Inertial spontaneous symmetry breaking and quantum scale invariance

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Weyl invariant theories of scalars and gravity can generate all mass scales spontaneously, initiated by a dynamical process of "inertial spontaneous symmetry breaking" that does not involve a potential. This is dictated by the structure of the Weyl current, K_μ, and a cosmological phase during which the Universe expands and the Einstein-Hilbert effective action is formed. Maintaining exact Weyl invariance in the renormalized quantum theory is straightforward when renormalization conditions are referred back to the VEV's of fields in the action of the theory, which implies a conserved Weyl current. We do not require scale invariant regulators. We illustrate the computation of a Weyl invariant Coleman-Weinberg potential.

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I. INTRODUCTION

The discovery of the Higgs boson with the appearance of a fundamental, pointlike, scalar field, unaccompanied by a natural custodial symmetry, has led many authors in search of new organising principles to turn to scale symmetry. In particular, Weyl symmetry [1] in conjunction with gravity may provide a modern context for fundamental scalar fields and a foundational symmetry for physics [2–7]. Scale or Weyl symmetry, like many of the flavor symmetries seen in nature, must be broken. Often this breaking is treated spontaneously, implemented for scale invariant potentials via the Coleman-Weinberg (CW) mechanism of dimensional transmutation [8].

In this paper we focus on the well-known Weyl current which has been studied by many of the previous authors listed above (e.g., see [2]). However, we emphasize that, underlying these ideas there is a new way to break scale symmetry that *does not employ a potential*. While this mechanism is implicit in the many approaches taken to spontaneously generating the Planck scale, it seems not to have been codified prior to Ref. [7]. This mechanism is a direct consequence of the structure of the Weyl scale current. We call this *inertial spontaneous scale symmetry breaking*.

By inertial spontaneous scale symmetry breaking, we presently follow the condensed matter parlance, where spontaneous symmetry breaking (SSB) represents the difference between an "ordered" state from a "disordered" one. Initially, we expect local fluctuations in fields that break scale, e.g., nonzero $\phi_i(x)$ or $\partial\phi_i(x)$, etc. These are analogous to local magnetic spins in a macroscopic spin system at high temperature and they do break scale symmetry locally, but they do not represent an ordered state. For that, we need an "order parameter" that evolves to become macroscopically constant in space and time, like the constant magnetization in the spin system as it cools to the ground state. The order parameter must capture any and all symmetry breaking.

In the present paper, working in a "Jordan frame," we identify the conserved Weyl current K_μ. We find that, in any Weyl invariant theory this current is always a derivative of a scalar quantity, $K_\mu = \partial_\mu K$ where K is the "kernel."[1] Owing to this structure of the conserved Weyl current, $D_\mu K^\mu = 0$, we are guaranteed that the system will dynamically evolve in an expanding universe such that scale charge density will evolve to zero, $K_0 \to 0$. This is just covariance, like the dilution of a conserved electric charge density or a magnetic field during general expansion.

It then follows that the kernel, $K \to \bar{K}$, is constant. Essentially all short distance initial scale fluctuations are

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[1]This is a theorem, and it isn't hard to see that it holds in R^2 and Weyl gravity generalizations, but the proof is beyond the scope of our present discussion.

stretched out to become a constant value of $K = \bar{K}$, and the scale symmetry is broken. K is the order parameter of the SSB since it intimately connects with the dynamics. At late times we have, for N-fields, $K(\{\phi_i\}) = \bar{K}(\{\phi_i\})\exp(\sigma/f)$. Here σ is the dilaton and $\bar{K}(\{\phi_i\})$ is a constraint that reduces the N fields $\{\phi_i\}$ to $N-1$ fields constrained to a locus in field space, such as the ellipse of [2].

We clearly see that $\bar{K}$ is the order parameter because the decay constant of the dilaton is precisely $f = \sqrt{2\bar{K}}$, in analogy to f_π in a chiral Lagrangian, or the VEV of the Higgs in the standard model. The constraint of constant $\bar{K}(\{\phi_i\})$ gurantees that the dilaton fluctuation is orthogonal in the kinetic terms to the other $N-1$ constrained fields and neatly factorizes.

We emphasize that this is a dynamical process. Just as steam can condense into water, a scale disordered phase can condense into a scale ordered one. All of this is tracked in a single frame, which begins as a Jordan frame. In this view the Universe is a physical system that starts in one phase, which has no scale ordering, and ends up in another in which the scale SSB defined by $\bar{K}$. This is treated in one set of "frame variables" with a Friedman-Roberston-Walker metric. In a sense, the approach of $K \to \bar{K}$ is just the relaxation of the dilaton $\sigma \to 0$, though the dilaton can only be defined in the broken phase of the theory.

Here, we need not do the Weyl transformation along the way and the SSB materializes dynamically. However, at late times the dynamically generated $\bar{K}$ can then be matched to the scale quantities, M_P, Λ, etc., in an Einstein-Hilbert action. Once these scales are identified, then it is useful to make a Weyl transformation, e.g., using $\bar{K}$, to isolate the dilaton. This guarantees that the dilaton factorizes and alleviates any putative messy kinetic term mixing issues. (in fact, this permits the dilaton to be "eaten" by a Higgs mechanism if we introduce Weyl's photon, $\hat{A}_\mu$ as in Sec. II D, allowing the Weyl photon and dilaton to decouple as very heavy states.) There does then remain a mixing issue amongst the remaining $N-1$ constrained fields, and these must be diagonalized to apply the low energy dynamics.

The advantage of phrasing things in terms of conserved currents is that the results are model independent. We never have to actually construct and solve difficult nonlinear partial differential equations of motion to see this; this will happen automatically, and the resulting mass scales, including the Planck mass, are generated spontaneously, controlled by the Weyl current. This mechanism does not depend upon a potential, (though the particular final vacuum state is dictated by a potential). The statements we make are general and model independent, similar to those of any traditional "current algebra."

A crucial aspect of this mechanism is that quantum theory should not break scale symmetry. We believe this is generally possible. To understand this, it is important that one does not conflate the procedure of regularization, which generally introduces arbitrary mass scales, with renormalization, which introduces counter-terms to define the final theory and its symmetries. Though it may be convenient, one need not deploy a regulator that is consistent with the symmetries of the renormalized theory. The nonexistence of a symmetry in the regulator does not imply the nonexistence of the symmetry in the renormalized theory. Furthermore, physics should not depend upon the choice of regulator [9].

In this view, Weyl symmetry is central and all mass scales must emerge by way of random initial conditions governing vacuum expectation values (VEVs) of fields that are entirely contained within the action. Essentially there exist no fundamental mass scales, and the mass of anything is defined only relative to field VEVs in the theory. For this to be phenomenologically acceptable it is necessary to explain how the spontaneous breaking of Weyl symmetry can lead to a period of inflation followed by a reheat phase and transition in the infrared to a theory describing the fundamental states of matter and their interactions with an hierarchically large difference between the Planck scale and the electroweak breaking scale.

Remarkably, it has been shown in a simplified model involving two scalar fields that this structure is possible [2,3,7]. The model has a scale invariant scalar potential and nonminimal coupling of the scalar fields to the Ricci scalar. When the fields develop VEVs the Planck scale is generated spontaneously in the Brans-Dicke manner. For a wide range of the nonminimal couplings and scalar interactions, there is an initial period of "slow-roll" inflation that can give acceptable values for the slow-roll parameters. This is followed by a "reheat" phase and a flow of the field VEVs to an infrared fixed point at which the ratio of the scalar field VEVs are determined by the dimensionless couplings of the theory. Thus it is possible to arrange an hierarchically large ratio for the VEVs and, interpreting the second scalar as modelling the standard model Higgs boson, this large ratio corresponds to the ratio between the Planck scale and the electroweak scale.

In Sec. II, we discuss the mechanism of inertial spontaneous symmetry breaking and conservation of the Weyl current in a toy model, and general N-scalar models. As it does not involve a potential the mechanism opens a new pathway to generating spontaneous scale symmetry breaking and the associated spontaneous breaking of other symmetries. As such it may be useful for novel aspects of model building. We also discuss a general feature of this mechanism, the origin of the dilaton and its intimate relationship to the current, We also briefly consider, as an aside, locally Weyl invariant models in which the dilaton will be eaten by a "Weyl photon," $\hat{A}_\mu$, to give it mass, i.e., the inertial symmetry breaking thus becomes a Weyl symmetry Higgs mechanism, and the dilaton disappears from low energy physics [10].

In Sec. III, we discuss how Weyl invariance is maintained at the quantum level and thus preserves the inertial spontaneous symmetry breaking mechanism. As a result the logarithmic corrections that normally break the scale invariance now automatically depend only on physically relevant ratios of field VEVs which preserve the underlying Weyl invariance of the theory. We compare this procedure to previous proposals for scale invariant regularization that require an arbitrary choice of regulator, a function of the scalar fields.

Finally, in Sec. IV, we present a summary of our results and the conclusions to be drawn.

II. INERTIAL SPONTANEOUS SYMMETRY BREAKING

A. A toy example

Consider a real scalar field theory action together with Einstein gravity and a cosmological constant [our metric signature convention is $(1,-1,-1,-1)$]:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \Lambda + \frac{1}{2}M_P^2 R\right). \quad (1)$$

This action provides a caricature of the cosmological world we live in.

We imagine an initial, ultra-high-temperature phase in which the massless scalar σ has the dominant energy density, $\rho_\sigma \propto T^4$. Consider a Friedman-Robertson-Walker (FRW) metric:

$$g_{\mu\nu} = [1, -a^2(t), -a^2(t), -a^2(t)] \quad H = \frac{\dot{a}}{a}. \quad (2)$$

In this theory, the Universe initially expands in a FRW phase, with the temperature red-shifting as $T \sim 1/a(t)$, and the scale factor growing as $a(t) \sim \sqrt{t}$. Eventually the σ thermal energy becomes smaller than the cosmological constant, $\rho_\sigma < \Lambda$, and we then enter a de Sitter phase with exponential growth, $a(t) \sim e^{t\sqrt{\Lambda/3M_P^2}}$. We can model the thermal phase as a preinflationary era, and the cosmological constant then represents a potential energy that drives inflation. In any case, the intuition that allows us to readily understand how this works is well honed.

Now consider a different action:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{4}\phi^4 - \frac{\alpha}{12}\phi^2 R\right). \quad (3)$$

This action is scale invariant, having no cosmological constant or Planck scale.

These two theories are classically equivalent, provided $\alpha < 1$. This equivalence follows from a Weyl transformation.

However, from our accumulated experience in inflationary cosmology, we understand the dynamics of Eq. (1) so well, then how could we directly understand the dynamics of the Weyl equivalent Eq. (3) without performing a Weyl transformation into Eq. (1)? What happens in the pure evolutionary dynamics intrinsic to Eq. (3) that produces the physical mass scales of M_P and Λ, as well as all other scales in nature?

At first, this doesn't look too hard. Indeed, if ϕ starts out in some very high-temperature phase, where the energy density is large compared to $\lambda\phi^4$ then we expect the scale factor will increase in a scale invariant way, $a(t) \sim t$. This follows by intuiting that the Hubble constant satisfies $H^2 \sim T^4/\phi^2$, where the ϕ^2 factor in the denominator replaces M_P^2. In thermal equilibrium we expect $\phi^2 \sim T^2$ and thus $H = \frac{\dot{a}}{a} \sim T \sim \frac{1}{t}$. Therefore, $a(t) \sim t$.

As the Universe cools, we expect $\phi(x)$ to settle into some spatially constant VEV $\langle\phi\rangle$. However, our intuition from conventional Einstein $M_P^2 R$ gravity tells us that this VEV will slow-roll in the potential, with $\langle\phi\rangle$ eventually becoming zero. In Eq. (3), this would then imply a vanishing M_P, and the details of the solution become less clear. It is plausible that the increasing strength of gravity will increase the Hubble damping, and halt the relaxation of $\langle\phi\rangle$, perhaps leading to a nonzero cosmological constant $\lambda\langle\phi\rangle^4$. If true, this would then match the cosmological constant case of Eq. (1), and it would imply a spontaneous breaking of scale symmetry. We could resort to a numerical solution, but how can we see what happens in a simple and intuitive way, without having to puzzle over the solutions of coupled nonlinear differential equations?

Indeed, from Eq. (3), we can directly obtain the Einstein equation:

$$\frac{1}{6}\alpha\phi^2 G_{\alpha\beta} = \left(\frac{3-\alpha}{3}\right)\partial_\alpha\phi\partial_\beta\phi - g_{\alpha\beta}\left(\frac{3-2\alpha}{6}\right)\partial^\mu\phi\partial_\mu\phi + \frac{1}{3}\alpha(g_{\alpha\beta}\phi D^2\phi - \phi D_\beta D_\alpha\phi) + g_{\alpha\beta}V(\phi). \quad (4)$$

The trace of the Einstein equation becomes:

$$-\frac{1}{6}\alpha\phi^2 R = (\alpha-1)\partial^\mu\phi\partial_\mu\phi + \alpha\phi D^2\phi + 4V(\phi). \quad (5)$$

We also have the Klein-Gordon (KG) equation for ϕ:

$$0 = \phi D^2\phi + \phi\frac{\delta}{\delta\phi}V(\phi) + \frac{1}{6}\alpha\phi^2 R. \quad (6)$$

We can combine the KG equation, Eq. (6), and trace equation, Eq. (5), to eliminate the $\alpha\phi^2 R$ term, and obtain:

$$0 = (1-\alpha)\phi D^2\phi + (1-\alpha)\partial^\mu\phi\partial_\mu\phi + \phi\frac{\delta}{\delta\phi}V(\phi) - 4V(\phi). \quad (7)$$

This can be written as a current divergence equation:

$$D^\mu K_\mu = 4V(\phi) - \phi\frac{\partial}{\partial\phi}V(\phi). \quad (8)$$

where

$$K_\mu = (1-\alpha)\phi\partial_\mu\phi \tag{9}$$

is the "Weyl current." For the scale invariant potential, $V(\phi) \propto \phi^4$, the *rhs* of Eq. (8) vanishes and the K_μ current is then covariantly conserved:

$$D^\mu K_\mu = 0. \tag{10}$$

We see that this is an "on-shell" conservation law, i.e., it assumes that the gravity satisfies Eq. (4). This is the global Weyl current and it can be derived by a Noether variation of the action under a Weyl transformation.

Note that the Weyl current, K_μ, is the derivative of a scalar, $K_\mu = \partial_\mu K$, where:

$$K = \frac{1}{2}(1-\alpha)\phi^2. \tag{11}$$

We refer to K as the "kernel." Using the conserved K-current with its kernel, we can easily understand the dynamics of this theory.

The form of the conservation law is $D^\mu K_\mu = D^2 K = 0$, and this holds in any frame. If we take ϕ to be functions of time t only, and consider a Friedman-Robertson-Walker universe $(g_{\mu\nu} = [1, -a^2(t), -a^2(t), -a^2(t)])$ the current conservation equation implies:

$$\ddot{K} + 3\left(\frac{\dot{a}}{a}\right)\dot{K} = 0. \tag{12}$$

This can be readily solved to give:

$$K(t) = c_1 + c_2\int_{t_0}^{t}\frac{dt'}{a^3(t')}, \tag{13}$$

where $c_{1,2}$ are constants. Therefore, in an expanding universe, *K will evolve to a constant value,* $K \to \bar{K}$.

In the single scalar case, as $K \to \bar{K}$ constant, the initial Jordan frame theory flows to an effective final Einstein-Hilbert theory with parameters $\Lambda = \frac{\lambda\bar{K}^2}{(1-\alpha)^2}$, $M_P^2 = -\frac{\alpha\bar{K}}{3(1-\alpha)}$, $f^2 = 2\bar{K}$ (dilaton decay constant, see II.B) [7]. The equivalence between the theories is achieved dynamically, without having performed a Weyl transformation, and it follows from the Weyl current algebra, and does not rely upon the solutions of complicated nonlinear differential equations of motion.

This is robust. If we consider a set of N scalars, $\{\phi_j\}$, with action given by[2]:

[2]It is straightforward to extend this effective Lagrangian to matter and gauge fields [2,10,12].

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\sum_i^N \partial_\mu\phi_i\partial^\mu\phi_i - W(\{\phi_j\}) - \frac{1}{12}F(\{\phi_j\})R\right]. \tag{14}$$

where we maintain scale invariance [i.e., $F(\{\phi_k\})$ and $W(\{\phi_k\})$ transform, respectively, as $F \to e^{2\epsilon}F$ and $W \to e^{4\epsilon}W$ under global Weyl transformations, as defined below in Eq. (20)]. The conserved Noether current kernel then generalizes to

$$K = \frac{1}{2}\left[\left(\sum_{i=1}^N \phi_i^2\right) - F(\{\phi_k\})\right]. \tag{15}$$

In particular, with $F(\{\phi_j\}) = \sum_i^N \alpha_i\phi_i^2$ the kernel takes the form [7]:

$$K = \frac{1}{2}\sum_{i=1}^N (1-\alpha_i)\phi_i^2. \tag{16}$$

In this case the N scalar fields will evolve such that their values will ultimately be constrained to lie on the N-dimensional locus by Eq. (15) with $K \to \bar{K}$, in particular an ellipsoid in the special case of Eq. (16).

Here we are "launching" the theory in an effective Jordan frame, with arbitrary initial values of the fields and their time derivatives $\{\phi_i, \dot{\phi}_j\}$. The initial expansion will be scale invariant, $a(t) \sim t$, but as $K \to \bar{K}$, the Planck scale becomes dynamically established, and we enter an effective Einstein frame where all mass scales are $\propto \sqrt{\bar{K}}$, and the expansion becomes de Sitter, $a(t) \sim \exp\sqrt{\bar{K}}t)$.

In a two-scalar model discussed in Ref. [7], we have checked numerically that the initial rate of approach to the ellipsoid is rapid and thereafter the fields precisely track the ellipsoid corresponding to constant $\bar{K}$. This is true for a wide range of initial conditions and readily allows for an inflationary period to commence. Since K has dimension of $(\text{mass})^2$, a constant vacuum value of K implies a spontaneous breaking of the scale symmetry in the theory has occurred. Note that this phase does not employ a potential but is driven solely by the initial conditions, and K is the order parameter of inertial spontaneous symmetry breaking.

In multi–scalar theories the flow $K \to \bar{K}$ does not fix the relative values of the scalar field VEVs, which initially end up at some random point on the locus (e.g., ellipse). It is here that the potential becomes important. In the infrared (IR), the fields constrained to the locus, flow towards an IR fixed point in which the ratios of the field VEVs are determined by the potential terms alone [7]. For the case that the potential has a flat direction, the vacuum energy vanishes at the minimum, corresponding to vanishing cosmological constant. The IR fixed point is then the intersection of the potential's flat direction with the locus.

The ratios of the VEV's is then determined by the scalar potential couplings, but constrained by the requirement the fields lie on the N-dimensonal ellipsoid.

For the case that the potential is positive definite, the IR fixed point corresponds to an eternally inflating de Sitter solution in which the ratio of the field VEV's is determined by the scalar potential couplings together with the couplings, α_i, of the scalars to the Ricci scalar.

B. General discussion

Inertial spontaneous symmetry breaking can be responsible for triggering the spontaneous breaking of symmetry in all sectors of the theory. As such it opens new possibilities for model building.

In summary, we found that the expansion of the Universe in a preinflationary phase drives the current charge density, K_0, to zero. The global Weyl current, K_μ, is *always* the derivative of a scalar, $K_\mu = \partial_\mu K$, and in particular $K_0 = \partial_0 K$, where K is the kernel. Hence, as the K_μ current density is diluted away, $K_0 \to 0$, the kernel K therefore evolves as $K \to \bar{K}$ constant. In a Weyl invariant theory, this implies that scale symmetry is broken, and the Planck mass is generated dynamically.

K plays the role of the symmetry breaking order parameter. While a potential may then be needed to engineer the final vacuum, and determine the ratios of individual fields $\langle\phi_i\rangle$, it plays no direct role in the inertial Weyl symmetry breaking phenomenon.

With a little thought, one might have guessed the structure of the order parameter K. Consider a set of N scalar fields $\{\phi_i\}$. If the fields are nonminimally coupled to gravity as $(-1/12)\sum_i \alpha_i \phi_i^2 R(g)$, then if any of the ϕ_i should develop a VEV, we would expect scale breaking, and a nonzero K. Hence, we expect that the order parameter takes the form, $K \sim c\sum_i \phi_i^2$. However, if any ϕ_i has $\alpha_i = 1$, then we can remove it from the action by a local Weyl transformation, absorbing it into the metric. We therefore expect $K = c'\sum_i (1-\alpha_i)\phi_i^2$. Indeed, we found that $K_\mu = \partial_\mu K$, with $c' = 1/2$, combining both the trace of the Einstein and KG equations, or by the Noether variation of the Jordan frame theory under a Weyl transformation, thus confirming our guess.

C. Factorization of the dilaton

We've seen the result that $K \to \bar{K}$ constant as the Universe expands implies that $N-1$ fields $\{\phi_i'\}$ will ultimately satisfy a constraint, such as in Eq. (16), $\bar{K} = (1/2)\sum_i (1-\alpha_i)\phi_i'^2$. Here the constrained fields, $\{\phi_i'\}$, lie on an ellipsoid in field space, but the constraint could be more general as in Eq. (15) with $F(\{\phi_i\})$, and the ellipsoid could be a more general locus in field space.

In any case, there remains one field unconstrained that becomes the dilaton. This is intimately related to the K_μ current. Let us perform a Weyl field redefinition on the N original fields,

$$\phi_i = \exp(\sigma/f)\phi_i' \qquad g_{\mu\nu} = \exp(-2\sigma/f) g'_{\mu\nu}. \tag{17}$$

We thus find the Weyl invariant action becomes

$$S(\phi, g) = S(\phi', g') + \int \sqrt{-g'}(\partial_\mu \bar{K}(\phi')\partial^\mu(\sigma/f) + \bar{K}(\phi')(\partial\sigma/f)^2) \tag{18}$$

Now using the constraint that $\bar{K}$ constant, and integrating by parts, we have

$$S(\phi, g) = S(\phi_i', g') + \frac{1}{2}\int \sqrt{-g'}(\partial\sigma)^2. \tag{19}$$

Here we identify $f^2 = 2\bar{K}$ so the dilaton is canonically normalized. From this we see that the dilaton, σ, describes a dilation of the ellipse, and fluctuates in field space orthogonally to the $N-1$ $\{\phi_i'\}$ fields. The dilaton decouples in the action from everything except gravity (this holds true for fermions and gauge bosons as well; decoupling implies that there are no direct couplings *in the action* to other fields).

This result is elegantly simple. There is no messy kinetic term mixing problem of the dilaton with the remaining ϕ' fields, as some authors have alluded to. Indeed, there is nontrivial mixing amongst the ϕ' that are subject to the constraint, but the dilaton is neatly factorized and does not mix with these other fields kinetically.

We further see that the current written in the unconstrained fields is equivalent to one written in the constrained fields by: $K_\mu = \partial_\mu K(\phi) = \partial_\mu(K(\phi')e^{2\sigma/f})$. Hence in the broken phase (Einstein frame) limit $K(\phi') \to \bar{K}$ constant, $K_\mu \to 2\bar{K}\partial_\mu\sigma/f = f\partial_\mu\sigma$. where $f = \sqrt{2\bar{K}}$. This is as we expect for a Nambu-Goldstone boson, e.g., the axial current of the pion takes the analogous form $f_\pi \partial_\mu \pi$. This implies that $\bar{K}$ is the order parameter of Weyl spontaneous symmetry breaking.

Why is this formulation important? Results following from the "current algebra" of Weyl invariant theories are general statements that are true, independent of the specific structure of the Lagrangian. The particular structure of K_μ and K is independent of the form of any scale invariant potential, but the detailed structure of K does depend upon the choice of the nonminimal couplings, e.g., $F(\phi_i)$ in Eq. (15) (and also any higher derivative gravitational terms can modify the simple forms we just discussed). The behavior of the current algebra will remain intact, since $K_\mu = \partial_\mu K$ is conserved, but the constraint defined by $\bar{K}$ could become a more general locus such as a hyperbola, etc., (such effects result from the renormalization group [7]). The survival of the general feature of inertial breaking with a stable goundstate, e.g., a stable M_{Planck}, requires that the quantum theory does not break Weyl symmetry through loops, as we discuss in Sec. III.

D. Local vs. global Weyl invariance: Eating the dilaton

Our main discussion is based upon globally Weyl invariant theories. However, we include the present section to indicate how it may be possible to promote these to locally Weyl invariant theories by introduction of Weyl's gauge field, i.e., "Weyl's photon." It is interesting that inertial symmetry breaking now becomes a Higgs mechanism, since the Weyl photon will "eat" the massless dilaton and thus remove it from the low energy spectrum, where it becomes the longitudinal degree of freedom of a massive Weyl photon. Hence, in this case the issue of long range fifth force limits becomes moot. The present section is classical, but it would be of interest to develop the full quantum (renormalization group) behavior of Weyl's photon.

Weyl's original idea was that, since coordinates are merely numbers invented by humans to account for events in spacetime, they should not carry length scale [1]. Rather, the concept of length should be relegated to the (covariant) metric, and (contravariant) coordinate differentials are scale free. Therefore, under a local Weyl scale transformation we would have:

$$\begin{aligned} &g_{\mu\nu}(x) \to e^{-2\epsilon(x)} g_{\mu\nu}(x) \qquad g^{\mu\nu}(x) \to e^{2\epsilon(x)} g^{\mu\nu}(x) \\ &\sqrt{-g} \to e^{-4\epsilon(x)}\sqrt{-g} \qquad \phi(x) \to e^{\epsilon(x)}\phi(x) \end{aligned} \tag{20}$$

Weyl transformations are distinct from coordinate diffeomorphisms that define scale transformations on coordinates, as $\delta x^\mu = \epsilon(x) x^\mu$, which we discuss below. The global Weyl symmetry corresponds as usual to $\epsilon =$ (constant in spacetime).

It is straightforward to construct a list of local Weyl invariants:

$$\begin{aligned} &\phi^2(x) g_{\mu\nu}(x); \qquad \phi^{-2}(x) g^{\mu\nu}(x); \qquad \sqrt{-g}(x)\phi^4(x); \\ &R(\phi^2 g_{\mu\nu}) = \phi^{-2} R(g_{\mu\nu}) + 6\phi^{-3} D^\mu(\partial_\mu\phi) \\ &\sqrt{-g}\phi^4 R(\phi^2 g_{\mu\nu}) = \sqrt{-g}(\phi^2 R(g_{\mu\nu}) + 6\phi D^\mu(\partial_\mu\phi)) \\ &\ldots \end{aligned} \tag{21}$$

Note that the computation of $R(\phi^2 g_{\mu\nu})$ above requires that any Christoffel symbols used in the definition of R be evaluated in the metric $\phi^2 g_{\mu\nu}$. Using these identities we can construct an action that is locally Weyl invariant:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left(-\frac{1}{12}\phi^4 R(\phi^2 g) - \frac{\lambda}{4}\phi^4\right) \\ &= \int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}\phi^2 R(g) - \frac{\lambda}{4}\phi^4\right) \end{aligned} \tag{22}$$

where we substituted the relationship of Eq. (21) and integrated by parts using the divergence rule $D_\mu V^\mu = \sqrt{-g}^{-1}\partial_\mu(\sqrt{-g}V^\mu)$. Here we obtain the famous locally Weyl invariant theory in which the nonminimal coupling of scalars to gravity is fixed by the coefficient $1/12$, needed to canonically normalize the ϕ kinetic term. This is a special and somewhat degenerate theory, since we can revert to the metric $\hat{g}_{\mu\nu} = \phi^2 g_{\mu\nu}$ and ϕ disappears from the action. The theory has a vanishing Weyl current [11].

We note that covariant gauge fields, such as the electromagnetic vector potential, A_μ, do not transform under the local Weyl transformation, since they are associated with derivatives $\partial_\mu - ieA_\mu$ which, like coordinates, do not transform. The electromagnetic fields that have the usual engineering scale $\sim(\text{mass})^2$, $\vec{E}$ and $\vec{B}$, are contained in the field strength with one covariant and one contravariant index, $F_\mu{}^\nu$, e.g., $\vec{E}_i = F_i^0$.

We can construct a covariant derivative of a scalar field under local Weyl transformations by introducing the "Weyl photon," $\tilde{A}_\mu$, as

$$\tilde{D}_\mu\phi = \partial_\mu\phi - \tilde{A}_\mu\phi \tag{23}$$

where $\phi(x) \to e^{\epsilon(x)}\phi(x)$ and $\tilde{A}_\mu(x) \to \tilde{A}_\mu(x) + \partial_\mu\epsilon(x)$ (note the major difference from electrodynamics in the absence of a factor of i in the coefficient of $\tilde{A}_\mu$: QED gauges phase, while the Weyl photon gauges scale). Armed with this we can construct another local Weyl invariant:

$$\sqrt{-g}g^{\mu\nu}\tilde{D}_\mu\phi(x)\tilde{D}_\nu\phi(x). \tag{24}$$

This is a locally Weyl invariant kinetic term.[3]

We can combine this with the previous invariants to define an action in which the Weyl symmetry is local, yet the nonminimal coupling of scalars to R is arbitrary:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\Big(\frac{1}{2}(1-\alpha)g^{\mu\nu}\tilde{D}_\mu\phi\tilde{D}_\nu\phi \\ &\quad - \frac{\lambda}{4}\phi^4 - \frac{\alpha}{12}\phi^4 R(\phi^2 g)\Big) \\ &= \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\alpha}{12}\phi^2 R(g) - \frac{\lambda}{4}\phi^4 \\ &\quad - \frac{1}{2}(1-\alpha)(\tilde{A}^\mu\partial_\mu(\phi^2) - \tilde{A}^\mu\tilde{A}_\mu\phi^2)\Big). \end{aligned} \tag{25}$$

[3]Here, there is a subtlety, as we must define the derivative of any conformal field as a commutator: $[D_\mu, \Phi] = \partial_\mu\Phi - A_\mu[W, \Phi]$ where $[W, \phi] = w\phi$ and w is the conformal charge of Φ. Hence $w = 1$ for ϕ. We also require $w = -2$ for $g_{\mu\nu}$, $w = +2$ for $g^{\mu\nu}$, $w = -4$ for det $-g$, etc. Note that $[D_\mu, g_{\rho\sigma}] = D_\mu g_{\rho\sigma} + 2\tilde{A}_\mu g_{\rho\sigma} = \tilde{A}_\mu g_{\rho\sigma}$ since $D_\mu g_{\rho\sigma} = 0$. This insures the invariance of the action with the Weyl covariant derivative under integration by parts. Note that we can alternatively define a restricted "pure gauge theory" with $A_\mu = \partial_\mu \ln(\chi)$, where χ is any massless scalar field.

Now, we want to pass to the Weyl broken phase. We write

$$\phi(x) \to f\exp(\sigma(x)/f)$$
$$g_{\mu\nu}(x) \to \exp(-2\sigma(x)/f) g_{\mu\nu}(x) \tag{26}$$

Note we do not at this stage do a gauge transformation, $A_\mu \to A_\mu + \partial_\mu\sigma(x)/f$. We obtain

$$S = \int \sqrt{-g}\bigg((1-\alpha)\frac{1}{2}g^{\mu\upsilon}\partial_\mu\sigma\partial_\nu\sigma - (1-\alpha)g^{\mu\upsilon}A_\mu f\partial_\nu\sigma + \frac{1}{2}(1-\alpha)g^{\mu\upsilon}A_\mu A_\nu f^2 - \frac{1}{12}\alpha f^2 R\bigg). \tag{27}$$

Note that the Weyl transformation cancelled the original $\frac{1}{2}\alpha g^{\mu\upsilon}\partial_\mu\phi\partial_\nu\phi$ piece since it was local. What is left is a perfect square;

$$S = \int \sqrt{-g}\bigg(\frac{1}{2}f^2(1-\alpha)g^{\mu\upsilon}(A_\mu - \partial_\mu\sigma/f)^2 - \frac{1}{12}\alpha f^2 R\bigg)$$
$$= \int \sqrt{-g}\bigg(\frac{1}{2}f^2(1-\alpha)g^{\mu\upsilon}B_\mu B_\nu - \frac{1}{12}\alpha f^2 R\bigg), \tag{28}$$

where we redefine $B_\mu = A_\mu - \partial_\mu\sigma/f$ which is a massive spin one field of mass $m = f\sqrt{(1-\alpha)}$. The dilaton has been eaten by the Weyl photon to become its longitudinal mode, and the massless dilaton has thus disappeared from the spectrum of the theory.

We can always have a kinetic term for A_μ with

$$F_{\mu\nu} = -[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{29}$$

and

$$S = \int \sqrt{-g}\bigg(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 g^{\mu\upsilon}B_\mu B_\nu - \frac{1}{12}\alpha f^2 R\bigg) \tag{30}$$

The equation of motion for B_μ is

$$\partial_\mu F^{\mu\nu} = D^2 B^\nu - \partial^\nu(D_\mu B^\mu) = m^2 B^\nu \tag{31}$$

This is mathematically analogous to a superconductor or the standard model Higgs mechanism. A gas of B_μ will freeze out and redshift away like matter once the temperature redshifts below m. It is also interesting to note that if we have N ϕ_i fields, the inertial symmetry breaking will yield the $N-1$ ϕ_i' fields and the dilaton which is again eaten to become the longitudinal component of B^μ, but we then find that the gauge field B^μ decouples from the ϕ_i'. It also has even charge conjugation and presumably decouples from fermions and gauge fields as well, and it cannot decay to a pair of gravitons (this is a variation on Yang's theorem which forbids decay of a vector meson to a photon pair). Therefore, relic B^μ fields are stable and could constitute a dark matter candidate if they are not inflated away.

From the action of Eq. (25), we see that the Weyl current is easily obtained:

$$K_\mu = -\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\tilde{A}^\mu} = (1-\alpha)(\phi\partial_\mu\phi - \tilde{A}_\mu\phi^2)$$
$$= (1-\alpha)\phi\tilde{D}_\mu\phi. \tag{32}$$

This still has the general form $K_\mu = D_\mu K$, where D_μ is a covariant Weyl derivative.

By setting $\tilde{A}_\mu = 0$ we obtain a globally invariant theory, and this current becomes the conserved Noether current for the global Weyl invariant theory:

$$K_\mu = (1-\alpha)\phi\partial_\mu\phi. \tag{33}$$

III. QUANTUM SCALE INVARIANCE AND REGULARIZATION

Up to now, our discussion has been confined to the classical action. For the scenario of inertial spontaneously broken scale symmetry to work, and lead to a stable Planck mass, it is essential the that Weyl current be identically conserved at the quantum level [5]:

$$D^\mu K_\mu = 0. \tag{34}$$

In what follows, we will refer to nonzero contributions coming from loops to the rhs of Eq. (34) as "Weyl anomalies." The trace anomalies of the scale current determined by diffeomorphisms are identical to those in K for the scalar sector of the theory.

Scale and Weyl symmetry of a theory appears *ab initio* to be broken by quantum loops. Loop divergences are subtle, however, and are often confused with physics. Here we adopt an operating principle that has been espoused by W. Bardeen [9]: The allowed symmetries of a renormalized quantum field heory are determined by anomalies, (or absence thereof). Quantum loop divergences are essentially unphysical artefacts of the method of calculation.

Weyl or scale symmetry is permitted if the renormalized theory has no Weyl anomalies. Since trace anomalies come from triangle diagrams they are necessarily associated with dimension-4 operators. Hence there is no Weyl anomaly in the standard model of the form $H^\dagger H$ where the Higgs mass is $m^2 H^\dagger H$. Thus there are no Weyl anomalies associated with quadratic or quartic divergences in quantum field theory in four dimensions. Another way of saying this is that divergent terms and counter terms are not separately measurable, only the renormalized mass is physical. In a variation of the standard model with no gravity, no grand unification and no Landau poles in the far UV the Higgs mass would be technically natural with no hierarchy problem.

A. The origin of Weyl anomalies

Our problem of maintaining Weyl symmetry requires that we build a theory that has no anomaly in K_μ. To understand this problem, and its solution, we turn to the CW potential. In computing CW potentials for massless scalar fields we encounter an infrared divergence that must be regularized [8,13]. To do so we often introduce explicit "external" mass scales into the theory by hand. These are mass scales that are not part of the defining action of the theory, and essentially define the RG trajectories of coupling constants. These externally injected mass scales lead directly to the Weyl anomaly.

We can see this in Eq. (3.7) of CW [8] where, to renormalize the quartic scalar coupling constant, λ, in an effective potential at one-loop level, $W(\phi)$, they introduce a mass scale M. Once one injects M into the theory, one has broken scale and Weyl symmetry, and the effective potential in the large $\frac{\phi}{M}$ limit then takes the form

$$W(\phi) = \frac{\beta_1}{4!}\phi^4 \ln\left(\frac{\phi}{M}\right) \tag{35}$$

Here, β_1 is the one-loop renormalization group coefficient, $d\lambda(\mu)/d\mu = \beta_1$. The manifestation of this is seen in the trace of the improved stress tensor [13], and in the divergence of the K_μ current:

$$\partial^\mu K_\mu = 4W(\phi) - \phi\frac{\delta}{\delta\phi}W(\phi) = -\frac{\beta_1}{4!}\phi^4 \tag{36}$$

Of course, there is nothing wrong with the CW potential, or with this procedure, if one is only treating the effective potential as a subsector of the larger theory. If, however, Weyl symmetry is to be maintained as an exact invariance of the world, then M must be replaced by an internal mass scale that is part of action, i.e., M must then be the VEV of a field, χ, or some combination of the fields, appearing in the extended action. We would then have the Coleman-Weinberg potential:

$$W(\phi,\chi) = \frac{\beta_1}{4!}\phi^4 \ln\left(\frac{\phi}{\chi}\right) \tag{37}$$

and, because we now have no external mass scales, the current divergence vanishes:

$$\partial^\mu K_\mu = 4W(\phi,\chi) - \phi\frac{\delta W(\phi,\chi)}{\delta\phi} - \chi\frac{\delta W(\phi,\chi)}{\delta\chi} = 0. \tag{38}$$

This defines the basic idea for maintaining scale symmetry in the quantum theory. It simply implements the notion that there are no fundamental mass scales, and masses are determined only as dimensionless ratios involving VEV's of scalar fields. In the next section, we illustrate this through a calculation of the one-loop correction to the scalar potential arising from the quartic scalar interaction. Of course, there will be further gravitational corrections but their calculation lies beyond the scope of this paper.

B. Weyl invariant Coleman-Weinberg calculation

How might we derive such a result as in Eq. (37) from first principles? We do so via a computation of a Coleman-Weinberg (CW) effective potential. It is important to realize that CW effective potentials themselves must have the full symmetry of the underlying theory. The symmetry is then broken spontaneously by the minimum of the potential.

In fact it is straightforward to show that the usual regularization procedure applied to the Weyl invariant theory of Eq. (14) *does* have a Weyl invariant form. For the simple two-scalar case, $N = 2$, with fields $\phi = \phi_1$ and $\chi = \phi_2$, it reduces to that of Eq. (37) when the ratio of VEV's is small, but the general form is applicable for arbitrary values of the ratio.

1. The two-scalar action

The case, $N = 2$, is the simplest model with "realistic" phenomenological properties. For reasonable parameter choices and initial conditions it can have an initial inflationary period followed by a "reheat" phase and subsequent evolution to an IR stable fixed point in which the ratio of the field VEVs is determined by the fudamental couplings of the theory. We will illustrate the regularization procedure applied to this model (in the limit of neglecting graviton loops) but we emphasize that the procedure immediately generalizes to the case with arbitrary N and indeed to the inclusion of fundamental fermions and vectors.

We start with the action given in Eq. (14) with $N = 2$. The Weyl invariance of the theory is spontaneously broken by the VEVs of the fields giving a massless Goldstone boson, the dilaton, σ. It was shown in [10] that the dilaton decouples and so, of the two initial scalar degrees of freedom, only one interacting one remains. To see how this happens in practice, we change variables to

$$\phi_i = e^{-\sigma/f}\hat{\phi}_i \qquad g_{\mu\nu} = e^{2\sigma/f}\hat{g}_{\mu\nu}, \tag{39}$$

where $\hat{\phi}_i$ are constrained to lie on the ellipse given by

$$2\bar{K} = \sum_{i=1}^{N}(1-\alpha_i)\hat{\phi}_i^2 = f^2, \tag{40}$$

where f^2 is a constant. It is important to note that f is invariant under scale transformations as the dilaton dependence of the original fields has been factored out.

To illustrate the regularization procedure it is sufficient to calculate the CW potential resulting from the $\frac{\lambda}{4!}\phi_1^4$ term in the potential. We first reparametrize the fields by

$$\hat{\phi}_1 = \frac{f}{\sqrt{1-\alpha_1}}\sin\theta, \qquad \hat{\phi}_2 = \frac{f}{\sqrt{1-\alpha_2}}\cos\theta. \tag{41}$$

After scaling out the dilaton, the relevant terms of Eq. (14) become

$$S = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2}f^2\left(\frac{\cos^2\theta}{(1-\alpha_1)} + \frac{\sin^2\theta}{(1-\alpha_2)}\right)\partial_\mu\theta\partial^\mu\theta \right.$$
$$\left. -\frac{\lambda}{4}f^4\frac{\sin^4\theta}{(1-\alpha_1)^2}\right]. \tag{42}$$

Performing the further redefinition $\Theta = F(\theta)$, where

$$F(\theta) = \int_0^\theta \sqrt{\frac{\cos^2\theta'}{(1-\alpha_1)} + \frac{\sin^2\theta'}{(1-\alpha_2)}}d\theta', \tag{43}$$

the action becomes

$$S = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2}f^2\partial_\mu\Theta\partial^\mu\Theta - \frac{\lambda}{4!}f^4\frac{\sin^4F^{-1}(\Theta)}{(1-\alpha_1)^2}\right]. \tag{44}$$

For the case when θ is small, the action approximates the simpler form,

$$S \approx \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{\lambda}{4!}\Phi^4\right], \tag{45}$$

where $\Phi = f\Theta$ and $\Theta \approx \frac{\theta}{\sqrt{1-\alpha_1}}$.

2. The CW potential

Here we demonstrate the derivation of the Weyl invariant CW potential for the case $\frac{\phi_1}{\phi_2} \ll 1$, starting with the action of Eq. (45). Adding a classical source term, $-J\Phi$, to the Lagrangian induces a shift in the Φ field:

$$\Phi = \Phi_c + \hbar^{1/2}\hat{\Phi}, \tag{46}$$

where $\hat{\Phi}$ is the small fluctuation about the classical minimum. Thus the potential has the form

$$W(\Phi) = \frac{\lambda}{4!}\Phi_c^4 + \hbar\frac{\lambda}{4}\Phi_c^2\hat{\Phi}^2 + \cdots, \tag{47}$$

where the linear term cancels due to the classical source term. Treating the quadratic term in $\hat{\Phi}$ as an interaction the one-loop potential with $\hat{\Phi}$ the propagating field is given by

$$\begin{aligned} W_{\text{eff}} &= \Omega + i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\Phi_c^2}{k^2+i\varepsilon}\right)^n \\ &= \Omega + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\Phi_c^2}{2k^2}\right) \\ &= \Omega + \frac{\lambda\Lambda^2}{128\pi^2}\Phi_c^2 - \frac{\lambda^2\Phi_c^4}{256\pi^2}\ln\left(\frac{\frac{1}{2}\lambda\Phi_c^2+\Lambda^2}{\frac{1}{2}\lambda\Phi_c^2}\right) \\ &\quad + \frac{\Lambda^4}{64\pi^2}\ln\left(\frac{\frac{1}{2}\lambda\Phi_c^2+\Lambda^2}{\Lambda^2}\right), \end{aligned} \tag{48}$$

where

$$\Omega = \frac{\lambda}{4!}\Phi_c^4 - \frac{1}{2}B\Phi_c^2 - \frac{\lambda}{4!}C\Phi_c^4. \tag{49}$$

Note that, at the intermediate stage, the UV divergences are regulated by introducing a cut-off, Λ^2, when performing the k^2 integration. Thus, in the $\Lambda \to \infty$ limit, we have the CW result:

$$W_{\text{eff}} = \Omega + \frac{\lambda\Lambda^2}{64\pi^2}\Phi_c^2 + \frac{\lambda^2\Phi_c^4}{256\pi^2}\left(\ln\frac{\lambda\Phi_c^2}{2\Lambda^2} - \frac{1}{2}\right). \tag{50}$$

Following CW, the renormalization conditions are

$$\left.\frac{d^2W_{\text{eff}}}{d\Phi_c^2}\right|_{\Phi_c=0} = 0, \quad \left.\frac{d^4W_{\text{eff}}}{d\Phi_c^4}\right|_{\Phi_c=M} = \lambda, \quad Z|_{\Phi_c=M} = 1. \tag{51}$$

Here, CW renormalizes at an "external" mass scale, M, to avoid the IR singularity. Implementing these conditions[4] determines the counter terms and gives the final CW result:

$$W = \frac{\lambda}{4!}\Phi_c^4 + \frac{\lambda^2\Phi_c^4}{256\pi^2}\left(\ln\frac{\Phi_c^2}{M^2} - \frac{25}{6}\right). \tag{52}$$

In terms of the original fields $\Phi = f\Theta$, $\Theta \approx \frac{\theta}{\sqrt{1-\alpha_1}}$ and $\theta \approx \hat{\phi}_1/\hat{\phi}_2$, the potential is given by

$$W \approx \frac{\lambda}{4!}\hat{\phi}_1^4 + \frac{\lambda^2\hat{\phi}_1^4}{256\pi^2}\left(\ln\left(\frac{C\hat{\phi}_{1c}^2}{\hat{\phi}_{2c}^2}\right) - \frac{25}{6}\right), \tag{53}$$

where $C = \frac{f^2}{M^2}\frac{1}{1-\alpha_2}$ is a constant invariant under scale changes. This is the Weyl invariant CW potential written in terms of the variables $(\hat{\phi}_1, \hat{\phi}_2)$ which are constrained by Eq. (41). In addition there is a dilaton, σ, with an isolated kinetic term. By performing a Weyl transformation that is the inverse of Eq. (39), we can relax the constraint Eq. (41) and obtain

[4]There is no wave-function renormalization at one-loop order.

$$W \approx \frac{\lambda}{4!}\phi_1^4 + \frac{\lambda^2\phi_1^4}{256\pi^2}\left(\ln\left(\frac{C\phi_{1c}^2}{\phi_{2c}^2}\right) - \frac{25}{6}\right). \quad (54)$$

which is Weyl invariant, and the fields $(\phi_1, \phi_2) = \exp(-\sigma/f)(\hat{\phi}_1, \hat{\phi}_2)$ are independent variables.

The reason Weyl invariance has been preserved is because the inertial spontaneous symmetry breaking has introduced the mass scale, f, that compensates for the appearance of the renormalization scale M under the log, leaving the logarithmic terms invariant. Note that the usual renormalization group equations still apply as a change in the renormalization scale M [a change in C in Eq. (53)] is compensated by a change in the couplings and wave function factors in the usual way.

3. Scale invariant regularization

The standard regularization described above clearly preserves Weyl invariance even away from the small $\frac{\phi_1}{\phi_2}$ limit because, on dimensional grounds, the spontaneous scale breaking factor, f, always compensates for the renormalization scale factor to give an overall constant under the log, together with a function of the scale invariant field $\Theta = f\Phi$.

Expanding Eq. (42) beyond leading order leads to higher-order terms in θ but these nonrenormalizable terms are small. The reason is that Planck scale is predominantly due to the VEV of ϕ_2 whereas the VEV of ϕ_1, which models the SM Higgs, is at the electroweak scale so that the nonrenormalizable terms are Planck suppressed. In order to generate the hierarchy in the VEV's at the IR fixed point, it is necessary that the only large coupling is λ while the other couplings associated with the other scale invariant quartic interactions are hierarchically small and can be neglected when calculating the radiative corrections.

Of course, there will be further terms when the gravitational interactions are included. Gravitational corrections require the addition of the Weyl tensor, W^2, and R^2 terms, which are induced by matter loops and have logarithmically running coefficients. An analysis of the full renormalization group equations appears in [14]. While the Weyl tensor term is locally invariant, the R^2 term is only globally invariant. Hence we expect to maintain a conserved current, K'_μ, however the current will be modified by the addition of a new term, $K'_\mu = K_\mu + c'\partial_\mu R/f_0^2$ in the notation of [14]. We expect that this is a small correction to the above scenario of a fixed ellipse, but may have some phenomenological implications that will be pursued elsewhere.

Another potentially challenging consequence of the gravitational corrections is that the λ_i become locked to the α_i by the renormalization group. This may necessitate some large fine-tunings to maintain a small cosmological constant and/or flat potentials. We feel that this requires a more sophisticated fundamental analysis since the RG equations computed in flat geometries amount ot a "gauge choice" for the Weyl symmetry and do not admit analysis of the Weyl transformation.

Finally, it is possible to maintain the local Weyl symmetry without choosing special values of the α_i, but rather by introducing the Weyl vector potential. When this is done, the dilaton is "eaten" to become the longitudinal part of a massive Weyl vector potential. The relationship of this to gravitational corrections and our general framework is unexplored.

4. Scale invariant dimensional regularization

Of course, regularization should not depend on the method used to control the intermediate divergences. Up to now we have used a momentum space cut-off but it is straightforward to use dimensional regularization. In this case one first continues the theory to d-dimensions and introduces an external mass scale, μ, to relate the four-dimensional dimensionless couplings to the dimension-full ones in d-dimensions. For the two-scalar theory discussed above, dimensional regularization leads straightforwardly to the form of Eq. (53) with M replaced by μ. In this case the quartic and quadratic terms are automatically absent. The dependence on the mass parameter, μ, needed to continue away from four dimensions, will always appear in the scale invariant ratio μ/f giving Eq. (53) as before.

5. Relation to previous regularization proposals

Scale invariant dimensional regularization that differs from the one just described has been considered by several authors [5,6]. The method generally adopted to maintain scale invariance in radiative order replaces μ by a function of the scalar fields, $\mu \to \mu(\phi_i)$, with the appropriate scaling behavior. In this case the d-dimensional tree-level potential $\tilde{V}$ has the form

$$\tilde{V}(\phi,\chi) \equiv \mu(\phi,\chi)^{4-d} V(\phi,\chi). \quad (55)$$

As a result, the tree-level potential introduced in Eq. (55) has *additional* interactions of the form

$$\tilde{W}(\phi,\chi) - W(\phi,\chi) = (4-d)W(\phi,\chi)\ln\mu(\phi,\chi) + O(4-d)^2. \quad (56)$$

Although these interactions vanish in four dimensions, they give a finite correction to $W_{\rm eff}$ at one-loop order because the underlying divergence in four dimensions cancels the $4-d$ factor in the additional term in Eq. (56). Thus, due to the additional interaction terms in Eq. (56) that depend on the choice of $\mu(\phi,\chi)$, the scale invariant d-dimensional theory is *not* the same as that defined purely in four dimensions. As a result the final regulated theory in 4-dimensional has additional terms that depend on the precise choice of the regulator $\mu(\phi,\chi)$. For the two-scalar case with potential given by Eq. (55) and the choice $\mu(\phi,\chi) = \chi$ the additional term at one-loop is of the form ϕ^6/χ^2. While this is still scale invariant it means the resulting four-dimensional potential is different from that obtained by the regularization procedure discussed above. The origin of this discrepancy is that the requirement that scale invariance be preserved in d-dimensions rather than regularization ambiguity requires such additional terms and defines a different theory.

In summary, we have shown that the standard regularization procedure preserves scale invariance. It does not involve the introduction of an arbitrary regularization function and, although it involves nonrenormalizable interactions, these are well defined. Of course, it is possible to add additional nonpolynomial terms to the theory while preserving scale invariance but we see no reason to do so.

IV. SUMMARY AND CONCLUSIONS

We have discussed how inflation and Planck scale generation can emerge from a dynamics associated with global Weyl symmetry and its current, K_μ. In the preinflationary universe, the Weyl current density, K_0, is driven to zero by general expansion. However, K_μ has a kernel structure, i.e., $K_\mu = \partial_\mu K$ and, as $K_0 \to 0$, the kernel evolves as $K \to \bar{K}$, constant. This resulting constant $\bar{K}$, that does not depend on the scalar potential, is the order parameter of the Weyl symmetry breaking; indeed, $\bar{K}$ directly defines the Planck mass.

In N-multiscalar-field theories, K has the general form $K = -\frac{1}{2}(F(\{\phi_j\}) - \sum_{i=1}^N \phi_i^2$ for nonminimal coupling $-(1/12)F(\{\phi_j\})R$. The fields become constrained to the manifold $K \to \bar{K}(\{\phi_j\})$. In detail we have studied $F(\{\phi_j\}) = \frac{1}{2}\sum_{i=1}^N \alpha_i \phi_i^2$. This defines an ellipsoidal constraint on the scalar field VEVs. An inflationary slow-roll period is then associated with the field VEVs migrating along the ellipse. Up to this point, the fate of scale symmetry is entirely controlled by the inertial symmetry breaking, $K \to \bar{K}(\{\phi_j\})$. A potential ultimately sculpts the ensuing slow roll on the manifold to the IR, and defines the ultimate vacuum (together with any quantum effects that may distort the K ellipse [7]) This fixes the relative value of the scalar field VEVs through quartic terms only. There is a harmless massless dilaton associated with the dynamical symmetry breaking which represents dilations of the ellipsoid. We emphasize that with more general choices of $F(\{\phi_j\})$, the constraint manifold can become a more general manifold in the field space, and it would be of interest to explore the possibilities in this case.

Any Weyl symmetry breaking effect at the quantum level is intolerable and will show up as a nonzero divergence in the K_μ current. We showed how, due to the decoupling of the dilaton, these quantum effects actually preserve the Weyl symmetry using the normal momentum space cutoff or dimensional regularization schemes. The potential scale dependence introduced by the "external" mass scale needed to regulate the logarithmic divergences is cancelled by the scale invariant order parameter responsible for spontaneous breaking of the Weyl symmetry. It would be of interest to study the local Wetyl invariant theories that involve the Weyl photon, as in Sec. II C, in great detail. This provides an example of an inertial Higgs mechanism, and the dilaton is eaten and completely removed from the low energy spectrum.

A strong motivation for considering such Weyl invariant theories is to provide a solution to the hierarchy problem of the standard model. In the absence of gravity or very massive states associated with the Landau pole of the standard model or of an extension of the standard model such as grand or string unification, the standard model is natural in the sense that the quadratic divergence found in radiative corrections to the Higgs mass is unphysical and is cancelled by the mass counter term. Requiring scale invariance ensures that the Higgs is massless but, of course, some mechanism to spontaneously break the scale symmetry is needed.

If gravity is included via the Weyl invariant extension discussed here, then the standard model *plus* gravity is natural in the sense just discussed. Of course, it is still necessary that there be no massive states strongly coupled to the Higgs with masses much larger than the electroweak scale. Moreover, the scale symmetry is now automatically spontaneously broken by the inertial mechanism. To obtain the hierarchy between the Planck scale and the electroweak breaking scale it is necessary to have hierarchically large ratios of the dimensionless couplings of the scalar potential. In the absence of gravitational radiative corrections, these ratios are only multiplicatively changed by radiative corrections and thus are natural. This may be seen from the underlying shift symmetry of the Weyl invariant Higgs potential.

This shift symmetry is broken by the Higgs coupling to the Ricci scalar. To determine whether the hierarchy is ultimately preserved requires a calculation of the gravitational radiative corrections which is beyond the scope of the present paper. In a Weyl invariant variation of the standard model with no gravity, no grand unification and no Landau poles in the far UV the Higgs mass is technically natural with no hierarchy problem.

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