Hair mass bound in the black hole with nonzero cosmological constants

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We study mass bounds of Maxwell fields in Reissner-Nordström black holes and genuine hair in Einstein-Born-Infeld black holes with various cosmological constants. It shows that the Maxwell field serves as a good probe to disclose the hair distribution described with the event horizon and the photonsphere. We find that the Hod's lower bound obtained in asymptotically flat space also holds in the asymptotically de Sitter Einstein-Born-Infeld hairy black holes. In contrast, the Hod's lower bound can be invaded in the asymptotically anti–de Sitter (AdS) Einstein-Born-Infeld hairy black holes. It implies that the AdS boundary could make the Born-Infeld hair easier to condense in the near horizon area.

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I. INTRODUCTION

The famous no hair conjecture of Wheeler [1–3] was motivated by research on uniqueness theorems that Einstein-Maxwell black holes are described by only three conserved parameters: mass, electric charge and angular momentum [4–8]. The belief in the no hair conjecture was based on a simple physical picture that matter fields outside black holes would eventually be radiated away to infinity or be swallowed by the black hole horizon except when those fields were associated with the three conserved parameters. In accordance with this logic, stationary black holes indeed exclude the existence of scalar fields, massive vector fields and spinor fields in the exterior spacetime of black holes [9–15].

However, nowadays we are faced with the surprising discovery of various types of hairy black holes, the first of which were Einstein-Yang-Mills black holes [16–18]. After that, other static hairy black hole solutions were also discovered in theories like Einstein-Skyrme, Einstein-non-Abelian-Proca, Einstein-Yang-Mills-Higgs and Einstein-Yang-Mills-dilaton and hair formation in nonstatic kerr black holes was investigated, for references see [19–32] and reviews [33,34]. The discovery of front hairy black holes provides a challenge to the validity of the classical no hair theorem. Now, it is clear that the formation of hair is due to the fact that self-interaction can bind together the hair in a region very close to the horizon and another region relatively distant from the horizon [35]. In accordance with this physical picture, a no short hair theorem was proposed as an alternative to the no hair conjecture based on the fact that the black hole hair of Einstein-Yang-Mills fields must extend above the photonsphere [35]. Shahar Hod also proved a no short scalar hair theorem that linearized massive scalar fields have no short hair behaviors in nonspherically symmetric nonstatic kerr black holes [36].

Along this line, it is interesting to study the distribution of hair mass. For the limit case of the linear Maxwell field, Hod showed that the region above the photonsphere contains at least half of the total mass of Maxwell fields and also found that this lower bound holds for various genuine hairy black holes in Einstein-Yang-Mills, Einstein-Skyrme, Einstein-non-Abelian-Proca, Einstein-Yang-Mills-Higgs and Einstein-Yang-Mills-dilaton systems [37]. It was found that the nonlinear Einstein-Born-Infeld black holes also satisfy this lower bound that half of the Born-Infeld hair is above the photonsphere [38]. In fact, it is reasonable to use Maxwell fields to study density distribution of genuine hair since the Maxwell field case is a linear limit of the nonlinear Einstein-Born-Infeld theory. The front studies of hair mass bounds were carried out in asymptotically flat backgrounds. As a further step, it is meaningful to extend the discussion in asymptotically flat black holes to spacetimes with nonzero cosmological constants.

In the following, we introduce black holes with nonzero cosmological constants and obtain bounds for linear hair mass ratio. We also disclose properties of genuine hair in Einstein-Born-Infeld black holes. We will summarize our main results at the last section.

II. ANALYTICAL STUDIES OF LINEAR HAIR MASS BOUNDS

In this paper, we use the Maxwell field as a linear limit to disclose properties of genuine hair similar to approaches in [37]. The four-dimensional Einstein-Maxwell black hole geometries with nonzero cosmological constant Λ are described by [39–42]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

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where metric functions $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2$ with M as the Arnowitt-Deser-Misner mass and Q as the charge.

The mass $m(r)$ of the Maxwell field above the radius r is given by

$$m(r) = \int_r^{+\infty} 4\pi r'^2 \rho(r') dr'. \tag{2}$$

For the Maxwell field, one has the energy density $\rho(r) = -T_t^t = \frac{Q^2}{8\pi r^4}$ [37]. It yields $m(r) = \frac{Q^2}{2r}$ for the mass function.

It was found that the photonsphere can be conveniently used to describe spatial distribution of the matter field [35,37]. According to the approach in [37], the radius r_γ of the null circular geodesic (photonsphere) in the Reissner-Nordström (RN) black hole is determined by the relation

$$2f(r_\gamma) - r_\gamma f'(r_\gamma) = 0. \tag{3}$$

From (3), one obtains the radius r_γ independent of the cosmological constants as

$$r_\gamma = \frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2}). \tag{4}$$

We define r_H as the black hole event horizon satisfying $f(r_H) = 0$. An interesting quantity which characterizes the spatial distribution of the hair is given by the dimensionless hair mass ratio $\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}}$, where

$$m^+_{\mathrm{hair}} = m(r_\gamma) \tag{5}$$

is the mass of the hair above the photonsphere and

$$m^-_{\mathrm{hair}} = m(r_H) - m(r_\gamma) \tag{6}$$

is the mass of the hair contained between the event horizon and the photonsphere. For the linear hair of Maxwell field outside the Reissner-Nordström (RN) black hole, Hod obtained bounds on the ratio $\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}} \geq 1$. The ratio can be expressed as

$$\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}} = \frac{\frac{Q^2}{2r_\gamma}}{\frac{Q^2}{2r_H} - \frac{Q^2}{2r_\gamma}} = \frac{1}{\frac{r_\gamma}{r_H} - 1}. \tag{7}$$

We have mentioned that $r_\gamma = \frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2})$ is independent of the cosmological constant Λ. In order to study the ratio $\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}}$, we try to research how the cosmological constant can affect the event horizon r_H. The event horizon r_H can be obtained from the equation $1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2 = \frac{1}{r^2}(r^2 - 2Mr + Q^2 - \Lambda r^4) = 0$ or the equation

$$r^2 - 2Mr + Q^2 - \Lambda r^4 = 0. \tag{8}$$

Case I: $\Lambda > 0$.—For the case of $\Lambda > 0$ or asymptotically de Sitter (dS) black hole spacetime, Eq. (8) has three real positive roots r_h, r_H and r_0, where r_h is the Cauchy horizon, r_H is the event horizon and r_0 is the cosmological horizon with $r_h < r_H < r_0$ [39,40]. For simplicity, we introduce a function $y = r^2 - 2Mr + Q^2 - \Lambda r^4$. The event horizon $r = r_H$ can be obtained from $y = 0$. In this work, we are interested in the case of $M \geq Q$. According to the relation $r_H^2 - 2Mr_H + Q^2 = \Lambda r_H^4 > 0$, there is $r_H > M + \sqrt{M^2 - Q^2}$ or $r_H < M - \sqrt{M^2 - Q^2}$. Since $y' = 2(r - M) - 4\Lambda r^3 < 0$ for $r \leq M$, at most one root of $y = 0$ is in the range of $r \leq M$ and there is $r_H > M$. Then we further have $r_H > M + \sqrt{M^2 - Q^2}$.

Considering the fact $r_H > M + \sqrt{M^2 - Q^2}$, the mass ratio can be expressed with $x = \frac{Q}{M} \in [0, 1]$ as

$$\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}} = \frac{1}{\frac{r_\gamma}{r_H} - 1} > \frac{1}{\frac{\frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2})}{M + \sqrt{M^2 - Q^2}} - 1} = \frac{1}{\frac{\frac{1}{2}(3 + \sqrt{9 - 8x^2})}{1 + \sqrt{1 - x^2}} - 1}. \tag{9}$$

According to the fact that

$$\left(\frac{1}{\frac{\frac{1}{2}(3+\sqrt{9-8x^2})}{1+\sqrt{1-x^2}} - 1}\right)'_x = -\frac{2x}{\left(\frac{\frac{1}{2}(3+\sqrt{9-8x^2})}{1+\sqrt{1-x^2}} - 1\right)^2} \frac{17 - 18x^2 + 3\sqrt{9 - 8x^2} + 8\sqrt{1 - x^2}}{3\sqrt{9 - 8x^2} + 8\sqrt{1 - x^2}} \tag{10}$$

and

$$17 - 18x^2 + 3\sqrt{9 - 8x^2} + 8\sqrt{1 - x^2} \geq 17 - 18 + 3 + 0 = 2 > 0, \tag{11}$$

we have

$$\left(\frac{1}{\frac{\frac{1}{2}(3+\sqrt{9-8x^2})}{1+\sqrt{1-x^2}} - 1}\right)'_x < 0$$

for all $x \in [0, 1]$. So we have

$$\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}} > 1 \tag{12}$$

and the lower bound is with $x = 1$ and $\Lambda \to 0$.

From the relation $r_H^2 - 2Mr_H + Q^2 - \Lambda r_H^4 = 0$, we arrive at $\Lambda = \frac{r_H^2 - 2Mr_H + Q^2}{r_H^4}$. Then there is $\frac{d\Lambda}{dr_H} = \frac{4(r_H^2 - 2Mr_H + Q^2)}{r_H^5} + \frac{2r_H - 2M}{r_H^4} > 0$ since we have proved $r_H > M + \sqrt{M^2 - Q^2}$. So Λ is an increasing function of r_H and the event horizon r_H also increases when we increase the value of Λ. For $r_H = r_\gamma$ or $\Lambda = \frac{r_\gamma^2 - 2Mr_\gamma + Q^2}{r_\gamma^4} > 0$ with $r_\gamma = \frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2})$, we have

$$\frac{m_{\rm hair}^+}{m_{\rm hair}^-} = \frac{1}{\frac{r_\gamma}{r_H} - 1} = \frac{1}{\frac{r_H}{r_H} - 1} = +\infty. \tag{13}$$

In all, the linear hair mass ratio satisfies the Hod's mass bound. We further conjecture that asymptotically dS genuine hairy black holes may also obey the Hod's hair mass bound.

Case II: $\Lambda < 0$.—For another case of $\Lambda < 0$ or asymptotically AdS charged black hole spacetime, Eq. (8) has two real positive roots r_h and r_H, where r_h is the Cauchy horizon and r_H is the event horizon [41]. From $r_H^2 - 2Mr_H + Q^2 = \Lambda r_H^4 < 0$, we have $r_H < M + \sqrt{M^2 - Q^2}$. The mass ratio satisfies the upper bound

$$\frac{m_{\rm hair}^+}{m_{\rm hair}^-} = \frac{1}{\frac{r_\gamma}{r_H} - 1} < \frac{1}{\frac{\frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2})}{M + \sqrt{M^2 - Q^2}} - 1} \leq 2. \tag{14}$$

The upper bound corresponds to the case of $Q \to 0$ and $\Lambda \to 0$. The existence of this upper bound is natural since the negative cosmological constant usually serves as a potential to confine the matter field around the horizon. Since we study the case of $0 < r_H \leq r_\gamma < +\infty$, there is $\frac{m_{\rm hair}^+}{m_{\rm hair}^-} = \frac{1}{\frac{r_\gamma}{r_H} - 1} > 0$. In all, we obtain bounds of the mass ratio

$$0 < \frac{m_{\rm hair}^+}{m_{\rm hair}^-} < 2. \tag{15}$$

Now we show that this lower bound can be approached as $\Lambda \to -\infty$. After choosing $Q \ll 1$, we solve $r^2 - 2Mr - \Lambda r^4 = 0$ to find the horizon r_H. Since $-2Mr$ is the leading term in $r^2 - 2Mr - \Lambda r^4$ around $r \approx 0$, we have $r^2 - 2Mr - \Lambda r^4 < 0$ for r a little larger than 0. There is also $r^2 - 2Mr - \Lambda r^4 \to +\infty$ as $r \to +\infty$. In the procedure of $\Lambda \to -\infty$, we divide the horizon into three cases: $r_H \to 0$, $r_H \to \infty$ and $r_H \to C$, where C is a nonzero constant.

In the cases of $r_H \to \infty$ and $\Lambda \to -\infty$, we have

$$r^2 - 2Mr \to +\infty \tag{16}$$

and

$$r^2 - 2Mr - \Lambda r^4 \to +\infty \tag{17}$$

in contradiction with the equation $r^2 - 2Mr - \Lambda r^4 = 0$.

In another case of $r_H \to C \neq 0$ and $\Lambda \to -\infty$, we have

$$r^2 - 2Mr - \Lambda r^4 \to C^2 - 2MC - \Lambda C^4 \to +\infty \tag{18}$$

in contradiction with the equation $r^2 - 2Mr - \Lambda r^4 = 0$.

In a word, we have $r_H \to 0$ as $\Lambda \to -\infty$ and $Q \to 0$. For the case of $Q \ll 1$ and M fixed, the lower bound can be approached as

$$\frac{m_{\rm hair}^+}{m_{\rm hair}^-} = \frac{1}{\frac{r_\gamma}{r_H} - 1} = \frac{1}{\frac{3M}{r_H} - 1} \to 0 \quad \text{as} \quad \Lambda \to -\infty. \tag{19}$$

Here the relation (19) shows that the linear hair of the Maxwell field can invade the Hod's lower bound $\frac{m_{\rm hair}^+}{m_{\rm hair}^-} \geq 1$. It implies that the Hod's bound may be invaded in the asymptotically AdS Einstein-Born-Infeld hairy black holes according to the fact that Born-Infeld field hair can be reduced to Maxwell field in the linear limit. We will further check this in the following part.

III. HAIR MASS BOUNDS OF EINSTEIN-BORN-INFELD BLACK HOLES

We should emphasize that the RN-(A)dS black hole is not hairy since the Maxwell field is associated with a Gauss law. In this part, we extend the discussion to Einstein-Born-Infeld hairy black holes with the Born-Infeld factor associated with no conserved charge [38,43]. The Lagrangian density for Born-Infeld theory is in the form

$$L_{\rm BI} = \frac{1}{b^2}\left(1 - \sqrt{1 + \frac{b^2 F^{\mu\nu}F_{\mu\nu}}{2}}\right). \tag{20}$$

Here b is the Born-Infeld factor parameter. We mention that in the limit $b \to 0$, this Lagrangian reduces to the Maxwell case and properties of the RN black holes may also hold in Born-Infeld hairy black holes at least for very small b.

Now we introduce the line element of Born-Infeld black holes with nonzero cosmological constant Λ as follows:

$$ds^2 = -f_{\rm EBI}(r)dt^2 + f(r)_{\rm EBI}^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{21}$$

where metric functions $f_{\rm EBI}(r) = 1 - \frac{2M}{r} - \Lambda r^2 + \frac{2b^2r^2}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r^4}}\right) + \frac{4Q^2}{3r^2}F[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; -\frac{Q^2}{b^2r^4}]$, where M is the Arnowitt-Deser-Misner mass, Q is the charge and F is the hypergeometric function satisfying $(\frac{1}{r}F[\frac{1}{4},\frac{1}{2},\frac{5}{4};-\frac{Q^2}{b^2r^4}])'_r = -\frac{1}{\sqrt{r^4+\frac{Q^2}{b^2}}}$ [44–46]. Around $x=0$, the hypergeometric function can be expanded as $F[a,b,c,x] = 1+\frac{abx}{c}+\frac{a(1+a)b(1+b)x^2}{2c(1+c)}+o[x^3]$ [see Eq. (15.7.1) of [47]]. So we have $F[\frac{1}{4},\frac{1}{2},\frac{5}{4};-\frac{Q^2}{b^2r^4}]\to 1$ and $f_{\mathrm{EBI}}(r)\to 1-\frac{2M}{r}+\frac{Q^2}{r^2}-\Lambda r^2$ as $r\to\infty$. The photon-sphere radius r_γ is determined by the relation $2f_{\mathrm{EBI}}(r_\gamma)-r_\gamma f'_{\mathrm{EBI}}(r_\gamma)=0$ [37,38] and the black hole event horizon r_H can be numerically obtained from $f(r_H)=0$ [39–41].

The energy density of the Born-Infeld hair is give by

$$\rho(r) = -T_t^t = -\frac{\left(\frac{2b^2r^3}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r^4}}\right)+\frac{4Q^2}{3r}F\left[\frac{1}{4},\frac{1}{2},\frac{5}{4};-\frac{Q^2}{b^2r^4}\right]\right)'}{8\pi r^2}.$$

The mass $m_{\mathrm{EBI}}(r)$ of the Born-Infeld hair above the radius r is given by

$$m_{\mathrm{EBI}}(r) = \int_r^{+\infty} 4\pi r'^2\rho(r')dr' = \frac{1}{2}\left(\frac{2b^2r^3}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r^4}}\right)+\frac{4Q^2}{3r}F\left[\frac{1}{4},\frac{1}{2},\frac{5}{4};-\frac{Q^2}{b^2r^4}\right]\right). \tag{22}$$

The hair mass ratio is

$$\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}} = \frac{m_{\mathrm{EBI}}(r_\gamma)}{m_{\mathrm{EBI}}(r_H)-m_{\mathrm{EBI}}(r_\gamma)} = \frac{1}{\frac{b^2r_H^3\left(1-\sqrt{1+\frac{Q^2}{b^2r_H^4}}\right)+\frac{2Q^2}{r_H}F\left[\frac{1}{4},\frac{1}{2},\frac{5}{4};-\frac{Q^2}{b^2r_H^4}\right]}{b^2r_\gamma^3\left(1-\sqrt{1+\frac{Q^2}{b^2r_\gamma^4}}\right)+\frac{2Q^2}{r_\gamma}F\left[\frac{1}{4},\frac{1}{2},\frac{5}{4};-\frac{Q^2}{b^2r_\gamma^4}\right]}-1}. \tag{23}$$

In the following, we calculate the hair mass ratio in the Einstein-Born-Infeld genuine hairy black holes. In the case of asymptotically flat space with $\Lambda=0$, we find that the Hod's hair mass ratio bound holds for various b and other parameters fixed. For example, in the case $M=1.5$, $Q=1$, $\Lambda=0$ and various b from 0.001 to 1000, we find that $\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}$ decreases as a function of b and the ratio approaches the limit value 1.895 for large b. We show part of the numerical data in Table I and it can be easily seen from the table that the mass ratio is above the Hod's lower bound. Since the mass ratio with $b=2$ almost reaches the lowest value, we fix $b=2$ to check the Hod's bound in the following.

We also find that the Hod's bound holds with different values of M. In Table II, we show that the ratio is above the Hod's lower bound in the case of $Q=1$, $b=2$, $\Lambda=0$ and various M. According to Table II, the ratio increases as a function of M and the smallest ratio $\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}\approx 1.218$ is obtained in the case of $M\approx 1.0$, which corresponds to the extremal black hole solution with $M=Q$.

We also show cases of $M=1.5$, $b=2$, $\Lambda=0$ and various Q in Table III. Here the ratio decreases with respect to the charge Q and in the case of the extremal black hole solution with $Q\approx 1.5$, the smallest ratio is $\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}\approx 1.147$ above the Hod's lower bound. We further numerically check for the parameters in a larger range and find that the Hod's bound $\frac{m^+_{\mathrm{hair}}}{m^-_{\mathrm{hair}}}\geq 1$ [37] holds in the asymptotically flat Einstein-Born-Infeld hairy black hole in accordance with results in [38].

TABLE I. The hair mass ratio $\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}$ with $M=1.5$, $Q=1$, $\Lambda=0$ and various b.

b	0.1	0.3	0.5	0.7	0.9	1.0	2.0	4.0	6.0
$\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}$	2.288	1.952	1.916	1.905	1.901	1.890	1.895	1.895	1.895

TABLE II. The hair mass ratio $\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}$ with $Q=1$, $b=2$, $\Lambda=0$ and various M.

M	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\frac{m^+_{\mathrm{EBI}}}{m^-_{\mathrm{EBI}}}$	1.218	1.668	1.780	1.837	1.872	1.895	1.913	1.926	1.936	1.934	1.950

TABLE III. The hair mass ratio $\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$ with $M = 1.5$, $b = 2$, $\Lambda = 0$ and various Q.

Q	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$	1.999	1.993	1.980	1.958	1.922	1.861	1.736	1.147

TABLE IV. The hair mass ratio $\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$ with $M = 1.5$, $Q = 1$, $b = 2$ and various positive Λ.

Λ	0	0.001	0.005	0.010	0.015	0.016	0.017	0.018	0.019	0.020
$\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$	1.895	1.942	2.169	2.633	3.724	4.190	4.907	6.241	10.422	117.500

TABLE V. The hair mass ratio $\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$ with $M = 1.5$, $Q = 1$, $b = 2$ and various negative Λ.

Λ	0	−0.02	−0.04	−0.06	−0.08	−0.10
$\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$	1.895	1.381	1.153	1.015	0.921	0.851

Now we will further show that the Hod's bound also holds in the asymptotically dS Einstein-Born-Infeld hairy black holes. The data in Table IV represents the mass ratio $\frac{m^{+}_{\mathrm{EBI}}}{m^{-}_{\mathrm{EBI}}}$ with respect to the positive cosmological constants Λ in the case of $M = 1.5$, $Q = 1$ and $b = 2$. We see that the mass ratio increases as we choose a larger cosmological constant with other parameters fixed and the smallest ratio can be obtained in the case of $\Lambda = 0$ or flat space. Since the Hod's lower bound holds in the asymptotically flat Einstein-Born-Infeld hairy black holes, we can conclude that the Hod's lower bound also holds in the asymptotically dS Einstein-Born-Infeld hairy black holes.

However, we find that the Hod's lower bound $\frac{m^{+}_{\mathrm{hair}}}{m^{-}_{\mathrm{hair}}} \geq 1$ can be invaded in the AdS background. With $M = 1.5$, $Q = 1$, $b = 2$ and different values of Λ, we find that the hair mass ratio is below the Hod's lower bound for very negative cosmological constants as can be seen in Table V.

We have further checked for the whole four-dimensional parameter space in a very large range and the properties are qualitatively the same as cases in Tables I–V. In summary, we show that the Hod's bound holds in Einstein-Born-Infeld hairy black holes with non-negative cosmological constants. In contrast, the Hod's bound can be invaded in the asymptotically AdS Einstein-Born-Infeld hairy black holes. Our results imply that the more negative cosmological constant makes the Born-Infeld hair more easier to condense in the near horizon region. Moreover, we conjecture that the Hod's bound may be also invaded in other AdS hairy black holes. For AdS black holes, a potential well in the near horizon region forms due to the AdS boundary [48], which provides the confinement of the scalar field and may make the scalar hair easier to condense in the near horizon well [49]. Another possible method to confine the scalar hair in the near horizon region is enclosing the black hole in a scalar reflecting box [50–54]. We also mention that Skyrme hairs with cosmological constants have been studied [55–57]. We plan to examine effects of cosmological constants on scalar hair and Skyrme hair distributions in the next work. Moreover, there is no scalar hair theorem in regular neutral reflecting stars [58,59] and static scalar fields can condense around charged reflecting stars [60–68]. So it is also very interesting to extend the discussion to the reflecting star background.

IV. CONCLUSIONS

We studied mass distribution of linear hair in RN black holes and genuine hair in Einstein-Born-Infeld theory with various cosmological constants. We used the event horizon and the photonsphere to divide the hair into two parts and obtained lower bounds for the mass ratio. We found that the Hod's lower bound obtained in asymptotically flat gravity also holds in the asymptotically dS Einstein-Born-Infeld hairy black holes. In contrast, the Hod's lower bound can be invaded in the asymptotically AdS Einstein-Born-Infeld hairy black holes. Our results showed that the more negative cosmological constants make the Born-Infeld hair easier to condense in the near horizon area. We further conjectured that effects of cosmological constants on hair distribution may be qualitatively the same in other hairy black holes.

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