

**Raychaudhuri equation in spacetimes with torsion and nonmetricity**Damianos Iosifidis,<sup>1</sup> Christos G. Tsagas,<sup>2,3</sup> and Anastasios C. Petkou<sup>1</sup><sup>1</sup>*Institute of Theoretical Physics Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece*<sup>2</sup>*Section of Astrophysics, Astronomy and Mechanics Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece*<sup>3</sup>*DAMTP, Centre for Mathematical Sciences, University of Cambridge Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

(Received 13 September 2018; published 26 November 2018)

We introduce and develop the  $1 + 3$  covariant approach to relativity and cosmology to spacetimes of arbitrary dimensions that have torsion and do not satisfy the metricity condition. Focusing on timelike observers, we identify and discuss the main differences between their kinematics and those of their counterparts in standard Riemannian spacetimes. At the center of our analysis lies the Raychaudhuri equation, which is the fundamental formula monitoring the convergence and divergence, namely the collapse and expansion, of timelike congruences. To the best of our knowledge, we provide the most general expression so far of the Raychaudhuri equation, with applications to an extensive range of nonstandard astrophysical and cosmological studies. Assuming that metricity holds, but allowing for nonzero torsion, we recover the results of analogous previous treatments. Focusing on nonmetricity alone, we identify a host of effects that depend on the nature of the timelike congruence and on the type of the adopted nonmetricity. We also demonstrate that in highly symmetric spaces one can recover the pure-torsion results from their pure nonmetricity analogues, and vice versa, via a simple ansatz between torsion and nonmetricity.

DOI: [10.1103/PhysRevD.98.104037](https://doi.org/10.1103/PhysRevD.98.104037)**I. INTRODUCTION**

Given a manifold of arbitrary dimensions, one can measure distances between points and angles between vectors once a metric has been introduced. On the other hand, for the parallel transport of vector and tensor fields on a manifold, a connection is needed. In general, these two spacetime features, namely the metric and the connection, do not need to be related, and (for the time being at least) there is no fundamental reason for them to be related apart from simplicity. In classical general relativity, however, the metric and the (Levi-Civita) connection are related to each other, with the latter been expressed in terms of the former and its derivatives. More specifically, one arrives at the aforementioned relation after assuming that the metric is covariantly constant (also known as the metricity condition) and that the connection is symmetric (also known as the torsionless condition). Even though these two assumptions greatly simplify any theoretical analysis, we are not as yet aware of any fundamental mathematical, or physical, reason for selecting the Levi-Civita connection. The effort to identify alternative connections dates back to the work Weyl and Cartan towards the beginning of the last century [1]—see also [2]. More specifically, Weyl considered torsionless spaces with nonmetricity in an attempt to unify gravity with electromagnetism, whereas Cartan considered spaces with torsion. In the literature, the study of non-

Riemannian contributions to gravity, is typically referred to as “metric-affine gravity” [3].

Motivated by the above, we extend the  $1 + 3$  covariant approach to general relativity and cosmology (see [4] for recent extensive reviews) to  $n$ -dimensional spacetimes that have nonzero torsion and do not satisfy the metricity condition. Our aim is to “exploit” the mathematical compactness and the geometrical and physical transparency of the covariant formalism in the ongoing quest for a deeper insight into these most general spacetimes. Torsion and nonmetricity introduce new features to their host spaces. Among others, nonzero torsion implies that the Ricci curvature tensor and the matter energy-momentum tensor are no longer necessarily symmetric. This asymmetry could be seen as a generic spacetime feature, but it may also reflect the nonzero spin of its material content. Nonmetricity, on the other hand, means that vectors and tensors do not maintain the same magnitude, as they are (parallelly) transported from one spacetime event to the next. As a result, the concepts of “proper length” and “proper time” lose their conventional meaning when the metricity condition is violated. In view of these complications, in the first three chapters of this work, we identify the key differences between our analysis and the standard treatments and also lay the foundations for extending the  $1 + 3$  formalism to general spacetimes with arbitrary dimensions, nonzero torsion and nonmetricity.

At the center of our study lies the Raychaudhuri equation, which has long been used to describe the mean kinematics of self-gravitating media (e.g., see [5]). In particular, Raychaudhuri's formula has been at the core of the gravitational collapse studies and the related singularity theorems. Also, alternative versions of the same equation are currently used in cosmology in search of an answer to the question posed by the recent universal acceleration. Here, we provide the most general (to the best of our knowledge) version of the Raychaudhuri equation, with no prior assumptions on the nature of the underlying gravitational theory. This ensures that our formula can be readily applied to a wide range of standard and nonstandard astrophysical and cosmological problems. Assuming that metricity holds, but allowing for nonzero torsion, we find perfect agreement with the earlier 1 + 3 study of [6]. On the other hand, switching the torsion off and turning the nonmetricity on reveals a rather intriguing resemblance between some (at least) of the torsion and nonmetricity effects. Motivated by this observation, as well as by analogous reports in the literature, we consider separately the simple cases of irrotational and shear-free autoparallel congruences residing in empty (i.e., Ricci-flat) spacetimes. In the first instance, we assume nonzero torsion with metricity, while in the second we have nonmetricity without torsion. Solving the Raychaudhuri equation in either case, we arrive at formally identical solutions. In particular, the pure-torsion solution can be recovered from its pure nonmetricity counterpart (and vice versa) after imposing a simple ansatz between these two spacetime features. We interpret this as clear demonstration of the so-called *duality* between torsion and nonmetricity (e.g., see also [7]), which in spaces of high symmetry seems able to make the two theories phenomenologically identical.

## II. SPACES WITH TORSION AND NONMETRICITY

Torsion and nonmetricity modify the familiar Riemannian relations between the metric tensor, the connection, and the curvature of the space. Here, we will briefly outline the main differences referring the reader to related reviews (e.g., see [3]) for further discussion.

### A. Torsion and nonmetricity tensors

In the presence of torsion, the connection of the space is generally asymmetric (i.e.,  $\Gamma^\mu_{\nu\lambda} \neq \Gamma^\mu_{(\nu\lambda)}$ ), with its anti-symmetric component giving the Cartan torsion tensor

$$S_{\mu\nu}{}^\lambda = \Gamma^\lambda_{[\mu\nu]}, \quad (1)$$

so that  $S_{\mu\nu}{}^\lambda = S_{[\mu\nu]}{}^\lambda$ .<sup>1</sup> At the same time, the metric is not covariantly conserved and the failure of the connection to do so is measured by the nonmetricity tensor

<sup>1</sup>Round brackets denote symmetrization, while square ones indicate antisymmetrization.

$$Q_{\lambda\mu\nu} = -\nabla_\lambda g_{\mu\nu}, \quad (2)$$

ensuring that  $Q_{\lambda\mu\nu} = Q_{\lambda(\mu\nu)}$ .<sup>2</sup> The geometrical effect of torsion is that the parallel transport of a pair of vectors, along each other's direction, does not lead to a closed parallelogram. Nonmetricity, on the other hand implies that the lengths of vectors are not preserved when they are parallelly transported in space.

Starting from the tensors defined above, one can construct two pairs of associated vectors. In particular, the torsion tensor leads to

$$S_\mu = S_{\mu\nu\lambda} g^{\nu\lambda} = S_{\mu\nu}{}^\nu \quad \text{and} \quad \tilde{S}_\mu = \varepsilon_{\mu\nu\lambda\beta} S^{\nu\lambda\beta}, \quad (3)$$

where  $\varepsilon_{\mu\nu\lambda\beta}$  is the associated alternating tensor (with  $\varepsilon_{\mu\nu\lambda\beta} = \varepsilon_{[\mu\nu\lambda\beta]}$ ). The former of these is the familiar torsion vector, while here we will refer to  $\tilde{S}_\mu$  as the torsion pseudovector. The latter vanishes in highly symmetric spacetimes, like those associated with the familiar Friedmann-Robertson-Walker (FRW) models, because it leads to parity violation. For the nonmetricity tensor, on the other hand, the related vectors are

$$Q_\mu = Q_{\mu\nu\lambda} g^{\nu\lambda} = Q_{\mu\nu}{}^\nu \quad \text{and} \quad \tilde{Q}_\mu = g^{\lambda\nu} Q_{\lambda\nu\mu} = Q^\nu{}_{\nu\mu}, \quad (4)$$

with  $Q_\mu$  representing the so-called Weyl vector. From here on, we will refer to  $\tilde{Q}_\mu$  as the second nonmetricity vector.

### B. Special types of torsion and nonmetricity

The simplest types of torsion and nonmetricity are of vector form. So, in an  $n$ -dimensional space, the associated torsion and nonmetricity tensors read

$$S_{\mu\nu}{}^\lambda = \frac{2}{n-1} S_{[\mu} \delta_{\nu]}{}^\lambda \quad \text{and} \quad Q_{\mu\nu\lambda} = \frac{1}{n} Q_\mu g_{\nu\lambda}, \quad (5)$$

respectively. Therefore, the torsion field is determined by the torsion vector ( $S_\mu$ ) and the nonmetricity by the Weyl vector ( $Q_\mu$ ), in which case we are dealing with the so-called Weyl nonmetricity. An additional interesting type of nonmetricity is one that allows for fixed-length vectors, in which case the nonmetricity tensor satisfies the constraint

$$Q_{\mu\nu\lambda} = v_\mu g_{\nu\lambda} - g_{\mu(\nu} v_{\lambda)}, \quad (6)$$

<sup>2</sup>Nonmetricity implies that raising and lowering the indices are no longer trivial operations when covariant differentiation is involved. For instance, starting from (2), one can show that  $\nabla^\lambda g^{\mu\nu} = Q^{\lambda\mu\nu}$ . Also note that  $\nabla^\mu = g^{\mu\nu} \nabla_\nu$  defines the contravariant counterpart of the covariant derivative operator.

where  $v_\mu$  is an arbitrary vector field. In what follows, we will first consider the implications of torsion and nonmetricity for the mean kinematics (i.e., for the volume expansion or contraction scalar—see Sec. IVB below) of the host spacetime, without imposing any restrictions on either of these two geometrical features. Then, we will apply our generalized equations to some of the specific forms of torsion and nonmetricity given in this section.<sup>3</sup>

### C. Curvature

As in conventional Riemannian geometry, the curvature of a space with torsion and nonmetricity reflects the fact that the covariant differentiation is not a commutative operation. This is manifested in the Ricci identity, which applied to the contravariant vector  $u^\mu$  reads

$$2\nabla_{[\mu}\nabla_{\nu]}u^\lambda = R^\lambda{}_{\beta\mu\nu}u^\beta + 2S_{\mu\nu}{}^\beta\nabla_\beta u^\lambda, \quad (7)$$

where  $R^\mu{}_{\nu\lambda\beta}$  is the curvature tensor of the space given by

$$R^\mu{}_{\nu\lambda\beta} = 2\partial_{[\lambda}\Gamma^\mu{}_{\nu]\beta} + 2\Gamma^\mu{}_{\alpha[\lambda}\Gamma^\alpha{}_{\nu]\beta}. \quad (8)$$

The above has only one symmetry, namely  $R_{\mu\nu\lambda\beta} = -R_{\mu\nu\beta\lambda}$ , in contrast to its purely Riemannian counterpart (i.e., to the Riemann curvature tensor itself).

The reduced symmetries of the curvature tensor ensure that there are three independent contractions, namely,

$$\tilde{R}_{\mu\nu} = g^{\lambda\beta}R_{\lambda\beta\mu\nu} = R^\lambda{}_{\lambda\mu\nu}, \quad \hat{R}_{\mu\nu} = g^{\lambda\beta}R_{\mu\lambda\beta\nu} = R_\mu{}^\lambda{}_{\lambda\nu}, \quad (9)$$

and

$$R_{\mu\nu} = g^{\lambda\beta}R_{\lambda\mu\beta\nu} = R^\lambda{}_{\mu\lambda\nu}, \quad (10)$$

respectively. The latter provides the familiar Ricci curvature tensor, while the former is usually referred to as the ‘‘homothetic’’ curvature tensor. One additional contraction leads to the Ricci scalar,

$$R = g^{\mu\nu}R_{\mu\nu}, \quad (11)$$

which is uniquely defined (since  $g^{\mu\nu}\tilde{R}_{\mu\nu} = 0$  and  $g^{\mu\nu}\hat{R}_{\mu\nu} = -R$ ).

An important relation is obtained by applying the Ricci identity to the metric tensor of the host space. Assuming that the latter is  $n$ -dimensional with torsion and nonmetricity, in addition to curvature, we arrive at

<sup>3</sup>An additional simple form of torsion has  $S_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\beta}\tilde{S}^\beta/3!$ , where  $\epsilon_{\mu\nu\lambda\beta}$  is the four-dimensional Levi-Civita tensor. Unlike vectorial torsion, however, this last form of torsion vanishes identically in spatially homogeneous and isotropic (FRW-type) cosmologies. Note that the latter spacetimes can naturally accommodate both the Weyl and the fixed-length forms of nonmetricity (see Sec. VI below).

$$2\nabla_{[\mu}\nabla_{\nu]}g_{\lambda\beta} = -2R_{(\lambda\beta)\mu\nu} + 2S_{\mu\nu}{}^\kappa\nabla_\kappa g_{\lambda\beta}. \quad (12)$$

Expanding this and then using definition (2) leads to

$$R_{(\mu\nu)\lambda\beta} = \nabla_{[\lambda}Q_{\beta]\mu\nu} - S_{\lambda\beta}{}^\kappa Q_{\kappa\mu\nu}, \quad (13)$$

which relates the curvature tensor with the torsion and the nonmetricity tensors of the space.

## III. SPACETIME SPLITTING

The 1 + 3 covariant approach to relativity and cosmology decomposes the four-dimensional spacetime into one temporal and three spatial dimensions, while it utilizes the Bianchi and the Ricci identities rather than the metric [4]. Over the years, this formalism has been extended to higher dimensions, and to spacetimes with nonzero torsion, but (to the best of our knowledge) it has never been applied to spaces where the metricity condition no longer applies (i.e., when  $\nabla_c g_{ab} \neq 0$ ). In what follows, we will attempt to take the first step in that direction.

### A. The timelike observes

In an  $n$ -dimensional spacetime, suppose that  $u^\mu$ , with  $u^\mu = dx^\mu/d\lambda$ , is the  $n$ -velocity vector tangent to a congruence of timelike curves. The latter also define the worldlines of a family of observers, known as the fundamental observers. In the absence of metricity, the magnitude of the  $n$ -velocity vector is no longer preserved and for this reason it cannot be normalized to  $-1$  (or in any other way). We may, therefore, write

$$u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu = -\ell^2 \equiv -\phi(x^\alpha), \quad (14)$$

where  $\phi(x^\alpha)$  is generally a function of both space and time.<sup>4</sup> As we will demonstrate throughout the rest of this manuscript, the spacetime dependence seen in Eq. (14) marks the starting point of a series of technical and conceptual differences between metric and nonmetric cosmologies. To begin with, the affine parameter  $\lambda$  does not necessarily coincide with the proper time ( $\tau$ ) measured along the observers’ timelike curves. In particular, setting  $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$ , applying the chain-rule of differentiation, and employing (14), we arrive at

$$\frac{d\tau}{d\lambda} = \pm\ell, \quad (15)$$

with  $\ell = \ell(x^\alpha)$  due to the nonmetricity of the spacetime. The above integrates to give the (nontrivial) relation  $\tau = \pm \int \ell(x^\alpha)d\lambda + \mathcal{C}$  between the proper time measured along a timelike worldline and any affine parameter of that curve.

<sup>4</sup>Greek indices takes values from 0 to  $n - 1$  and Latin indices run from 1 to  $n - 1$  throughout this article.

Therefore, here onwards, we will use overdots to indicate differentiation with respect to the affine parameter (i.e.,  $\dot{\phantom{x}} = d/d\lambda$ ) and primes to denote derivatives in terms of proper time (i.e.,  $\prime = d/d\tau$ ).<sup>5</sup>

### B. Temporal and spatial derivatives

The nonmetricity of the host spacetime also affects the (spatial) hypersurfaces orthogonal to the timelike  $u^\mu$ -field (e.g., see [4] for a comparison). More specifically, the associated projection tensor is now given by

$$h_{\mu\nu} = g_{\mu\nu} + \frac{1}{\ell^2} u_\mu u_\nu, \quad (16)$$

recalling that  $g_{\mu\nu} g^{\mu\nu} = \delta_\mu^\mu = n$ . The above guarantees that  $h_{\mu\nu} = h_{\nu\mu}$ , that  $h_{\mu\nu} u^\mu = 0$  and that  $h_{\mu\nu} h^{\mu\nu} = n - 1$ . In addition, following definition (16), we obtain

$$h_{\mu\lambda} h^{\lambda\nu} = h_\mu^\nu = \delta_\mu^\nu + \frac{1}{\ell^2} u_\mu u^\nu. \quad (17)$$

Overall, the timelike  $n$ -velocity field and the projector defined above, introduce an  $1 + (n - 1)$  splitting of the spacetime into one temporal direction and  $n - 1$  spatial counterparts. We may therefore define the temporal and spatial derivatives of a general tensor field  $T_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_m}$  as

$$\dot{T}_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_m} = u^\mu \nabla_\mu T_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_m} \quad (18)$$

and

$$D_\mu T_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_m} = h_\mu^\lambda h_{\alpha_1}^{\gamma_1} \dots h_{\alpha_n}^{\gamma_n} h_{\delta_1}^{\beta_1} \dots h_{\delta_m}^{\beta_m} \nabla_\lambda T_{\gamma_1 \dots \gamma_n}^{\delta_1 \dots \delta_m}, \quad (19)$$

respectively. On using the above, every spacetime variable, equation, and operator can be decomposed into their temporal and spatial components.

## IV. KINEMATICS

Torsion and nonmetricity complicate considerably the kinematics of the timelike observers introduced in the previous section. For example, some of the standard kinematic variables are no longer uniquely defined. Here, we will attempt to address these issues and also set up the mathematical formalism that will be used in the rest of this study.

### A. Path and hyper $n$ -acceleration

The fact that the metric tensor is not covariantly conserved (i.e., nonmetricity) means that the processes of covariant differentiation and of index raising and lowering

are not commutative. This, in turn, implies that there are two different  $n$ -acceleration vectors, namely a contravariant and a covariant one, defined by

$$A^\mu \equiv \dot{u}^\mu \equiv u^\lambda \nabla_\lambda u^\mu \quad (20)$$

and

$$a_\mu \equiv \dot{u}_\mu \equiv u^\lambda \nabla_\lambda u_\mu, \quad (21)$$

respectively. Given that  $\nabla^\lambda g^{\mu\nu} = Q^{\lambda\mu\nu}$ , with  $Q^{\lambda\mu\nu}$  representing the nonmetricity of the host spacetime, we deduce that  $A^\mu \neq g^{\mu\nu} a_\nu$ . More specifically, definitions (20) and (21) ensure that

$$A^\mu = a^\mu + Q^{\nu\lambda\mu} u_\nu u_\lambda, \quad (22)$$

in direct contrast to metric spacetimes where  $A^\mu = a^\mu$  [4]. It is then imperative to distinguish between these two types of  $n$ -acceleration. So, hereafter, we will name the  $A^\mu$  path  $n$ -acceleration, since it vanishes along autoparallel trajectories or paths, while we will refer to  $a_\mu$  as the hyper  $n$ -acceleration, because it remains nonzero on autoparallel curves. In particular, Eq. (22) ensures that  $a_\mu = -Q_{\lambda\mu\nu} u^\lambda u^\nu \neq 0$  when  $A_\mu = 0$ .<sup>6</sup>

An additional key difference between metric and non-metric spacetimes is that none of the two  $n$ -acceleration vectors defined above is normal to their associated  $n$ -velocity vector. Indeed, given that  $u_\mu u^\mu = -\ell^2$ , with  $\ell = \ell(x^\alpha)$ , differentiating in terms of the affine parameter ( $\lambda$ —see Sec. III B before) leads to

$$A^\mu u_\mu = -\frac{1}{2} (\ell^2) \cdot + \frac{1}{2} Q_{\mu\nu\lambda} u^\mu u^\nu u^\lambda. \quad (23)$$

Similarly, recalling that  $\nabla_\lambda g_{\mu\nu} = -Q_{\lambda\mu\nu}$ , we arrive at

$$a^\mu u_\mu = -\frac{1}{2} (\ell^2) \cdot - \frac{1}{2} Q_{\mu\nu\lambda} u^\mu u^\nu u^\lambda. \quad (24)$$

The last two relations combine to give

$$(A^\mu + a^\mu) u_\mu = -(\ell^2) \cdot \quad (25)$$

and

$$(A^\mu - a^\mu) u_\mu = Q_{\mu\nu\lambda} u^\mu u^\nu u^\lambda. \quad (26)$$

Finally, we should note that in the case of autoparallel “motion” (i.e., when  $A_\mu = 0$ ), expressions (25) and (26) guarantee that  $\ell^2 = \int Q_{\mu\nu\beta} u^\mu u^\nu u^\beta d\lambda + C$ .

<sup>5</sup>The coordinate time ( $x^0$ ) measured by a comoving observers (those with  $u^a = 0$ ) relates to the affine parameter of their timelike worldlines by means of  $dx^0/d\lambda = \pm \ell / \sqrt{-g_{00}}$ . Then, setting  $g_{00} = -1$  and using Eq. (15) we deduce that  $dx^0 = d\tau$ . In other words, proper and coordinate time still coincide for comoving observers.

<sup>6</sup>By definition, autoparallel curves have zero path acceleration, that is  $A^\mu = \dot{u}^\mu = u^\lambda \nabla_\lambda u^\mu = 0$ . Autoparallel and geodesic trajectories coincide in Riemannian spaces, equipped with the Levi-Civita connection (i.e., when  $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$ ), but not in the presence of torsion (i.e., when  $\Gamma^\lambda_{[\mu\nu]} \neq 0$ ), or nonmetricity (i.e., when  $\nabla_\lambda g_{\mu\nu} \neq 0$ ).

### B. Volume scalar, shear and vorticity tensors

The irreducible kinematics of the  $u_a$ -field are determined by decomposing the associated covariant derivative into its temporal and spatial components, according to

$$\nabla_\nu u_\mu = D_\nu u_\mu - \frac{1}{\ell^2}(u_\mu \xi_\nu + a_\mu u_\nu) - \frac{1}{\ell^4}(u^\lambda a_\lambda)u_\mu u_\nu, \quad (27)$$

where  $D_\nu u_\mu = h_\nu^\beta h_\mu^\lambda \nabla_\beta u_\lambda$  [see definition (19)]. Also,  $\xi_\mu = u^\nu \nabla_\mu u_\nu$  by definition with  $\xi_\mu u^\mu = a_\mu u^\mu$  by construction.<sup>7</sup> Moreover, the projected covariant derivative decomposes further into

$$D_\nu u_\mu = \frac{1}{n-1} \left( \Theta + \frac{1}{\ell^2} a_\lambda u^\lambda \right) h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (28)$$

with

$$\Theta = g^{\mu\nu} \nabla_\nu u_\mu = D^\mu u_\mu - \frac{1}{\ell^2} a_\mu u^\mu, \quad (29)$$

representing a uniquely defined ‘‘volume’’ scalar (where  $D^\mu u_\mu = h^{\mu\nu} \nabla_\nu u_\mu$ ).<sup>8</sup> When the latter is positive, the curves tangent to the  $u_\mu$ -field move apart and we have expansion. In the opposite case, on the other hand, the curves approach each other and there is contraction. Also, the variables

$$\sigma_{\mu\nu} = D_{\langle\nu} u_{\mu\rangle} \quad \text{and} \quad \omega_{\mu\nu} = D_{[\nu} u_{\mu]}, \quad (31)$$

define the shear tensor and the vorticity tensor, respectively.<sup>9</sup> The former monitors kinematic anisotropies, namely ‘‘shape’’ distortions under constant ‘‘volume,’’ while a nonzero vorticity implies that the  $u_\mu$ -field rotates.<sup>10</sup> Note that, by construction,  $\sigma_\mu{}^\mu = 0 = \omega_\mu{}^\mu$  and  $\sigma_{\mu\nu} u^\nu = 0 = \omega_{\mu\nu} u^\nu$ . In other words, both the shear and the vorticity ‘‘live’’ in the observers  $(n-1)$ -dimensional rest space. Finally, expressions (27) and (28), combine to the following decomposition,

<sup>7</sup>By construction  $\xi_\mu = u^\nu \nabla_\mu u_\nu$  and  $\xi^\mu = u^\nu \nabla^\mu u_\nu$ . Note, however, that nonmetricity guarantees that  $u^\nu \nabla_\mu u_\nu \neq u_\nu \nabla_\mu u^\nu$  in general, since  $u^\nu \nabla_\mu u_\nu - u_\nu \nabla_\mu u^\nu = -Q_{\mu\nu\lambda} u^\nu u^\lambda$ .

<sup>8</sup>In the absence of metricity  $D^\mu u_\mu \neq D_\mu u^\mu$ , which implies that the ‘‘spatial’’ divergence of the  $u_\mu$ -field is not uniquely defined. More specifically, using definitions (2), (4b), (20), and (21), recalling that  $Q_{\mu\nu\lambda} = Q_{\mu(\nu\lambda)}$  and employing the auxiliary relation (26), we find that

$$D^\mu u_\mu = D_\mu u^\mu - \frac{1}{\ell^2} Q_{\mu\nu\lambda} u^\mu u^\nu u^\lambda - \tilde{Q}_\mu u^\mu. \quad (30)$$

The above explain/justify our choice for defining  $\Theta = g^{\mu\nu} \nabla_\nu u_\mu$  as our volume scalar. We should also point out that  $\Theta \neq \nabla^\mu u_\mu$ , since the divergence of the  $n$ -velocity is also not uniquely defined (i.e.,  $\nabla^\mu u_\mu \neq \tilde{\nabla}_\mu u^\mu$ ).

<sup>9</sup>Angled brackets indicate the symmetric and trace-free part of a second-rank tensor. For instance, the shear tensor is constructed as  $\sigma_{\mu\nu} = h_{(\nu}^\beta h_{\mu)}^\lambda \nabla_\beta u_\lambda - [\Theta/(n-1)]h_{\mu\nu}$ .

<sup>10</sup>Each of the three kinematic variables splits in its Riemannian and non-Riemannian parts. For instance the expansion scalar decomposes as  $\Theta = \tilde{\Theta} + (\tilde{Q}_\mu - Q_\mu/2 - 2S_\mu)u^\mu$ , with  $\tilde{\Theta}$  representing the Riemannian component.

$$\begin{aligned} \nabla_\nu u_\mu &= \frac{1}{n-1} \left( \Theta + \frac{1}{\ell^2} a_\lambda u^\lambda \right) h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \\ &\quad - \frac{1}{\ell^2} (u_\mu \xi_\nu + a_\mu u_\nu) - \frac{1}{\ell^4} (u^\lambda a_\lambda) u_\mu u_\nu, \end{aligned} \quad (32)$$

of the covariant form  $(\nabla_\nu u_\mu)$  of the  $n$ -velocity gradient into the irreducible kinematic variables of the motion.

Given that  $g^{\nu\beta} g^{\mu\lambda} \nabla_\beta u_\lambda = \nabla^\nu (g^{\mu\lambda} u_\lambda) - u_\lambda \nabla^\nu g^{\mu\lambda}$  and recalling that  $\nabla^\lambda g^{\mu\nu} = Q^{\lambda\mu\nu}$ —see footnote 2 in Sec. II A), one can show that the contravariant form  $(\nabla^\nu u^\mu)$  of the velocity gradient accepts the following irreducible decomposition

$$\begin{aligned} \nabla^\nu u^\mu &= \frac{1}{n-1} \left( \Theta + \frac{1}{\ell^2} a_\lambda u^\lambda \right) h^{\mu\nu} + \sigma^{\mu\nu} + \omega^{\mu\nu} \\ &\quad - \frac{1}{\ell^2} (u^\mu \xi^\nu + a^\mu u^\nu) - \frac{1}{\ell^4} (a_\lambda u^\lambda) u^\mu u^\nu \\ &\quad + Q^{\nu\mu\lambda} u_\lambda, \end{aligned} \quad (33)$$

where  $\Theta = g^{\mu\nu} \nabla_\nu u_\mu$  as in Eq. (32) above. Also,  $\sigma^{\mu\nu} = g^{\mu\lambda} g^{\nu\beta} \sigma_{\lambda\beta}$  and  $\omega^{\mu\nu} = g^{\mu\lambda} g^{\nu\beta} \omega_{\lambda\beta}$  are the contravariant components of the shear and vorticity tensors, respectively. Note, however, that  $\sigma^{\mu\nu} \neq D^{(\nu} u^{\mu)}$  and  $\omega^{\mu\nu} \neq D^{[\nu} u^{\mu]}$  due to the nonmetricity of the spacetime.

## V. THE RAYCHAUDHURI EQUATION

The Raychaudhuri equation monitors the expansion, or the contraction, of a self-gravitating medium. It plays a fundamental role both in astrophysics and in cosmology and has been at the center of all the singularity theorems. In what follows, we will provide an expression for Raychaudhuri’s formula in  $n$ -dimensional spaces with torsion and nonmetricity.<sup>11</sup>

### A. Deriving Raychaudhuri’s formula

Raychaudhuri’s formula is purely geometrical by nature and follows from a set of (also purely geometrical) relations, known as the Ricci identities. Applied to the  $n$ -velocity vector  $u_\mu$  defined in Sec. III A, the latter read

$$2\nabla_{[\mu} \nabla_{\nu]} u_\lambda = -R_{\beta\lambda\mu\nu} u^\beta + 2S_{\mu\nu}{}^\beta \nabla_\beta u_\lambda, \quad (34)$$

with  $S_{\mu\nu\lambda}$  representing the torsion tensor and  $R_{\mu\nu\lambda\beta}$  being the curvature tensor of the spacetime (so that  $R_{\mu\nu\lambda\beta} = R_{\mu\nu[\lambda\beta]}$ —see Sec. II A and Sec. II C earlier). Contracting (34) along  $g^{\lambda\nu} u^\mu$  gives

$$\begin{aligned} g^{\lambda\nu} u^\mu (\nabla_\mu \nabla_\nu u_\lambda - \nabla_\nu \nabla_\mu u_\lambda) &= -R_{\beta\lambda\mu\nu} u^\beta u^\mu g^{\lambda\nu} \\ &\quad + 2S_{\mu\nu}{}^{\lambda\nu} u^\mu \nabla_\lambda u_\nu. \end{aligned} \quad (35)$$

<sup>11</sup>Versions of the Raychaudhuri equation in spacetimes with nonzero torsion and/or spin have a fairly long history (e.g., see [8] for a representative list). Here we adopt the formalism developed in [6]. Recently, there was also an attempt to extend Raychaudhuri’s formula to spaces with Weyl geometry [9].

where the velocity gradient  $\nabla_\nu u_\mu$  satisfies decomposition (32). Using the latter, recalling that  $\Theta = g^{\mu\nu}\nabla_\nu u_\mu$  and  $\nabla_\mu Q^{\nu\lambda} = Q_{\mu}{}^{\nu\lambda}$  (see Sec. II A earlier), while employing definition (4a) together with the symmetry property  $Q_{\mu\nu\lambda} = Q_{\mu(\nu\lambda)}$  of the nonmetricity tensor, the first term on the left-hand side of the above evaluates to

$$g^{\lambda\nu}u^\mu\nabla_\mu\nabla_\nu u_\lambda = \dot{\Theta} - \frac{1}{n-1}\left(\Theta + \frac{1}{\ell^2}a_\nu u^\nu\right)Q_\mu u^\mu - \frac{1}{\ell^2(n-1)}\left(\Theta - \frac{n-2}{\ell^2}a_\beta u^\beta\right)Q_{\mu\nu\lambda}u^\mu u^\nu u^\lambda + \frac{1}{\ell^2}Q_{\mu\nu\lambda}u^\mu u^\nu (a^\lambda + \xi^\lambda) - Q_{\mu\nu\lambda}u^\mu \sigma^{\nu\lambda}. \quad (36)$$

Employing decompositions (32) and (33), while keeping in mind that  $h_{\mu\nu}h^{\mu\nu} = n-1$ , that  $h_{\mu\nu}u^\nu = 0 = \sigma_{\mu\nu}u^\nu = \omega_{\mu\nu}u^\nu$ , that  $\sigma_{\mu\nu}h^{\mu\nu} = 0 = \omega_{\mu\nu}h^{\mu\nu} = \sigma_{\mu\nu}\omega^{\mu\nu}$ , that  $\xi_\mu u^\mu = a_\mu u^\mu$  and also using definition (4b), the second term on the left-hand side of (35) becomes

$$g^{\lambda\nu}u^\mu\nabla_\nu\nabla_\mu u_\lambda = -\frac{1}{n-1}\Theta^2 - 2(\sigma^2 - \omega^2) + D^\mu a_\mu + \frac{1}{\ell^2}a_\mu A^\mu - \frac{1}{\ell^2}(a_\mu u^\mu)^\cdot - \frac{2\Theta}{\ell^2(n-1)}a_\mu u^\mu + \frac{n-2}{\ell^4(n-1)}(a_\mu u^\mu)^2 + \frac{2}{\ell^2}a_\mu \xi^\mu - \frac{1}{n-1}\left(\Theta + \frac{1}{\ell^2}a_\beta u^\beta\right)\tilde{Q}_\mu u^\mu - \frac{1}{\ell^2(n-1)}\left(\Theta - \frac{n-2}{\ell^2}a_\beta u^\beta\right)Q_{\mu\nu\lambda}u^\mu u^\nu u^\lambda - Q_{\mu\nu\lambda}(\sigma^{\mu\nu} + \omega^{\mu\nu})u^\lambda + \frac{1}{\ell^2}Q_{\mu\nu\lambda}(u^\mu \xi^\nu + a^\mu u^\nu)u^\lambda. \quad (37)$$

Note that the scalars  $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}/2$  and  $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}/2$  measure the magnitude of the shear and the vorticity tensors, respectively.<sup>12</sup>

Let us now turn our attention to the right-hand side of Eq. (35). Starting from relation (13) that was obtained in Sec. II C earlier, while recalling that  $Q_{\mu\nu\lambda} = Q_{\mu(\nu\lambda)}$  and  $S_{\mu\nu\lambda} = S_{[\mu\nu]\lambda}$ , the first term on the right-hand side of expression (35) reads

$$R_{\beta\lambda\mu\nu}u^\beta u^\mu g^{\lambda\nu} = R_{\mu\nu}u^\mu u^\nu + \dot{\tilde{Q}}_\mu u^\mu - u^\mu u^\nu \nabla^\lambda Q_{\mu\nu\lambda} - Q_{\mu}{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu - 2S_{\mu}{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu, \quad (39)$$

with  $R_{\mu\nu} = g^{\lambda\beta}R_{\lambda\mu\beta\nu}$  defining the Ricci curvature tensor. In addition, substituting decomposition (32) and putting

together definition (3a) and the symmetry property  $S_{\mu\nu\lambda} = S_{[\mu\nu]\lambda}$  of the torsion tensor, the second term on the right-hand side of (35) recasts into

$$S_{\mu}{}^{\nu\lambda}u^\mu\nabla_\lambda u_\nu = \frac{1}{n-1}\left(\Theta + \frac{1}{\ell^2}a_\nu u^\nu\right)S_\mu u^\mu + S_{\mu\nu\lambda}u^\mu(\sigma^{\nu\lambda} + \omega^{\nu\lambda}) + \frac{1}{\ell^2}S_{\mu\nu\lambda}a^\mu u^\nu u^\lambda. \quad (40)$$

Finally, combining the intermediate relations (36), (37), (39), and (40), we obtain the generalization of the Raychaudhuri equation to  $n$ -dimensional spaces with torsion and nonmetricity, in addition to curvature, namely,

$$\begin{aligned} \dot{\Theta} = & -\frac{1}{n-1}\Theta^2 - R_{\mu\nu}u^\mu u^\nu - 2(\sigma^2 - \omega^2) + D^\mu a_\mu + \frac{1}{\ell^2}a_\mu A^\mu + \frac{2}{n-1}\left(\Theta + \frac{1}{\ell^2}a_\nu u^\nu\right)S_\mu u^\mu \\ & + 2S_{\mu\nu\lambda}u^\mu(\sigma^{\nu\lambda} + \omega^{\nu\lambda}) + \frac{2}{\ell^2}S_{\mu\nu\lambda}a^\mu u^\nu u^\lambda - \frac{1}{\ell^2}(a_\mu u^\mu)^\cdot - \frac{2\Theta}{\ell^2(n-1)}a_\mu u^\mu + \frac{n-2}{\ell^4(n-1)}(a_\mu u^\mu)^2 \\ & + \frac{2}{\ell^2}a_\mu \xi^\mu - \dot{\tilde{Q}}_\mu u^\mu + \frac{1}{n-1}\left(\Theta + \frac{1}{\ell^2}a_\nu u^\nu\right)(Q_\mu - \tilde{Q}_\mu)u^\mu - Q_{\mu\nu\lambda}(\sigma^{\mu\nu} + \omega^{\mu\nu})u^\lambda - \frac{1}{\ell^2}Q_{\mu\nu\lambda}u^\mu u^\nu (a^\lambda + \xi^\lambda) \\ & + Q_{\mu\nu\lambda}u^\mu \sigma^{\nu\lambda} + \frac{1}{\ell^2}Q_{\mu\nu\lambda}(u^\mu \xi^\nu + a^\mu u^\nu)u^\lambda + u^\mu u^\nu \nabla^\lambda Q_{\mu\nu\lambda} + Q_{\mu}{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu + 2S_{\mu}{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu. \end{aligned} \quad (41)$$

Note that only the first five terms on the right-hand side of the above have Riemannian analogues. More specifically, in the absence of torsion and in the presence of metricity

(i.e., when  $S_{\mu\nu\lambda} \equiv 0 \equiv Q_{\mu\nu\lambda}$ ), the rest of the terms on the right-hand side of (41) vanish identically. Then, setting  $n = 4$ , we recover the standard form of the Raychaudhuri

<sup>12</sup>In deriving expression (37), we have also used the auxiliary relation

$$\nabla^\mu a_\mu = D^\mu a_\mu + \frac{1}{\ell^2}A^\mu a_\mu - \frac{1}{\ell^2}(a_\mu u^\mu)^\cdot. \quad (38)$$

equation (e.g., see [4] and also keep in mind that  $a_\mu \equiv A_\mu$ , with  $a_\mu u^\mu = 0 = A_\mu u^\mu$ , and that  $\xi_\mu \equiv 0$  when metricity holds).

The Raychaudhuri equation derived in this section, as well as its reduced expressions given in the following sections (see Sec. VB and Sec. VC next), is a purely geometrical relation. As yet, no matter sources have been introduced and no assumption has been made about the nature of the gravitational field. One could add physical context to these geometrical expressions by introducing a set of field equations, like the Einstein, or the Einstein-Cartan, equations for example. In principle, Eq. (41) should be compatible with any geometrical theory of gravity.

$$\Theta' = -\frac{1}{n-1}\Theta^2 - R_{\mu\nu}u^\mu u^\nu - 2(\sigma^2 - \omega^2) + D^\mu A_\mu + A^\mu A_\mu + \frac{2}{n-1}\Theta S_\mu u^\mu + 2S_{\mu\nu\lambda}u^\mu(\sigma^{\nu\lambda} + \omega^{\nu\lambda}) + 2S_{\mu\nu\lambda}A^\mu u^\nu u^\lambda, \quad (42)$$

with primes indicating proper-time derivatives (see Sec. III A earlier). Applying the above to a four-dimensional spacetime, one recovers the Raychaudhuri equation of the Riemann-Cartan geometry derived in [6]. Note that, when doing the aforementioned identification, one should also take into account the differences in the definitions of the torsion tensor and of the torsion vector between the two studies.

Following (42), torsion affects the convergence or divergence of a timelike congruence in a variety of ways, which depend on whether these worldlines are geodesics or not, as well as on whether they have nonzero shear or vorticity. The most straightforward effect of torsion propagates via the first term in the second line on the right-hand side of the above. More specifically, torsion enhances or inhibits the expansion or contraction of the worldline congruence depending on the sign

Finally, it is worth stressing that the torsion terms on the right-hand side of Eq. (41) share a certain “resemblance” with some of the nonmetricity terms seen in the same formula. This analogy between torsion and nonmetricity, will become more apparent in Sec. VIA and Sec. VIB below.

## B. The case of pure torsion

When dealing with a  $n$ -dimensional spacetime that has nonzero torsion but satisfies the metricity condition, expression (41) reduces to

of the inner product ( $S_\mu u^\mu$ ) between the torsion vector and the  $n$ -velocity (i.e., on the relative orientation of the two vector fields—see also [6] for further discussion).

As we mentioned in the previous section, Eq. (42) is of a purely geometrical nature, since no matter fields have been introduced yet. In order to investigate the effects of gravity, we need to relate both the Ricci tensor and the torsion tensor to the material component of the spacetime. This can be done by means of, say, the Einstein-Cartan and the Cartan field equations [6].

## C. The case of pure nonmetricity

In the presence of nonmetricity, but in the absence of torsion, expression (41) recasts as

$$\begin{aligned} \dot{\Theta} = & -\frac{1}{n-1}\Theta^2 - R_{\mu\nu}u^\mu u^\nu - 2(\sigma^2 - \omega^2) + D^\mu a_\mu + \frac{1}{\ell^2}a_\mu A^\mu - \frac{1}{\ell^2}(a_\mu u^\mu)' - \frac{2\Theta}{\ell^2(n-1)}a_\mu u^\mu + \frac{n-2}{\ell^4(n-1)}(a_\mu u^\mu)^2 \\ & + \frac{2}{\ell^2}a_\mu \xi^\mu - \dot{Q}_\mu u^\mu + \frac{1}{n-1}\left(\Theta + \frac{1}{\ell^2}a_\nu u^\nu\right)(Q_\mu - \tilde{Q}_\mu)u^\mu - Q_{\mu\nu\lambda}(\sigma^{\mu\nu} + \omega^{\mu\nu})u^\lambda - \frac{1}{\ell^2}Q_{\mu\nu\lambda}u^\mu u^\nu (a^\lambda + \xi^\lambda) \\ & + Q_{\mu\nu\lambda}u^\mu \sigma^{\nu\lambda} + \frac{1}{\ell^2}Q_{\mu\nu\lambda}(u^\mu \xi^\nu + a^\mu u^\nu)u^\lambda + u^\mu u^\nu \nabla^\lambda Q_{\mu\nu\lambda} + Q_\mu{}^{\lambda\beta}Q_{\beta\lambda\nu}u^\mu u^\nu. \end{aligned} \quad (43)$$

Here, in contrast to Eq. (42), the overdot implies differentiation in terms of the affine parameter (i.e., relative to  $\lambda$ —see Sec. III A). According to the above, the implications of nonmetricity for the convergence or divergence of a timelike congruence are multiple and not straightforward to decode. Similarly to the case of pure torsion seen before, the most transparent effects are those depending on the orientation of the nonmetricity vectors and their derivatives (i.e.,  $Q_\mu$ ,  $\tilde{Q}_\mu$  and  $\dot{\tilde{Q}}_\mu$ ) relative to the  $u_\mu$ -field.

Before closing this section, we should point out that the Raychaudhuri formulas given in expressions (41)–(43), are

purely geometrical relations, which acquire physical relevance after the energy-momentum and the hypermomentum tensors are introduced. The former gives rise to spacetime curvature, while the latter leads to both torsion and nonmetricity through the field equations and the Palatini equations, respectively. Also note that the nature of the observers’ worldlines, namely of the curves tangent to the  $n$ -velocity vector  $u_\mu$ , has so far been left unspecified. Assuming, e.g., motion along autoparallel curves the path-acceleration vanishes (i.e.,  $A_\mu = 0$ —see Sec. IVA earlier).

## VI. CHARACTERISTIC CASES

According to Eq. (41), torsion and nonmetricity affect the mean expansion and contraction of the host spacetime in a variety of intricate ways. In this section, we will try to reveal the role of torsion and nonmetricity in some characteristic cases.

### A. Vectorial torsion

The kinematic effects of torsion (and spin) have been investigated primarily within the framework of the Einstein-Cartan theory. Assuming that the metricity condition holds (i.e., setting  $Q_{\mu\nu\lambda} = 0$ ), let us consider the case of vectorial torsion with

$$S_{\mu\nu\lambda} = \frac{2}{n-1} S_{[\mu} g_{\nu]\lambda}, \quad (44)$$

where  $S_\mu = S_{\mu\nu}{}^\nu$  defines the associated torsion vector (e.g., see [10]). In this case, the connection decomposes as  $\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + 2(S_\mu \delta^\lambda{}_\nu - S^\lambda g_{\mu\nu})/(n-1)$ , with  $\tilde{\Gamma}^\lambda{}_{\mu\nu}$  being the Christoffel symbols. Then, the second-last term on the right-hand side of (42) vanishes, while the last one reduces to  $2S_{\mu\nu\lambda} A^\mu u^\nu u^\lambda = -2S_\mu A^\mu/(n-1)$ . As a result, the Raychaudhuri equation recasts into

$$\Theta' = -\frac{1}{n-1} \Theta^2 - R_{\mu\nu} u^\mu u^\nu - 2(\sigma^2 - \omega^2) + D_\mu A^\mu + A_\mu A^\mu + \frac{2}{n-1} \Theta S_\mu u^\mu - \frac{2}{n-1} S_\mu A^\mu. \quad (45)$$

Consequently, the effects of vectorial torsion on the mean expansion or contraction of the host spacetime, depend on the orientation of the torsion vector relative to the observer's velocity and acceleration. In particular, when  $S_\mu$  is purely timelike, we have  $S_\mu A^\mu = 0$  (recall that  $A_\mu u^\mu = 0$  when metricity holds). For purely spacelike torsion vector, on the other hand,  $S_\mu u^\mu = 0$ .

Suppose now that the  $u_\mu$ -field is tangent to a congruence of autoparallel curves in a four-dimensional spacetime (i.e., set  $A_\mu = 0$  and  $n = 4$ ). Assume also a Ricci-flat (i.e., empty) spacetime with homogeneous and isotropic spatial hypersurfaces (i.e., set  $R_{\mu\nu} = 0 = \sigma_{\mu\nu} = \omega_{\mu\nu}$ ). In such an FRW-like environment, expression (45) reduces to

$$\Theta' = -\frac{1}{3} \Theta^2 + \frac{2}{3} \Theta S_\mu u^\mu = -\frac{1}{3} \Theta (\Theta - 2S_\mu u^\mu), \quad (46)$$

while the torsion vector becomes purely timelike (to preserve the isotropy of the 3-space).<sup>13</sup> Therefore, the

<sup>13</sup>In FRW-type models with torsion the associated torsion tensor is conveniently given by the ansatz  $S_{\mu\nu\lambda} = 2\phi u_{[\mu} h_{\nu]\lambda}$ , where  $\phi$  is a scalar function that depends only on time [11]. It is then straightforward to show that  $S_\mu = S_{\mu\nu}{}^\nu = 3\phi u_\mu$ . Note that the aforementioned torsion ansatz is a special case of definition (44).

vectorial torsion increases or decreases the rate of the mean expansion or contraction of a timelike congruence, depending on whether the torsion vector is, respectively, parallel or antiparallel to the  $u_\mu$ -field. We may take a qualitative look by employing the relation  $\Theta = \tilde{\Theta} + 2S_\mu u^\mu$ , where  $\tilde{\Theta}$  represents the purely Riemannian (i.e., the torsionless) counterpart of the expansion or contraction scalar (e.g., see [6]). Recalling that  $\tilde{\Theta}/3 = a'/a$ , with  $a = a(\tau)$  being the associated scale factor, solving the above relation for  $S_\mu u^\mu$  and then substituting the resulting expression into the right-hand side of Eq. (46), we find that  $\Theta'/\Theta = -a'/a$ . The latter integrates immediately to give

$$\Theta = \Theta_0 \left( \frac{a_0}{a} \right), \quad (47)$$

with the zero suffix marking a given initial time. According to the above solution, in an expanding spacetime (with  $\Theta_0 > 0$ ), we find that  $\Theta \rightarrow 0^+$  at late times (i.e., as  $a \rightarrow +\infty$ ). When dealing with contracting models, on the other hand, we have  $\Theta_0 < 0$ . In this case, solution (47) ensures that  $\Theta \rightarrow -\infty$  as  $a \rightarrow 0^+$ . In the former example the expansion comes (asymptotically) to a halt, while in the latter the (autoparallel) worldline congruence focuses at a point.<sup>14</sup>

Not surprisingly, the quantitative effect of vectorial torsion on the mean kinematics of the host spacetime depends on the specific form of the associated torsion vector. We can demonstrate this dependence by solving Eq. (47) for the cosmological scale factor [ $a = a(\tau)$ ]. More specifically, using the result  $a\Theta = a_0\Theta_0 = \text{constant}$  and the splitting  $\Theta = \tilde{\Theta} + 2S_\mu u^\mu$ , of the volume scalar into its purely Riemannian and torsional parts, we arrive at

$$a' + \frac{2}{3} (S_\mu u^\mu) a = a_0 \Theta_0 = C_0. \quad (48)$$

Keeping in mind that the torsion vector is purely timelike due to the spatial symmetry and homogeneity of the Friedmann-like spacetimes, the above accepts the solution

$$a = a(\tau) = e^{-\frac{2}{3} \int S_\mu u^\mu d\tau} \left[ C_1 + C_0 \int e^{\frac{2}{3} \int S_\mu u^\mu d\tau} d\tau \right], \quad (49)$$

where the integration constant  $C_0$  and  $C_1$  are decided by the initial conditions. Therefore, in metric-compatible FRW-type spacetimes with nonzero torsion, the scale factor evolution is decided by the product  $S_\mu u^\mu$ , namely by the orientation of the torsion vector relative to the  $u_\mu$ -field. Interestingly, solution (49) also allows for the exponential

<sup>14</sup>Generally speaking, a singularity in the volume scalar (i.e.,  $\Theta \rightarrow -\infty$ ) means that caustics will develop in the worldline congruence and does not necessarily imply a singularity in the spacetime structure [5].



increase of the scale factor. This can happen, e.g., when the scalar  $S_\mu u^\mu$  equals a negative constant.

### B. Weyl nonmetricity

The Weyl nonmetricity is also of vectorial form, since  $Q_{\mu\nu\lambda} = Q_\mu g_{\nu\lambda}/n$ , where  $Q_\mu = Q_{\mu\nu}{}^\nu$  is the associated Weyl vector (see definition (4a) in Sec. II B earlier). Then,  $\tilde{Q}_\mu = Q^\nu{}_{\nu\mu} = Q_\mu/n$  [see definition (4b)], which means that there is only one independent nonmetricity vector. Also note that the connection splits as  $\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + (2\delta^\lambda{}_{(\mu}Q_{\nu)} - Q^\lambda g_{\mu\nu})/2n$ , with  $\tilde{\Gamma}^\lambda{}_{\mu\nu}$  giving the Christoffel symbols. Therefore, assuming zero torsion, Weyl nonmetricity and confining to autoparallel curves (i.e., setting  $A_a = 0$ ), the Raychaudhuri equation (see (43) in Sec. V C) reduces to

$$\left(\Theta - 2\frac{\dot{\ell}}{\ell}\right)' = -\frac{1}{n-1}\left(\Theta - 2\frac{\dot{\ell}}{\ell}\right)^2 - R_{\mu\nu}u^\mu u^\nu - 2(\sigma^2 - \omega^2). \quad (50)$$

Note that, in deriving the above, we have utilized the relation  $a_\mu = -(Q_\nu u^\nu/n)u_\mu$ , which connects the hyper acceleration to the Weyl vector in the case of autoparallel motion (see Eq. (22) in Sec. IV A). Then, one can immediately obtain the auxiliary results  $a_\mu u^\mu = -2\dot{\ell}/\ell$  and  $Q_\mu u^\mu = -2n\dot{\ell}/\ell$ , which also hold for Weyl nonmetricity and for zero path acceleration. In addition, we have  $\sigma_{\mu\nu}g^{\mu\nu} = \sigma_\mu{}^\mu = 0$  and  $\xi_\mu u^\mu = a_\mu u^\mu$  by construction.

Confining to a four-dimensional spacetime and assuming an autoparallel congruence that is also irrotational and shear free, namely, setting  $n = 4$  and  $\omega = 0 = \sigma$  in Eq. (50), the latter leads to

$$\left(\Theta - 2\frac{\dot{\ell}}{\ell}\right)' + \frac{1}{3}\left(\Theta - 2\frac{\dot{\ell}}{\ell}\right)^2 \leq 0, \quad (51)$$

provided that  $R_{\mu\nu}u^\mu u^\nu \geq 0$ . This last constraint on the Ricci tensor may be seen as the generalization of the familiar “weak energy condition” to spacetimes with (Weyl) nonmetricity. It is then straightforward to show (e.g., see [5] for details) that (51) integrates to

$$\left(\Theta - 2\frac{\dot{\ell}}{\ell}\right)^{-1} \geq \left[\Theta_0 - 2\left(\frac{\dot{\ell}}{\ell}\right)_0\right]^{-1} + \frac{1}{3}\lambda, \quad (52)$$

with the zero suffix marking a given initial affine value. Starting from the above and following [5], we deduce that  $\Theta - 2\dot{\ell}/\ell \rightarrow -\infty$  within finite affine length (i.e., for  $\lambda \leq [\Theta_0 - 2(\dot{\ell}/\ell)_0]/3$ ), assuming that  $\Theta_0 - 2(\dot{\ell}/\ell)_0 < 0$  initially. Put another way, provided that  $\Theta_0 < 2(\dot{\ell}/\ell)_0$ , the volume scalar of the congruence will develop a caustic singularity (i.e.,  $\Theta \rightarrow -\infty$ ), unless  $\dot{\ell}/\ell \rightarrow +\infty$

simultaneously. An interesting deviation from the standard Riemannian studies is that, when  $(\dot{\ell}/\ell)_0 > 0$ , caustic formation seems now possible even for initially expanding congruences, namely for those with  $0 < \Theta_0 < 2(\dot{\ell}/\ell)_0$ .

Before attempting to solve Eq. (50), it helps to decompose the volume scalar into a purely Riemannian component and the nonmetricity contribution. Recalling that  $\Theta = g^{\mu\nu}\nabla_\nu u_\mu$  by definition, we find that  $\Theta = \nabla_\mu u^\mu - Q_\mu u^\mu/n = \nabla_\mu u^\mu + 2\dot{\ell}/\ell$  in the case of Weyl nonmetricity. In addition, we have  $\nabla_\mu u^\mu = \tilde{\nabla}_\mu u^\mu + Q_\mu u^\mu/2 = \partial_\mu u^\mu + \tilde{\Gamma}^\mu{}_{\nu\mu}u^\nu - \dot{\ell}/\ell$ , with the latter equality also holding for Weyl nonmetricity. Combining all the above gives

$$\Theta = \partial_0 \ell + \tilde{\Gamma}^\mu{}_{0\mu} \ell + \frac{\dot{\ell}}{\ell} = \tilde{\Gamma}^\mu{}_{0\mu} \ell + 2\frac{\dot{\ell}}{\ell}, \quad (53)$$

since  $u^\mu = \delta^\mu{}_0 \ell$  and  $\partial_0 \ell = \dot{\ell} = \dot{\ell}/\ell$ . Finally, keeping in mind that  $\tilde{\Gamma}^0{}_{00} = 0$  and  $\tilde{\Gamma}^1{}_{01} = \tilde{\Gamma}^2{}_{02} = \tilde{\Gamma}^3{}_{03} = a'/a$  in a flat FRW spacetime, we arrive at

$$\Theta = 3\frac{\dot{a}}{a} - \frac{\dot{\ell}}{\ell}. \quad (54)$$

Let us now apply Eq. (50) to a congruence of irrotational and shear-free autoparallel curves “living” in a Ricci-flat four-dimensional spacetime. Then, a straightforward integration of the remaining differential equation leads to

$$\Theta - 2\frac{\dot{\ell}}{\ell} = \left(\frac{1}{3}\lambda + C\right)^{-1}, \quad (55)$$

where the integration constant ( $C$ ) depends on the initial conditions. In addition, keeping in mind that  $\Theta = 3\dot{a}/a - \dot{\ell}/\ell$  (see Eq. (54) above), the left-hand side of (55) reads  $[\ln(a/\ell)^3]'$  and we arrive at the following expression

$$a = a(\lambda) = \ell(C_1 + C_2\lambda), \quad (56)$$

for the scale factor in terms of the affine parameter. To proceed further, recall that  $\ell^2 = \int Q_{\mu\nu\lambda}u^\mu u^\nu u^\lambda d\lambda + C$  when dealing with autoparallel curves (see Sec. IV A earlier). Therefore, for Weyl nonmetricity in a four-dimensional spacetime, we find

$$\ell = \ell_0 e^{-\frac{1}{3}\int Q_\mu u^\mu d\lambda}. \quad (57)$$

Furthermore, substituting the above expression into Eq. (15) and integrating leads to

$$\lambda = \pm \frac{1}{\ell_0} \int e^{-\frac{1}{3}\int Q_\mu u^\mu d\lambda}. \quad (58)$$

Finally, on using the auxiliary relations (57) and (58), expression (56) recasts into

$$a = a(\lambda) = e^{-\frac{1}{8} \int Q_\mu u^\mu d\lambda} \left( C_3 + C_4 \int e^{\frac{1}{8} \int Q_\mu u^\mu d\lambda} \right). \quad (59)$$

The above solution provides the scale factor in terms of the affine parameter of an autoparallel congruence of irrotational and shear-free worldlines, which reside in a four-dimensional, Ricci-flat spacetime “equipped” with Weyl nonmetricity. In close analogy with the case of pure (vectorial) torsion (see Sec. VI A before), when the host spacetime is FRW-like, the nonmetricity vector is purely timelike. Also, the scale-factor evolution is decided by the scalar product  $Q_\mu u^\mu$ , that is by the orientation (parallel or antiparallel) of the nonmetricity vector relative to the  $u_\mu$ -field. What is most intriguing, however, is that expression (59) is formally identical to the pure-torsion solution (49). In fact, the two expressions are indistinguishable, provided we make the simple exchange  $Q_\mu \leftrightarrow 16S_\mu/3$  and interchange proper time with the affine parameter in the related integrals.<sup>15</sup> This apparent “duality” between torsion and nonmetricity has been observed and reported in earlier works as well [7]. Here, we see that in highly symmetric (Friedmann-like) spacetimes where only the vector components of torsion and nonmetricity survive, the effects of the aforementioned two geometrical agents are phenomenologically indistinguishable.

## VII. DISCUSSION

Classical general relativity combines theoretical elegance and observational success at the highest level. Nevertheless, modifications and extensions of Einstein’s theory have been proposed and investigated ever since relativity was introduced in the early years of the last century. The motivation behind these efforts are multiple, ranging from the quest for quantum gravity and the existence of singular solutions for key relativistic equations, to the awareness of the intrinsic limitations of the theory and its apparent inability to explain certain observations. Violating the metricity condition and including spacetime torsion have long been suggested as possible ways of “improving” standard general relativity. Technically speaking the noncompatibility of the metric and the asymmetry of the connection imply that the latter is no longer uniquely defined by the former. In other words, the metric and the connection are treated as independent

<sup>15</sup>In analogy with its pure-torsion analogue derived in the previous section, solution (59) also allows for the exponential increase of the scale factor. Similarly to the torsion case seen in Sec. VI A, this could happen when the scalar product  $Q_\mu u^\mu$  equals a negative constant. This result, which requires further scrutiny, seems to support earlier claims made in the literature about the theoretical possibility of a nonmetricity driven inflation [12].

geometrical fields, an approach that is often referred to as the “Palatini formalism”, although a more precise terminology is metric-affine formalism.

Historically speaking, nonmetricity was first introduced to unify gravity with electromagnetism and torsion to incorporate the nonzero spin of the matter into the gravitational field. In the literature there are several suggestions, as well as a debate, on the possibility of experimentally testing torsion [13]. Although less frequent, there is also discussion on potentially measurable effects from nonmetricity [14]. In this work we have considered a generalized spacetime with  $n$  dimensions, nonzero torsion, and general nonmetricity. Our aim was to study the mean kinematics of timelike worldlines and see how these are affected by the aforementioned two extra spacetime features. We did so, by employing and extending the 1 + 3 covariant formalism, which combines both mathematical compactness and physical clarity, to spaces with torsion and nonmetricity. After adapting the covariant approach to the new environment and clarifying several subtle issues, we derived and provided the most general (to the best of our knowledge) version of Raychaudhuri’s formula. The latter is known to monitor the mean kinematics of timelike observers and has been the key formula for studying self-gravitating media.

Not surprisingly, the introduction of extra degrees of freedom into the host spacetime added several new effects to the Raychaudhuri equation. This in turn made the kinematics of the residing observers considerably more involved and therefore more difficult to decode. Nevertheless, by treating torsion and nonmetricity separately and by confining to highly symmetric (Friedmann-like) spacetimes, we were able to obtain both qualitative results and analytical solutions. In particular, assuming vectorial torsion and Weyl nonmetricity, we found that the solutions of the associated Raychaudhuri equations were formally identical. More specifically, it was shown that one could recover the former solution from the latter (and vice versa), by merely imposing a simple ansatz between the torsion and the Weyl vectors. Analogous reports of such a “duality” relation between these two geometrical agents are not uncommon in the literature. We attribute ours to the high symmetry of the host spacetimes, which appears to make the effects of torsion and nonmetricity macroscopically indistinguishable.

## ACKNOWLEDGMENTS

We would like to thank Lavinia Heisenberg, Alan Kostelecky, Iarley Lobo, and Simone Speziale for their helpful comments. C.G.T. acknowledges support from Clare Hall College and from the Department of Applied Mathematics and Theoretical Physics (DAMTP) at Cambridge University, where part of this work was done.

- [1] H. Weyl, Sitz. Berichte d. Preuss. Akad. d. Wissenschaften 465 (1918); E. Cartan, *J. Math. Pure Appl.* **1**, 141 (1922).
- [2] E. Schrodinger, *Space-Time Structure* (Cambridge University Press, Cambridge, England, 1950).
- [3] J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954); F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, *Phys. Rep.* **258**, 1 (1995); T. P. Sotiriou and S. Liberati, *Ann. Phys. (Amsterdam)* **322**, 935 (2007); V. Vitagliano, T. P. Sotiriou, and S. Liberati, *Ann. Phys. (Amsterdam)* **326**, 1259 (2011); V. Vitagliano, *Classical Quantum Gravity* **31**, 045006 (2014); L. Heisenberg, arXiv:1807.01725 [*Phys. Rep. (to be published)*].
- [4] C. G. Tsagas, A. Challinor, and R. Maartens, *Phys. Rep.* **465**, 61 (2008); G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology* (Cambridge University Press, Cambridge, England, 2012).
- [5] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984); E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, England, 2004).
- [6] K. Pasmatsiou, C. G. Tsagas, and J. D. Barrow, *Phys. Rev. D* **95**, 104007 (2017).
- [7] J.-P. Berthias and B. Shahid-Saless, *Classical Quantum Gravity* **10**, 1039 (1993); S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, *Classical Quantum Gravity* **24**, 6417 (2007); T. P. Sotiriou, *Classical Quantum Gravity* **26**, 152001 (2009); S. Capozziello and S. Vignolo, *Ann. Phys. (Berlin)* **19**, 238 (2010); G. J. Olmo, *Int. J. Mod. Phys. D* **20**, 413 (2011); L. Jarv, M. Runkla, M. Saal, and O. Vilson, *Phys. Rev. D* **97**, 124025 (2018).
- [8] J. Tafel, *Phys. Lett. A* **45**, 341 (1973); A. J. Fennelly, J. P. Krisch, J. R. Ray, and L. L. Smalley, *J. Math. Phys. (N.Y.)* **32**, 485 (1991); D. Palle, *Nuovo Cimento B* **114**, 853 (1999); S. Kar and S. Sengupta, *Pramana* **69**, 49 (2007); S. D. Brechet, M. P. Hobson, and A. N. Lasenby, *Classical Quantum Gravity* **24**, 6329 (2007); P. Luz and V. Vitagliano, *Phys. Rev. D* **96**, 024021 (2017); S. Speziale, *Phys. Rev. D* **98**, 084029 (2018).
- [9] I. P. Lobo, A. B. Barreto, and C. Romero, *Eur. Phys. J. C* **75**, 448 (2015).
- [10] G. J. Olmo, *Int. J. Mod. Phys. D* **20**, 413 (2011); S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, *Phys. Scr.* **78**, 065010 (2008).
- [11] M. Tsamparlis, *Phys. Rev. D* **24**, 1451 (1981).
- [12] J. Stelmach, *Classical Quantum Gravity* **8**, 897 (1991); J. B. Jimenez and T. S. Koivisto, *Phys. Lett. B* **756**, 400 (2016).
- [13] Y. Mao, M. Tegmark, A. H. Guth, and S. Cabi, *Phys. Rev. D* **76**, 104029 (2007); V. A. Kostelecky, N. Russell, and J. D. Tasson, *Phys. Rev. Lett.* **100**, 111102 (2008); F. W. Hehl, Y. N. Obukhov, and D. Puetzfeld, *Phys. Lett. A* **377**, 1775 (2013); D. Puetzfeld and Y. N. Obukhov, *Phys. Rev. D* **88**, 064025 (2013); L. Iorio, N. Radicella, and M. L. Ruggiero, *J. Cosmol. Astropart. Phys.* **08** (2015) 021; R. Lehnert, W. M. Snow, and H. Yan, *Phys. Lett. B* **744**, 415 (2015).
- [14] J. Foster, V. A. Kostelecky, and R. Xu, *Phys. Rev. D* **95**, 084033 (2017); R. Lehnert, W. M. Snow, Z. Xiao, and R. Xu, *Phys. Lett. B* **772**, 865 (2017); A. Delhome-Latorre, G. J. Olmo, and M. Ronco, *Phys. Lett. B* **780**, 294 (2018); I. P. Lobo and C. Romero, *Phys. Lett. B* **783**, 306 (2018).