigg)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'-\left(\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^\tau\Bigg(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\\&-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\Bigg)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'+\frac{1}{k\tau}\int^\tau\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau',\end{aligned} \tag{5.16}$$

where G_1, G_2, G_3 are $\mathbf{k}$-dependent constants and correspond to the homogeneous solution, having the same form as the 1st-order solution (3.33). (The G_1 term is a gauge mode as shall be seen in the next section).

The general solution of 2nd-order density contrast in $\mathbf{k}$-space follows from (5.5) as the following

$$\begin{aligned}\delta^{(2)}_{S\mathbf{k}}=&-4v^{\|(2)'}_{S\mathbf{k}}+3F^{\|}_{S\mathbf{k}}\\=&\frac{4G_1}{k\tau^2}+G_2\left(\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}}+G_3\left(\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}}+3F^{\|}_{S\mathbf{k}}+\Bigg(-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\\&-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\Bigg)\int^\tau\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'+\Bigg(-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\\&+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\Bigg)\int^\tau\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\&+\frac{1}{k\tau^2}\int^\tau\frac{12(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau'.\end{aligned} \tag{5.17}$$

The general solution of scalar $\phi^{(2)}_{S\mathbf{k}}$ is given by integrating (5.6),

$$
\begin{aligned}
\phi^{(2)}_{S\mathbf{k}} &= -v^{\|(2)\prime}_{S\mathbf{k}} - \int^{\tau} \frac{k^2}{3} v^{\|(2)}_{S\mathbf{k}} d\tau' + \frac{3}{4} F^{\|}_{S\mathbf{k}} - \frac{1}{4}\int^{\tau} A_{S\mathbf{k}} d\tau' + G_4 \\
&= G_1\left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right) + G_2\left(\frac{2}{k\tau^2} + \frac{2i}{\sqrt{3}\tau}\right) e^{-ik\tau/\sqrt{3}} + G_3\left(\frac{2}{k\tau^2} - \frac{2i}{\sqrt{3}\tau}\right) e^{ik\tau/\sqrt{3}} + G_4 \\
&\quad - \frac{2k}{3}\int^{\tau} [G_2 e^{-ik\tau'/\sqrt{3}} + G_3 e^{ik\tau'/\sqrt{3}}] \frac{d\tau'}{\tau'} + \frac{3}{4} F^{\|}_{S\mathbf{k}}(\tau) - \frac{1}{4}\int^{\tau} A_{S\mathbf{k}}(\tau') d\tau' \\
&\quad + \int^{\tau} \frac{(k^2\tau'^2 + 6)\ln\tau' + 3}{k^2\tau'} Z_{S\mathbf{k}}(\tau') d\tau' + \left(\frac{1}{k\tau^2} - \frac{k\ln\tau}{3}\right)\int^{\tau} \frac{3(k^2\tau'^2+6)}{k^3\tau'} Z_{S\mathbf{k}}(\tau') d\tau' \\
&\quad - \left(\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'} d\tau' \\
&\quad - \left(\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'} d\tau' \\
&\quad + \int^{\tau}\left[\frac{2}{k\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'} d\tau'\right] d\tau'' \\
&\quad + \int^{\tau}\left[\frac{2}{k\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'} d\tau'\right] d\tau'', \qquad (5.18)
\end{aligned}
$$

where G_4 is a $\mathbf{k}$-dependent constant, which is also a gauge term, as shall be seen in the next section. Finally, plugging (5.17) and (5.18) into (4.1) in $\mathbf{k}$-space, one obtains the general solution for scalar $\chi^{\|(2)}_{S\mathbf{k}}$ as the following:

$$
\begin{aligned}
\chi^{\|(2)}_{S\mathbf{k}} &= \frac{18}{k^4\tau}\phi^{(2)\prime}_{S\mathbf{k}} + \frac{6}{k^2}\phi^{(2)}_{S\mathbf{k}} + \frac{9}{k^4\tau^2}\delta^{(2)}_{S\mathbf{k}} + \frac{3}{k^4}E_{S\mathbf{k}} \\
&= -G_1\frac{2\ln\tau}{k} + G_2\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - G_3\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} + \frac{6G_4}{k^2} - \frac{4}{k}\int^{\tau}[G_2 e^{-ik\tau'/\sqrt{3}} + G_3 e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} \\
&\quad - \frac{2\ln\tau}{k}\int^{\tau}\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau' + \int^{\tau}\frac{6(k^2\tau'^2+6)\ln\tau' + 18}{k^4\tau'}Z_{S\mathbf{k}}(\tau')d\tau' \\
&\quad - \frac{4\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + 3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad + \frac{4\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - 3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' \\
&\quad + \int^{\tau}\left[\frac{12}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right) + \sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \int^{\tau}\left[\frac{12}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right) - \sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'' \\
&\quad + \frac{3}{k^4}E_{S\mathbf{k}} + \frac{27}{2k^4\tau}F^{\|\prime}_{S\mathbf{k}} + \frac{27}{k^4\tau^2}F^{\|}_{S\mathbf{k}} + \frac{9}{2k^2}F^{\|}_{S\mathbf{k}} - \frac{9}{2k^4\tau}A_{S\mathbf{k}} - \frac{3}{2k^2}\int^{\tau}A_{S\mathbf{k}}(\tau')d\tau'. \qquad (5.19)
\end{aligned}
$$

We have checked that the scalar solutions (5.16), (5.17), (5.18), and (5.19) satisfy the scalar parts of the evolution equations (4.11) and (4.15). Thus far, all the solutions of the 2nd-order perturbations have been given.

The above 2nd-order solutions involve many terms of integrals of the scalar-scalar coupling terms, each of which contains $\int d^3k$ integrations and time integrations $\int d\tau$. In the $\mathbf{k}$-integrations, two functions $D_2(\mathbf{k})$ and $D_3(\mathbf{k})$ appear, which depend upon the concrete initial condition at the beginning of RD stage and should be practically determined by the precedent inflation or reheating stages. Various models of inflation will give different $D_2(\mathbf{k})$, $D_3(\mathbf{k})$. Moreover, in actually doing integration, one should avoid IR and UV divergences [20] which may arise from the lower and upper limits of $\int d^3k$, so that $D_2(\mathbf{k})$, $D_3(\mathbf{k})$ may be required to satisfy certain conditions. As an illustration, suppose $D_2(\mathbf{k}) \propto k^{N_1}$ and $D_3(\mathbf{k}) \propto k^{N_2}$. Then we shall have the following typical integration terms:

$$\int d^3k_1 k_1^{n_1}|\mathbf{k}-\mathbf{k_1}|^{n_2} \propto \int_{K_1}^{K_2} dk_1 k_1^{n_1+2}\frac{(k+k_1)^{n_2+2}-|k-k_1|^{n_2+2}}{kk_1},$$

where n_1 and n_2 are linearly related to N_1 and N_2, and the integration limits K_1 and K_2 are introduced as possible cutoffs to ensure IR and UV convergence. All other terms can be treated similarly.

The time integrations can also be carried out. The types of time integrations of $A_{S\mathbf{k}}$ in (5.18) and (5.19) are already contained in those of $Z_{S\mathbf{k}}(\tau)$, the latter has four types of terms: $\frac{1}{\tau^n}$, $\frac{1}{\tau^n}e^{-i\frac{k\tau}{\sqrt{3}}}$, $\int^\tau \frac{d\tau'}{\tau'}e^{-i\frac{k\tau'}{\sqrt{3}}}$, and $(\int^\tau \frac{d\tau'}{\tau'}e^{-i\frac{k_1\tau'}{\sqrt{3}}})(\int^\tau \frac{d\tau''}{\tau''}e^{-i\frac{k_2\tau''}{\sqrt{3}}})$. The single time integrations of $Z_{S\mathbf{k}}(\tau)$ have the following nontrivial terms:

$$\int^\tau \frac{d\tau'}{\tau'^n}e^{-i\frac{k\tau'}{\sqrt{3}}} \propto k^{n-1}\Gamma\left(1-n,\frac{ik\tau}{\sqrt{3}}\right),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}\int^{\tau'}\frac{d\tau''}{\tau''}e^{-i\frac{k\tau''}{\sqrt{3}}} \propto \tau^{1-n}\mathrm{Ei}\left(-\frac{ik\tau}{\sqrt{3}}\right)+3^{\frac{1}{2}-\frac{n}{2}}(ik)^{n-1}\Gamma\left(1-n,\frac{ik\tau}{\sqrt{3}}\right),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}\left(\int^{\tau'}\frac{d\tau''}{\tau''}e^{-i\frac{k_1\tau''}{\sqrt{3}}}\right)\left(\int^{\tau'}\frac{d\tau'''}{\tau'''}e^{-i\frac{k_2\tau'''}{\sqrt{3}}}\right) \equiv z_1(\tau;n;k_1,k_2),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}e^{-i\frac{k_1\tau'}{\sqrt{3}}}\int^{\tau'}\frac{d\tau''}{\tau''}e^{-i\frac{k_2\tau''}{\sqrt{3}}} \equiv z_2(\tau;n;k_1,k_2),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}e^{i\frac{k_3\tau'}{\sqrt{3}}}\left(\int^{\tau'}\frac{d\tau''}{\tau''}e^{-i\frac{k_1\tau''}{\sqrt{3}}}\right)\left(\int^{\tau'}\frac{d\tau'''}{\tau'''}e^{-i\frac{k_2\tau'''}{\sqrt{3}}}\right) \equiv z_3(\tau;n;k_1,k_2,k_3),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}\ln\tau' e^{-i\frac{k\tau'}{\sqrt{3}}} \propto 3^{\frac{1-n}{2}}(ik)^{n-1}\ln\tau\Gamma\left(1-n,0,\frac{ik\tau}{\sqrt{3}}\right)-(1-n)^{-2}\tau^{1-n}{}_2F_2\left(1-n,1-n;2-n,2-n;-\frac{ik\tau}{\sqrt{3}}\right),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}\ln\tau'\int^{\tau'}\frac{d\tau''}{\tau''}e^{-i\frac{k\tau''}{\sqrt{3}}} \propto -(n-1)^{-3}\tau^{1-n}{}_2F_2\left(1-n,1-n;2-n,2-n;-\frac{ik\tau}{\sqrt{3}}\right)$$
$$-(n-1)^{-2}\tau^{1-n}(1+(n-1)\ln\tau)\mathrm{Ei}\left(-\frac{ik\tau}{\sqrt{3}}\right)+(n-1)^{-2}3^{\frac{1}{2}-\frac{n}{2}}i^{n+1}k^{n-1}\left(\Gamma\left(1-n,\frac{ik\tau}{\sqrt{3}}\right)-(n-1)\ln\tau\Gamma\left(1-n,0,\frac{ik\tau}{\sqrt{3}}\right)\right),$$

$$\int^\tau \frac{d\tau'}{\tau'^n}\ln\tau'\left(\int^{\tau'}\frac{d\tau''}{\tau''}e^{-i\frac{k_1\tau''}{\sqrt{3}}}\right)\left(\int^{\tau'}\frac{d\tau'''}{\tau'''}e^{-i\frac{k_2\tau'''}{\sqrt{3}}}\right) \equiv z_4(\tau;n;k_1,k_2).$$

The double time integrations of $Z_{S\mathbf{k}}$ have the following nontrivial terms:

$$\int^\tau \frac{d\tau'}{\tau'}e^{i\frac{k_1\tau'}{\sqrt{3}}}\int^{\tau'}\frac{d\tau''}{\tau''^n}e^{i\frac{k_2\tau''}{\sqrt{3}}} \equiv z_5(\tau;n;k_1,k_2),$$

$$\int^\tau \frac{d\tau'}{\tau'}e^{i\frac{k_3\tau'}{\sqrt{3}}}\int^{\tau'}\frac{d\tau''}{\tau''^n}e^{i\frac{k_1\tau''}{\sqrt{3}}}\int^{\tau''}\frac{d\tau'''}{\tau'''}e^{-i\frac{k_2\tau'''}{\sqrt{3}}} \equiv z_6(\tau;n;k_1,k_2,k_3),$$

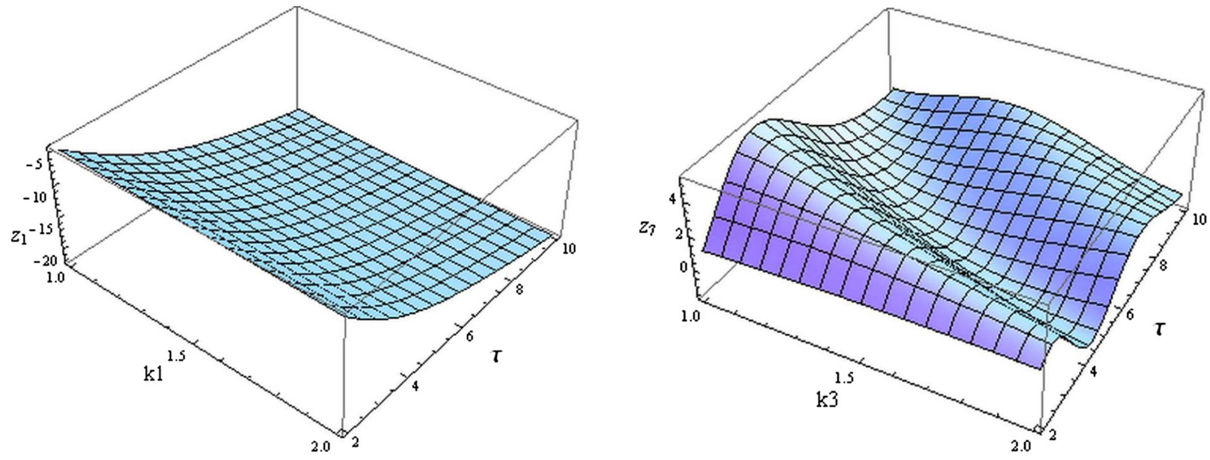
$$\int^\tau \frac{d\tau'}{\tau'}e^{i\frac{k_3\tau'}{\sqrt{3}}}\int^{\tau'}\frac{d\tau''}{\tau''^n}e^{i\frac{k_4\tau''}{\sqrt{3}}}\left(\int^{\tau''}\frac{d\tau'''}{\tau'''}e^{-i\frac{k_1\tau'''}{\sqrt{3}}}\right)\left(\int^{\tau''}\frac{d\tau''''}{\tau''''}e^{-i\frac{k_2\tau''''}{\sqrt{3}}}\right) \equiv z_7(\tau;n;k_1,k_2,k_3,k_4).$$

In actual computing, $z_1, ..., z_7$ in the above can be defined as functions and recalled. In our test computing, the triple time integrals, z_6 and z_7, take more computing time than $z_1, ...z_5$. As an illustration, we plot real parts of z_1 and z_7 in Fig. 1.

We mention that in the MD model [46,47], the scalar modes are not a wave and do not contain the oscillating factors $e^{\pm\frac{ik\tau}{\sqrt{3}}}$, so that the time integrations consist of powers of time τ, which are simpler than the RD model.

VI. 2ND-ORDER RESIDUAL GAUGE MODES

The 2nd-order perturbation solutions in the last section have the residual degrees of gauge freedom. A general 2nd-order coordinate transformation can involve a 2nd-order transformation vector $\xi^{(2)\mu}$, and the square of a 1st-order transformation vector $\xi^{(1)\mu}$ as well. For synchronous-to-synchronous coordinate transformations in a general RW spacetime, we list $\xi^{(1)\mu}$ in (C12) and (C13), $\xi^{(2)\mu}$ in (C27) and (C28) in Appendix C (see also Ref. [46]). For the RD stage and the scalar-scalar coupling, $\xi^{(2)\mu}$ is given in (C35), (C36), and (C37). The 2nd-order transformations of the 2nd-order metric perturbations for



(a) (b)

FIG. 1. Left: real part of $z_1(\tau;1;k_1,k_2)$ at fixed k_2; right: real part of $z_7(\tau;1;k_1,k_2,k_3,k_4)$ with fixed k_1, k_2, and k_4.

a general RW spacetime are given by (C31), (C32), (C33), and (C34), which, for the RD stage and the scalar-scalar coupling, reduce to

$$\begin{aligned}\bar{\phi}_S^{(2)} &= \phi_S^{(2)} + \frac{1}{\tau^2}\left[-\frac{2}{3}A^{(1)}\nabla^2 A^{(1)} - \frac{1}{3}A^{(1),l}A^{(1)}_{,l} - 4\phi^{(1)}A^{(1)}\right] - \frac{2}{\tau}\phi^{(1)\prime}A^{(1)} + \frac{\ln\tau}{\tau^2}\left[-\frac{4}{3}A^{(1)}\nabla^2 A^{(1)} - A^{(1)}_{,l}A^{(1),l}\right]\\
&\quad - \frac{(\ln\tau)^2}{3}A^{(1)}_{,lm}A^{(1),lm} + \ln\tau\left[\frac{2}{3}A^{(1),lm}C^{\|(1)}_{,lm} + \frac{2}{3}A^{(1),l}\nabla^2 C^{\|(1)}_{,l} - \frac{4}{3}\phi^{(1)}\nabla^2 A^{(1)} - 2\phi^{(1)}_{,l}A^{(1),l} + \frac{2}{3}A^{(1),lm}D_{lm}\chi^{\|(1)}\right]\\
&\quad + \nabla^2 A^{(1)}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})}{3\tau'}d\tau' + A^{(1)}_{,l}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})^{,l}}{3\tau'}d\tau' - A^{(1),lm}\int^\tau \frac{2D_{lm}\chi^{\|(1)}(\tau',\mathbf{x})}{3\tau'}d\tau'\\
&\quad - A^{(1)}_{,l}\int^\tau \frac{4\nabla^2\chi^{\|(1)}(\tau',\mathbf{x})^{,l}}{9\tau'}d\tau' + \frac{A^{(2)}}{\tau^2} + \frac{\ln\tau}{3}\nabla^2 A^{(2)} + \frac{1}{3}\nabla^2 C^{\|(2)},\end{aligned} \tag{6.1}$$

where $A^{(2)}$ and $C^{\|(2)}$ in the last line are due to the vector $\xi^{(2)\mu}$, and

$$\begin{aligned}\bar{\chi}_S^{\|(2)} &= \chi_S^{\|(2)} + \frac{1}{\tau^2}[A^{(1)}A^{(1)} + \nabla^{-2}(2A^{(1)}\nabla^2 A^{(1)} + 8A^{(1)}\nabla^2 C^{\|(1)}) + \nabla^{-2}\nabla^{-2}(3A^{(1),lm}A^{(1)}_{,lm} - 3\nabla^2 A^{(1)}\nabla^2 A^{(1)}\\
&\quad - 4A^{(1)}\nabla^2\nabla^2\chi^{\|(1)} - 6A^{(1),lm}D_{lm}\chi^{\|(1)} - 8A^{(1),l}\nabla^2\chi^{\|(1)}_{,l} + 12A^{(1),lm}C^{\|(1)}_{,lm} - 12\nabla^2 A^{(1)}\nabla^2 C^{\|(1)})]\\
&\quad + \frac{4\ln\tau}{\tau^2}[2\nabla^{-2}(A^{(1)}\nabla^2 A^{(1)}) + 3\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2 A^{(1)}\nabla^2 A^{(1)})] - \frac{1}{\tau}\nabla^{-2}\nabla^{-2}[2A^{(1)}\nabla^2\nabla^2\chi^{\|(1)\prime}\\
&\quad + 3A^{(1),lm}D_{lm}\chi^{\|(1)\prime} + 4A^{(1),m}\nabla^2\chi^{(1)\prime}_{,m}] + (\ln\tau)^2[\nabla^{-2}(2A^{(1)}_{,lm}A^{(1),lm}) + 3\nabla^{-2}\nabla^{-2}(\nabla^2 A^{(1),l}\nabla^2 A^{(1)}_{,l} - A^{(1),lmn}A^{(1)}_{,lmn})]\\
&\quad + \ln\tau[2A^{(1),l}C^{\|(1)}_{,l} + 2\nabla^{-2}(4\phi^{(1)}\nabla^2 A^{(1)} + A^{(1),lm}D_{lm}\chi^{\|(1)} - 2A^{(1),l}\nabla^2 C^{\|(1)}_{,l}) + \nabla^{-2}\nabla^{-2}(-2A^{(1)}_{,l}\nabla^2\nabla^2\chi^{\|(1),l}\\
&\quad - 9A^{(1),lmn}D_{lm}\chi^{\|(1)}_{,n} - 8A^{(1),lm}\nabla^2\chi^{\|(1)}_{,lm} - 4\nabla^2\chi^{\|(1)}_{,l}\nabla^2 A^{(1),l} - 6D_{lm}\chi^{\|(1)}\nabla^2 A^{(1),lm} + 12\phi^{(1),lm}A^{(1)}_{,lm} - 12\nabla^2\phi^{(1)}\nabla^2 A^{(1)}\\
&\quad + 6\nabla^2 A^{(1),l}\nabla^2 C^{\|(1)}_{,l} - 6A^{(1),lmn}C^{\|(1)}_{,lmn})] + [2C^{\|(1)}_{,l}C^{\|(1),l} + \nabla^{-2}(8\phi^{(1)}\nabla^2 C^{\|(1)} + 2C^{\|(1),lm}D_{lm}\chi^{\|(1)} - 2C^{\|(1),l}\nabla^2 C^{\|(1)}_{,l})\\
&\quad + \nabla^{-2}\nabla^{-2}(12\phi^{(1),lm}C^{\|(1)}_{,lm} - 12\nabla^2\phi^{(1)}\nabla^2 C^{\|(1)} - 2C^{\|(1)}_{,l}\nabla^2\nabla^2\chi^{\|(1),l} - 9C^{\|(1),lmn}D_{lm}\chi^{\|(1)}_{,n} - 8C^{\|(1),lm}\nabla^2\chi^{\|(1)}_{,lm}\\
&\quad - 4\nabla^2\chi^{\|(1)}_{,l}\nabla^2 C^{\|(1),l} - 6D_{lm}\chi^{\|(1)}\nabla^2 C^{\|(1),lm} + 3\nabla^2 C^{\|(1),l}\nabla^2 C^{\|(1)}_{,l} - 3C^{\|(1),lmn}C^{\|(1)}_{,lmn})]\\
&\quad - 4\nabla^{-2}\left[\nabla^2 A^{(1)}\int^\tau \frac{2\phi^{(1)}(\tau',\mathbf{x})}{\tau'}d\tau' + A^{(1)}_{,l}\int^\tau \frac{2\phi^{(1)}(\tau',\mathbf{x})^{,l}}{\tau'}d\tau' - A^{(1),lm}\int^\tau \frac{D_{lm}\chi^{\|(1)}(\tau',\mathbf{x})}{\tau'}d\tau'\right.\\
&\quad \left. - A^{(1)}_{,l}\int^\tau \frac{2\nabla^2\chi^{\|(1)}(\tau',\mathbf{x})^{,l}}{3\tau'}d\tau'\right] - 2A^{(2)}\ln\tau - 2C^{\|(2)},\end{aligned} \tag{6.2}$$

and

$$
\begin{aligned}
\bar{\chi}^{\perp(2)}_{Sij} = {} & \chi^{\perp(2)}_{Sij} + \frac{1}{\tau^2}\Big[2\partial_i\nabla^{-2}\Big(-A^{(1)}_{,j}\nabla^2A^{(1)} + 4A^{(1),l}C^{\|(1)}_{,lj} - 4A^{(1)}_{,j}\nabla^2C^{\|(1)} - \frac{4}{3}A^{(1)}\nabla^2\chi^{\|(1)}_{,j} - 2A^{(1),l}D_{lj}\chi^{\|(1)}\Big) \\
& + \partial_i\partial_j\nabla^{-2}(A^{(1),l}A^{(1)}_{,l}) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}\Big(-A^{(1),lm}A^{(1)}_{,lm} + \nabla^2A^{(1)}\nabla^2A^{(1)} + 4\nabla^2A^{(1)}\nabla^2C^{\|(1)} - 4A^{(1),lm}C^{\|(1)}_{,lm} \\
& + \frac{4}{3}A^{(1)}\nabla^2\nabla^2\chi^{\|(1)} + 2A^{(1),lm}D_{lm}\chi^{\|(1)} + \frac{8}{3}A^{(1),m}\nabla^2\chi^{\|(1)}_{,m}\Big)\Big] + \frac{4\ln\tau}{\tau^2}[-2\partial_i\nabla^{-2}(A^{(1)}_{,j}\nabla^2A^{(1)}) + \partial_i\partial_j\nabla^{-2}(A^{(1),l}A^{(1)}_{,l}) \\
& - 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2A^{(1)}\nabla^2A^{(1)})] - \frac{1}{\tau}\Big[2\partial_i\nabla^{-2}\Big(\frac{2}{3}A^{(1)}\nabla^2\chi^{\|(1)\prime}_{,j} + A^{(1),l}D_{lj}\chi^{\|(1)\prime}\Big) \\
& - 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}\Big(\frac{2}{3}A^{(1)}\nabla^2\nabla^2\chi^{\|(1)\prime} + A^{(1),lm}D_{lm}\chi^{\|(1)\prime} + \frac{4}{3}A^{(1),m}\nabla^2\chi^{\|(1)\prime}_{,m}\Big)\Big] + [\ln\tau]^2[2\partial_i\nabla^{-2}(A^{(1)}_{,lj}\nabla^2A^{(1),l}) \\
& - \partial_i\partial_j\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm}) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lmn}A^{(1)}_{,lmn} - \nabla^2A^{(1),l}\nabla^2A^{(1)}_{,l})] - \ln\tau\Big[2\partial_i\nabla^{-2}\Big(4\phi^{(1)}_{,j}\nabla^2A^{(1)} \\
& - 4\phi^{(1),l}A^{(1)}_{,lj} + \frac{2}{3}A^{(1),l}\nabla^2\chi^{\|(1)}_{,lj} + 2A^{(1),lm}D_{lj}\chi^{\|(1)}_{,m} + \frac{2}{3}A^{(1),l}_{,j}\nabla^2\chi^{\|(1)}_{,l} + A^{(1),lm}_{,j}D_{lm}\chi^{\|(1)} + D_{lj}\chi^{\|(1)}\nabla^2A^{(1),l} \\
& + 2A^{(1),lm}C^{\|(1)}_{,lmj} - 2A^{(1)}_{,lj}\nabla^2C^{\|(1),l}\Big) - 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}\Big(4\nabla^2\phi^{(1)}\nabla^2A^{(1)} - 4\phi^{(1),lm}A^{(1)}_{,lm} + \frac{2}{3}A^{(1),l}\nabla^2\nabla^2\chi^{\|(1)}_{,l} \\
& + 3A^{(1),lmn}D_{lm}\chi^{\|(1)}_{,n} + \frac{8}{3}A^{(1),lm}\nabla^2\chi^{\|(1)}_{,lm} + \frac{4}{3}\nabla^2\chi^{\|(1)}_{,m}\nabla^2A^{(1),m} + 2D_{lm}\chi^{\|(1)}\nabla^2A^{(1),lm} + 2A^{(1),lmn}C^{\|(1)}_{,lmn} \\
& - 2\nabla^2A^{(1),l}\nabla^2C^{\|(1)}_{,l}\Big)\Big] + \Big[-2\partial_i\nabla^{-2}\Big(4\phi^{(1)}_{,j}\nabla^2C^{\|(1)} - 4\phi^{(1),l}C^{\|(1)}_{,lj} + \frac{2}{3}C^{\|(1),l}\nabla^2\chi^{\|(1)}_{,lj} + 2C^{\|(1),lm}D_{lj}\chi^{\|(1)}_{,m} \\
& + \frac{2}{3}C^{\|(1),l}_{,j}\nabla^2\chi^{\|(1)}_{,l} + C^{\|(1),lm}_{,j}D_{lm}\chi^{\|(1)} + D_{lj}\chi^{\|(1)}\nabla^2C^{\|(1),l} + C^{\|(1),ln}C^{\|(1)}_{,lnj} - C^{\|(1)}_{,lj}\nabla^2C^{\|(1),l}\Big) \\
& + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}\Big(4\nabla^2\phi^{(1)}\nabla^2C^{\|(1)} - 4\phi^{(1),lm}C^{\|(1)}_{,lm} + \frac{2}{3}C^{\|(1),l}\nabla^2\nabla^2\chi^{\|(1)}_{,l} + \frac{8}{3}C^{\|(1),lm}\nabla^2\chi^{\|(1)}_{,lm} + 3C^{\|(1),lmn}D_{lm}\chi^{\|(1)}_{,n} \\
& + \frac{4}{3}\nabla^2\chi^{\|(1)}_{,m}\nabla^2C^{\|(1),m} + 2D_{lm}\chi^{\|(1)}\nabla^2C^{\|(1),lm} + C^{\|(1),lmn}C^{\|(1)}_{,lmn} - \nabla^2C^{\|(1),l}\nabla^2C^{\|(1)}_{,l}\Big)\Big] \\
& + \partial_i\Big[A^{(1),l}\int^{\tau}\frac{2D_{lj}\chi^{\|(1)}(\tau',\mathbf{x})}{\tau'}d\tau' - A^{(1)}_{,j}\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})}{\tau'}d\tau'\Big] \\
& + \partial_i\partial_j\nabla^{-2}\Big[\nabla^2A^{(1)}\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})}{\tau'}d\tau' + A^{(1)}_{,l}\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})^{,l}}{\tau'}d\tau' \\
& - A^{(1),lm}\int^{\tau}\frac{2D_{lm}\chi^{\|(1)}(\tau',\mathbf{x})}{\tau'}d\tau - A^{(1)}_{,l}\int^{\tau}\frac{4\nabla^2\chi^{\|(1)}(\tau',\mathbf{x})^{,l}}{3\tau'}d\tau'\Big] - C^{\perp(2)}_{i,j} + (i\leftrightarrow j),
\end{aligned}
\tag{6.3}
$$

and

$$
\begin{aligned}
\bar{\chi}^{\top(2)}_{Sij} = {} & \chi^{\top(2)}_{Sij} + \frac{1}{\tau^2}\Big[-\delta_{ij}\nabla^{-2}\Big(-A^{(1),lm}A^{(1)}_{,lm} + \nabla^2A^{(1)}\nabla^2A^{(1)} + \frac{4}{3}A^{(1)}\nabla^2\nabla^2\chi^{\|(1)} + 2A^{(1),lm}D_{lm}\chi^{\|(1)} + \frac{8}{3}A^{(1),m}\nabla^2\chi^{\|(1)}_{,m} \\
& + 4\nabla^2A^{(1)}\nabla^2C^{\|(1)} - 4A^{(1),lm}C^{\|(1)}_{,lm}\Big) - 4A^{(1)}D_{ij}\chi^{\|(1)} + 4\partial_i\nabla^{-2}\Big(\frac{2}{3}A^{(1)}\nabla^2\chi^{\|(1)}_{,j} + A^{(1),l}D_{lj}\chi^{\|(1)}\Big) \\
& + 4\partial_j\nabla^{-2}\Big(\frac{2}{3}A^{(1)}\nabla^2\chi^{\|(1)}_{,i} + A^{(1),l}D_{li}\chi^{\|(1)}\Big) + 4\nabla^{-2}(A^{(1)}_{,ij}\nabla^2A^{(1)} - A^{(1),l}_{,i}A^{(1)}_{,lj} + 2A^{(1)}_{,ij}\nabla^2C^{\|(1)} + 2C^{\|(1)}_{,ij}\nabla^2A^{(1)} \\
& - 2A^{(1),l}_{,i}C^{\|(1)}_{,lj} - 2A^{(1),l}_{,j}C^{\|(1)}_{,li}) - \partial_i\partial_j\nabla^{-2}\nabla^{-2}\Big(-A^{(1),lm}A^{(1)}_{,lm} + \nabla^2A^{(1)}\nabla^2A^{(1)} + \frac{4}{3}A^{(1)}\nabla^2\nabla^2\chi^{\|(1)} \\
& + 2A^{(1),lm}D_{lm}\chi^{\|(1)} + \frac{8}{3}A^{(1),m}\nabla^2\chi^{\|(1)}_{,m} + 4A^{(1),lm}C^{\|(1)}_{,lm} - 4\nabla^2A^{(1)}\nabla^2C^{\|(1)}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\ln\tau}{\tau^2}[\delta_{ij}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm}-\nabla^2A^{(1)}\nabla^2A^{(1)})+4\nabla^{-2}(A^{(1)}_{,ij}\nabla^2A^{(1)}-A^{(1),l}_{,i}A^{(1)}_{,lj})+\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm}\\
&-\nabla^2A^{(1)}\nabla^2A^{(1)})]-\frac{1}{\tau}\left[\delta_{ij}\nabla^{-2}\left(\frac{2}{3}A^{(1)}\nabla^2\nabla^2\chi^{\|(1)'}+A^{(1),lm}D_{lm}\chi^{\|(1)'}+\frac{4}{3}A^{(1),m}\nabla^2\chi^{\|(1)'}_{,m}\right)+2A^{(1)}D_{ij}\chi^{\|(1)'}\right.\\
&-2\partial_i\nabla^{-2}\left(\frac{2}{3}A^{(1)}\nabla^2\chi^{\|(1)'}_{,j}+A^{(1),l}D_{lj}\chi^{\|(1)'}\right)-2\partial_j\nabla^{-2}\left(\frac{2}{3}A^{(1)}\nabla^2\chi^{\|(1)'}_{,i}+A^{(1),l}D_{li}\chi^{\|(1)'}\right)\\
&\left.+\partial_i\partial_j\nabla^{-2}\nabla^{-2}\left(\frac{2}{3}A^{(1)}\nabla^2\nabla^2\chi^{\|(1)'}+A^{(1),lm}D_{lm}\chi^{\|(1)'}+\frac{4}{3}A^{(1),m}\nabla^2\chi^{\|(1)'}_{,m}\right)\right]-(\ln\tau)^2[\delta_{ij}\nabla^{-2}(A^{(1),lmn}A^{(1)}_{,lmn}\\
&-\nabla^2A^{(1),l}\nabla^2A^{(1)}_{,l})+2\nabla^{-2}(-2A^{(1),lm}_{,i}A^{(1)}_{,lmj}+2A^{(1)}_{,lij}\nabla^2A^{(1),l})+\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lmn}A^{(1)}_{,lmn}-\nabla^2A^{(1),l}\nabla^2A^{(1)}_{,l})]\\
&+\ln\tau\left[\delta_{ij}\nabla^{-2}\left(4\phi^{(1),lm}A^{(1)}_{,lm}-4\nabla^2\phi^{(1)}\nabla^2A^{(1)}-\frac{2}{3}A^{(1),l}\nabla^2\nabla^2\chi^{\|(1)}_{,l}-\frac{8}{3}A^{(1),lm}\nabla^2\chi^{\|(1)}_{,lm}+A^{(1),lmn}D_{lm}\chi^{\|(1)}_{,n}\right.\right.\\
&\left.-\frac{4}{3}\nabla^2\chi^{\|(1)}_{,m}\nabla^2A^{(1),m}+2A^{(1),lm}\nabla^2D_{lm}\chi^{\|(1)}-2A^{(1),lmn}C^{\|(1)}_{,lmn}+2\nabla^2A^{(1),l}\nabla^2C^{\|(1)}_{,l}\right)-2A^{(1),l}D_{ij}\chi^{\|(1)}_{,l}-2A^{(1),l}_{,j}D_{li}\chi^{\|(1)}\\
&-2A^{(1),l}_{,i}D_{lj}\chi^{\|(1)}-4\nabla^{-2}(2\phi^{(1),l}_{,i}A^{(1)}_{,lj}+2\phi^{(1),l}_{,j}A^{(1)}_{,li}-2\phi^{(1)}_{,ij}\nabla^2A^{(1)}-2A^{(1)}_{,ij}\nabla^2\phi^{(1)}-A^{(1),lm}_{,i}C^{\|(1)}_{,lmj}-A^{(1),lm}_{,j}C^{\|(1)}_{,lmi}\\
&+A^{(1)}_{,lij}\nabla^2C^{\|(1),l}+C^{\|(1)}_{,lij}\nabla^2A^{(1),l})+2\partial_i\nabla^{-2}\left(\frac{2}{3}A^{(1),m}\nabla^2\chi^{\|(1)}_{,mj}+2A^{(1),lm}D_{lj}\chi^{\|(1)}_{,m}+\frac{2}{3}A^{(1),l}_{,j}\nabla^2\chi^{\|(1)}_{,l}+A^{(1),lm}_{,j}D_{lm}\chi^{\|(1)}\right.\\
&\left.+D_{lj}\chi^{\|(1)}\nabla^2A^{(1),l}\right)+2\partial_j\nabla^{-2}\left(\frac{2}{3}A^{(1),m}\nabla^2\chi^{\|(1)}_{,mi}+2A^{(1),lm}D_{li}\chi^{\|(1)}_{,m}+\frac{2}{3}A^{(1),l}_{,i}\nabla^2\chi^{\|(1)}_{,l}+A^{(1),lm}_{,i}D_{lm}\chi^{\|(1)}\right.\\
&\left.+D_{li}\chi^{\|(1)}\nabla^2A^{(1),l}\right)+\partial_i\partial_j\nabla^{-2}\nabla^{-2}\left(4\phi^{(1),lm}A^{(1)}_{,lm}-4\nabla^2\phi^{(1)}\nabla^2A^{(1)}-\frac{2}{3}A^{(1),l}\nabla^2\nabla^2\chi^{\|(1)}_{,l}-\frac{8}{3}A^{(1),lm}\nabla^2\chi^{\|(1)}_{,lm}\right.\\
&-7A^{(1),lmn}D_{lm}\chi^{\|(1)}_{,n}-\frac{4}{3}\nabla^2\chi^{\|(1)}_{,m}\nabla^2A^{(1),m}-4D_{lm}\chi^{\|(1)}\nabla^2A^{(1),lm}-2A^{(1),lm}\nabla^2D_{lm}\chi^{\|(1)}-2A^{(1),lmn}C^{\|(1)}_{,lmn}\\
&\left.\left.+2\nabla^2A^{(1),l}\nabla^2C^{\|(1)}_{,l}\right)\right]+\left[\delta_{ij}\nabla^{-2}\left(4\phi^{(1),lm}C^{\|(1)}_{,lm}-4\nabla^2\phi^{(1)}\nabla^2C^{\|(1)}-\frac{2}{3}C^{\|(1),l}\nabla^2\nabla^2\chi^{\|(1)}_{,l}-\frac{8}{3}C^{\|(1),lm}\nabla^2\chi^{\|(1)}_{,lm}\right.\right.\\
&\left.+C^{\|(1),lmn}D_{lm}\chi^{\|(1)}_{,n}-\frac{4}{3}\nabla^2\chi^{\|(1)}_{,m}\nabla^2C^{\|(1),m}+2C^{\|(1),lm}\nabla^2D_{lm}\chi^{\|(1)}+\nabla^2C^{\|(1),l}\nabla^2C^{\|(1)}_{,l}-C^{\|(1),lmn}C^{\|(1)}_{,lmn}\right)\\
&-2C^{\|(1),l}D_{ij}\chi^{\|(1)}_{,l}-2C^{\|(1),l}_{,j}D_{li}\chi^{\|(1)}-2C^{\|(1),l}_{,i}D_{lj}\chi^{\|(1)}-4\nabla^{-2}(2\phi^{(1),l}_{,i}C^{\|(1)}_{,lj}+2\phi^{(1),l}_{,j}C^{\|(1)}_{,li}-2\phi^{(1)}_{,ij}\nabla^2C^{\|(1)}\\
&-2C^{\|(1)}_{,ij}\nabla^2\phi^{(1)}-C^{\|(1),lm}_{,i}C^{\|(1)}_{,lmj}+C^{\|(1)}_{,lij}\nabla^2C^{\|(1),l})+2\partial_i\nabla^{-2}\left(\frac{2}{3}C^{\|(1),m}\nabla^2\chi^{\|(1)}_{,mj}+2C^{\|(1),lm}D_{lj}\chi^{\|(1)}_{,m}\right.\\
&\left.+\frac{2}{3}C^{\|(1),l}_{,j}\nabla^2\chi^{\|(1)}_{,l}+C^{\|(1),lm}_{,j}D_{lm}\chi^{\|(1)}+D_{lj}\chi^{\|(1)}\nabla^2C^{\|(1),l}\right)+2\partial_j\nabla^{-2}\left(\frac{2}{3}C^{\|(1),m}\nabla^2\chi^{\|(1)}_{,mi}+2C^{\|(1),lm}D_{li}\chi^{\|(1)}_{,m}\right.\\
&\left.+\frac{2}{3}C^{\|(1),l}_{,i}\nabla^2\chi^{\|(1)}_{,l}+C^{\|(1),lm}_{,i}D_{lm}\chi^{\|(1)}+D_{li}\chi^{\|(1)}\nabla^2C^{\|(1),l}\right)-\partial_i\partial_j\nabla^{-2}\nabla^{-2}\left(4\nabla^2\phi^{(1)}\nabla^2C^{\|(1)}-4\phi^{(1),lm}C^{\|(1)}_{,lm}\right.\\
&+\frac{2}{3}C^{\|(1),l}\nabla^2\nabla^2\chi^{\|(1)}_{,l}+\frac{8}{3}C^{\|(1),lm}\nabla^2\chi^{\|(1)}_{,lm}+7C^{\|(1),lmn}D_{lm}\chi^{\|(1)}_{,n}+\frac{4}{3}\nabla^2\chi^{\|(1)}_{,l}\nabla^2C^{\|(1),l}+4D_{lm}\chi^{\|(1)}\nabla^2C^{\|(1),lm}\\
&\left.\left.+2C^{\|(1),lm}\nabla^2D_{lm}\chi^{\|(1)}+C^{\|(1),lmn}C^{\|(1)}_{,lmn}-\nabla^2C^{\|(1),l}\nabla^2C^{\|(1)}_{,l}\right)\right]. \qquad (6.4)
\end{aligned}
$$

The 2nd-order transformation of the 2nd-order density, density contrast, and velocity for a general RW spacetime are give in (C38), (C39), (C42), (C43), and (C44). For the RD stage and the scalar-scalar coupling, (C39) reduces to the following transformation of the density contrast:

$$
\bar{\delta}^{(2)}_S=\delta^{(2)}_S+\frac{24}{\tau^4}A^{(1)}A^{(1)}-\frac{4\ln\tau}{\tau^2}A^{(1)}_{,l}A^{(1),l}+\frac{1}{\tau^2}[-4A^{(1)}_{,l}C^{\|(1),l}+8\delta^{(1)}A^{(1)}]-\frac{2}{\tau}\delta^{(1)'}A^{(1)}-2\delta^{(1)}_{,l}C^{\|(1),l}-2\delta^{(1)}_{,l}A^{(1),l}\ln\tau+\frac{4}{\tau^2}A^{(2)}, \qquad (6.5)
$$

and (C43) and (C44) reduce to the following transformation of velocity

$$\bar{v}_S^{\parallel(2)} = v_S^{\parallel(2)} + \frac{2}{\tau^3}A^{(1)}A^{(1)} + \frac{2}{\tau^2}\nabla^{-2}[v^{\parallel(1),l}A^{(1)}_{,l} + A^{(1)}\nabla^2 v^{\parallel(1)}] + \frac{1}{\tau}\bigg[A^{(1)}_{,l}C^{\parallel(1),l} + \nabla^{-2}\bigg(-2C^{\parallel(1),l}\nabla^2 A^{(1)}_{,l} + 2A^{(1)}_{,l}\nabla^2 C^{\parallel(1),l} + 4A^{(1),l}\phi^{(1)}_{,l} + 4\phi^{(1)}\nabla^2 A^{(1)} - 2A^{(1)}_{,lm}D^{lm}\chi^{\parallel(1)} - \frac{4}{3}A^{(1)}_{,l}\nabla^2\chi^{\parallel(1),l}\bigg)\bigg] + \frac{\ln\tau}{\tau}A^{(1)}_{,l}A^{(1),l} + [2v^{\parallel(1),l}C^{\parallel(1)}_{,l} + \nabla^{-2}(-4v^{\parallel(1)}_{,lm}C^{\parallel(1),lm} - 4C^{\parallel(1),l}\nabla^2 v^{\parallel(1)}_{,l})] + \ln\tau[2v^{\parallel(1),l}A^{(1)}_{,l} + \nabla^{-2}(-4v^{\parallel(1)}_{,lm}A^{(1),lm} - 4A^{(1),l}\nabla^2 v^{\parallel(1)}_{,l})] + \frac{A^{(2)}}{\tau}, \tag{6.6}$$

$$\bar{v}_{Si}^{\perp(2)} = v_{Si}^{\perp(2)} + \frac{2}{\tau^2}[v^{\parallel(1)}_{,i}A^{(1)} + \partial_i\nabla^{-2}(-v^{\parallel(1),l}A^{(1)}_{,l} - A^{(1)}\nabla^2 v^{\parallel(1)})] + \frac{1}{\tau}\bigg[-2A^{(1)}_{,li}C^{\parallel(1),l} + 2A^{(1),l}C^{\parallel(1)}_{,li} + 4A^{(1)}_{,i}\phi^{(1)} - 2A^{(1),l}D_{li}\chi^{\parallel(1)} + \partial_i\nabla^{-2}\bigg(2C^{\parallel(1),l}\nabla^2 A^{(1)}_{,l} - 2A^{(1)}_{,l}\nabla^2 C^{\parallel(1),l} - 4A^{(1),l}\phi^{(1)}_{,l} - 4\phi^{(1)}\nabla^2 A^{(1)} + 2A^{(1)}_{,lm}D^{lm}\chi^{\parallel(1)} + \frac{4}{3}A^{(1)}_{,l}\nabla^2\chi^{\parallel(1),l}\bigg)\bigg] + [-4v^{\parallel(1)}_{,li}C^{\parallel(1),l} + \partial_i\nabla^{-2}(4v^{\parallel(1)}_{,lm}C^{\parallel(1),lm} + 4C^{\parallel(1),l}\nabla^2 v^{\parallel(1)}_{,l})] + \ln\tau[-4v^{\parallel(1)}_{,li}A^{(1),l} + \partial_i\nabla^{-2}(4v^{\parallel(1)}_{,lm}A^{(1),lm} + 4A^{(1),l}\nabla^2 v^{\parallel(1)}_{,l})]. \tag{6.7}$$

The above synchronous-to-synchronous transformations are general, in the sense that two vector fields $\xi^{(1)}$ and $\xi^{(2)}$ are involved simultaneously. These expressions are lengthy due to the parameters $A^{(1)}$, $C^{\parallel(1)}$, and $C_i^{\perp(1)}$ of the 1st-order vector $\xi^{(1)}$, and only a few terms are due to the parameters $A^{(2)}$, $C^{\parallel(2)}$, and $C_i^{\perp(2)}$ of $\xi^{(2)}$. In particular, (6.4) and (6.7) tell that the transformation of 2nd-order tensor and curl velocity involve only $\xi^{(1)}$, independent of $A^{(2)}$, $C^{\parallel(2)}$, and $C_i^{\perp(2)}$. Recall that the transformations are similar in the dust model for MD stage [46], but there is no velocity.

However, distinctions should be made between $\xi^{(2)}$ and $\xi^{(1)}$, as pointed out in Ref. [46]. In applications we are often interested in the following case: the 2nd-order solutions are transformed [52] at the same time the 1st-order solutions are fixed. That is, we just transform the 2nd-order solutions without altering the 1st-order ones. This is referred to as the effective 2nd-order transformations, which requires

$$\xi^{(1)} = 0, \quad \text{but} \quad \xi^{(2)} \neq 0. \tag{6.8}$$

Contrarily, if one would set $\xi^{(2)} = 0$ [40,53], only $\xi^{(1)}$ remains, one would have no freedom to make $\bar{g}^{(2)}_{00} = 0$ and $\bar{g}^{(2)}_{0i} = 0$ anymore, because $\xi^{(1)}$ has been already fixed in ensuring $\bar{g}^{(1)}_{00} = 0$ and $\bar{g}^{(1)}_{0i} = 0$.

For the effective 2nd-order transformations, (C35), (C36), and (C37) reduce to

$$\alpha^{(2)}(\tau,\mathbf{x}) = \frac{A^{(2)}(\mathbf{x})}{\tau}, \tag{6.9}$$

$$\beta^{(2)}(\tau,\mathbf{x}) = A^{(2)}(\mathbf{x})\ln\tau + C^{\parallel(2)}(\mathbf{x}), \tag{6.10}$$

$$d_i^{(2)}(\mathbf{x}) = C_i^{\perp(2)}(\mathbf{x}), \tag{6.11}$$

and (6.1)–(6.7) reduce to

$$\bar{\phi}_S^{(2)}(\tau,\mathbf{x}) = \phi_S^{(2)}(\tau,\mathbf{x}) + \frac{A^{(2)}(\mathbf{x})}{\tau^2} + \frac{\ln\tau}{3}\nabla^2 A^{(2)}(\mathbf{x}) + \frac{1}{3}\nabla^2 C^{\parallel(2)}(\mathbf{x}), \tag{6.12}$$

$$\bar{\chi}_S^{\parallel(2)}(\tau,\mathbf{x}) = \chi_S^{\parallel(2)}(\tau,\mathbf{x}) - 2A^{(2)}(\mathbf{x})\ln\tau - 2C^{\parallel(2)}(\mathbf{x}), \tag{6.13}$$

$$\bar{\chi}_{Sij}^{\perp(2)}(\tau,\mathbf{x}) = \chi_{Sij}^{\perp(2)}(\tau,\mathbf{x}) - \partial_j C_i^{\perp(2)}(\mathbf{x}) - \partial_i C_j^{\perp(2)}(\mathbf{x}), \tag{6.14}$$

$$\bar{\chi}_{Sij}^{\top(2)}(\tau,\mathbf{x}) = \chi_{Sij}^{\top(2)}(\tau,\mathbf{x}), \tag{6.15}$$

$$\bar{\delta}_S^{(2)}(\tau,\mathbf{x}) = \delta_S^{(2)}(\tau,\mathbf{x}) + \frac{4}{\tau^2}A^{(2)}(\mathbf{x}). \tag{6.16}$$

$$\bar{v}_S^{\parallel(2)}(\tau,\mathbf{x}) = v_S^{\parallel(2)}(\tau,\mathbf{x}) + \frac{A^{(2)}(\mathbf{x})}{\tau}, \tag{6.17}$$

$$\bar{v}_{Si}^{\perp(2)}(\tau,\mathbf{x}) = v_{Si}^{\perp(2)}(\tau,\mathbf{x}), \tag{6.18}$$

which has the same structure as the 1st-order residual transformations (3.37)–(3.42). From (6.15) and (6.18) we see that the 2nd-order tensor and curl velocity are invariant under the 2nd-order transformation within synchronous coordinates, so the solution $\chi^{\top(2)}_{Sij}$ of (5.1) and $v^{\perp(2)}_{Si}$ of (5.4) are gauge-invariant modes. Equation (6.14) tells us that c_{1ij} in the solution $\chi^{\perp(2)}_{Sij}$ of (5.3) is a gauge term and can be eliminated, so that the gauge-invariant vector mode is

$$\chi^{\perp(2)}_{Sij}(\mathbf{x},\tau) = \frac{c_{2ij}(\mathbf{x})}{\tau} + 2\int^{\tau}\frac{d\tau'}{\tau'^2}\int^{\tau'}\tau''^2 V_{Sij}(\mathbf{x},\tau'')d\tau'', \tag{6.19}$$

where V_{Sij} is in (4.19) and does not change under this effective 2nd-order residual transformation with $\xi^{(2)\mu}\neq 0$, but $\xi^{(1)\mu}=0$.

To identify the residual gauge modes in the 2nd-order scalar solutions, we write (6.12), (6.13), (6.16), and (6.17) in **k**-space

$$\bar{\phi}^{(2)}_{S\mathbf{k}}(\tau) = \phi^{(2)}_{S\mathbf{k}}(\tau) + A^{(2)}_{\mathbf{k}}\left(\frac{1}{\tau^2} - \frac{k^2}{3}\ln\tau\right) - \frac{k^2}{3}C^{\parallel(2)}_{\mathbf{k}}, \tag{6.20}$$

$$\bar{\chi}^{\parallel(2)}_{S\mathbf{k}}(\tau) = \chi^{\parallel(2)}_{S\mathbf{k}}(\tau) - 2A^{(2)}_{\mathbf{k}}\ln\tau - 2C^{\parallel(2)}_{\mathbf{k}}, \tag{6.21}$$

$$\bar{\delta}^{(2)}_{S\mathbf{k}}(\tau) = \delta^{(2)}_{S\mathbf{k}}(\tau) + \frac{4}{\tau^2}A^{(2)}_{\mathbf{k}}, \tag{6.22}$$

$$\bar{v}^{\parallel(2)}_{S\mathbf{k}}(\tau) = v^{\parallel(2)}_{S\mathbf{k}}(\tau) + \frac{A^{(2)}_{\mathbf{k}}}{\tau}. \tag{6.23}$$

Comparing them with the solutions (5.18), (5.19), (5.17), and (5.16), respectively, tells us that G_1 and G_4 terms in the solutions are gauge terms, which can be removed simultaneously by choosing

$$A^{(2)}_{\mathbf{k}} = -\frac{G_1}{k}, \tag{6.24}$$

$$C^{\parallel(2)}_{\mathbf{k}} = \frac{3G_4}{k^2}. \tag{6.25}$$

Thus, the gauge-invariant modes of the 2nd-order scalar perturbations are

$$\begin{aligned}
\phi^{(2)}_{S\mathbf{k}} =\;& G_2\left(\frac{2}{k\tau^2}+\frac{2i}{\sqrt{3}\tau}\right)e^{-ik\tau/\sqrt{3}} + G_3\left(\frac{2}{k\tau^2}-\frac{2i}{\sqrt{3}\tau}\right)e^{ik\tau/\sqrt{3}} - \frac{2k}{3}\int^{\tau}[G_2e^{-ik\tau'/\sqrt{3}}+G_3e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} + \frac{3}{4}F^{\parallel}_{S\mathbf{k}}(\tau)\\
&-\frac{1}{4}\int^{\tau}A_{S\mathbf{k}}(\tau')d\tau' + \int^{\tau}\frac{(k^2\tau'^2+6)\ln\tau'+3}{k^2\tau'}Z_{S\mathbf{k}}(\tau')d\tau' + \left(\frac{1}{k\tau^2}-\frac{k\ln\tau}{3}\right)\int^{\tau}\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau'\\
&-\left(\frac{2}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&-\left(\frac{2}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{2}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\int^{\tau}\left[\frac{2}{k\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\
&+\int^{\tau}\left[\frac{2}{k\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau'',
\end{aligned} \tag{6.26}$$

$$\begin{aligned}
\chi^{\parallel(2)}_{S\mathbf{k}} =\;& G_2\frac{4\sqrt{3}i}{k^2\tau}e^{-ik\tau/\sqrt{3}} - G_3\frac{4\sqrt{3}i}{k^2\tau}e^{ik\tau/\sqrt{3}} - \frac{4}{k}\int^{\tau}[G_2e^{-ik\tau'/\sqrt{3}}+G_3e^{ik\tau'/\sqrt{3}}]\frac{d\tau'}{\tau'} - \frac{2\ln\tau}{k}\int^{\tau}\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau'\\
&+\int^{\tau}\frac{6(k^2\tau'^2+6)\ln\tau'+18}{k^4\tau'}Z_{S\mathbf{k}}(\tau')d\tau' - \frac{4\sqrt{3}}{k^2\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\frac{4\sqrt{3}}{k^2\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau'\\
&+\int^{\tau}\left[\frac{12}{k^3\tau''}\cos\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\
&+\int^{\tau}\left[\frac{12}{k^3\tau''}\sin\left(\frac{k\tau''}{\sqrt{3}}\right)\int^{\tau''}\left(3\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k\tau'}d\tau'\right]d\tau''\\
&+\frac{3}{k^4}E_{S\mathbf{k}} + \frac{27}{2k^4\tau}F^{\parallel\prime}_{S\mathbf{k}}(\tau) + \frac{27}{k^4\tau^2}F^{\parallel}_{S\mathbf{k}} + \frac{9}{2k^2}F^{\parallel}_{S\mathbf{k}}(\tau) - \frac{9}{2k^4\tau}A_{S\mathbf{k}}(\tau) - \frac{3}{2k^2}\int^{\tau}A_{S\mathbf{k}}(\tau')d\tau',
\end{aligned} \tag{6.27}$$

$$\delta_{S\mathbf{k}}^{(2)} = G_2\left(\frac{8}{k\tau^2}+\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{-ik\tau/\sqrt{3}} + G_3\left(\frac{8}{k\tau^2}-\frac{8i}{\sqrt{3}\tau}-\frac{4k}{3}\right)e^{ik\tau/\sqrt{3}} + 3F^{\parallel}_{S\mathbf{k}} + \left(-\frac{8}{k\tau^2}\cos\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{8}{\sqrt{3}\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right.$$
$$\left.+\frac{4k}{3}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \left(-\frac{8}{k\tau^2}\sin\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{8}{\sqrt{3}\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right.$$
$$\left.+\frac{4k}{3}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \frac{1}{k\tau^2}\int^{\tau}\frac{12(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau', \tag{6.28}$$

$$v_{\mathbf{k}}^{\parallel(2)} = G_2\left(\frac{2}{k\tau}+\frac{i}{\sqrt{3}}\right)e^{-ik\tau/\sqrt{3}} + G_3\left(\frac{2}{k\tau}-\frac{i}{\sqrt{3}}\right)e^{ik\tau/\sqrt{3}} - \left(\frac{2}{k\tau}\cos\left(\frac{k\tau}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right.$$
$$\left.+3\sqrt{3}k\tau'\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' - \left(\frac{2}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right)-\frac{1}{\sqrt{3}}\cos\left(\frac{k\tau}{\sqrt{3}}\right)\right)\int^{\tau}\left(9\sin\left(\frac{k\tau'}{\sqrt{3}}\right)\right.$$
$$\left.-3\sqrt{3}k\tau'\cos\left(\frac{k\tau'}{\sqrt{3}}\right)\right)\frac{Z_{S\mathbf{k}}(\tau')}{k^3\tau'}d\tau' + \frac{1}{k\tau}\int^{\tau}\frac{3(k^2\tau'^2+6)}{k^3\tau'}Z_{S\mathbf{k}}(\tau')d\tau', \tag{6.29}$$

where $Z_S, E_S, A_S, F_S^{\parallel}$ are in (5.14), (4.2), (4.24), and (4.28), which do not change under this effective 2nd-order transformation induced by $\xi^{(2)\mu}$. So far, we have obtained all the gauge invariant solutions of the 2nd-order perturbations in Eqs. (5.1), (5.4), (6.19), (6.26), (6.27), (6.28), and (6.29).

VII. CONCLUSION AND DISCUSSION

We present a systematic study of the 2nd-order cosmological perturbations in RD stage in synchronous coordinates based on the Einstein equation. The dominant radiation is modeled by a relativistic fluid with the energy density ρ, the pressure $p = c_s^2\rho$ with $c_s^2 = \frac{1}{3}$ and the velocity U^μ, and we assume that there is no shear for the fluid and the 1st-order velocity is curlless. The model has more complications than the pressureless dust model in the MD stage [43,46,47] since the spatial components T_{ij} of stress tensor are nonvanishing and have to be specified nontrivially.

We give a detailed analysis of the structure and the 1st-order solutions and residual gauge transformations, on which the 2nd-order perturbations are based. We have demonstrated that, during RD stage, the 1st-order scalar modes, density contrast, and longitudinal velocity all propagate at the sound speed $\frac{1}{\sqrt{3}}$, instead of the speed of light. The 1st-order vector modes and curl velocity are not a wave and do not propagate, and they simply decrease with time. In contrast, the tensor modes are waves and propagate at the speed of light. Compare this with the MD stage during which only the tensor modes propagate at the speed of light, whereas all other perturbations are not a wave and do not propagate. Hence, we conclude that the tensor modes are truly radiative as dynamic d.o.f., regardless of the background matter, but the scalar and vector modes are not. Sometimes in the literature all metric perturbations were misleadingly referred to as six gravitational waves (GW). Our analysis suggests that the term gravitational waves should be reserved for the tensor modes only.

The 2nd-order perturbed Einstein equation contains various couplings of 1st-order metric perturbations derived. The case of scalar-scalar coupling has been considered in this paper. The 1st-order vector metric perturbations is taken to be vanishing, consistent with the curlless 1st-order velocity. When these coupling terms are moved to the rhs of the Einstein equation, they together with $T_{\mu\nu}$ of the fluid serve as the effective source for the 2nd-order metric perturbations. The resulting 2nd-order Einstein equation has a similar structure to the 1st-order Einstein equation, except that the effective source is now more complicated. The (00) component of Einstein equation is the energy constraint, the $(0i)$ components are the momentum constraints which are decomposed into longitudinal and transverse parts, and the (ij) components are decomposed into the respective evolution equations of 2nd-order scalar, vector, and tensor metric perturbations. Moreover, to specify ρ, p, and U^μ that appear in the source, the equation of covariant conservation $T^{\mu\nu}{}_{;\nu} = 0$ up to the 2nd-order needs to be solved. The presence of velocity U^μ makes the calculations algebraically more involved than the dust model. We have solved the set of equations for the 2nd-order metric perturbations, density contrast, and velocity analytically, and obtained all the 2nd-order solutions in the integral form. They consist of the homogeneous part for general initial conditions and the inhomogeneous part due to the effective source. We have analyzed also the general 2nd-order residual gauge transformations in synchronous coordinates, which involve both the 2nd-order vector $\xi^{(2)}$ and the 1st-order vector $\xi^{(1)}$. We have obtained the explicit expressions of transformations for all the 2nd-order perturbations, which are lengthy and have many terms due to the square of $\xi^{(1)}$ and the products of 1st-order perturbations with $\xi^{(1)}$. In particular, we also have distinguished the transformations due to the 2nd-order vector $\xi^{(2)}$ from those due to the combinations of the 1st-order vector $\xi^{(1)}$. The 1st-order solutions are often fixed in actual applications, only those transformations due to $\xi^{(2)}$ are effective. In this case the effective transformation of the 2nd-order perturbations have similar structure to the 1st-order perturbations. After this analysis, we have identified the gauge-invariant modes of the 2nd-order solutions.

These 2nd-order analytical results of RD stage, in conjunction with the 2nd-order results of MD stage [46,47], can be used to study nonlinear effects of cosmological perturbations. As an advantage of analytical results, one will be able to focus on certain aspect of the nonlinearity of 2nd-order perturbation. For example, one can study separately the tensor modes, as well as the vector modes, besides the scalar modes. And one can examine the individual contribution of each k-mode to the nonlinearity, the transfer of perturbation power among different modes, the interference of positive and negative frequency modes represented by D_2 and D_3, the influences by initial conditions, the separate influence by the growing and decaying modes. Furthermore, one can pick up a particular period of evolution and estimate the dynamic behavior there. These aspects can provide possible advantages that other methods often lack. To use these solutions in specific applications, one needs to carry out numerical integrals, such as $z_1, ..., z_7$, and choose the appropriate initial conditions. These possible applications will need more work.

To improve the above work, one can calculate the couplings involving the 1st-order tensor mode, which may have effects comparable to the scalar-scalar. As possible extensions, one can study 2nd-order perturbations for inflationary models. These will be left for future studies.

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APPENDIX A: PERTURBED QUANTITIES

The quantities listed in this appendix are valid for a general flat RW spacetime in synchronous coordinate. By (2.1) and (2.2), the nonvanishing Christopher symbols up to 2nd order are

$$\Gamma^0_{00} = \frac{a'}{a}, \tag{A1}$$

$$\Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} + \left(\frac{a'}{a}\gamma^{(1)}_{ij} + \frac{1}{2}\gamma^{(1)'}_{ij}\right) + \left(\frac{a'}{2a}\gamma^{(2)}_{ij} + \frac{1}{4}\gamma^{(2)'}_{ij}\right), \tag{A2}$$

$$\Gamma^i_{j0} = \frac{a'}{a}\delta^i_j + \left(\frac{1}{2}\gamma^{(1)'i}_{j}\right) + \left(\frac{1}{4}\gamma^{(2)'i}_{j} - \frac{1}{2}\gamma^{(1)ik}\gamma^{(1)'}_{jk}\right), \tag{A3}$$

$$\begin{aligned}\Gamma^i_{jk} = {}& \left(\frac{1}{2}\gamma^{(1)i}_{j,k} + \frac{1}{2}\gamma^{(1)i}_{k,j} - \frac{1}{2}\gamma^{(1),i}_{jk}\right) \\ &+ \left(-\frac{1}{2}\gamma^{(1)im}\gamma^{(1)}_{mj,k} - \frac{1}{2}\gamma^{(1)im}\gamma^{(1)}_{mk,j} + \frac{1}{2}\gamma^{(1)im}\gamma^{(1)}_{jk,m}\right. \\ &\left. + \frac{1}{4}\gamma^{(2)i}_{j,k} + \frac{1}{4}\gamma^{(2)i}_{k,j} - \frac{1}{4}\gamma^{(2),i}_{jk}\right).\end{aligned} \tag{A4}$$

The Ricci tensor are $R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\nu\alpha,\mu} + \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\alpha\mu}$. One calculates the 0th-order Ricci tensors are $R^{(0)}_{00} = -\frac{3a''}{a} + 3(\frac{a'}{a})^2$, $R^{(0)}_{0i} = 0$, $R^{(0)}_{ij} = \delta_{ij}(\frac{a''}{a} + (\frac{a'}{a})^2)$, $R^{(0)} = \frac{6}{a^2}\frac{a''}{a}$, and the 1st-order perturbed Ricci tensor

$$R^{(1)}_{00} = 3\phi^{(1)''} + 3\frac{a'}{a}\phi^{(1)'}, \tag{A5}$$

$$R^{(1)}_{0i} = 2\phi^{(1)'}_{,i} + \frac{1}{2}D_{ij}\chi^{\|(1)',j} + \frac{1}{2}\chi^{\perp(1)',j}_{ij}, \tag{A6}$$

$$\begin{aligned}R^{(1)}_{ij} = {}& -5\frac{a'}{a}\phi^{(1)'}\delta_{ij} - 2\frac{a''}{a}\phi^{(1)}\delta_{ij} - 2\left(\frac{a'}{a}\right)^2\phi^{(1)}\delta_{ij} - \phi^{(1)''}\delta_{ij} + \nabla^2\phi^{(1)}\delta_{ij} + \phi^{(1)}_{,ij} + \frac{1}{2}D_{ij}\chi^{\|(1)''} + \frac{a'}{a}D_{ij}\chi^{\|(1)'} \\ &+ \frac{a''}{a}D_{ij}\chi^{\|(1)} + \left(\frac{a'}{a}\right)^2 D_{ij}\chi^{\|(1)} - \frac{1}{2}\nabla^2 D_{ij}\chi^{\|(1)} + \frac{1}{2}D^k_j\chi^{\|(1)}_{,ik} + \frac{1}{2}D^k_i\chi^{\|(1)}_{,jk} \\ &+ \frac{1}{2}\chi^{\perp(1)''}_{ij} + \frac{a'}{a}\chi^{\perp(1)'}_{ij} + \frac{a''}{a}\chi^{\perp(1)}_{ij} + \left(\frac{a'}{a}\right)^2\chi^{\perp(1)}_{ij} + \frac{1}{2}[\chi^{\perp(1),k}_{kj,i} + \chi^{\perp(1),k}_{ki,j} - \nabla^2\chi^{\perp(1)}_{ij}] \\ &+ \frac{a'}{a}\chi^{\top(1)'}_{ij} + \frac{a''}{a}\chi^{\top(1)}_{ij} + \left(\frac{a'}{a}\right)^2\chi^{\top(1)}_{ij} + \frac{1}{2}\chi^{\top(1)''}_{ij} - \frac{1}{2}\nabla^2\chi^{\top(1)}_{ij}.\end{aligned} \tag{A7}$$

The 1st-order Ricci scalar is

$$R^{(1)} = \frac{1}{a^2}\left(-6\phi^{(1)''} - 18\frac{a'}{a}\phi^{(1)'} + 4\nabla^2\phi^{(1)} + D_{ij}\chi^{\parallel(1),ij}\right), \tag{A8}$$

which is independent of the 1st-order vector and tensor.

The 1st-order Einstein tensor is

$$G_{00}^{(1)} \equiv R_{00}^{(1)} - \frac{1}{2}g_{00}^{(0)}R^{(1)} = -6\frac{a'}{a}\phi^{(1)'} + 2\nabla^2\phi^{(1)} + \frac{1}{3}\nabla^2\nabla^2\chi^{\parallel(1)}, \tag{A9}$$

$$G_{0i}^{(1)} \equiv R_{0i}^{(1)}, \tag{A10}$$

$$\begin{aligned} G_{ij}^{(1)} &\equiv R_{ij}^{(1)} - \frac{1}{2}\delta_{ij}a^2R^{(1)} - \frac{1}{2}a^2\gamma_{ij}^{(1)}R^{(0)} \\ &= 2\phi^{(1)''}\delta_{ij} + 4\frac{a'}{a}\phi^{(1)'}\delta_{ij} + \phi^{(1)}_{,ij} - \nabla^2\phi^{(1)}\delta_{ij} + \left[4\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2\right]\phi^{(1)}\delta_{ij} + \frac{1}{2}D_{ij}\chi^{\parallel(1)''} + \frac{a'}{a}D_{ij}\chi^{\parallel(1)'} \\ &\quad + \left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi^{\parallel(1)} + \left[\frac{1}{2}D^k_j\chi^{\parallel(1)}_{,ik} + \frac{1}{2}D^k_i\chi^{\parallel(1)}_{,jk} - \frac{1}{2}\delta_{ij}D_{kl}\chi^{\parallel(1),kl}\right] - \frac{1}{2}\nabla^2D_{ij}\chi^{\parallel(1)} \\ &\quad + \frac{1}{2}\chi_{ij}^{\perp(1)''} + \frac{a'}{a}\chi_{ij}^{\perp(1)'} + \left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{ij}^{\perp(1)} + \frac{1}{2}[\chi^{\perp(1),k}_{kj,i} + \chi^{\perp(1),k}_{ki,j} - \nabla^2\chi_{ij}^{\perp(1)}] \\ &\quad + \frac{1}{2}\chi_{ij}^{\top(1)''} + \frac{a'}{a}\chi_{ij}^{\top(1)'} + \left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{ij}^{\top(1)} - \frac{1}{2}\nabla^2\chi_{ij}^{\top(1)}. \end{aligned} \tag{A11}$$

The 2nd-order Ricci tensors are

$$\begin{aligned} R_{00}^{(2)} &= \frac{3a'}{2a}\phi^{(2)'} + \frac{3}{2}\phi^{(2)''} + 6\frac{a'}{a}\phi^{(1)}\phi^{(1)'} + 6\phi^{(1)}\phi^{(1)''} + 3\phi^{(1)'}\phi^{(1)'} + \frac{1}{2}D^{ij}\chi^{\parallel(1)}D_{ij}\chi^{\parallel(1)''} \\ &\quad + \frac{1}{4}D^{ij}\chi^{\parallel(1)'}D_{ij}\chi^{\parallel(1)'} + \frac{a'}{2a}D^{ij}\chi^{\parallel(1)}D_{ij}\chi^{\parallel(1)'} + \frac{1}{2}\chi^{\top(1)ij}\chi_{ij}^{\top(1)''} + \frac{1}{4}\chi^{\top(1)'ij}\chi_{ij}^{\top(1)'} \\ &\quad + \frac{a'}{2a}\chi^{\top(1)ij}\chi_{ij}^{\top(1)'} + \frac{1}{2}\chi^{\top(1)ij}D_{ij}\chi^{\parallel(1)''} + \frac{1}{2}\chi^{\top(1)'ij}D_{ij}\chi^{\parallel(1)'} + \frac{a'}{2a}\chi^{\top(1)ij}D_{ij}\chi^{\parallel(1)'} \\ &\quad + \frac{1}{2}\chi_{ij}^{\top(1)''}D^{ij}\chi^{\parallel(1)} + \frac{a'}{2a}\chi_{ij}^{\top(1)'}D^{ij}\chi^{\parallel(1)}, \end{aligned} \tag{A12}$$

$$\begin{aligned} R_{0i}^{(2)} &= \phi^{(2)'}_{,i} + \frac{1}{4}D_{ij}\chi^{\parallel(2)',j} + \frac{1}{4}\chi_{ij}^{\perp(2)',j} + 4\phi^{(1)'}\phi^{(1)}_{,i} + 4\phi^{(1)}\phi^{(1)'}_{,i} + \phi^{(1)}D_{ij}\chi^{\parallel(1)',j} \\ &\quad + \phi^{(1)'}D_{ij}\chi^{\parallel(1),j} - \frac{1}{2}\phi^{(1),j}D_{ij}\chi^{\parallel(1)'} + \phi^{(1)',j}D_{ij}\chi^{\parallel(1)} - \frac{1}{2}\phi^{(1),j}\chi_{ij}^{\top(1)'} \\ &\quad + \phi^{(1)',j}\chi_{ij}^{\top(1)} - \frac{1}{2}D^k_j\chi^{\parallel(1),j}D_{ik}\chi^{\parallel(1)'} - \frac{1}{2}D^k_j\chi^{\parallel(1)}D_{ik}\chi^{\parallel(1)',j} + \frac{1}{2}D^{jk}\chi^{\parallel(1)'}_{,i}D_{jk}\chi^{\parallel(1)} \\ &\quad + \frac{1}{4}D^{jk}\chi^{\parallel(1)'}D_{jk}\chi^{\parallel(1)}_{,i} - \frac{1}{2}\chi_j^{\top(1)k}D_{ik}\chi^{\parallel(1)',j} + \frac{1}{2}\chi^{\top(1)jk}D_{jk}\chi^{\parallel(1)'}_{,i} + \frac{1}{4}\chi^{\top(1)jk}_{,i}D_{jk}\chi^{\parallel(1)'} \\ &\quad - \frac{1}{2}\chi_{ik}^{\top(1)',j}D^k_j\chi^{\parallel(1)} + \frac{1}{2}\chi_{jk,i}^{\top(1)'}D^{jk}\chi^{\parallel(1)} + \frac{1}{4}\chi_{jk}^{\top(1)'}D^{jk}\chi^{\parallel(1)}_{,i} - \frac{1}{2}\chi_{ik}^{\top(1)'}D^k_j\chi^{\parallel(1),j} \\ &\quad - \frac{1}{2}\chi_j^{\top(1)k}\chi_{ik}^{\top(1)',j} + \frac{1}{2}\chi^{\top(1)jk}\chi_{jk,i}^{\top(1)'} + \frac{1}{4}\chi^{\top(1)jk}_{,i}\chi_{jk}^{\top(1)'}, \end{aligned} \tag{A13}$$

$$
\begin{aligned}
R_{ij}^{(2)} = {} & \delta_{ij}\Big[-\frac{5}{2}\frac{a'}{a}\phi^{(2)\prime} - \left(\frac{a'}{a}\right)^2\phi^{(2)} - \frac{a''}{a}\phi^{(2)} - \frac{1}{2}\phi^{(2)\prime\prime} + \frac{1}{2}\nabla^2\phi^{(2)} + (\phi^{(1)\prime})^2 + \phi^{(1),k}\phi^{(1)}_{,k} + 2\phi^{(1)}\nabla^2\phi^{(1)} - \phi^{(1)}_{,k}D^{lk}\chi^{\parallel(1)}_{,l} \\
& - \phi^{(1)}_{,kl}D^{kl}\chi^{\parallel(1)} - \phi^{(1)}_{,kl}\chi^{\top(1)kl} - \frac{a'}{2a}D^{kl}\chi^{\parallel(1)}D_{kl}\chi^{\parallel(1)\prime} - \frac{a'}{2a}\chi^{\top(1)\prime}_{kl}D^{kl}\chi^{\parallel(1)} - \frac{a'}{2a}\chi^{\top(1)kl}D_{kl}\chi^{\parallel(1)\prime} - \frac{a'}{2a}\chi^{\top(1)kl}\chi^{\top(1)\prime}_{kl}\Big] \\
& + \frac{1}{2}\phi^{(2)}_{,ij} + \frac{1}{4}D_{ij}\chi^{\parallel(2)\prime\prime} + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a}\right]D_{ij}\chi^{\parallel(2)} + \frac{a'}{2a}D_{ij}\chi^{\parallel(2)\prime} + \frac{1}{4}D^k_j\chi^{\parallel(2)}_{,ki} + \frac{1}{4}D^k_i\chi^{\parallel(2)}_{,kj} - \frac{1}{4}\nabla^2 D_{ij}\chi^{\parallel(2)} + \frac{1}{4}\chi^{\perp(2)\prime\prime}_{ij} \\
& + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a}\right]\chi^{\perp(2)}_{ij} + \frac{a'}{2a}\chi^{\perp(2)\prime}_{ij} + \frac{1}{4}\chi^{\perp(2),k}_{kj,i} + \frac{1}{4}\chi^{\perp(2),k}_{ki,j} - \frac{1}{4}\nabla^2\chi^{\perp(2)}_{ij} + \frac{1}{4}\chi^{\top(2)\prime\prime}_{ij} + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a}\right]\chi^{\top(2)}_{ij} \\
& + \frac{a'}{2a}\chi^{\top(2)\prime}_{ij} - \frac{1}{4}\nabla^2\chi^{\top(2)}_{ij} + 3\phi^{(1)}_{,i}\phi^{(1)}_{,j} + 2\phi^{(1)}\phi^{(1)}_{,ij} - \frac{3a'}{a}\phi^{(1)\prime}D_{ij}\chi^{\parallel(1)} + \frac{1}{2}\phi^{(1)\prime}D_{ij}\chi^{\parallel(1)\prime} + \frac{1}{2}\phi^{(1)}_{,k}D^k_j\chi^{\parallel(1)}_{,i} \\
& + \frac{1}{2}\phi^{(1)}_{,k}D^k_i\chi^{\parallel(1)}_{,j} - \frac{3}{2}\phi^{(1)}_{,k}D_{ij}\chi^{\parallel(1),k} + \phi^{(1)}D^k_j\chi^{\parallel(1)}_{,ik} + \phi^{(1)}D^k_i\chi^{\parallel(1)}_{,jk} - \phi^{(1)}\nabla^2 D_{ij}\chi^{\parallel(1)} + \phi^{(1)}_{,i}D^k_j\chi^{\parallel(1)}_{,k} + \phi^{(1)}_{,j}D^k_i\chi^{\parallel(1)}_{,k} \\
& + \phi^{(1)}_{,ki}D^k_j\chi^{\parallel(1)} + \phi^{(1)}_{,kj}D^k_i\chi^{\parallel(1)} - \frac{3a'}{a}\phi^{(1)\prime}\chi^{\top(1)}_{ij} + \frac{1}{2}\phi^{(1)\prime}\chi^{\top(1)\prime}_{ij} + \frac{1}{2}\phi^{(1)}_{,k}\chi^{\top(1)k}_{j,i} + \frac{1}{2}\phi^{(1)}_{,k}\chi^{\top(1)k}_{i,j} - \frac{3}{2}\phi^{(1)}_{,k}\chi^{\top(1),k}_{ij} \\
& - \phi^{(1)}\nabla^2\chi^{\top(1)}_{ij} + \phi^{(1)}_{,ki}\chi^{\top(1)k}_{j} + \phi^{(1)}_{,kj}\chi^{\top(1)k}_{i} - \frac{1}{2}D^k_i\chi^{\parallel(1)\prime}D_{kj}\chi^{\parallel(1)\prime} - \frac{1}{2}D^{kl}\chi^{\parallel(1)}_{,l}D_{kj}\chi^{\parallel(1)}_{,i} - \frac{1}{2}D^{kl}\chi^{\parallel(1)}_{,l}D_{ki}\chi^{\parallel(1)}_{,j} \\
& + \frac{1}{2}D^{lk}\chi^{\parallel(1)}_{,l}D_{ij}\chi^{\parallel(1)}_{,k} - \frac{1}{2}D^{kl}\chi^{\parallel(1)}D_{lj}\chi^{\parallel(1)}_{,ik} - \frac{1}{2}D^{kl}\chi^{\parallel(1)}D_{li}\chi^{\parallel(1)}_{,jk} + \frac{1}{2}D^{kl}\chi^{\parallel(1)}D_{ij}\chi^{\parallel(1)}_{,kl} + \frac{1}{2}D^{kl}\chi^{\parallel(1)}D_{kl}\chi^{\parallel(1)}_{,ij} \\
& + \frac{1}{4}D^{kl}\chi^{\parallel(1)}_{,i}D_{kl}\chi^{\parallel(1)}_{,j} - \frac{1}{2}D^l_i\chi^{\parallel(1)}_{,k}D^k_j\chi^{\parallel(1)}_{,l} + \frac{1}{2}D^k_i\chi^{\parallel(1)}_{,l}D_{kj}\chi^{\parallel(1),l} - \frac{1}{2}\chi^{\top(1)\prime}_{kj}D^k_i\chi^{\parallel(1)\prime} - \frac{1}{2}\chi^{\top(1)\prime k}_{i}D_{kj}\chi^{\parallel(1)\prime} - \frac{1}{2}\chi^{\top(1)}_{kj,i}D^{kl}\chi^{\parallel(1)}_{,l} \\
& - \frac{1}{2}\chi^{\top(1)}_{ki,j}D^{kl}\chi^{\parallel(1)}_{,l} + \frac{1}{2}\chi^{\top(1)}_{ij,k}D^{lk}\chi^{\parallel(1)}_{,l} - \frac{1}{2}\chi^{\top(1)}_{lj,ik}D^{kl}\chi^{\parallel(1)} - \frac{1}{2}\chi^{\top(1)kl}D_{lj}\chi^{\parallel(1)}_{,ik} - \frac{1}{2}\chi^{\top(1)}_{li,jk}D^{kl}\chi^{\parallel(1)} - \frac{1}{2}\chi^{\top(1)kl}D_{li}\chi^{\parallel(1)}_{,jk} \\
& + \frac{1}{2}\chi^{\top(1)}_{ij,kl}D^{kl}\chi^{\parallel(1)} + \frac{1}{2}\chi^{\top(1)kl}D_{ij}\chi^{\parallel(1)}_{,kl} + \frac{1}{2}\chi^{\top(1)}_{kl,ij}D^{kl}\chi^{\parallel(1)} + \frac{1}{2}\chi^{\top(1)kl}D_{kl}\chi^{\parallel(1)}_{,ij} + \frac{1}{4}\chi^{\top(1)}_{kl,j}D^{kl}\chi^{\parallel(1)}_{,i} + \frac{1}{4}\chi^{\top(1)kl}_{,i}D_{kl}\chi^{\parallel(1)}_{,j} \\
& - \frac{1}{2}\chi^{\top(1)k}_{j,l}D^l_i\chi^{\parallel(1)}_{,k} - \frac{1}{2}\chi^{\top(1)l}_{i,k}D^k_j\chi^{\parallel(1)}_{,l} + \frac{1}{2}\chi^{\top(1),l}_{kj}D^k_i\chi^{\parallel(1)}_{,l} + \frac{1}{2}\chi^{\top(1)k}_{i,l}D_{kj}\chi^{\parallel(1),l} - \frac{1}{2}\chi^{\top(1)\prime k}_{i}\chi^{\top(1)\prime}_{kj} - \frac{1}{2}\chi^{\top(1)kl}\chi^{\top(1)}_{lj,ik} \\
& - \frac{1}{2}\chi^{\top(1)kl}\chi^{\top(1)}_{li,jk} + \frac{1}{2}\chi^{\top(1)kl}\chi^{\top}(1)_{ij,kl} + \frac{1}{2}\chi^{\top(1)kl}\chi^{\top(1)}_{kl,ij} + \frac{1}{4}\chi^{\top(1)kl}_{,i}\chi^{\top(1)}_{kl,j} - \frac{1}{2}\chi^{\top(1)l}_{i,k}\chi^{\top(1)k}_{j,l} + \frac{1}{2}\chi^{\top(1)k}_{i,l}\chi^{\top(1),l}_{kj}. \qquad \text{(A14)}
\end{aligned}
$$

The above expression of $R_{ij}^{(2)}$ corrects some typos in (B3) of Ref. [46] where a factor $\frac{1}{2}$ was missed in the three terms of $[(\frac{a'}{a})^2 + \frac{a''}{a}]$. The 2nd-order Ricci scalar is

$$
\begin{aligned}
R^{(2)} = {} & \frac{1}{a^2}\Big[2\nabla^2\phi^{(2)} - 9\frac{a'}{a}\phi^{(2)\prime} - 3\phi^{(2)\prime\prime} + \frac{1}{2}D^{kl}\chi^{\parallel(2)}_{,kl} - 12\phi^{(1)}\phi^{(1)\prime\prime} - 36\frac{a'}{a}\phi^{(1)\prime}\phi^{(1)} + 6\phi^{(1)}_{,k}\phi^{(1),k} + 16\phi^{(1)}\nabla^2\phi^{(1)} \\
& + 4\phi^{(1)}D^{kl}\chi^{\parallel(1)}_{,kl} - 2\phi^{(1)}_{,kl}D^{kl}\chi^{\parallel(1)} - 2\phi^{(1)}_{,kl}\chi^{\top(1)kl} - D^{kl}\chi^{\parallel(1)}D_{kl}\chi^{\parallel(1)\prime\prime} - \frac{3}{4}D^{kl}\chi^{\parallel(1)\prime}D_{kl}\chi^{\parallel(1)\prime} - 3\frac{a'}{a}D^{kl}\chi^{\parallel(1)}D_{kl}\chi^{\parallel(1)\prime} \\
& - 2D_{ml}\chi^{\parallel(1),l}_{,k}D^{km}\chi^{\parallel(1)} + D^{km}\chi^{\parallel(1)}\nabla^2 D_{km}\chi^{\parallel(1)} - D^{km}\chi^{\parallel(1)}_{,k}D_{ml}\chi^{\parallel(1),l} + \frac{3}{4}D^{km}\chi^{\parallel(1)}_{,l}D_{km}\chi^{\parallel(1),l} - \frac{1}{2}D^{km}\chi^{\parallel(1)}_{,l}D^l_k\chi^{\parallel(1)}_{,m} \\
& - D^{kl}\chi^{\parallel(1)}\chi^{\top(1)\prime\prime}_{kl} - \chi^{\top(1)kl}D_{kl}\chi^{\parallel(1)\prime\prime} - \frac{3}{2}\chi^{\top(1)\prime}_{kl}D^{kl}\chi^{\parallel(1)\prime} - 3\frac{a'}{a}\chi^{\top(1)\prime}_{kl}D^{kl}\chi^{\parallel(1)} - 3\frac{a'}{a}\chi^{\top(1)kl}D_{kl}\chi^{\parallel(1)\prime} - 2\chi^{\top(1)km}D_{ml}\chi^{\parallel(1),l}_{,k} \\
& + \chi^{\top(1)km}\nabla^2 D_{km}\chi^{\parallel(1)} + D^{km}\chi^{\parallel(1)}\nabla^2\chi^{\top(1)}_{km} + \frac{3}{2}\chi^{\top(1),l}_{km}D^{km}\chi^{\parallel(1)}_{,l} - \chi^{\top(1)l}_{k,m}D^{km}\chi^{\parallel(1)}_{,l} - \chi^{\top(1)kl}\chi^{\top(1)\prime\prime}_{kl} - \frac{3}{4}\chi^{\top(1)\prime kl}\chi^{\top(1)\prime}_{kl} \\
& - 3\frac{a'}{a}\chi^{\top(1)kl}\chi^{\top(1)\prime}_{kl} + \chi^{\top(1)km}\nabla^2\chi^{\top(1)}_{km} + \frac{3}{4}\chi^{\top(1)km}_{,l}\chi^{\top(1),l}_{km} - \frac{1}{2}\chi^{\top(1)km}_{,l}\chi^{\top(1)l}_{k,m}\Big]. \qquad \text{(A15)}
\end{aligned}
$$

The 2nd-order perturbed Einstein tensors are

$$
\begin{aligned}
G_{00}^{(2)} &\equiv R_{00}^{(2)} - \frac{1}{2} g_{00}^{(0)} R^{(2)} \\
&= \nabla^2\phi^{(2)} - \frac{3a'}{a}\phi^{(2)'} + \frac{1}{4} D^{kl}\chi^{\|(2)}_{,kl} \\
&\quad - 12\frac{a'}{a}\phi^{(1)'}\phi^{(1)} + 3\phi^{(1)'}\phi^{(1)'} + 3\phi^{(1)}_{,k}\phi^{(1),k} + 8\phi^{(1)}\nabla^2\phi^{(1)} + 2\phi^{(1)}D^{kl}\chi^{\|(1)}_{,kl} \\
&\quad - \phi^{(1)}_{,kl}D^{kl}\chi^{\|(1)} - \frac{1}{8}D^{kl}\chi^{\|(1)'}D_{kl}\chi^{\|(1)'} - \frac{a'}{a}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)'} \\
&\quad - D_{ml}\chi^{\|(1),l}_{,k}D^{km}\chi^{\|(1)} + \frac{1}{2}D^{km}\chi^{\|(1)}\nabla^2 D_{km}\chi^{\|(1)} - \frac{1}{2}D^{km}\chi^{\|(1)}_{,k}D_{ml}\chi^{\|(1),l} \\
&\quad + \frac{3}{8}D^{km}\chi^{\|(1)}_{,l}D_{km}\chi^{\|(1),l} - \frac{1}{4}D^{km}\chi^{\|(1)}_{,l}D^l_k\chi^{\|(1)}_{,m} - \phi^{(1)}_{,kl}\chi^{\top(1)kl} \\
&\quad - \frac{1}{4}\chi^{\top(1)'}_{kl}D^{kl}\chi^{\|(1)'} - \frac{a'}{a}\chi^{\top(1)kl}D_{kl}\chi^{\|(1)'} - \frac{a'}{a}\chi^{\top(1)'}_{kl}D^{kl}\chi^{\|(1)} \\
&\quad - \chi^{\top(1)km}D_{ml}\chi^{\|(1),l}_{,k} + \frac{1}{2}\chi^{\top(1)km}\nabla^2 D_{km}\chi^{\|(1)} + \frac{1}{2}D^{km}\chi^{\|(1)}\nabla^2\chi^{\top(1)}_{km} \\
&\quad + \frac{3}{4}\chi^{\top(1),l}_{km}D^{km}\chi^{\|(1)}_{,l} - \frac{1}{2}\chi^{\top(1)l}_{k,m}D^{km}\chi^{\|(1)}_{,l} - \frac{1}{8}\chi^{\top(1)'kl}\chi^{\top(1)'}_{kl} - \frac{a'}{a}\chi^{\top(1)kl}\chi^{\top(1)'}_{kl} \\
&\quad + \frac{1}{2}\chi^{\top(1)km}\nabla^2\chi^{\top(1)}_{km} + \frac{3}{8}\chi^{\top(1)km}_{,l}\chi^{\top(1),l}_{km} - \frac{1}{4}\chi^{\top(1)km}_{,l}\chi^{\top(1)l}_{k,m},
\end{aligned}
\tag{A16}
$$

$$
G_{0i}^{(2)} \equiv R_{0i}^{(2)},
\tag{A17}
$$

$$
\begin{aligned}
G_{ij}^{(2)} &= R_{ij}^{(2)} - \frac{1}{2}a^2\left(\frac{1}{2}\gamma_{ij}^{(2)}\right)R^{(0)} - \frac{1}{2}a^2\gamma_{ij}^{(1)}R^{(1)} - \frac{1}{2}a^2\delta_{ij}R^{(2)} \\
&= \delta_{ij}\bigg[-\frac{1}{2}\nabla^2\phi^{(2)} + \phi^{(2)''} + 2\frac{a'}{a}\phi^{(2)'} + \left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]\phi^{(2)} - \frac{1}{4}D^{kl}\chi^{\|(2)}_{,kl} \\
&\quad - 2\phi^{(1),k}\phi^{(1)}_{,k} - 2\phi^{(1)}\nabla^2\phi^{(1)} + (\phi^{(1)'})^2 - \phi^{(1)}_{,k}D^{lk}\chi^{\|(1)}_{,l} - \phi^{(1)}D^{kl}\chi^{\|(1)}_{,kl} \\
&\quad + \frac{a'}{a}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)'} + D_{ml}\chi^{\|(1),l}_{,k}D^{km}\chi^{\|(1)} - \frac{1}{2}D^{km}\chi^{\|(1)}\nabla^2 D_{km}\chi^{\|(1)} \\
&\quad - \frac{3}{8}D^{km}\chi^{\|(1),l}D_{km}\chi^{\|(1)}_{,l} + \frac{1}{4}D^{km}\chi^{\|(1),l}D_{kl}\chi^{\|(1)}_{,m} + \frac{1}{2}D^{kl}\chi^{\|(1)}_{,l}D^m_k\chi^{\|(1)}_{,m} \\
&\quad + \frac{3}{8}D^{kl}\chi^{\|(1)'}D_{kl}\chi^{\|(1)'} + \frac{1}{2}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)''} + \frac{a'}{a}\chi^{\top(1)kl}D_{kl}\chi^{\|(1)'} + \frac{a'}{a}D^{kl}\chi^{\|(1)}\chi^{\top(1)'}_{kl} \\
&\quad + \chi^{\top(1)km}D_{ml}\chi^{\|(1),l}_{,k} - \frac{1}{2}\chi^{\top(1)km}\nabla^2 D_{km}\chi^{\|(1)} - \frac{1}{2}D^{km}\chi^{\|(1)}\nabla^2\chi^{\top(1)}_{km} \\
&\quad - \frac{3}{4}D^{km}\chi^{\|(1),l}\chi^{\top(1)}_{km,l} + \frac{1}{2}D^{km}\chi^{\|(1),l}\chi^{\top(1)}_{kl,m} + \frac{3}{4}\chi^{\top(1)'kl}D_{kl}\chi^{\|(1)'} + \frac{1}{2}\chi^{\top(1)''}_{kl}D^{kl}\chi^{\|(1)} \\
&\quad + \frac{1}{2}\chi^{\top(1)kl}D_{kl}\chi^{\|(1)''} + \frac{a'}{a}\chi^{\top(1)kl}\chi^{\top(1)'}_{kl} - \frac{1}{2}\chi^{\top(1)km}\nabla^2\chi^{\top(1)}_{km} - \frac{3}{8}\chi^{\top(1)km,l}\chi^{\top(1)}_{km,l} \\
&\quad + \frac{1}{4}\chi^{\top(1)km,l}\chi^{\top(1)}_{kl,m} + \frac{3}{8}\chi^{\top(1)'kl}\chi^{\top(1)'}_{kl} + \frac{1}{2}\chi^{\top(1)kl}\chi^{\top(1)''}_{kl}\bigg] \\
&\quad + \frac{1}{2}\phi^{(2)}_{,ij} + \frac{a'}{2a}D_{ij}\chi^{\|(2)'} + \frac{1}{4}D^k_j\chi^{\|(2)}_{,ki} + \frac{1}{4}D^k_i\chi^{\|(2)}_{,kj} - \frac{1}{4}\nabla^2 D_{ij}\chi^{\|(2)} + \frac{1}{4}D_{ij}\chi^{\|(2)''} \\
&\quad + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi^{\|(2)} + \frac{a'}{2a}\chi^{\perp(2)'}_{ij} + \frac{1}{4}\chi^{\perp(2)k}_{j,ki} + \frac{1}{4}\chi^{\perp(2)k}_{i,kj} - \frac{1}{4}\nabla^2\chi^{\perp(2)}_{ij} + \frac{1}{4}\chi^{\perp(2)''}_{ij}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left[\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\chi_{ij}^{\perp(2)}+\frac{a'}{2a}\chi_{ij}^{\top(2)\prime}-\frac{1}{4}\nabla^2\chi_{ij}^{\top(2)}+\frac{1}{4}\chi_{ij}^{\top(2)\prime\prime}+\frac{1}{2}\left[\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\chi_{ij}^{\top(2)}\\
&+3\phi^{(1)}_{,i}\phi^{(1)}_{,j}+2\phi^{(1)}\phi^{(1)}_{,ij}+\frac{1}{2}\phi^{(1)\prime}D_{ij}\chi^{\|(1)\prime}+\frac{1}{2}\phi^{(1)}_{,k}D^k_j\chi^{\|(1)}_{,i}+\frac{1}{2}\phi^{(1)}_{,k}D^k_i\chi^{\|(1)}_{,j}\\
&-\frac{3}{2}\phi^{(1)}_{,k}D_{ij}\chi^{\|(1),k}+\phi^{(1)}D^k_j\chi^{\|(1)}_{,ik}+\phi^{(1)}D^k_i\chi^{\|(1)}_{,jk}-\phi^{(1)}\nabla^2D_{ij}\chi^{\|(1)}-2D_{ij}\chi^{\|(1)}\nabla^2\phi^{(1)}\\
&+\phi^{(1)}_{,i}D^k_j\chi^{\|(1)}_{,k}+\phi^{(1)}_{,j}D^k_i\chi^{\|(1)}_{,k}+\phi^{(1)}_{,ki}D^k_j\chi^{\|(1)}+\phi^{(1)}_{,kj}D^k_i\chi^{\|(1)}+3\phi^{(1)\prime\prime}D_{ij}\chi^{\|(1)}\\
&+6\frac{a'}{a}\phi^{(1)\prime}D_{ij}\chi^{\|(1)}+\frac{1}{2}\phi^{(1)\prime}\chi_{ij}^{\top(1)\prime}+\frac{1}{2}\phi^{(1)}_{,k}\chi_{j,i}^{\top(1)k}+\frac{1}{2}\phi^{(1)}_{,k}\chi_{i,j}^{\top(1)k}-\frac{3}{2}\phi^{(1)}_{,k}\chi_{ij}^{\top(1),k}\\
&-\phi^{(1)}\nabla^2\chi_{ij}^{\top(1)}-2\chi_{ij}^{\top(1)}\nabla^2\phi^{(1)}+\phi^{(1)}_{,ki}\chi_j^{\top(1)k}+\phi^{(1)}_{,kj}\chi_i^{\top(1)k}+3\phi^{(1)\prime\prime}\chi_{ij}^{\top(1)}\\
&+6\frac{a'}{a}\phi^{(1)\prime}\chi_{ij}^{\top(1)}-\frac{1}{2}D^k_i\chi^{\|(1)\prime}D_{kj}\chi^{\|(1)\prime}-\frac{1}{2}D^{kl}\chi^{\|(1)}_{,l}D_{kj}\chi^{\|(1)}_{,i}-\frac{1}{2}D^{kl}\chi^{\|(1)}_{,l}D_{ki}\chi^{\|(1)}_{,j}\\
&+\frac{1}{2}D^{lk}\chi^{\|(1)}_{,l}D_{ij}\chi^{\|(1)}_{,k}-\frac{1}{2}D^{kl}\chi^{\|(1)}D_{lj}\chi^{\|(1)}_{,ik}-\frac{1}{2}D^{kl}\chi^{\|(1)}D_{li}\chi^{\|(1)}_{,jk}\\
&+\frac{1}{2}D^{kl}\chi^{\|(1)}D_{ij}\chi^{\|(1)}_{,kl}+\frac{1}{2}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)}_{,ij}+\frac{1}{4}D^{kl}\chi^{\|(1)}_{,i}D_{kl}\chi^{\|(1)}_{,j}\\
&-\frac{1}{2}D_{kl}\chi^{\|(1),kl}D_{ij}\chi^{\|(1)}-\frac{1}{2}D^l_i\chi^{\|(1)}_{,k}D^k_j\chi^{\|(1)}_{,l}+\frac{1}{2}D^k_i\chi^{\|(1)}_{,l}D_{kj}\chi^{\|(1),l}-\frac{1}{2}\chi_{kj}^{\top(1)\prime}D^k_i\chi^{\|(1)\prime}\\
&-\frac{1}{2}\chi_i^{\top(1)\prime k}D_{kj}\chi^{\|(1)\prime}-\frac{1}{2}\chi_{kj,i}^{\top(1)}D^{kl}\chi^{\|(1)}_{,l}-\frac{1}{2}\chi_{ki,j}^{\top(1)}D^{kl}\chi^{\|(1)}_{,l}+\frac{1}{2}\chi_{ij,k}^{\top(1)}D^{lk}\chi^{\|(1)}_{,l}\\
&-\frac{1}{2}\chi_{lj,ik}^{\top(1)}D^{kl}\chi^{\|(1)}-\frac{1}{2}\chi^{\top(1)kl}D_{lj}\chi^{\|(1)}_{,ik}-\frac{1}{2}\chi_{li,jk}^{\top(1)}D^{kl}\chi^{\|(1)}-\frac{1}{2}\chi^{\top(1)kl}D_{li}\chi^{\|(1)}_{,jk}\\
&+\frac{1}{2}\chi_{ij,kl}^{\top(1)}D^{kl}\chi^{\|(1)}+\frac{1}{2}\chi^{\top(1)kl}D_{ij}\chi^{\|(1)}_{,kl}+\frac{1}{2}\chi_{kl,ij}^{\top(1)}D^{kl}\chi^{\|(1)}+\frac{1}{2}\chi^{\top(1)kl}D_{kl}\chi^{\|(1)}_{,ij}\\
&+\frac{1}{4}\chi_{kl,j}^{\top(1)}D^{kl}\chi^{\|(1)}_{,i}+\frac{1}{4}\chi_{,i}^{\top(1)kl}D_{kl}\chi^{\|(1)}_{,j}-\frac{1}{2}\chi_{ij}^{\top(1)}D_{kl}\chi^{\|(1),kl}-\frac{1}{2}\chi_{j,l}^{\top(1)k}D^l_i\chi^{\|(1)}_{,k}\\
&-\frac{1}{2}\chi_{i,k}^{\top(1)l}D^k_j\chi^{\|(1)}_{,l}+\frac{1}{2}\chi_{kj}^{\top(1),l}D^k_i\chi^{\|(1)}_{,l}+\frac{1}{2}\chi_{i,l}^{\top(1)k}D_{kj}\chi^{\|(1),l}-\frac{1}{2}\chi_i^{\top(1)\prime k}\chi_{kj}^{\top(1)\prime}\\
&-\frac{1}{2}\chi^{\top(1)kl}\chi_{lj,ik}^{\top(1)}-\frac{1}{2}\chi^{\top(1)kl}\chi_{li,jk}^{\top(1)}+\frac{1}{2}\chi^{\top(1)kl}\chi_{ij,kl}^{\top(1)}+\frac{1}{2}\chi^{\top(1)kl}\chi_{kl,ij}^{\top(1)}\\
&+\frac{1}{4}\chi_{,i}^{\top(1)kl}\chi_{kl,j}^{\top(1)}-\frac{1}{2}\chi_{i,k}^{\top(1)l}\chi_{j,l}^{\top(1)k}+\frac{1}{2}\chi_{i,l}^{\top(1)k}\chi_{kj}^{\top(1),l}.
\end{aligned}
\tag{A18}
$$

The above expressions are valid for a general expansion stage and corrects some typos in (B7) of Ref. [46] where four terms of $[2\frac{a''}{a}-(\frac{a'}{a})^2]$ were missed and which was valid only for MD stage.

The 1st-order energy-momentum tensor is

$$
T_{00}^{(1)}=(\rho^{(1)}+p^{(1)})[U_0U_0]^{(0)}+g_{00}^{(0)}p^{(1)}=a^2\rho^{(0)}\delta^{(1)},
\tag{A19}
$$

$$
T_{0i}^{(1)}=(\rho^{(0)}+p^{(0)})[U_0U_i]^{(1)}=-(1+c_s^2)\rho^{(0)}a^2v_i^{(1)},
\tag{A20}
$$

$$
T_{ij}^{(1)}=[(\rho+p)U_iU_j+g_{ij}p]^{(1)}=a^2\rho^{(0)}(c_s^2\gamma_{ij}^{(1)}+\delta_{ij}c_L^2\delta^{(1)}).
\tag{A21}
$$

The 2nd-order energy-momentum tensor is

$$T_{00}^{(2)} = \left(\frac{1}{2}\rho^{(2)} + \frac{1}{2}p^{(2)}\right)[U_0U_0]^{(0)} + (\rho^{(0)} + p^{(0)})[U_0U_0]^{(2)} + g_{00}^{(0)}\frac{1}{2}p^{(2)}$$
$$= \frac{1}{2}a^2\delta^{(2)}\rho^{(0)} + (1+c_s^2)a^2v^{(1)m}v_m^{(1)}\rho^{(0)}, \tag{A22}$$

$$T_{0i}^{(2)} = (\rho^{(0)} + p^{(0)})[U_0]^{(0)}[U_i]^{(2)} + (\rho^{(1)} + p^{(1)})[U_0]^{(0)}[U_i]^{(1)}$$
$$= -\rho^{(0)}(1+c_s^2)a^2\left(\gamma_{ij}^{(1)}v^{(1)j} + \frac{1}{2}v_i^{(2)}\right) - \delta^{(1)}\rho^{(0)}(1+c_L^2)a^2v_i^{(1)}, \tag{A23}$$

$$T_{ij}^{(2)} = (\rho^{(0)} + p^{(0)})[U_i]^{(1)}[U_j]^{(1)} + g_{ij}^{(0)}\frac{1}{2}p^{(2)} + \frac{1}{2}g_{ij}^{(2)}p^{(0)} + g_{ij}^{(1)}p^{(1)}$$
$$= \rho^{(0)}(1+c_s^2)a^2v_i^{(1)}v_j^{(1)} + a^2\delta_{ij}\frac{1}{2}c_N^2\rho^{(0)}\delta^{(2)} + \frac{1}{2}a^2\gamma_{ij}^{(2)}c_s^2\rho^{(0)}$$
$$+ a^2\gamma_{ij}^{(1)}c_L^2\rho^{(0)}\delta^{(1)}. \tag{A24}$$

APPENDIX B: 2ND-ORDER PERTURBED EINSTEIN EQUATIONS FOR A GENERAL RW SPACETIME

In this Appendix we list the perturbed Einstein equations and the equation of covariant conservation for a flat RW spacetime with a general scale factor $a(\tau)$.

The (00) component of 1st-order perturbed Einstein equation, i.e., the 1st-order energy constraint is

$$-6\frac{a'}{a}\phi^{(1)'} + 2\nabla^2\phi^{(1)} + \frac{1}{3}\nabla^2\nabla^2\chi^{\|(1)} = 3\left(\frac{a'}{a}\right)^2\delta^{(1)}. \tag{B1}$$

The $(0i)$ component of the 1st-order perturbed Einstein equation, i.e., the 1st-order momentum constraint is

$$2\phi_{,i}^{(1)'} + \frac{1}{2}D_{ij}\chi^{\|(1)',j} + \frac{1}{2}\chi_{ij}^{\perp(1)',j} = -3(1+c_s^2)\left(\frac{a'}{a}\right)^2v_i^{(1)}. \tag{B2}$$

The (ij) component of 1st-order perturbed Einstein equation, i.e., the 1st-order evolution equation is

$$2\phi^{(1)''}\delta_{ij} + 4\frac{a'}{a}\phi^{(1)'}\delta_{ij} + \phi_{,ij}^{(1)} - \nabla^2\phi^{(1)}\delta_{ij}$$
$$+ \frac{1}{2}D_{ij}\chi^{\|(1)''} + \frac{a'}{a}D_{ij}\chi^{\|(1)'} + \frac{1}{6}\nabla^2D_{ij}\chi^{\|(1)}$$
$$- \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\|(1)} + \frac{1}{2}\chi_{ij}^{\perp(1)''} + \frac{a'}{a}\chi_{ij}^{\perp(1)'}$$
$$+ \frac{1}{2}\chi_{ij}^{\top(1)''} + \frac{a'}{a}\chi_{ij}^{\top(1)'} - \frac{1}{2}\nabla^2\chi_{ij}^{\top(1)} = 3c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\delta_{ij}. \tag{B3}$$

The 1st-order energy conservation is

$$\delta^{(1)'} + \frac{2a''}{a'}\delta^{(1)} + \frac{3a'}{a}\left(c_L^2 - \frac{1}{3}\right)\delta^{(1)}$$
$$= 3(1+c_s^2)\phi^{(1)'} - (1+c_s^2)\nabla^2v^{\|(1)}. \tag{B4}$$

The 1st-order momentum conservation is

$$c_L^2\delta^{(1),i} + \frac{2a''}{a'}(1+c_s^2)v^{(1)i} + (1+c_s^2)v^{(1)'i} = 0. \tag{B5}$$

The following are the 2nd-order perturbed equations for the scalar-scalar coupling of 1st-order perturbations. The (00) component of 2nd-order perturbed Einstein equation is

$$G_{00}^{(2)} = 8\pi GT_{00}^{(2)}, \tag{B6}$$

where $G_{00}^{(2)}$ is given by (A16) and $T_{00}^{(2)}$ is given by (A22). For the scalar-scalar coupling, this gives the 2nd-order energy constraint

$$\nabla^2\phi_S^{(2)} - \frac{3a'}{a}\phi_S^{(2)'} + \frac{1}{4}D^{kl}\chi_{S,kl}^{\|(2)} - 12\frac{a'}{a}\phi^{(1)'}\phi^{(1)} + 3\phi^{(1)'}\phi^{(1)'}$$
$$+ 3\phi_{,k}^{(1)}\phi^{(1),k} + 8\phi^{(1)}\nabla^2\phi^{(1)} + 2\phi^{(1)}D^{kl}\chi_{,kl}^{\|(1)}$$
$$- \phi_{,kl}^{(1)}D^{kl}\chi^{\|(1)} - \frac{1}{8}D^{kl}\chi^{\|(1)'}D_{kl}\chi^{\|(1)'} - \frac{a'}{a}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)'}$$
$$- D_{ml}\chi_{,k}^{\|(1),l}D^{km}\chi^{\|(1)} + \frac{1}{2}D^{km}\chi^{\|(1)}\nabla^2D_{km}\chi^{\|(1)}$$
$$- \frac{1}{2}D^{km}\chi_{,k}^{\|(1)}D_{ml}\chi^{\|(1),l}$$
$$+ \frac{3}{8}D^{km}\chi_{,l}^{\|(1)}D_{km}\chi^{\|(1),l} - \frac{1}{4}D^{km}\chi_{,l}^{\|(1)}D_k^l\chi_{,m}^{\|(1)}$$
$$= 8\pi G\left[\frac{1}{2}a^2\delta_S^{(2)}\rho^{(0)} + (1+c_s^2)a^2v^{\|(1),m}v_{,m}^{\|(1)}\rho^{(0)}\right], \tag{B7}$$

where a subscript "S" in $\phi_S^{(2)}$, etc., indicates the case of scalar-scalar coupling, and the 1st-order scalar perturbations $\phi^{(1)}$, $\chi^{\|(1)}$, $v^{\|(1)}$ are already shown in Sec. III. Moving the scalar-scalar coupling terms to the rhs gives the following 2nd-order energy constraint:

$$-\frac{6a'}{a}\phi_S^{(2)\prime}+2\nabla^2\phi_S^{(2)}+\frac{1}{3}\nabla^2\nabla^2\chi_S^{\|(2)}=3\left(\frac{a'}{a}\right)^2\delta_S^{(2)}+E_S, \tag{B8}$$

where

$$\begin{aligned}E_S={}&6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),k}v^{\|(1)}_{,k}+24\frac{a'}{a}\phi^{(1)\prime}\phi^{(1)}-6\phi^{(1)\prime}\phi^{(1)\prime}-6\phi^{(1)}_{,k}\phi^{(1),k}-16\phi^{(1)}\nabla^2\phi^{(1)}\\&-\frac{8}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}+2\phi^{(1),kl}\chi^{\|(1)}_{,kl}-\frac{2}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}+\frac{1}{4}\chi^{\|(1)\prime,kl}\chi^{\|(1)\prime}_{,kl}\\&-\frac{1}{12}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}+\frac{2a'}{a}\chi^{\|(1),kl}\chi^{\|(1)\prime}_{,kl}-\frac{2a'}{3a}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)\prime}+\frac{1}{3}\chi^{\|(1),kl}\nabla^2\chi^{\|(1)}_{,kl}\\&-\frac{1}{9}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}-\frac{1}{4}\chi^{\|(1),klm}\chi^{\|(1)}_{,klm}+\frac{5}{12}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\chi^{\|(1),k}.\end{aligned} \tag{B9}$$

The $(0i)$ component of the 2nd-order perturbed Einstein equation is

$$G_{0i}^{(2)}=R_{0i}^{(2)}=8\pi G T_{0i}^{(2)}, \tag{B10}$$

where $R_{0i}^{(2)}$ is given by (A13) and $T_{0i}^{(2)}$ by (A23). For the scalar-scalar coupling it is

$$\begin{aligned}&\phi^{(2)\prime}_{S,i}+\frac{1}{4}D_{ij}\chi_S^{\|(2)\prime,j}+\frac{1}{4}\chi^{\perp(2)\prime,j}_{Sij}+4\phi^{(1)\prime}\phi^{(1)}_{,i}+4\phi^{(1)}\phi^{(1)\prime}_{,i}+\phi^{(1)}D_{ij}\chi^{\|(1)\prime,j}\\&\quad+\phi^{(1)\prime}D_{ij}\chi^{\|(1),j}-\frac{1}{2}\phi^{(1),j}D_{ij}\chi^{\|(1)\prime}+\phi^{(1)\prime,j}D_{ij}\chi^{\|(1)}-\frac{1}{2}D^k_j\chi^{\|(1),j}D_{ik}\chi^{\|(1)\prime}\\&\quad-\frac{1}{2}D^k_j\chi^{\|(1)}D_{ik}\chi^{\|(1)\prime,j}+\frac{1}{2}D^{jk}\chi^{\|(1)\prime}_{,i}D_{jk}\chi^{\|(1)}+\frac{1}{4}D^{jk}\chi^{\|(1)\prime}D_{jk}\chi^{\|(1)}_{,i}\\&=8\pi G\left[-a^2\rho^{(0)}(1+c_s^2)\left(-2\phi^{(1)}v^{\|(1)}_{,i}+v^{\|(1),m}D_{im}\chi^{\|(1)}+\frac{1}{2}v^{(2)}_{Si}\right)-a^2\rho^{(0)}(1+c_L^2)\delta^{(1)}v^{\|(1)}_{,i}\right].\end{aligned} \tag{B11}$$

Moving the couplings to the rhs, one has the 2nd-order momentum constraint

$$2\phi^{(2)\prime}_{S,i}+\frac{1}{2}D_{ij}\chi_S^{\|(2)\prime,j}+\frac{1}{2}\chi^{\perp(2)\prime,j}_{Sij}=-3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{(2)}_{Si}+M_{Si}, \tag{B12}$$

where

$$\begin{aligned}M_{Si}\equiv{}&12(1+c_s^2)\left(\frac{a'}{a}\right)^2\phi^{(1)}v^{\|(1)}_{,i}-6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),k}\chi^{\|(1)}_{,ki}\\&+2(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1)}_{,i}\nabla^2\chi^{\|(1)}-6(1+c_L^2)\left(\frac{a'}{a}\right)^2\delta^{(1)}v^{\|(1)}_{,i}\\&-8\phi^{(1)\prime}\phi^{(1)}_{,i}-8\phi^{(1)}\phi^{(1)\prime}_{,i}-\frac{4}{3}\phi^{(1)}\nabla^2\chi^{\|(1)\prime}_{,i}-\frac{4}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}_{,i}\\&+\phi^{(1),k}\chi^{\|(1)\prime}_{,ki}-\frac{1}{3}\phi^{(1)}_{,i}\nabla^2\chi^{\|(1)\prime}-2\phi^{(1)\prime,k}\chi^{\|(1)}_{,ki}+\frac{2}{3}\phi^{(1)\prime}_{,i}\nabla^2\chi^{\|(1)}\\&+\frac{2}{3}\chi^{\|(1)\prime}_{,ki}\nabla^2\chi^{\|(1),k}-\frac{1}{18}\nabla^2\chi^{\|(1)}_{,i}\nabla^2\chi^{\|(1)\prime}-\frac{1}{3}\chi^{\|(1)}_{,ki}\nabla^2\chi^{\|(1)\prime,k}\\&+\frac{1}{9}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)\prime}_{,i}-\frac{1}{2}\chi^{\|(1)\prime,kl}\chi^{\|(1)}_{,kli}.\end{aligned} \tag{B13}$$

The longitudinal part of momentum constraint (B12) is

$$2\phi_S^{(2)\prime}+\frac{1}{3}\nabla^2\chi_S^{\|(2)\prime}=-3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v_S^{\|(2)}+\nabla^{-2}M_{Sl}^{,l}, \tag{B14}$$

and the transverse part is

$$\frac{1}{2}\chi_{Sij}^{\perp(2)\prime,j}=-3(1+c_s^2)\left(\frac{a'}{a}\right)^2 v_{Si}^{\perp(2)}+(M_{Si}-\partial_i\nabla^{-2}M_{Sl}^{,l}), \tag{B15}$$

where

$$\begin{aligned}
M_{Sl}^{,l}=&\nabla^2\bigg[-8\phi^{(1)\prime}\phi^{(1)}-\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)\prime}-\frac{4}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}-\frac{1}{18}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)\prime}\\
&-\frac{1}{6}\chi_{,k}^{\|(1)}\nabla^2\chi^{\|(1)\prime,k}\bigg]+12(1+c_s^2)\left(\frac{a'}{a}\right)^2\phi^{(1),k}v_{,k}^{\|(1)}+12(1+c_s^2)\left(\frac{a'}{a}\right)^2\phi^{(1)}\nabla^2 v^{\|(1)}\\
&-6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),kl}\chi_{,kl}^{\|(1)}-4(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),k}\nabla^2\chi_{,k}^{\|(1)}\\
&+2(1+c_s^2)\left(\frac{a'}{a}\right)^2\nabla^2 v^{\|(1)}\nabla^2\chi^{\|(1)}-6(1+c_L^2)\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2 v^{\|(1)}\\
&-6(1+c_L^2)\left(\frac{a'}{a}\right)^2\delta_{,k}^{(1)}v^{\|(1),k}+\phi^{(1),kl}\chi_{,kl}^{\|(1)\prime}-\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)\prime}+2\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)}\\
&-2\phi^{(1)\prime,kl}\chi_{,kl}^{\|(1)}+\frac{1}{6}\chi_{,k}^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)\prime,k}+\frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)\prime}\\
&+\frac{1}{6}\chi_{,kl}^{\|(1)\prime}\nabla^2\chi^{\|(1),kl}+\frac{2}{3}\nabla^2\chi_{,k}^{\|(1)\prime}\nabla^2\chi^{\|(1),k}-\frac{1}{2}\chi^{\|(1)\prime,klm}\chi_{,klm}^{\|(1)},
\end{aligned} \tag{B16}$$

and

$$\begin{aligned}
(M_{Si}-\partial_i\nabla^{-2}M_{Sl}^{,l})=&12(1+c_s^2)\left(\frac{a'}{a}\right)^2\phi^{(1)}v_{,i}^{\|(1)}-6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),k}\chi_{,ki}^{\|(1)}\\
&+2(1+c_s^2)\left(\frac{a'}{a}\right)^2 v_{,i}^{\|(1)}\nabla^2\chi^{\|(1)}-6(1+c_L^2)\left(\frac{a'}{a}\right)^2\delta^{(1)}v_{,i}^{\|(1)}\\
&-\phi^{(1)}\nabla^2\chi_{,i}^{\|(1)\prime}+2\phi_{,i}^{(1)\prime}\nabla^2\chi^{\|(1)}+\phi^{(1),k}\chi_{,ki}^{\|(1)\prime}-2\phi^{(1)\prime,k}\chi_{,ki}^{\|(1)}+\frac{2}{3}\chi_{,ki}^{\|(1)\prime}\nabla^2\chi^{\|(1),k}\\
&-\frac{1}{6}\chi_{,ki}^{\|(1)}\nabla^2\chi^{\|(1)\prime,k}+\frac{1}{6}\chi_{,k}^{\|(1)}\nabla^2\chi_{,i}^{\|(1)\prime,k}+\frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\chi_{,i}^{\|(1)\prime}-\frac{1}{2}\chi^{\|(1)\prime,kl}\chi_{,kli}^{\|(1)}\\
&+\partial_i\nabla^{-2}\bigg[-12(1+c_s^2)\left(\frac{a'}{a}\right)^2\phi^{(1),k}v_{,k}^{\|(1)}-12(1+c_s^2)\left(\frac{a'}{a}\right)^2\phi^{(1)}\nabla^2 v^{\|(1)}\\
&+6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),kl}\chi_{,kl}^{\|(1)}+4(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1),k}\nabla^2\chi_{,k}^{\|(1)}\\
&-2(1+c_s^2)\left(\frac{a'}{a}\right)^2\nabla^2 v^{\|(1)}\nabla^2\chi^{\|(1)}+6(1+c_L^2)\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2 v^{\|(1)}\\
&+6(1+c_L^2)\left(\frac{a'}{a}\right)^2\delta_{,k}^{(1)}v^{\|(1),k}-\phi^{(1),kl}\chi_{,kl}^{\|(1)\prime}+\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)\prime}-2\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)}\\
&+2\phi^{(1)\prime,kl}\chi_{,kl}^{\|(1)}-\frac{1}{6}\chi_{,k}^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)\prime,k}-\frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)\prime}-\frac{1}{6}\chi_{,kl}^{\|(1)\prime}\nabla^2\chi^{\|(1),kl}\\
&-\frac{2}{3}\nabla^2\chi_{,k}^{\|(1)\prime}\nabla^2\chi^{\|(1),k}+\frac{1}{2}\chi^{\|(1)\prime,klm}\chi_{,klm}^{\|(1)}\bigg].
\end{aligned} \tag{B17}$$

The (ij) component of 2nd-order perturbed Einstein equation is

$$G_{ij}^{(2)} = 8\pi G T_{ij}^{(2)}, \tag{B18}$$

where $G_{ij}^{(2)}$ is given in (A18) and $T_{ij}^{(2)}$ in (A24). For the scalar-scalar coupling, (B18) gives the 2nd-order evolution equation

$$\begin{aligned}
&\delta_{ij}\Bigg[-\frac{1}{2}\nabla^2\phi_S^{(2)} + \phi_S^{(2)''} + 2\frac{a'}{a}\phi_S^{(2)'} + \left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]\phi_S^{(2)} - \frac{1}{4}D^{kl}\chi_{S,kl}^{\|(2)} - 2\phi^{(1),k}\phi_{,k}^{(1)} \\
&\quad - 2\phi^{(1)}\nabla^2\phi^{(1)} + (\phi^{(1)'})^2 - \phi_{,k}^{(1)}D^{lk}\chi_{,l}^{\|(1)} - \phi^{(1)}D^{kl}\chi_{,kl}^{\|(1)} + \frac{a'}{a}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)'} \\
&\quad + D_{ml}\chi_{,k}^{\|(1),l}D^{km}\chi^{\|(1)} - \frac{1}{2}D^{km}\chi^{\|(1)}\nabla^2 D_{km}\chi^{\|(1)} - \frac{3}{8}D^{km}\chi^{\|(1),l}D_{km}\chi_{,l}^{\|(1)} \\
&\quad + \frac{1}{4}D^{km}\chi^{\|(1),l}D_{kl}\chi_{,m}^{\|(1)} + \frac{1}{2}D^{kl}\chi_{,l}^{\|(1)}D_k^m\chi_{,m}^{\|(1)} + \frac{3}{8}D^{kl}\chi^{\|(1)'}D_{kl}\chi^{\|(1)'} \\
&\quad + \frac{1}{2}D^{kl}\chi^{\|(1)}D_{kl}\chi^{\|(1)''}\Bigg] + \frac{1}{2}\phi_{S,ij}^{(2)} + \frac{a'}{2a}D_{ij}\chi_S^{\|(2)'} + \frac{1}{4}D_j^k\chi_{S,ki}^{\|(2)} + \frac{1}{4}D_i^k\chi_{S,kj}^{\|(2)} \\
&\quad - \frac{1}{4}\nabla^2 D_{ij}\chi_S^{\|(2)} + \frac{1}{4}D_{ij}\chi_S^{\|(2)''} + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi_S^{\|(2)} + \frac{a'}{2a}\chi_{Sij}^{\perp(2)'} + \frac{1}{4}\chi_{Sj,ki}^{\perp(2)k} \\
&\quad + \frac{1}{4}\chi_{Si,kj}^{\perp(2)k} - \frac{1}{4}\nabla^2\chi_{Sij}^{\perp(2)} + \frac{1}{4}\chi_{Sij}^{\perp(2)''} + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Sij}^{\perp(2)} + \frac{a'}{2a}\chi_{Sij}^{\top(2)'} - \frac{1}{4}\nabla^2\chi_{Sij}^{\top(2)} \\
&\quad + \frac{1}{4}\chi_{Sij}^{\top(2)''} + \frac{1}{2}\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Sij}^{\top(2)} + 3\phi_{,i}^{(1)}\phi_{,j}^{(1)} + 2\phi^{(1)}\phi_{,ij}^{(1)} + \frac{1}{2}\phi^{(1)'}D_{ij}\chi^{\|(1)'} \\
&\quad + \frac{1}{2}\phi_{,k}^{(1)}D_j^k\chi_{,i}^{\|(1)} + \frac{1}{2}\phi_{,k}^{(1)}D_i^k\chi_{,j}^{\|(1)} - \frac{3}{2}\phi_{,k}^{(1)}D_{ij}\chi^{\|(1),k} + \phi^{(1)}D_j^k\chi_{,ik}^{\|(1)} + \phi^{(1)}D_i^k\chi_{,jk}^{\|(1)} \\
&\quad - \phi^{(1)}\nabla^2 D_{ij}\chi^{\|(1)} - 2D_{ij}\chi^{\|(1)}\nabla^2\phi^{(1)} + \phi_{,i}^{(1)}D_j^k\chi_{,k}^{\|(1)} + \phi_{,j}^{(1)}D_i^k\chi_{,k}^{\|(1)} + \phi_{,ki}^{(1)}D_j^k\chi^{\|(1)} \\
&\quad + \phi_{,kj}^{(1)}D_i^k\chi^{\|(1)} + 3\phi^{(1)''}D_{ij}\chi^{\|(1)} + 6\frac{a'}{a}\phi^{(1)'}D_{ij}\chi^{\|(1)} - \frac{1}{2}D_i^k\chi^{\|(1)'}D_{kj}\chi^{\|(1)'} \\
&\quad - \frac{1}{2}D^{kl}\chi_{,l}^{\|(1)}D_{kj}\chi_{,i}^{\|(1)} - \frac{1}{2}D^{kl}\chi_{,l}^{\|(1)}D_{ki}\chi_{,j}^{\|(1)} + \frac{1}{2}D^{lk}\chi_{,l}^{\|(1)}D_{ij}\chi_{,k}^{\|(1)} - \frac{1}{2}D^{kl}\chi^{\|(1)}D_{lj}\chi_{,ik}^{\|(1)} \\
&\quad - \frac{1}{2}D^{kl}\chi^{\|(1)}D_{li}\chi_{,jk}^{\|(1)} + \frac{1}{2}D^{kl}\chi^{\|(1)}D_{ij}\chi_{,kl}^{\|(1)} + \frac{1}{2}D^{kl}\chi^{\|(1)}D_{kl}\chi_{,ij}^{\|(1)} + \frac{1}{4}D^{kl}\chi_{,i}^{\|(1)}D_{kl}\chi_{,j}^{\|(1)} \\
&\quad - \frac{1}{2}D_{kl}\chi^{\|(1),kl}D_{ij}\chi^{\|(1)} - \frac{1}{2}D_i^l\chi_{,k}^{\|(1)}D_j^k\chi_{,l}^{\|(1)} + \frac{1}{2}D_i^k\chi_{,l}^{\|(1)}D_{kj}\chi^{\|(1),l} \\
&= 8\pi G a^2\rho^{(0)}\Bigg\{(1+c_s^2)v_{,i}^{\|(1)}v_{,j}^{\|(1)} + \frac{1}{2}c_N^2\delta_S^{(2)}\delta_{ij} + \frac{1}{2}c_s^2[-2\phi_S^{(2)}\delta_{ij} + D_{ij}\chi_S^{\|(2)} \\
&\quad + \chi_{Sij}^{\perp(2)} + \chi_{Sij}^{\top(2)}] + c_L^2[-2\delta^{(1)}\phi^{(1)}\delta_{ij} + \delta^{(1)}D_{ij}\chi^{\|(1)}]\Bigg\}.
\end{aligned} \tag{B19}$$

Moving the coupling terms to the rhs yields the following 2nd-order evolution equation:

$$\begin{aligned}
&2\phi_S^{(2)''}\delta_{ij} + 4\frac{a'}{a}\phi_S^{(2)'}\delta_{ij} + \phi_{S,ij}^{(2)} - \nabla^2\phi_S^{(2)}\delta_{ij} + \left[4\frac{a''}{a} + 6\left(c_s^2 - \frac{1}{3}\right)\left(\frac{a'}{a}\right)^2\right]\phi_S^{(2)}\delta_{ij} + \frac{1}{2}D_{ij}\chi_S^{\|(2)''} + \frac{a'}{a}D_{ij}\chi_S^{\|(2)'} \\
&\quad + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi_S^{\|(2)} + \frac{1}{6}\nabla^2 D_{ij}\chi_S^{\|(2)} - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi_S^{\|(2)} + \frac{1}{2}\chi_{Sij}^{\perp(2)''} + \frac{a'}{a}\chi_{Sij}^{\perp(2)'} \\
&\quad + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Sij}^{\perp(2)} + \frac{1}{2}\chi_{Sij}^{\top(2)''} + \frac{a'}{a}\chi_{Sij}^{\top(2)'} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Sij}^{\top(2)} - \frac{1}{2}\nabla^2\chi_{Sij}^{\top(2)} \\
&= 3c_N^2\left(\frac{a'}{a}\right)^2\delta_S^{(2)}\delta_{ij} + S_{Sij},
\end{aligned} \tag{B20}$$

where

$$
\begin{aligned}
S_{Sij} = {} & 6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v_{,i}^{\parallel(1)} v_{,j}^{\parallel(1)} - 12c_L^2\left(\frac{a'}{a}\right)^2 \delta^{(1)}\phi^{(1)}\delta_{ij} + 6c_L^2\left(\frac{a'}{a}\right)^2 \delta^{(1)}\chi_{,ij}^{\parallel(1)} \\
& - 2c_L^2\left(\frac{a'}{a}\right)^2 \delta^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij} - 6\phi_{,i}^{(1)}\phi_{,j}^{(1)} + 4\phi^{(1),k}\phi_{,k}^{(1)}\delta_{ij} + 4\phi^{(1)}\nabla^2\phi^{(1)}\delta_{ij} - 4\phi^{(1)}\phi_{,ij}^{(1)} \\
& - 2\phi^{(1)\prime}\phi^{(1)\prime}\delta_{ij} - 12\frac{a'}{a}\phi^{(1)\prime}\chi_{,ij}^{\parallel(1)} + 4\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)}\delta_{ij} - \phi^{(1)\prime}\chi_{,ij}^{\parallel(1)\prime} + \frac{1}{3}\phi^{(1)\prime}\nabla^2\chi^{\parallel(1)\prime}\delta_{ij} \\
& - 6\phi^{(1)\prime\prime}\chi_{,ij}^{\parallel(1)} + 2\phi^{(1)\prime\prime}\nabla^2\chi^{\parallel(1)}\delta_{ij} - \phi_{,j}^{(1)}\nabla^2\chi_{,i}^{\parallel(1)} - \phi_{,i}^{(1)}\nabla^2\chi_{,j}^{\parallel(1)} + \phi_{,k}^{(1)}\chi_{,ij}^{\parallel(1),k} \\
& - \frac{2}{3}\phi^{(1)}\nabla^2\chi_{,ij}^{\parallel(1)} + \frac{1}{3}\phi_{,k}^{(1)}\nabla^2\chi^{\parallel(1),k}\delta_{ij} + \frac{2}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)}\delta_{ij} + 4\chi_{,ij}^{\parallel(1)}\nabla^2\phi^{(1)} \\
& - \frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\parallel(1)}\delta_{ij} - 2\phi_{,ki}^{(1)}\chi_{,j}^{\parallel(1),k} - 2\phi_{,kj}^{(1)}\chi_{,i}^{\parallel(1),k} + \frac{4}{3}\phi_{,ij}^{(1)}\nabla^2\chi^{\parallel(1)} \\
& + \frac{1}{4}\chi^{\parallel(1),klm}\chi_{,klm}^{\parallel(1)}\delta_{ij} - \frac{11}{36}\nabla^2\chi^{\parallel(1),k}\nabla^2\chi_{,k}^{\parallel(1)}\delta_{ij} - \frac{2}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)}\delta_{ij} \\
& - \frac{1}{6}\nabla^2\chi_{,i}^{\parallel(1)}\nabla^2\chi_{,j}^{\parallel(1)} - \frac{1}{3}\chi_{,j}^{\parallel(1),k}\nabla^2\chi_{,ik}^{\parallel(1)} - \frac{1}{3}\chi_{,i}^{\parallel(1),k}\nabla^2\chi_{,jk}^{\parallel(1)} + \frac{2}{3}\chi_{,ij}^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)} \\
& + \frac{2}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\chi_{,ij}^{\parallel(1)} - \frac{1}{2}\chi_{,i}^{\parallel(1),kl}\chi_{,klj}^{\parallel(1)} + \frac{2}{3}\chi_{,kij}^{\parallel(1)}\nabla^2\chi^{\parallel(1),k} - \frac{2a'}{a}\chi^{\parallel(1),kl}\chi_{,kl}^{\parallel(1)\prime}\delta_{ij} \\
& + \frac{2a'}{3a}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime}\delta_{ij} - \frac{3}{4}\chi^{\parallel(1)\prime,kl}\chi_{,kl}^{\parallel(1)\prime}\delta_{ij} - \chi^{\parallel(1),kl}\chi_{,kl}^{\parallel(1)\prime\prime}\delta_{ij} \\
& + \frac{1}{3}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime\prime}\delta_{ij} + \frac{13}{36}\nabla^2\chi^{\parallel(1)\prime}\nabla^2\chi^{\parallel(1)\prime}\delta_{ij} + \chi_{,i}^{\parallel(1)\prime,k}\chi_{,kj}^{\parallel(1)\prime} - \frac{2}{3}\chi_{,ij}^{\parallel(1)\prime}\nabla^2\chi^{\parallel(1)\prime}.
\end{aligned} \tag{B21}
$$

The trace part of 2nd-order evolution equation (B20) is

$$
2\phi_S^{(2)\prime\prime} + 4\frac{a'}{a}\phi_S^{(2)\prime} - \frac{2}{3}\nabla^2\phi_S^{(2)} + \left[4\frac{a''}{a} + 6\left(c_s^2 - \frac{1}{3}\right)\left(\frac{a'}{a}\right)^2\right]\phi_S^{(2)} - \frac{1}{9}\nabla^2\nabla^2\chi_S^{\parallel(2)} = 3c_N^2\left(\frac{a'}{a}\right)^2\delta_S^{(2)} + \frac{1}{3}S_{Sk}^k, \tag{B22}
$$

where

$$
\begin{aligned}
S_{Sk}^k = {} & 6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v_{,k}^{\parallel(1)} v^{\parallel(1),k} - 36c_L^2\left(\frac{a'}{a}\right)^2 \delta^{(1)}\phi^{(1)} + 6\phi_{,k}^{(1)}\phi^{(1),k} + 8\phi^{(1)}\nabla^2\phi^{(1)} \\
& - 6\phi^{(1)\prime}\phi^{(1)\prime} - 4\phi_{,kl}^{(1)}\chi^{\parallel(1),kl} + \frac{4}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\parallel(1)} + \frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\parallel(1)} \\
& + \frac{2}{9}\nabla^2\chi^{\parallel(1)}\nabla^2\nabla^2\chi^{\parallel(1)} - \frac{2}{3}\chi^{\parallel(1),kl}\nabla^2\chi_{,kl}^{\parallel(1)} - \frac{5}{12}\nabla^2\chi^{\parallel(1),k}\nabla^2\chi_{,k}^{\parallel(1)} \\
& + \frac{1}{4}\chi^{\parallel(1),klm}\chi_{,klm}^{\parallel(1)} - \frac{6a'}{a}\chi^{\parallel(1),kl}\chi_{,kl}^{\parallel(1)\prime} + \frac{2a'}{a}\nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime} - \frac{5}{4}\chi^{\parallel(1)\prime,kl}\chi_{,kl}^{\parallel(1)\prime} \\
& - 3\chi^{\parallel(1),kl}\chi_{,kl}^{\parallel(1)\prime\prime} + \nabla^2\chi^{\parallel(1)}\nabla^2\chi^{\parallel(1)\prime\prime} + \frac{5}{12}\nabla^2\chi^{\parallel(1)\prime}\nabla^2\chi^{\parallel(1)\prime}.
\end{aligned} \tag{B23}
$$

The traceless part of 2nd-order evolution equation (B20) is

$$
\begin{aligned}
& D_{ij}\phi_S^{(2)} + \frac{1}{2}D_{ij}\chi_S^{\parallel(2)\prime\prime} + \frac{a'}{a}D_{ij}\chi_S^{\parallel(2)\prime} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]D_{ij}\chi_S^{\parallel(2)} + \frac{1}{6}\nabla^2 D_{ij}\chi_S^{\parallel(2)} \\
& + \frac{1}{2}\chi_{Sij}^{\perp(2)\prime\prime} + \frac{a'}{a}\chi_{Sij}^{\perp(2)\prime} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Sij}^{\perp(2)} \\
& + \frac{1}{2}\chi_{Sij}^{\top(2)\prime\prime} + \frac{a'}{a}\chi_{Sij}^{\top(2)\prime} + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi_{Sij}^{\top(2)} - \frac{1}{2}\nabla^2\chi_{Sij}^{\top(2)} = \bar{S}_{Sij},
\end{aligned} \tag{B24}
$$

where

$$\begin{aligned}
\bar{S}_{Sij} \equiv{}& S_{Sij} - \frac{1}{3} S^k_{Sk}\delta_{ij} \\
={}& 6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1)}_{,i} v^{\|(1)}_{,j} - 2(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1)}_{,k} v^{\|(1),k}\delta_{ij} + 6c_L^2\left(\frac{a'}{a}\right)^2 \delta^{(1)}\chi^{\|(1)}_{,ij} \\
& - 2c_L^2\left(\frac{a'}{a}\right)^2 \delta^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij} - 6\phi^{(1)}_{,i}\phi^{(1)}_{,j} + 2\phi^{(1),k}\phi^{(1)}_{,k}\delta_{ij} - 4\phi^{(1)}\phi^{(1)}_{,ij} + \frac{4}{3}\phi^{(1)}\nabla^2\phi^{(1)}\delta_{ij} \\
& - 12\frac{a'}{a}\phi^{(1)\prime}\chi^{\|(1)}_{,ij} + 4\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}\delta_{ij} - \phi^{(1)\prime}\chi^{\|(1)\prime}_{,ij} + \frac{1}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}\delta_{ij} - 6\phi^{(1)\prime\prime}\chi^{\|(1)}_{,ij} \\
& + 2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}\delta_{ij} - \phi^{(1)}_{,j}\nabla^2\chi^{\|(1)}_{,i} - \phi^{(1)}_{,i}\nabla^2\chi^{\|(1)}_{,j} + \phi^{(1)}_{,k}\chi^{\|(1),k}_{,ij} + \frac{1}{3}\phi^{(1)}_{,k}\nabla^2\chi^{\|(1),k}\delta_{ij} \\
& - \frac{2}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}_{,ij} + \frac{2}{9}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}\delta_{ij} + 4\chi^{\|(1)}_{,ij}\nabla^2\phi^{(1)} - \frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij} \\
& - 2\phi^{(1)}_{,ki}\chi^{\|(1),k}_{,j} - 2\phi^{(1)}_{,kj}\chi^{\|(1),k}_{,i} + \frac{4}{3}\phi^{(1)}_{,kl}\chi^{\|(1),kl}\delta_{ij} + \frac{4}{3}\phi^{(1)}_{,ij}\nabla^2\chi^{\|(1)} - \frac{4}{9}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij} \\
& + \frac{2}{3}\chi^{\|(1)}_{,kij}\nabla^2\chi^{\|(1),k} - \frac{1}{6}\nabla^2\chi^{\|(1)}_{,i}\nabla^2\chi^{\|(1)}_{,j} - \frac{1}{6}\nabla^2\chi^{\|(1),k}\nabla^2\chi^{\|(1)}_{,k}\delta_{ij} - \frac{1}{3}\chi^{\|(1),k}_{,j}\nabla^2\chi^{\|(1)}_{,ik} \\
& - \frac{1}{3}\chi^{\|(1),k}_{,i}\nabla^2\chi^{\|(1)}_{,jk} + \frac{2}{9}\chi^{\|(1),kl}\nabla^2\chi^{\|(1)}_{,kl}\delta_{ij} + \frac{2}{3}\chi^{\|(1)}_{,ij}\nabla^2\nabla^2\chi^{\|(1)} + \frac{2}{9}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)}_{,ij} \\
& - \frac{8}{27}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}\delta_{ij} - \frac{1}{2}\chi^{\|(1),kl}_{,i}\chi^{\|(1)}_{,klj} + \frac{1}{6}\chi^{\|(1),klm}\chi^{\|(1)}_{,klm}\delta_{ij} + \chi^{\|(1)\prime,k}_{,i}\chi^{\|(1)\prime}_{,kj} \\
& - \frac{1}{3}\chi^{\|(1)\prime,kl}\chi^{\|(1)\prime}_{,kl}\delta_{ij} - \frac{2}{3}\chi^{\|(1)\prime}_{,ij}\nabla^2\chi^{\|(1)\prime} + \frac{2}{9}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}\delta_{ij}.
\end{aligned} \tag{B25}$$

The scalar part of (B24) is

$$\phi^{(2)}_S + \frac{1}{2}\chi^{\|(2)\prime\prime}_S + \frac{a'}{a}\chi^{\|(2)\prime}_S + \left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\chi^{\|(2)}_S + \frac{1}{6}\nabla^2\chi^{\|(2)}_S = \frac{3}{2}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl}, \tag{B26}$$

where

$$\begin{aligned}
\bar{S}^{,ij}_{Sij} ={}& \frac{2}{3}\nabla^2\nabla^2\left[-\frac{9}{4}\phi^{(1)}\phi^{(1)} - \frac{2}{3}\phi^{(1)}\nabla^2\chi^{\|(1)} - \frac{1}{18}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)} - \frac{1}{16}\chi^{\|(1)}_{,kl}\chi^{\|(1),kl}\right] \\
& + \nabla^2\Bigg[(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1)}_{,k}v^{\|(1),k} + 4c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)} - \frac{5}{3}\phi^{(1)}\nabla^2\phi^{(1)} - 8\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)} - \frac{2}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime} \\
& - 4\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)} - \frac{4}{9}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,k} + \frac{4}{3}\phi^{(1)}_{,kl}\chi^{\|(1),kl} + \frac{1}{54}\nabla^2\chi^{\|(1),k}\nabla^2\chi^{\|(1)}_{,k} + \frac{11}{36}\chi^{\|(1)}_{,kl}\nabla^2\chi^{\|(1),kl} \\
& + \frac{1}{6}\chi^{\|(1)\prime,kl}\chi^{\|(1)\prime}_{,kl} - \frac{1}{9}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}\Bigg] + 6(1+c_s^2)\left(\frac{a'}{a}\right)^2\nabla^2 v^{\|(1)}\nabla^2 v^{\|(1)} + 6(1+c_s^2)\left(\frac{a'}{a}\right)^2 v^{\|(1)}_{,k}\nabla^2 v^{\|(1),k} \\
& - 6c_L^2\left(\frac{a'}{a}\right)^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)} + 6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),kl}\chi^{\|(1)}_{,kl} + 2\phi^{(1),k}\nabla^2\phi^{(1)}_{,k} \\
& + 2\phi^{(1)}\nabla^2\nabla^2\phi^{(1)} - 12\frac{a'}{a}\phi^{(1)\prime,kl}\chi^{\|(1)}_{,kl} + 12\frac{a'}{a}\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)} - \phi^{(1)\prime,kl}\chi^{\|(1)\prime}_{,kl} \\
& + \nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime} - 6\phi^{(1)\prime\prime,kl}\chi^{\|(1)}_{,kl} + 6\nabla^2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)} + \frac{4}{3}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\phi^{(1)} \\
& + \frac{11}{3}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\phi^{(1),k} + \frac{2}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k} - 3\phi^{(1)}_{,klm}\chi^{\|(1),klm} + \frac{7}{9}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\nabla^2\chi^{\|(1),k} + \frac{5}{18}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)} \\
& - \frac{1}{2}\chi^{\|(1)}_{,klm}\nabla^2\chi^{\|(1),klm} + \frac{1}{3}\chi^{\|(1)\prime}_{,kl}\nabla^2\chi^{\|(1)\prime,kl} + \frac{1}{3}\nabla^2\chi^{\|(1)\prime}_{,k}\nabla^2\chi^{\|(1)\prime,k}.
\end{aligned} \tag{B27}$$

The vector part of the equation (B24) is

$$\frac{1}{2}\chi^{\perp(2)''}_{Sij}+\frac{a'}{a}\chi^{\perp(2)'}_{Sij}+\left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\chi^{\perp(2)}_{Sij}=(\nabla^{-2}\bar{S}^{,k}_{Skj,i}+\nabla^{-2}\bar{S}^{,k}_{Ski,j}-2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl,ij}), \tag{B28}$$

where the rhs of the above is

$$\begin{aligned}
&(\nabla^{-2}\bar{S}^{,k}_{Skj,i}+\nabla^{-2}\bar{S}^{,k}_{Ski,j}-2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl,ij})\\
&=\partial_i\left[\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}_{,j}+2\phi^{(1),k}\chi^{\|(1)}_{,kj}+\frac{1}{3}\chi^{\|(1)}_{,kj}\nabla^2\chi^{\|(1),k}\right]\\
&\quad-\partial_i\partial_j\nabla^{-2}\left[6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}-12\frac{a'}{a}\phi^{(1)'}\nabla^2\chi^{\|(1)}-\phi^{(1)'}\nabla^2\chi^{\|(1)'}-6\phi^{(1)''}\nabla^2\chi^{\|(1)}\right.\\
&\quad\left.+\frac{1}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}+2\phi^{(1),kl}\chi^{\|(1)}_{,kl}+\frac{1}{12}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\chi^{\|(1),k}+\frac{1}{3}\chi^{\|(1)}_{,kl}\nabla^2\chi^{\|(1),kl}\right]\\
&\quad+\partial_i\nabla^{-2}\left[6(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,j}\nabla^2v^{\|(1)}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}_{,j}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),k}\chi^{\|(1)}_{,kj}\right.\\
&\quad+2\phi^{(1)}\nabla^2\phi^{(1)}_{,j}-12\frac{a'}{a}\phi^{(1)',k}\chi^{\|(1)}_{,kj}-12\frac{a'}{a}\phi^{(1)'}\nabla^2\chi^{\|(1)}_{,j}-\phi^{(1)'}\nabla^2\chi^{\|(1)'}_{,j}-\phi^{(1)',k}\chi^{\|(1)'}_{,kj}\\
&\quad-6\phi^{(1)''}\nabla^2\chi^{\|(1)}_{,j}-6\phi^{(1)'',k}\chi^{\|(1)}_{,kj}+\frac{4}{3}\nabla^2\chi^{\|(1)}_{,j}\nabla^2\phi^{(1)}-\frac{5}{3}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,kj}-3\phi^{(1)}_{,kl}\chi^{\|(1),kl}_{,j}\\
&\quad\left.+\frac{5}{18}\nabla^2\chi^{\|(1)}_{,j}\nabla^2\nabla^2\chi^{\|(1)}-\frac{1}{2}\chi^{\|(1)}_{,klj}\nabla^2\chi^{\|(1),kl}+\frac{1}{3}\chi^{\|(1)'}_{,kj}\nabla^2\chi^{\|(1)',k}\right]\\
&\quad-\partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[6(1+c_s^2)\left(\frac{a'}{a}\right)^2\nabla^2v^{\|(1)}\nabla^2v^{\|(1)}+6(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,k}\nabla^2v^{\|(1),k}\right.\\
&\quad-6c_L^2\left(\frac{a'}{a}\right)^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),kl}\chi^{\|(1)}_{,kl}+2\phi^{(1),k}\nabla^2\phi^{(1)}_{,k}+2\phi^{(1)}\nabla^2\nabla^2\phi^{(1)}\\
&\quad-12\frac{a'}{a}\phi^{(1)',kl}\chi^{\|(1)}_{,kl}+12\frac{a'}{a}\nabla^2\phi^{(1)'}\nabla^2\chi^{\|(1)}-\phi^{(1)',kl}\chi^{\|(1)'}_{,kl}+\nabla^2\phi^{(1)'}\nabla^2\chi^{\|(1)'}\\
&\quad-6\phi^{(1)'',kl}\chi^{\|(1)}_{,kl}+6\nabla^2\phi^{(1)''}\nabla^2\chi^{\|(1)}+\frac{4}{3}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\phi^{(1)}+\frac{11}{3}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\phi^{(1),k}\\
&\quad+\frac{2}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k}-3\phi^{(1)}_{,klm}\chi^{\|(1),klm}+\frac{7}{9}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\nabla^2\chi^{\|(1),k}-\frac{1}{2}\chi^{\|(1)}_{,klm}\nabla^2\chi^{\|(1),klm}\\
&\quad\left.+\frac{5}{18}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}+\frac{1}{3}\chi^{\|(1)'}_{,kl}\nabla^2\chi^{\|(1)',kl}+\frac{1}{3}\nabla^2\chi^{\|(1)'}_{,k}\nabla^2\chi^{\|(1)',k}\right]+(i\leftrightarrow j).
\end{aligned} \tag{B29}$$

The tensor part (GW) of the equation (B24) is

$$\begin{aligned}
&\frac{1}{2}\chi^{\top(2)''}_{Sij}+\frac{a'}{a}\chi^{\top(2)'}_{Sij}+\left[(1-3c_s^2)\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\chi^{\top(2)}_{Sij}-\frac{1}{2}\nabla^2\chi^{\top(2)}_{Sij}\\
&\quad=\bar{S}_{Sij}-\frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl}-\nabla^{-2}\bar{S}^{,k}_{Skj,i}-\nabla^{-2}\bar{S}^{,k}_{Ski,j}+2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl,ij},
\end{aligned} \tag{B30}$$

where the rhs is the effective source for the 2nd-order tensor

$$
\begin{aligned}
&\left(\bar{S}_{Sij}-\frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl}-\nabla^{-2}\bar{S}^{,k}_{Skj,i}-\nabla^{-2}\bar{S}^{,k}_{Ski,j}+2\nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Skl,ij}\right)\\
&=D_{ij}\bigg[-\frac{3}{4}\phi^{(1)}\phi^{(1)}-\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}-2\phi^{(1)}_{,k}\chi^{\|(1),k}+\frac{1}{6}\nabla^2\chi^{\|(1)}\nabla^2\chi^{\|(1)}+\frac{1}{16}\chi^{\|(1)}_{,kl}\chi^{\|(1),kl}\\
&\quad-\frac{1}{3}\chi^{\|(1)}_{,k}\nabla^2\chi^{\|(1),k}\bigg]+6(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,i}v^{\|(1)}_{,j}-2(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,k}v^{\|(1),k}\delta_{ij}\\
&\quad+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\chi^{\|(1)}_{,ij}-2c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij}+2\phi^{(1)}\phi^{(1)}_{,ij}-\frac{2}{3}\phi^{(1)}\nabla^2\phi^{(1)}\delta_{ij}\\
&\quad-12\frac{a'}{a}\phi^{(1)\prime}\chi^{\|(1)}_{,ij}+4\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}\delta_{ij}-\phi^{(1)\prime}\chi^{\|(1)\prime}_{,ij}+\frac{1}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}\delta_{ij}-6\phi^{(1)\prime\prime}\chi^{\|(1)}_{,ij}\\
&\quad+2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}\delta_{ij}+\frac{7}{3}\phi^{(1)}_{,ij}\nabla^2\chi^{\|(1)}-\frac{7}{9}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij}+\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}_{,ij}\\
&\quad-\frac{1}{9}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}\delta_{ij}+3\phi^{(1)}_{,k}\chi^{\|(1),k}_{,ij}-\phi^{(1)}_{,k}\nabla^2\chi^{\|(1),k}\delta_{ij}+2\chi^{\|(1),k}\phi^{(1)}_{,kij}\\
&\quad-\frac{2}{3}\chi^{\|(1),k}\nabla^2\phi^{(1)}_{,k}\delta_{ij}+4\chi^{\|(1)}_{,ij}\nabla^2\phi^{(1)}-\frac{4}{3}\nabla^2\phi^{(1)}\nabla^2\chi^{\|(1)}\delta_{ij}+\frac{2}{3}\chi^{\|(1)}_{,ij}\nabla^2\nabla^2\chi^{\|(1)}\\
&\quad-\frac{2}{9}\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}\delta_{ij}-\frac{7}{18}\nabla^2\chi^{\|(1)}_{,i}\nabla^2\chi^{\|(1)}_{,j}+\chi^{\|(1)}_{,kij}\nabla^2\chi^{\|(1),k}\\
&\quad-\frac{11}{54}\nabla^2\chi^{\|(1),k}\nabla^2\chi^{\|(1)}_{,k}\delta_{ij}-\frac{1}{9}\chi^{\|(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k}\delta_{ij}+\frac{1}{3}\chi^{\|(1)}_{,k}\nabla^2\chi^{\|(1),k}_{,ij}\\
&\quad-\frac{1}{2}\chi^{\|(1),kl}_{,i}\chi^{\|(1)}_{,klj}+\frac{1}{6}\chi^{\|(1),klm}\chi^{\|(1)}_{,klm}\delta_{ij}+\chi^{\|(1)\prime,k}_{,i}\chi^{\|(1)\prime}_{,kj}-\frac{1}{3}\chi^{\|(1)\prime,kl}\chi^{\|(1)\prime}_{,kl}\delta_{ij}\\
&\quad-\frac{2}{3}\chi^{\|(1)\prime}_{,ij}\nabla^2\chi^{\|(1)\prime}+\frac{2}{9}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}\delta_{ij}\\
&\quad-\frac{3}{2}D_{ij}\nabla^{-2}\bigg[(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,k}v^{\|(1),k}+4c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}-\frac{5}{3}\phi^{(1)}\nabla^2\phi^{(1)}\\
&\quad-8\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}-\frac{2}{3}\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}-4\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}-\frac{4}{9}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,k}+\frac{4}{3}\phi^{(1)}_{,kl}\chi^{\|(1),kl}\\
&\quad+\frac{1}{54}\nabla^2\chi^{\|(1),k}\nabla^2\chi^{\|(1)}_{,k}+\frac{11}{36}\chi^{\|(1)}_{,kl}\nabla^2\chi^{\|(1),kl}+\frac{1}{6}\chi^{\|(1)\prime,kl}\chi^{\|(1)\prime}_{,kl}-\frac{1}{9}\nabla^2\chi^{\|(1)\prime}\nabla^2\chi^{\|(1)\prime}\bigg]\\
&\quad-\frac{3}{2}D_{ij}\nabla^{-2}\nabla^{-2}\bigg[6(1+c_s^2)\left(\frac{a'}{a}\right)^2\nabla^2v^{\|(1)}\nabla^2v^{\|(1)}+6(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,k}\nabla^2v^{\|(1),k}\\
&\quad-6c_L^2\left(\frac{a'}{a}\right)^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),kl}\chi^{\|(1)}_{,kl}+2\phi^{(1),k}\nabla^2\phi^{(1)}_{,k}+2\phi^{(1)}\nabla^2\nabla^2\phi^{(1)}\\
&\quad-12\frac{a'}{a}\phi^{(1)\prime,kl}\chi^{\|(1)}_{,kl}+12\frac{a'}{a}\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)}-\phi^{(1)\prime,kl}\chi^{\|(1)\prime}_{,kl}+\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}\\
&\quad-6\phi^{(1)\prime\prime,kl}\chi^{\|(1)}_{,kl}+6\nabla^2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}+\frac{4}{3}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\phi^{(1)}+\frac{11}{3}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\phi^{(1),k}\\
&\quad+\frac{2}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k}-3\phi^{(1)}_{,klm}\chi^{\|(1),klm}+\frac{7}{9}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\nabla^2\chi^{\|(1),k}+\frac{5}{18}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}\\
&\quad-\frac{1}{2}\chi^{\|(1)}_{,klm}\nabla^2\chi^{\|(1),klm}+\frac{1}{3}\chi^{\|(1)\prime}_{,kl}\nabla^2\chi^{\|(1)\prime,kl}+\frac{1}{3}\nabla^2\chi^{\|(1)\prime}_{,k}\nabla^2\chi^{\|(1)\prime,k}\bigg]\\
&\quad-\partial_i\left[\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}_{,j}+2\phi^{(1),k}\chi^{\|(1)}_{,kj}+\frac{1}{3}\chi^{\|(1)}_{,kj}\nabla^2\chi^{\|(1),k}\right]\\
&\quad-\partial_j\left[\frac{1}{3}\phi^{(1)}\nabla^2\chi^{\|(1)}_{,i}+2\phi^{(1),k}\chi^{\|(1)}_{,ki}+\frac{1}{3}\chi^{\|(1)}_{,ki}\nabla^2\chi^{\|(1),k}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\partial_i\partial_j\nabla^{-2}\bigg[12c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}-24\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}-2\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}-12\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}\\
&+\frac{2}{3}\phi^{(1)}\nabla^2\nabla^2\chi^{\|(1)}+4\phi^{(1),kl}\chi^{\|(1)}_{,kl}+\frac{1}{6}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\chi^{\|(1),k}+\frac{2}{3}\chi^{\|(1)}_{,kl}\nabla^2\chi^{\|(1),kl}\bigg]\\
&-\partial_i\nabla^{-2}\bigg[6(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,j}\nabla^2v^{\|(1)}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}_{,j}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),k}\chi^{\|(1)}_{,kj}\\
&+2\phi^{(1)}\nabla^2\phi^{(1)}_{,j}-12\frac{a'}{a}\phi^{(1)\prime,k}\chi^{\|(1)}_{,kj}-12\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}_{,j}-\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}_{,j}-\phi^{(1)\prime,k}\chi^{\|(1)\prime}_{,kj}\\
&-6\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}_{,j}-6\phi^{(1)\prime\prime,k}\chi^{\|(1)}_{,kj}+\frac{4}{3}\nabla^2\chi^{\|(1)}_{,j}\nabla^2\phi^{(1)}-\frac{5}{3}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,kj}-3\phi^{(1)}_{,kl}\chi^{\|(1),kl}_{,j}\\
&+\frac{5}{18}\nabla^2\chi^{\|(1)}_{,j}\nabla^2\nabla^2\chi^{\|(1)}-\frac{1}{2}\chi^{\|(1)}_{,klj}\nabla^2\chi^{\|(1),kl}+\frac{1}{3}\chi^{\|(1)\prime}_{,kj}\nabla^2\chi^{\|(1)\prime,k}\bigg]\\
&-\partial_j\nabla^{-2}\bigg[6(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,i}\nabla^2v^{\|(1)}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1)}\nabla^2\chi^{\|(1)}_{,i}+6c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),k}\chi^{\|(1)}_{,ki}\\
&+2\phi^{(1)}\nabla^2\phi^{(1)}_{,i}-12\frac{a'}{a}\phi^{(1)\prime,k}\chi^{\|(1)}_{,ki}-12\frac{a'}{a}\phi^{(1)\prime}\nabla^2\chi^{\|(1)}_{,i}-\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}_{,i}-\phi^{(1)\prime,k}\chi^{\|(1)\prime}_{,ki}\\
&-6\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}_{,i}-6\phi^{(1)\prime\prime,k}\chi^{\|(1)}_{,ki}+\frac{4}{3}\nabla^2\chi^{\|(1)}_{,i}\nabla^2\phi^{(1)}-\frac{5}{3}\phi^{(1),k}\nabla^2\chi^{\|(1)}_{,ki}-3\phi^{(1)}_{,kl}\chi^{\|(1),kl}_{,i}\\
&+\frac{5}{18}\nabla^2\chi^{\|(1)}_{,i}\nabla^2\nabla^2\chi^{\|(1)}-\frac{1}{2}\chi^{\|(1)}_{,kli}\nabla^2\chi^{\|(1),kl}+\frac{1}{3}\chi^{\|(1)\prime}_{,ki}\nabla^2\chi^{\|(1)\prime,k}\bigg]\\
&+\partial_i\partial_j\nabla^{-2}\nabla^{-2}\bigg[12(1+c_s^2)\left(\frac{a'}{a}\right)^2\nabla^2v^{\|(1)}\nabla^2v^{\|(1)}+12(1+c_s^2)\left(\frac{a'}{a}\right)^2v^{\|(1)}_{,k}\nabla^2v^{\|(1),k}\\
&-12c_L^2\left(\frac{a'}{a}\right)^2\nabla^2\delta^{(1)}\nabla^2\chi^{\|(1)}+12c_L^2\left(\frac{a'}{a}\right)^2\delta^{(1),kl}\chi^{\|(1)}_{,kl}+4\phi^{(1),k}\nabla^2\phi^{(1)}_{,k}+4\phi^{(1)}\nabla^2\nabla^2\phi^{(1)}\\
&-24\frac{a'}{a}\phi^{(1)\prime,kl}\chi^{\|(1)}_{,kl}+24\frac{a'}{a}\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)}-2\phi^{(1)\prime,kl}\chi^{\|(1)\prime}_{,kl}+2\nabla^2\phi^{(1)\prime}\nabla^2\chi^{\|(1)\prime}\\
&-12\phi^{(1)\prime\prime,kl}\chi^{\|(1)}_{,kl}+12\nabla^2\phi^{(1)\prime\prime}\nabla^2\chi^{\|(1)}+\frac{8}{3}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\phi^{(1)}+\frac{22}{3}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\phi^{(1),k}\\
&+\frac{4}{3}\phi^{(1),k}\nabla^2\nabla^2\chi^{\|(1)}_{,k}-6\phi^{(1)}_{,klm}\chi^{\|(1),klm}+\frac{14}{9}\nabla^2\chi^{\|(1)}_{,k}\nabla^2\nabla^2\chi^{\|(1),k}-\chi^{\|(1)}_{,klm}\nabla^2\chi^{\|(1),klm}\\
&+\frac{5}{9}\nabla^2\nabla^2\chi^{\|(1)}\nabla^2\nabla^2\chi^{\|(1)}+\frac{2}{3}\chi^{\|(1)\prime}_{,kl}\nabla^2\chi^{\|(1)\prime,kl}+\frac{2}{3}\nabla^2\chi^{\|(1)\prime}_{,k}\nabla^2\chi^{\|(1)\prime,k}\bigg].
\end{aligned}
\tag{B31}
$$

The equations of covariant conservation of energy and momentum are given by (2.22) and (2.23). Substituting $\Gamma^{\alpha}_{\mu\nu}$ in (A1)–(A4) and U^{μ} in (2.13) and (2.14) into (2.22) and (2.23) and only keeping scalar-scalar couplings, we arrive at the 2nd-order energy conservation

$$
\begin{aligned}
&\delta_S^{(2)\prime}+2a''(a')^{-1}\delta_S^{(2)}+(-1+3c_N^2)a'a^{-1}\delta_S^{(2)}\\
&\quad+(1+c_s^2)v^{(2)k}_{S,k}-3(1+c_s^2)\phi_S^{(2)\prime}+4(1+c_s^2)a''(a')^{-1}v^{\|(1),k}v^{\|(1)}_{,k}\\
&\quad+4(1+c_s^2)v^{\|(1)\prime,k}v^{\|(1)}_{,k}+2(1+c_L^2)\delta^{(1)}_{,k}v^{\|(1),k}+2(1+c_L^2)\delta^{(1)}\nabla^2v^{\|(1)}-6(1+c_L^2)\delta^{(1)}\phi^{(1)\prime}\\
&\quad-12(1+c_s^2)\phi^{(1)\prime}\phi^{(1)}-(1+c_s^2)D_{kl}\chi^{\|(1)\prime}D^{kl}\chi^{\|(1)}-6(1+c_s^2)\phi^{(1)}_{,k}v^{\|(1),k}=0,
\end{aligned}
\tag{B32}
$$

and the 2nd-order momentum conservation equation

$$
\begin{aligned}
&c_N^2\delta_S^{(2),i} + 2(1+c_s^2)a''(a')^{-1}v_S^{(2)i} + (1+c_s^2)v_S^{(2)'i} \\
&\quad + 4(1+c_L^2)a''(a')^{-1}\delta^{(1)}v^{\|(1),i} + 2(1+c_L^2)\delta^{(1)'}v^{\|(1),i} \\
&\quad + 2(1+c_L^2)\delta^{(1)}v^{\|(1)',i} + 2(1+c_s^2)v^{\|(1),i}_{,k}v^{\|(1),k} \\
&\quad + 2(1+c_s^2)v^{\|(1),i}\nabla^2 v^{\|(1)} + 4c_L^2\delta^{(1),i}\phi^{(1)} \\
&\quad - 10(1+c_s^2)v^{\|(1),i}\phi^{(1)'} - 2c_L^2\delta^{(1)}_{,k}D^{ik}\chi^{\|(1)} \\
&\quad + 2(1+c_s^2)v^{\|(1),k}D^i_k\chi^{\|(1)'} = 0,
\end{aligned} \tag{B33}
$$

which are for a general RW spacetime.

APPENDIX C: GAUGE TRANSFORMATIONS FROM SYNCHRONOUS TO SYNCHRONOUS

The formulae for the gauge transform between two general coordinates for a flat RW spacetime, as well as between two synchronous coordinates, have been analyzed in Appendix C in Ref. [46]. But the transformation of the stress tensor of a relativistic fluid has not been given. Here we shall add some new results on ρ and U^μ.

A general coordinate transformation is given by [42,43,46,48]

$$
x^\mu \to \bar{x}^\mu = x^\mu + \xi^{(1)\mu} + \frac{1}{2}\xi^{(1)\mu}_{,\alpha}\xi^{(1)\alpha} + \frac{1}{2}\xi^{(2)\mu}, \tag{C1}
$$

where $\xi^{(1)\mu}$ is a 1st-order vector field, and $\xi^{(2)\mu}$ is a 2nd-order vector field which is independent of $\xi^{(1)\mu}$. They can be denoted by the parameters

$$
\xi^{(A)0} = \alpha^{(A)}, \qquad \xi^{(A)i} = \partial^i\beta^{(A)} + d^{(A)i}, \quad \text{with} \quad A = 1, 2. \tag{C2}
$$

with the constraint $\partial_i d^{(A)i} = 0$. The transformation rules of a tensor, such as a metric, are $g_{\mu\nu}(x) = \frac{\partial\bar{x}^\alpha}{\partial x^\mu}\frac{\partial\bar{x}^\beta}{\partial x^\nu}\bar{g}_{\alpha\beta}(\bar{x})$ [32,48]. Writing $g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + \frac{1}{2}g^{(2)}_{\mu\nu}$ to the 2nd order, and similar for $\bar{g}_{\mu\nu}$, one has

$$
g^{(0)}_{\mu\nu}(x) = \bar{g}^{(0)}_{\mu\nu}(x), \tag{C3}
$$

$$
\bar{g}^{(1)}_{\mu\nu}(x) = g^{(1)}_{\mu\nu}(x) - \mathcal{L}_{\xi^{(1)}}g^{(0)}_{\mu\nu}(x), \tag{C4}
$$

$$
\begin{aligned}
\bar{g}^{(2)}_{\mu\nu}(x) = g^{(2)}_{\mu\nu}(x) - 2\mathcal{L}_{\xi^{(1)}}g^{(1)}_{\mu\nu}(x) + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}g^{(0)}_{\mu\nu}(x)) \\
- \mathcal{L}_{\xi^{(2)}}g^{(0)}_{\mu\nu}(x),
\end{aligned} \tag{C5}
$$

where the Lie derivative along $\xi^{(1)\mu}$ is defined as $\mathcal{L}_{\xi^{(1)}}g^{(0)}_{\mu\nu} \equiv g^{(0)}_{\mu\nu,\alpha}\xi^{(1)\alpha} + g^{(0)}_{\mu\alpha}\xi^{(1)\alpha}_{,\nu} + g^{(0)}_{\nu\alpha}\xi^{(1)\alpha}_{,\mu}$, and others are similarly defined. Under (C1), a scalar function transforms as $f(x) = \bar{f}(\bar{x})$. By writing $f(x) = f^{(0)}(x) + f^{(1)}(x) + \frac{1}{2}f^{(2)}(x)$, one has

$$
\bar{f}^{(0)}(x) = f^{(0)}(x), \tag{C6}
$$

$$
\bar{f}^{(1)}(x) = f^{(1)}(x) - \mathcal{L}_{\xi^{(1)}}f^{(0)}(x), \tag{C7}
$$

$$
\begin{aligned}
\bar{f}^{(2)}(x) = f^{(2)}(x) - 2\mathcal{L}_{\xi^{(1)}}f^{(1)}(x) + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}f^{(0)}(x)) \\
- \mathcal{L}_{\xi^{(2)}}f^{(0)}(x),
\end{aligned} \tag{C8}
$$

where $\mathcal{L}_\xi f \equiv f_{,\alpha}\xi^\alpha$. A 4-vector Z^μ transforms as $\bar{Z}^\mu(\bar{x}) = \frac{\partial\bar{x}^\mu}{\partial x^\alpha}Z^\alpha(x)$. Writing $Z^\mu(x) = Z^{(0)\mu}(x) + Z^{(1)\mu}(x) + \frac{1}{2}Z^{(2)\mu}(x)$, one has

$$
\bar{Z}^{(0)\mu}(x) = Z^{(0)\mu}(x), \tag{C9}
$$

$$
\bar{Z}^{(1)\mu}(x) = Z^{(1)\mu}(x) - \mathcal{L}_{\xi^{(1)}}Z^{(0)\mu}(x), \tag{C10}
$$

$$
\begin{aligned}
\bar{Z}^{(2)\mu}(x) = Z^{(2)\mu}(x) - 2\mathcal{L}_{\xi^{(1)}}Z^{(1)\mu}(x) + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}Z^{(0)\mu}(x)) \\
- \mathcal{L}_{\xi^{(2)}}Z^{(0)\mu}(x),
\end{aligned} \tag{C11}
$$

where $\mathcal{L}_\xi Z^\mu \equiv Z^\mu_{,\alpha}\xi^\alpha - \xi^\mu_{,\alpha}Z^\alpha$ and $\mathcal{L}_\xi Z_\mu \equiv Z_{\mu,\alpha}\xi^\alpha + \xi_{\mu,\alpha}Z^\alpha$.

The above transformations are general for any two coordinates. In this paper we are only concerned with the ones from synchronous to synchronous coordinates. Moreover, for the 2nd-order transformations (C5), (C8), and (C11), we should distinguish the role of $\xi^{(2)}$ from that of $\xi^{(1)}$. The real interesting case of 2nd-order gauge transformations is that the 2nd-order perturbations are transformed while the 1st-order perturbations are fixed. As we mentioned around Eq. (6.8), this is the effective 2nd-order transformations, which require that $\xi^{(1)} = 0$, but $\xi^{(2)} \neq 0$ in (C5), (C8), and (C11).

First consider the 1st-order transformation. In the synchronous coordinate, requiring $\bar{g}_{00}(x) = -a^2(\tau)$ and $\bar{g}_{0i}(x) = 0$ leads to $\xi^{(1)\mu}$ as the following:

$$
\xi^{(1)0}(\tau, \mathbf{x}) = \frac{A^{(1)}(\mathbf{x})}{a(\tau)}, \tag{C12}
$$

$$
\xi^{(1)i}(\tau, \mathbf{x}) = A^{(1)}(\mathbf{x})^{,i}\int^\tau \frac{d\tau'}{a(\tau')} + C^{(1)i}(\mathbf{x}), \tag{C13}
$$

where $A^{(1)}$ and $C^{(1)i}$ are small, arbitrary functions depending on $\mathbf{x}$ only, and $C^{(1)i}$ can be decomposed into

$$
C^{(1)i}(\mathbf{x}) = C^{\|(1),i}(\mathbf{x}) + C^{\perp(1)i}(\mathbf{x}), \tag{C14}
$$

where the transverse part satisfies $\partial_i C^{\perp(1)i} = 0$. By Eq. (C4), the residual gauge transform of the metric g_{ij} within the synchronous coordinates is

$$\bar{g}^{(1)}_{ij} = g^{(1)}_{ij} + a^2\left(-2\frac{a'}{a}\delta_{ij}\frac{A^{(1)}}{a} - 2A^{(1)}_{,ij}\int^\tau \frac{d\tau}{a(\tau)} - C^{(1)}_{i,j} - C^{(1)}_{j,i}\right), \tag{C15}$$

from which the residual gauge transformations of each mode are identified as the following:

$$\bar{\phi}^{(1)} = \phi^{(1)} + \frac{1}{3}\nabla^2 A^{(1)}\int \frac{d\tau}{a(\tau)} + \frac{1}{3}\nabla^2 C^{\parallel(1)} + \frac{a'}{a^2}A^{(1)}, \tag{C16}$$

$$\bar{\chi}^{\parallel(1)} = \chi^{\parallel(1)} - 2A^{(1)}\int \frac{d\tau}{a(\tau)} - 2C^{\parallel(1)}, \tag{C17}$$

$$\bar{\chi}^{\perp(1)}_{ij} = \chi^{\perp(1)}_{ij} - C^{\perp(1)}_{i,j} - C^{\perp(1)}_{j,i}, \tag{C18}$$

$$\bar{\chi}^{\top(1)}_{ij} = \chi^{\top(1)}_{ij}. \tag{C19}$$

By (C6), the 0th-order energy density transforms as $\bar{\rho}^{(0)} = \rho^{(0)}$. By (C7), the density perturbation transforms as

$$\bar{\rho}^{(1)} = \rho^{(1)} - \rho^{(0)}_{,0}\frac{A^{(1)}}{a}, \tag{C20}$$

which leads to the transform of the density contrast

$$\bar{\delta}^{(1)} = \delta^{(1)} - \left[2\frac{a''}{a'a} - 4\frac{a'}{a^2}\right]A^{(1)}. \tag{C21}$$

By (C9), the 0th-order 4-velocity transforms between synchronous coordinates as $\bar{U}^{(0)0} = U^{(0)0} = a^{-1}$ and $\bar{U}^{(0)i} = U^{(0)i} = 0$. By (C10), the 1st-order velocity transforms as

$$\bar{U}^{(1)0} = U^{(1)0} = 0, \tag{C22}$$

$$\bar{U}^{(1)i} = U^{(1)i} + \frac{A^{(1),i}}{a^2}. \tag{C23}$$

Using Eq. (2.14) and $\bar{U}^{(1)i} = a^{-1}\bar{v}^{(1)i}$, (C23) can written as transformation of the 1st-order 3-velocity

$$\bar{v}^{(1)i} = v^{(1)i} + \frac{A^{(1),i}}{a}, \tag{C24}$$

whose transverse and longitudinal parts are

$$\bar{v}^{\perp(1)i} = v^{\perp(1)i}, \tag{C25}$$

$$\bar{v}^{\parallel(1)} = v^{\parallel(1)} + \frac{A^{(1)}}{a}. \tag{C26}$$

Now we determine the 2nd-order synchronous-to-synchronous gauge transform for a general RW spacetime. First we determine the 2nd-order vector $\xi^{(2)}$. By the requirements $\bar{g}^{(2)}_{00}(x) = g^{(2)}_{00}(x) = 0$, $\bar{g}^{(2)}_{0i}(x) = g^{(2)}_{0i}(x) = 0$, and (C12) and (C13), the formula (C5) gives

$$\xi^{(2)0} = \alpha^{(2)} = \frac{A^{(2)}(\mathbf{x})}{a(\tau)}, \tag{C27}$$

$$\begin{aligned}\xi^{(2)}_i(\tau,\mathbf{x}) = {}& 4A^{(1)}(\mathbf{x})_{,i}\int^\tau \frac{\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')}d\tau' - 2A^{(1)}(\mathbf{x})^{,k}\int^\tau \frac{\chi^{(1)}_{ki}(\tau',\mathbf{x})}{a(\tau')}d\tau' - \frac{1}{a^2}A^{(1)}(\mathbf{x})A^{(1)}(\mathbf{x})_{,i}\\ &+ 2A^{(1)}(\mathbf{x})^{,k}A^{(1)}(\mathbf{x})_{,ki}\int^\tau \frac{d\tau'}{a(\tau')}\int^{\tau'}\frac{d\tau''}{a(\tau'')} + 2A^{(1)}(\mathbf{x})^{,k}C^{\parallel(1)}(\mathbf{x})_{,ki}\int^\tau \frac{d\tau'}{a(\tau')} + A^{(2)}(\mathbf{x})_{,i}\int^\tau \frac{d\tau'}{a(\tau')} + C^{(2)}_i(\mathbf{x}),\end{aligned} \tag{C28}$$

where $A^{(2)}$ is an arbitrary function of 2nd-order, $C^{(2)}_i$ is an arbitrary 3-vector of 2nd order and can be decomposed into $C^{(2)}_i = C^{\parallel(2)}_{,i} + C^{\perp(2)}_i$. We remark that the transformation (C28) is general as $\chi^{(1)}_{ki}$ in the integration term contains the 1st-order scalar as well as the 1st-order tensor. Equation (C28) can be also written in terms of the parameters

$$\begin{aligned}\beta^{(2)} = {}& \nabla^{-2}\Bigg[\nabla^2 A^{(1)}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')}d\tau' + A^{(1)}_{,k}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})^{,k}}{a(\tau')}d\tau' - A^{(1),ki}\int^\tau \frac{2\chi^{(1)}_{ki}(\tau',\mathbf{x})}{a(\tau')}d\tau' - A^{(1),k}\int^\tau \frac{2\chi^{(1)}_{ki}(\tau',\mathbf{x})^{,i}}{a(\tau')}d\tau'\\ &+ 2A^{(1),ki}C^{\parallel(1)}_{,ki}\int^\tau \frac{d\tau'}{a(\tau')} + 2A^{(1),k}\nabla^2 C^{\parallel(1)}(\mathbf{x})_{,k}\int^\tau \frac{d\tau'}{a(\tau')}\Bigg] - \frac{1}{2a^2(\tau)}A^{(1)}A^{(1)} + A^{(1),k}A^{(1)}_{,k}\int^\tau \frac{d\tau'}{a(\tau')}\int^{\tau'}\frac{d\tau''}{a(\tau'')}\\ &+ A^{(2)}\int^\tau \frac{d\tau'}{a(\tau')} + C^{\parallel(2)},\end{aligned} \tag{C29}$$

$$
\begin{aligned}
d_i^{(2)} &= \xi_i^{(2)} - \beta_{,i}^{(2)} \\
&= \partial_i \nabla^{-2}\Bigg[-\nabla^2 A^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')} d\tau' - A_{,k}^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})^{,k}}{a(\tau')} d\tau' + 2A^{(1),kl} \int^\tau \frac{\chi_{kl}^{(1)}(\tau',\mathbf{x})}{a(\tau')} d\tau' \\
&\quad + 2A^{(1),k} \int^\tau \frac{\chi_{kl}^{(1)}(\tau',\mathbf{x})^{,l}}{a(\tau')} d\tau' - 2A^{(1),kl} C^{\|(1)}(\mathbf{x})_{,kl} \int^\tau \frac{d\tau'}{a(\tau')} - 2A^{(1),k} \nabla^2 C_{,k}^{\|(1)} \int^\tau \frac{d\tau'}{a(\tau')}\Bigg] + 4A_{,i}^{(1)} \int^\tau \frac{\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')} d\tau' \\
&\quad - 2A^{(1),k} \int^\tau \frac{\chi_{ki}^{(1)}(\tau',\mathbf{x})}{a(\tau')} d\tau' + 2A^{(1),k} C_{,ki}^{\|(1)} \int^\tau \frac{d\tau'}{a(\tau')} + C_i^{\perp(2)}.
\end{aligned} \tag{C30}
$$

With this, we now determine the transformation of 2nd-order metric perturbations (see also Ref. [46])

$$
\begin{aligned}
\bar{\phi}^{(2)} &= \phi^{(2)} - \frac{a''}{a^3} A^{(1)} A^{(1)} + \frac{1}{a^2}\left[-\frac{2}{3} A^{(1)} \nabla^2 A^{(1)} - \frac{1}{3} A^{(1),l} A_{,l}^{(1)}\right] - 4\frac{a'}{a^2} \phi^{(1)} A^{(1)} - \frac{2}{a} \phi^{(1)\prime} A^{(1)} \\
&\quad + \frac{a'}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]\left[-\frac{4}{3} A^{(1)} \nabla^2 A^{(1)} - A_{,l}^{(1)} A^{(1),l}\right] + \left[\int^\tau \frac{d\tau'}{a(\tau')} \int^{\tau'} \frac{d\tau''}{a(\tau'')}\right]\left[\frac{2}{3} A^{(1),l} \nabla^2 A_{,l}^{(1)} + \frac{2}{3} A^{(1),lm} A_{,lm}^{(1)}\right] \\
&\quad + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right]^2\left[-\frac{1}{3} A^{(1),l} \nabla^2 A_{,l}^{(1)} - \frac{2}{3} A_{,lm}^{(1)} A^{(1),lm}\right] + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right]\Bigg[\frac{2}{3} A^{(1),lm} C_{,lm}^{\|(1)} + \frac{2}{3} A^{(1),l} \nabla^2 C_{,l}^{\|(1)} - \frac{4}{3} \phi^{(1)} \nabla^2 A^{(1)} \\
&\quad - 2\phi_{,l}^{(1)} A^{(1),l} + \frac{2}{3} \chi_{lm}^{(1)} A^{(1),lm}\Bigg] + \nabla^2 A^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})}{3a(\tau')} d\tau' + A_{,l}^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})^{,l}}{3a(\tau')} d\tau' - A^{(1),lm} \int^\tau \frac{2\chi_{lm}^{(1)}(\tau',\mathbf{x})}{3a(\tau')} d\tau' \\
&\quad - A^{(1),l} \int^\tau \frac{2\chi_{lm}^{(1)}(\tau',\mathbf{x})^{,m}}{3a(\tau')} d\tau' + \frac{a'}{a^2} A^{(2)} + \frac{1}{3} \nabla^2 A^{(2)} \int^\tau \frac{d\tau'}{a(\tau')} + \frac{1}{3} \nabla^2 C^{\|(2)}.
\end{aligned} \tag{C31}
$$

The residual gauge transforms of the $\chi^{\|(2)}, \chi_{ij}^{\perp(2)}, \chi_{ij}^{\top(2)}$ are given in (C56), (C57), and (C58) of Ref. [46], which are listed as following:

$$
\begin{aligned}
\bar{\chi}^{\|(2)} &= \chi^{\|(2)} + \frac{1}{a^2}[A^{(1)} A^{(1)} + 2\nabla^{-2}(A^{(1)} \nabla^2 A^{(1)}) + 3\nabla^{-2}\nabla^{-2}(A^{(1),lm} A_{,lm}^{(1)} - \nabla^2 A^{(1)} \nabla^2 A^{(1)})] \\
&\quad + \frac{4a'}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right][2\nabla^{-2}(A^{(1)} \nabla^2 A^{(1)}) + 3\nabla^{-2}\nabla^{-2}(A^{(1),lm} A_{,lm}^{(1)} - \nabla^2 A^{(1)} \nabla^2 A^{(1)})] + \frac{a'}{a^2}[8\nabla^{-2}(A^{(1)} \nabla^2 C^{\|(1)}) \\
&\quad + 6\nabla^{-2}\nabla^{-2}(-\chi_{lm}^{(1),lm} A^{(1)} - \chi_{lm}^{(1)} A^{(1),lm} - 2\chi_{lm}^{(1),l} A^{(1),m} + 2A^{(1),lm} C_{,lm}^{\|(1)} - 2\nabla^2 A^{(1)} \nabla^2 C^{\|(1)})] - \frac{3}{a} \nabla^{-2}\nabla^{-2}[\chi_{lm}^{(1)\prime,lm} A^{(1)} \\
&\quad + \chi_{lm}^{(1)\prime} A^{(1),lm} + 2\chi_{lm}^{(1)\prime,l} A^{(1),l}] - 2\left[\int^\tau \frac{d\tau'}{a(\tau')} \int^{\tau'} \frac{d\tau''}{a(\tau'')}\right] A^{(1),l} A_{,l}^{(1)} + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right]^2 [2A_{,l}^{(1)} A^{(1),l} - 2\nabla^{-2}(A^{(1),l} \nabla^2 A_{,l}^{(1)}) \\
&\quad + 3\nabla^{-2}\nabla^{-2}(\nabla^2 A^{(1),l} \nabla^2 A_{,l}^{(1)} - A^{(1),lmn} A_{,lmn}^{(1)})] + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right][2A^{(1),l} C_{,l}^{\|(1)} + 2\nabla^{-2}(4\phi^{(1)} \nabla^2 A^{(1)} + \chi_{lm}^{(1)} A^{(1),lm} \\
&\quad - 2A^{(1),l} \nabla^2 C_{,l}^{\|(1)}) + 3\nabla^{-2}\nabla^{-2}(-\chi_{lm}^{(1),lmn} A_{,n}^{(1)} - 3\chi_{lm,n}^{(1)} A^{(1),lmn} - 4\chi_{lm,n}^{(1),l} A^{(1),mn} - 2\chi_{lm}^{(1),l} \nabla^2 A^{(1),m} - 2\chi_{lm}^{(1)} \nabla^2 A^{(1),lm} \\
&\quad + 4\phi^{(1),lm} A_{,lm}^{(1)} - 4\nabla^2 \phi^{(1)} \nabla^2 A^{(1)} + 2\nabla^2 A^{(1),l} \nabla^2 C_{,l}^{\|(1)} - 2A^{(1),lmn} C_{,lmn}^{\|(1)})] + [2C_{,l}^{\|(1)} C^{\|(1),l} + \nabla^{-2}(8\phi^{(1)} \nabla^2 C^{\|(1)} \\
&\quad + 2\chi_{lm}^{(1)} C^{\|(1),lm} - 2C^{\|(1),l} \nabla^2 C_{,l}^{\|(1)}) + 3\nabla^{-2}\nabla^{-2}(4\phi^{(1),lm} C_{,lm}^{\|(1)} - 4\nabla^2 \phi^{(1)} \nabla^2 C^{\|(1)} - \chi_{lm}^{(1),lmn} C_{,n}^{\|(1)} - 3\chi_{lm,n}^{(1)} C^{\|(1),lmn} \\
&\quad - 4\chi_{lm,n}^{(1),l} C^{\|(1),mn} - 2\chi_{lm}^{(1),l} \nabla^2 C^{\|(1),m} - 2\chi_{lm}^{(1)} \nabla^2 C^{\|(1),lm} + \nabla^2 C^{\|(1),l} \nabla^2 C_{,l}^{\|(1)} - C^{\|(1),lmn} C_{,lmn}^{\|(1)})] \\
&\quad - 4\nabla^{-2}\Bigg[\nabla^2 A^{(1)} \int^\tau \frac{2\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')} d\tau' + A_{,l}^{(1)} \int^\tau \frac{2\phi^{(1)}(\tau',\mathbf{x})^{,l}}{a(\tau')} d\tau' - A^{(1),lm} \int^\tau \frac{\chi_{lm}^{(1)}(\tau',\mathbf{x})}{a(\tau')} d\tau' \\
&\quad - A^{(1),l} \int^\tau \frac{\chi_{lm}^{(1)}(\tau',\mathbf{x})^{,m}}{a(\tau')} d\tau'\Bigg] - 2A^{(2)} \int^\tau \frac{d\tau'}{a(\tau')} - 2C^{\|(2)},
\end{aligned} \tag{C32}
$$

and

$$
\begin{aligned}
\bar{\chi}^{\perp(2)}_{ij} &= \chi^{\perp(2)}_{ij} + \frac{1}{a^2}[-2\partial_i\nabla^{-2}(A^{(1)}_{,j}\nabla^2 A^{(1)}) + \partial_i\partial_j\nabla^{-2}(A^{(1),l}A^{(1)}_{,l}) - 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2 A^{(1)}\nabla^2 A^{(1)})] \\
&+ \frac{4a'}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right][-2\partial_i\nabla^{-2}(A^{(1)}_{,j}\nabla^2 A^{(1)}) + \partial_i\partial_j\nabla^{-2}(A^{(1),l}A^{(1)}_{,l}) - 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2 A^{(1)}\nabla^2 A^{(1)})] \\
&+ \frac{2a'}{a^2}[2\partial_i\nabla^{-2}(2A^{(1),l}C^{\|(1)}_{,lj} - 2A^{(1)}_{,j}\nabla^2 C^{\|(1)} - \chi^{(1),l}_{lj}A^{(1)} - \chi^{(1)}_{lj}A^{(1),l}) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(2\nabla^2 A^{(1)}\nabla^2 C^{\|(1)} \\
&- 2A^{(1),lm}C^{\|(1)}_{,lm} + \chi^{(1),lm}_{lm}A^{(1)} + \chi^{(1)}_{lm}A^{(1),lm} + 2\chi^{(1),l}_{lm}A^{(1),m})] - \frac{1}{a}[2\partial_i\nabla^{-2}(\chi^{(1)\prime,l}_{lj}A^{(1)} + \chi^{(1)\prime}_{lj}A^{(1),l}) \\
&- 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{(1)\prime,lm}_{lm}A^{(1)} + \chi^{(1)\prime}_{lm}A^{(1),lm} + 2\chi^{(1)\prime,l}_{lm}A^{(1),m})] + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right]^2[2\partial_i\nabla^{-2}(A^{(1)}_{,lj}\nabla^2 A^{(1),l}) \\
&- \partial_i\partial_j\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm}) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lmn}A^{(1)}_{,lmn} - \nabla^2 A^{(1),l}\nabla^2 A^{(1)}_{,l})] - \left[\int^\tau \frac{d\tau'}{a(\tau')}\right][2\partial_i\nabla^{-2}(4\phi^{(1)}_{,j}\nabla^2 A^{(1)} \\
&- 4\phi^{(1),l}A^{(1)}_{,lj} + \chi^{(1),l}_{lj,n}A^{(1),n} + 2\chi^{(1)}_{lj,n}A^{(1),ln} + \chi^{(1),m}_{lm}A^{(1),l}_{,j} + \chi^{(1)}_{lm}A^{(1),lm}_{,j} + \chi^{(1)}_{lj}\nabla^2 A^{(1),l} + 2A^{(1),ln}C^{\|(1)}_{,lnj} - 2A^{(1)}_{,lj}\nabla^2 C^{\|(1),l}) \\
&- 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(4\nabla^2\phi^{(1)}\nabla^2 A^{(1)} - 4\phi^{(1),lm}A^{(1)}_{,lm} + \chi^{(1),lm}_{lm,n}A^{(1),n} + 3\chi^{(1)}_{lm,n}A^{(1),lmn} + 4\chi^{(1),l}_{lm,n}A^{(1),mn} + 2\chi^{(1),l}_{lm}\nabla^2 A^{(1),m} \\
&+ 2\chi^{(1)}_{lm}\nabla^2 A^{(1),lm} + 2A^{(1),lmn}C^{\|(1)}_{,lmn} - 2\nabla^2 A^{(1),l}\nabla^2 C^{\|(1)}_{,l})] + [-2\partial_i\nabla^{-2}(4\phi^{(1)}_{,j}\nabla^2 C^{\|(1)} - 4\phi^{(1),l}C^{\|(1)}_{,lj} + \chi^{(1),l}_{lj,n}C^{\|(1),n} \\
&+ 2\chi^{(1)}_{lj,n}C^{\|(1),ln} + \chi^{(1),m}_{lm}C^{\|(1),l}_{,j} + \chi^{(1)}_{lm}C^{\|(1),lm}_{,j} + \chi^{(1)}_{lj}\nabla^2 C^{\|(1),l} + C^{\|(1),ln}C^{\|(1)}_{,lnj} - C^{\|(1)}_{,lj}\nabla^2 C^{\|(1),l}) \\
&+ 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(4\nabla^2\phi^{(1)}\nabla^2 C^{\|(1)} - 4\phi^{(1),lm}C^{\|(1)}_{,lm} + \chi^{(1),lm}_{lm,n}C^{\|(1),n} + 4\chi^{(1),l}_{lm,n}C^{\|(1),mn} + 3\chi^{(1)}_{lm,n}C^{\|(1),lmn} \\
&+ 2\chi^{(1),l}_{lm}\nabla^2 C^{\|(1),m} + 2\chi^{(1)}_{lm}\nabla^2 C^{\|(1),lm} + C^{\|(1),lmn}C^{\|(1)}_{,lmn} - \nabla^2 C^{\|(1),l}\nabla^2 C^{\|(1)}_{,l})] + \partial_i\left[A^{(1),l}\int^\tau \frac{2\chi^{(1)}_{lj}(\tau',\mathbf{x})}{a(\tau')}d\tau'\right. \\
&\left.- A^{(1)}_{,j}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')}d\tau'\right] + \partial_i\partial_j\nabla^{-2}\left[\nabla^2 A^{(1)}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})}{a(\tau')}d\tau' + A^{(1)}_{,l}\int^\tau \frac{4\phi^{(1)}(\tau',\mathbf{x})^{,l}}{a(\tau')}d\tau'\right. \\
&\left.- A^{(1),lm}\int^\tau \frac{2\chi^{(1)}_{lm}(\tau',\mathbf{x})}{a(\tau')}d\tau - A^{(1),l}\int^\tau \frac{2\chi^{(1)}_{ln}(\tau',\mathbf{x})^{,n}}{a(\tau')}d\tau'\right] - C^{\perp(2)}_{i,j} + (i\leftrightarrow j),
\end{aligned} \tag{C33}
$$

which shows that transformation of $\chi^{\perp(2)}_{ij}$ depends on $\xi^{(2)}$ only through $C^{\perp(2)}_{(i,j)}$,

$$
\begin{aligned}
\bar{\chi}^{\top(2)}_{ij} &= \chi^{\top(2)}_{ij} + \left[\frac{1}{a^2}\right][\delta_{ij}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2 A^{(1)}\nabla^2 A^{(1)}) + 4\nabla^{-2}(A^{(1)}_{,ij}\nabla^2 A^{(1)} - A^{(1),l}_{,i}A^{(1)}_{,lj}) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} \\
&- \nabla^2 A^{(1)}\nabla^2 A^{(1)})] + \frac{4a'}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right][\delta_{ij}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2 A^{(1)}\nabla^2 A^{(1)}) + 4\nabla^{-2}(A^{(1)}_{,ij}\nabla^2 A^{(1)} - A^{(1),l}_{,i}A^{(1)}_{,lj}) \\
&+ \partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lm}A^{(1)}_{,lm} - \nabla^2 A^{(1)}\nabla^2 A^{(1)})] + \frac{2a'}{a^2}[-\delta_{ij}\nabla^{-2}(\chi^{(1),lm}_{lm}A^{(1)} + \chi^{(1)}_{lm}A^{(1),lm} + 2\chi^{(1),l}_{lm}A^{(1),m} \\
&+ 2\nabla^2 A^{(1)}\nabla^2 C^{\|(1)} - 2A^{(1),lm}C^{\|(1)}_{,lm}) - 2\chi^{(1)}_{ij}A^{(1)} + 2\partial_i\nabla^{-2}(\chi^{(1),l}_{lj}A^{(1)} + \chi^{(1)}_{lj}A^{(1),l}) + 2\partial_j\nabla^{-2}(\chi^{(1),l}_{li}A^{(1)} \\
&+ \chi^{(1)}_{li}A^{(1),l}) + 4\nabla^{-2}(A^{(1)}_{,ij}\nabla^2 C^{\|(1)} + C^{\|(1)}_{,ij}\nabla^2 A^{(1)} - A^{(1),l}_{,i}C^{\|(1)}_{,lj} - A^{(1),l}_{,j}C^{\|(1)}_{,li}) - \partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{(1),lm}_{lm}A^{(1)} + \chi^{(1)}_{lm}A^{(1),lm} \\
&+ 2\chi^{(1),l}_{lm}A^{(1),m} + 2A^{(1),lm}C^{\|(1)}_{,lm} - 2\nabla^2 A^{(1)}\nabla^2 C^{\|(1)})] - \frac{1}{a}[\delta_{ij}\nabla^{-2}(\chi^{(1)\prime,lm}_{lm}A^{(1)} + \chi^{(1)\prime}_{lm}A^{(1),lm} + 2\chi^{(1)\prime,l}_{lm}A^{(1),m}) \\
&+ 2\chi^{(1)\prime}_{ij}A^{(1)} - 2\partial_i\nabla^{-2}(\chi^{(1)\prime,l}_{lj}A^{(1)} + \chi^{(1)\prime}_{lj}A^{(1),l}) - 2\partial_j\nabla^{-2}(\chi^{(1)\prime,l}_{li}A^{(1)} + \chi^{(1)\prime}_{li}A^{(1),l}) \\
&+ \partial_i\partial_j\nabla^{-2}\nabla^{-2}(\chi^{(1)\prime,lm}_{lm}A^{(1)} + \chi^{(1)\prime}_{lm}A^{(1),lm} + 2\chi^{(1)\prime,l}_{lm}A^{(1),m})] - \left[\int^\tau \frac{d\tau'}{a(\tau')}\right]^2[\delta_{ij}\nabla^{-2}(A^{(1),lmn}A^{(1)}_{,lmn} - \nabla^2 A^{(1),l}\nabla^2 A^{(1)}_{,l}) \\
&+ 2\nabla^{-2}(-2A^{(1),lm}_{,i}A^{(1)}_{,lmj} + 2A^{(1)}_{,lij}\nabla^2 A^{(1),l}) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),lmn}A^{(1)}_{,lmn} - \nabla^2 A^{(1),l}\nabla^2 A^{(1)}_{,l})]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\int^{\tau}\frac{d\tau'}{a(\tau')}\right][\delta_{ij}\nabla^{-2}(4\phi^{(1),lm}A^{(1)}_{,lm}-4\nabla^2\phi^{(1)}\nabla^2A^{(1)}-\chi^{(1),lm}_{lm,n}A^{(1),n}-4\chi^{(1),l}_{lm,n}A^{(1),mn}+\chi^{(1)}_{lm,n}A^{(1),lmn}-2\chi^{(1),l}_{lm}\nabla^2A^{(1),m}\\
&+2A^{(1),lm}\nabla^2\chi^{(1)}_{lm}-2A^{(1),lmn}C^{\|(1)}_{,lmn}+2\nabla^2A^{(1),l}\nabla^2C^{\|(1)}_{,l})-2\chi^{(1)}_{ij,l}A^{(1),l}-2\chi^{(1)}_{li}A^{(1),l}_{,j}-2\chi^{(1)}_{lj}A^{(1),l}_{,i}-4\nabla^{-2}(2\phi^{(1),l}_{,i}A^{(1)}_{,lj}\\
&+2\phi^{(1),l}_{,j}A^{(1)}_{,li}-2\phi^{(1)}_{,ij}\nabla^2A^{(1)}-2A^{(1)}_{,ij}\nabla^2\phi^{(1)}-A^{(1),lm}_{,i}C^{\|(1)}_{,lmj}-A^{(1),lm}_{,j}C^{\|(1)}_{,lmi}+A^{(1)}_{,lij}\nabla^2C^{\|(1),l}+C^{\|(1)}_{,lij}\nabla^2A^{(1),l})\\
&+2\partial_i\nabla^{-2}(\chi^{(1),l}_{lj,m}A^{(1),m}+2\chi^{(1)}_{lj,m}A^{(1),lm}+\chi^{(1),m}_{lm}A^{(1),l}_{,j}+\chi^{(1)}_{lm}A^{(1),lm}_{,j}+\chi^{(1)}_{lj}\nabla^2A^{(1),l})+2\partial_j\nabla^{-2}(\chi^{(1),l}_{li,m}A^{(1),m}+2\chi^{(1)}_{li,m}A^{(1),lm}\\
&+\chi^{(1),m}_{lm}A^{(1),l}_{,i}+\chi^{(1)}_{lm}A^{(1),lm}_{,i}+\chi^{(1)}_{li}\nabla^2A^{(1),l})+\partial_i\partial_j\nabla^{-2}\nabla^{-2}(4\phi^{(1),lm}A^{(1)}_{,lm}-4\nabla^2\phi^{(1)}\nabla^2A^{(1)}-\chi^{(1),lm}_{lm,n}A^{(1),n}\\
&-4\chi^{(1),l}_{lm,n}A^{(1),mn}-7\chi^{(1)}_{lm,n}A^{(1),lmn}-2\chi^{(1),l}_{lm}\nabla^2A^{(1),m}-4\chi^{(1)}_{lm}\nabla^2A^{(1),lm}-2A^{(1),lm}\nabla^2\chi^{(1)}_{lm}-2A^{(1),lmn}C^{\|(1)}_{,lmn}\\
&+2\nabla^2A^{(1),l}\nabla^2C^{\|(1)}_{,l})]+[\delta_{ij}\nabla^{-2}(4\phi^{(1),lm}C^{\|(1)}_{,lm}-4\nabla^2\phi^{(1)}\nabla^2C^{\|(1)}-\chi^{(1),lm}_{lm,n}C^{\|(1),n}-4\chi^{(1),l}_{lm,n}C^{\|(1),mn}+\chi^{(1)}_{lm,n}C^{\|(1),lmn}\\
&-2\chi^{(1),l}_{lm}\nabla^2C^{\|(1),m}+2C^{\|(1),lm}\nabla^2\chi^{(1)}_{lm}+\nabla^2C^{\|(1),l}\nabla^2C^{\|(1)}_{,l}-C^{\|(1),lmn}C^{\|(1)}_{,lmn})-2\chi^{(1)}_{ij,l}C^{\|(1),l}-2\chi^{(1)}_{li}C^{\|(1),l}_{,j}\\
&-2\chi^{(1)}_{lj}C^{\|(1),l}_{,i}-4\nabla^{-2}(2\phi^{(1),l}_{,i}C^{\|(1)}_{,lj}+2\phi^{(1),l}_{,j}C^{\|(1)}_{,li}-2\phi^{(1)}_{,ij}\nabla^2C^{\|(1)}-2C^{\|(1)}_{,ij}\nabla^2\phi^{(1)}-C^{\|(1),lm}_{,i}C^{\|(1)}_{,lmj}+C^{\|(1)}_{,lij}\nabla^2C^{\|(1),l})\\
&+2\partial_i\nabla^{-2}(\chi^{(1),l}_{lj,m}C^{\|(1),m}+2\chi^{(1)}_{lj,m}C^{\|(1),lm}+\chi^{(1),m}_{lm}C^{\|(1),l}_{,j}+\chi^{(1)}_{lm}C^{\|(1),lm}_{,j}+\chi^{(1)}_{lj}\nabla^2C^{\|(1),l})\\
&+2\partial_j\nabla^{-2}(\chi^{(1),l}_{li,m}C^{\|(1),m}+2\chi^{(1)}_{li,m}C^{\|(1),lm}+\chi^{(1),m}_{lm}C^{\|(1),l}_{,i}+\chi^{(1)}_{lm}C^{\|(1),lm}_{,i}+\chi^{(1)}_{li}\nabla^2C^{\|(1),l})-\partial_i\partial_j\nabla^{-2}\nabla^{-2}(4\nabla^2\phi^{(1)}\nabla^2C^{\|(1)}\\
&-4\phi^{(1),lm}C^{\|(1)}_{,lm}+\chi^{(1),lm}_{lm,n}C^{\|(1),n}+4\chi^{(1),l}_{lm,n}C^{\|(1),mn}+7\chi^{(1)}_{lm,n}C^{\|(1),lmn}+2\chi^{(1),l}_{lm}\nabla^2C^{\|(1),m}\\
&+4\chi^{(1)}_{lm}\nabla^2C^{\|(1),lm}+2C^{\|(1),lm}\nabla^2\chi^{(1)}_{lm}+C^{\|(1),lmn}C^{\|(1)}_{,lmn}-\nabla^2C^{\|(1),l}\nabla^2C^{\|(1)}_{,l})]. \qquad \text{(C34)}
\end{aligned}
$$

Equation (C34) states that transformation of $\chi^{\top(2)}_{ij}$ involves the 1st-order transformation vector $\xi^{(1)}$ only, but not the 2nd-order vector $\xi^{(2)}$.

In (C28), (C31), (C32), (C33), and (C34), $\chi^{(1)}_{ij}$ contains the tensor $\chi^{\top(1)}_{ij}$, which belongs to the type of scalar-tensor coupling and will not be considered. For RD stage and the scalar-scalar coupling, the transformation vector (C27), (C29), and (C30) reduces to the following:

$$\alpha^{(2)}=\frac{A^{(2)}(\mathbf{x})}{\tau}, \qquad \text{(C35)}$$

$$
\begin{aligned}
\beta^{(2)}=\nabla^{-2}\Bigg[&\nabla^2A^{(1)}(\mathbf{x})\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})}{\tau'}d\tau'+A^{(1)}(\mathbf{x})_{,k}\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})^{,k}}{\tau'}d\tau'-A^{(1)}(\mathbf{x})^{,ki}\int^{\tau}\frac{2D_{ki}\chi^{\|(1)}(\tau',\mathbf{x})}{\tau'}d\tau'\\
&-A^{(1)}(\mathbf{x})^{,k}\int^{\tau}\frac{4\nabla^2\chi^{\|(1)}(\tau',\mathbf{x})_{,k}}{3\tau'}d\tau'+2\ln\tau A^{(1)}(\mathbf{x})^{,ki}C^{\|(1)}(\mathbf{x})_{,ki}+2\ln\tau A^{(1)}(\mathbf{x})^{,k}\nabla^2C^{\|(1)}(\mathbf{x})_{,k}\Bigg]-\frac{1}{2\tau^2}A^{(1)}(\mathbf{x})A^{(1)}(\mathbf{x})\\
&+\frac{(\ln\tau)^2}{2}A^{(1)}(\mathbf{x})^{,k}A^{(1)}(\mathbf{x})_{,k}+A^{(2)}(\mathbf{x})\ln\tau+C^{\|(2)}(\mathbf{x}), \qquad \text{(C36)}
\end{aligned}
$$

$$
\begin{aligned}
d^{(2)}_i=\partial_i\nabla^{-2}\Bigg[&-\nabla^2A^{(1)}(\mathbf{x})\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})}{\tau'}d\tau'-A^{(1)}(\mathbf{x})_{,k}\int^{\tau}\frac{4\phi^{(1)}(\tau',\mathbf{x})^{,k}}{\tau'}d\tau'+2A^{(1)}(\mathbf{x})^{,kl}\int^{\tau}\frac{D_{kl}\chi^{\|(1)}(\tau',\mathbf{x})}{\tau'}d\tau'\\
&+A^{(1)}(\mathbf{x})^{,k}\int^{\tau}\frac{4\nabla^2\chi^{\|(1)}(\tau',\mathbf{x})_{,k}}{3\tau'}d\tau'-2\ln\tau A^{(1)}(\mathbf{x})^{,kl}C^{\|(1)}(\mathbf{x})_{,kl}-2\ln\tau A^{(1)}(\mathbf{x})^{,k}\nabla^2C^{\|(1)}(\mathbf{x})_{,k}\Bigg]\\
&+4A^{(1)}(\mathbf{x})_{,i}\int^{\tau}\frac{\phi^{(1)}(\tau',\mathbf{x})}{\tau'}d\tau'-2A^{(1)}(\mathbf{x})^{,k}\int^{\tau}\frac{D_{ki}\chi^{\|(1)}(\tau',\mathbf{x})}{\tau'}d\tau'+2A^{(1)}(\mathbf{x})^{,k}C^{\|(1)}(\mathbf{x})_{,ki}\ln\tau+C^{\perp(2)}_i(\mathbf{x}), \qquad \text{(C37)}
\end{aligned}
$$

and the transformations of the 2nd-order perturbations are given by (6.1), (6.2), (6.3), (6.4), (6.5), (6.6), and (6.7) in the context.

Next we give the residual transform of $\rho^{(2)}$ and $U^{(2)\mu}$ for a general RW spacetime, which has not been given in Ref. [46]. By using Eqs. (C8), (C12), (C13), and (C27), and omitting the 1st-order curl vector, one gets the residual gauge transform of the 2nd-order density perturbation

$$\bar\rho^{(2)} = \rho^{(2)} - \frac{a'}{a^3}\rho^{(0)\prime}A^{(1)}A^{(1)} + \frac{1}{a^2}\rho^{(0)\prime\prime}A^{(1)}A^{(1)} + \frac{1}{a}[-2\rho^{(1)\prime}A^{(1)} + \rho^{(0)\prime}A^{(1)}_{,l}C^{\parallel(1),l}] + \frac{1}{a}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]\rho^{(0)\prime}A^{(1)}_{,l}A^{(1),l}$$
$$- 2\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]\rho^{(1)}_{,l}A^{(1),l} - 2\rho^{(1)}_{,l}C^{\parallel(1),l} - \frac{1}{a}\rho^{(0)\prime}A^{(2)}, \tag{C38}$$

which can be written in terms of the 2nd-order density contrast

$$\bar\delta^{(2)} = \delta^{(2)} - 22\frac{a''}{a^3}A^{(1)}A^{(1)} + 2\frac{a'''}{a'a^2}A^{(1)}A^{(1)} + 2\frac{a''^2}{a'^2a^2}A^{(1)}A^{(1)} + 24\frac{a'^2}{a^4}A^{(1)}A^{(1)} + \frac{a''}{a'a}[-4\delta^{(1)}A^{(1)} + 2A^{(1)}_{,l}C^{\parallel(1),l}]$$
$$- 4\frac{a'}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]A^{(1)}_{,l}A^{(1),l} + \frac{a'}{a^2}[-4A^{(1)}_{,l}C^{\parallel(1),l} + 8\delta^{(1)}A^{(1)}] - \frac{2}{a}\delta^{(1)\prime}A^{(1)} - 2\delta^{(1)}_{,l}C^{\parallel(1),l} - 2\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]\delta^{(1)}_{,l}A^{(1),l}$$
$$+ 2\frac{a''}{a'a}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]A^{(1)}_{,l}A^{(1),l} - 2\frac{a''}{a'a}A^{(2)} + 4\frac{a'}{a^2}A^{(2)}. \tag{C39}$$

From (C11), (C12), (C13), and (C27), the 0-component of the 4-velocity transforms between synchronous as the following:

$$\bar U^{(2)0}(x) = \frac{1}{a}\left(v^{(1)l} + \frac{A^{(1),l}}{a}\right)\left(v^{(1)}_l + \frac{A^{(1)}_{,l}}{a}\right) = a^{-1}\bar v^{(1)l}\bar v^{(1)}_l. \tag{C40}$$

From (C24), since $\bar v^{(1)i} = v^{(1)i} + a^{-1}A^{(1),i}$, the above is the definition of $\bar U^{(2)0}(x)$ in the new synchronous coordinate. By using (2.13), (2.14), (C12), (C13), (C27), and (C28), and omitting the 1st-order curl-vectors, one has the transformation of i-component

$$\bar U^{(2)i} = U^{(2)i} + \frac{4a'}{a^4}A^{(1),i}A^{(1)} + \frac{2a'}{a^3}v^{\parallel(1),i}A^{(1)} + \frac{1}{a^2}[-A^{(1),i}_{,l}C^{\parallel(1),l} + 3A^{(1)}_{,l}C^{\parallel(1),li} + 4A^{(1),i}\phi^{(1)} - 2A^{(1)}_{,l}\chi^{(1)li}]$$
$$+ \frac{1}{a}[-2v^{\parallel(1),i}_{,l}C^{\parallel(1),l} + 2v^{\parallel(1),l}C^{\parallel(1),i}_{,l}] + \frac{2}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]A^{(1)}_{,l}A^{(1),li}$$
$$+ \frac{1}{a}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right][-2v^{\parallel(1),i}_{,l}A^{(1),l} + 2v^{\parallel(1),l}A^{(1),i}_{,l}] + \frac{1}{a^2}A^{(2),i}. \tag{C41}$$

By $\bar U^{(2)i} = a^{-1}\bar v^{(2)i}$ and $U^{(2)i} = a^{-1}v^{(2)i}$, the above is written as

$$\bar v^{(2)i} = v^{(2)i} + \frac{4a'}{a^3}A^{(1),i}A^{(1)} + \frac{2a'}{a^2}v^{\parallel(1),i}A^{(1)} + \frac{1}{a}[-A^{(1),i}_{,l}C^{\parallel(1),l} + 3A^{(1)}_{,l}C^{\parallel(1),li} + 4A^{(1),i}\phi^{(1)} - 2A^{(1)}_{,l}\chi^{(1)li}]$$
$$+ [-2v^{\parallel(1),i}_{,l}C^{\parallel(1),l} + 2v^{\parallel(1),l}C^{\parallel(1),i}_{,l}] + \frac{2}{a}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]A^{(1)}_{,l}A^{(1),li} + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right][-2v^{\parallel(1),i}_{,l}A^{(1),l} + 2v^{\parallel(1),l}A^{(1),i}_{,l}] + \frac{1}{a}A^{(2),i}. \tag{C42}$$

This can be decomposed further. Taking $\nabla^{-2}\partial_i$ upon (C42) to eliminate $v^{\perp(2)}$ and $\bar v^{\perp(2)}$, one obtains the transform of $v^{\parallel(2)}$ as the following:

$$\bar v^{\parallel(2)} = v^{\parallel(2)} + \frac{2a'}{a^3}A^{(1)}A^{(1)} + \frac{2a'}{a^2}\nabla^{-2}[v^{\parallel(1),l}A^{(1)}_{,l} + A^{(1)}\nabla^2 v^{\parallel(1)}] + \frac{1}{a}[A^{(1)}_{,l}C^{\parallel(1),l} + \nabla^{-2}(-2C^{\parallel(1),l}\nabla^2 A^{(1)}_{,l} + 2A^{(1)}_{,l}\nabla^2 C^{\parallel(1),l}$$
$$+ 4A^{(1),l}\phi^{(1)}_{,l} + 4\phi^{(1)}\nabla^2 A^{(1)} - 2A^{(1)}_{,lm}\chi^{(1)lm} - 2A^{(1)}_{,l}\chi^{(1)lm}_{,m})] + [2v^{\parallel(1),l}C^{\parallel(1)}_{,l} + \nabla^{-2}(-4v^{\parallel(1)}_{,lm}C^{\parallel(1),lm}$$
$$- 4C^{\parallel(1),l}\nabla^2 v^{\parallel(1)}_{,l})] + \frac{1}{a}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right]A^{(1)}_{,l}A^{(1),l} + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right][2v^{\parallel(1),l}A^{(1)}_{,l} + \nabla^{-2}(-4v^{\parallel(1)}_{,lm}A^{(1),lm}$$
$$- 4A^{(1),l}\nabla^2 v^{\parallel(1)}_{,l})] + \frac{A^{(2)}}{a}. \tag{C43}$$

Then, [(C42) $-\partial^i$ (C43)] gives the transform for $v^{\perp(2)i}$ as the following:

$$\bar{v}^{\perp(2)i} = v^{\perp(2)i} + \frac{2a'}{a^2}[v^{\|(1),i}A^{(1)} + \partial^i\nabla^{-2}(-v^{\|(1),l}A^{(1)}_{,l} - A^{(1)}\nabla^2 v^{\|(1)})] + \frac{1}{a}[-2A^{(1),i}_{,l}C^{\|(1),l} + 2A^{(1)}_{,l}C^{\|(1),li} + 4A^{(1),i}\phi^{(1)}$$
$$- 2A^{(1)}_{,l}\chi^{(1)li} + \partial^i\nabla^{-2}(2C^{\|(1),l}\nabla^2 A^{(1)}_{,l} - 2A^{(1)}_{,l}\nabla^2 C^{\|(1),l} - 4A^{(1),l}\phi^{(1)}_{,l} - 4\phi^{(1)}\nabla^2 A^{(1)} + 2A^{(1)}_{,lm}\chi^{(1)lm} + 2A^{(1)}_{,l}\chi^{(1)lm}_{,m})]$$
$$+ [-4v^{\|(1),i}_{,l}C^{\|(1),l} + \partial^i\nabla^{-2}(4v^{\|(1)}_{,lm}C^{\|(1),lm} + 4C^{\|(1),l}\nabla^2 v^{\|(1)}_{,l})]$$
$$+ \left[\int^\tau \frac{d\tau'}{a(\tau')}\right][-4v^{\|(1),i}_{,l}A^{(1),l} + \partial^i\nabla^{-2}(4v^{\|(1)}_{,lm}A^{(1),lm} + 4A^{(1),l}\nabla^2 v^{\|(1)}_{,l})], \tag{C44}$$

which does not depend on $\xi^{(2)\mu}$, similar to the transformation of $v^{\perp(1)i}$ that does not depend on $\xi^{(1)\mu}$ in (3.43).

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