Nonrelativistic limit of Einstein-Cartan-Dirac equations

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We derive the Schrödinger-Newton equation as the nonrelativistic limit of the Einstein-Dirac equations. Our analysis relaxes the assumption of spherical symmetry, made in an earlier work in the literature, while deriving this limit. Since the spin of the Dirac field couples naturally to torsion, we generalize our analysis to the Einstein-Cartan-Dirac equations, again recovering the Schrödinger-Newton equation.

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I. INTRODUCTION

The Schrödinger-Newton (SN) equation has been proposed in the literature as a model for investigating the effects of self-gravity on the motion of a nonrelativistic (NR) quantum particle [1–4] (specifically as a model for gravitational localization of macroscopic objects). It is a nonlinear modification to the Schrödinger equation with a Newtonian gravitational potential ϕ,

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + m\phi\psi(\mathbf{r},t), \qquad (1)$$

where the self-gravitating potential ϕ is assumed to be classical and obeys the semiclassical Poisson equation

$$\nabla^2\phi = 4\pi G m|\psi|^2. \qquad (2)$$

The coupled system of the above two equations in integrodifferential form is given by

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) - Gm^2 \int \frac{|\psi(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|} d^3r'\psi(\mathbf{r},t), \qquad (3)$$

and is known as the SN equation. There are two broad (complementary) viewpoints under which the SN equation has been dealt with in the literature, amongst others. In one of them, it is considered as a hypothesis and the ways to falsify it are studied through theoretical and (or) experimental considerations, e.g., the localization of wave packets for macroscopic objects [5], with a gravitationally induced inhibition of quantum dispersion. The second approach focuses on whether the SN equation can be understood as a consequence of the known principles of physics. It is viewed as a model for self-interaction of matter waves. Notable work in this context [6] shows that the SN equation is the nonrelativistic limit of the Einstein-Klein-Gordon system and the Einstein-Dirac system for a spherically symmetric space-time. Our present paper follows the second approach. We relax the assumption of a spherically symmetric space-time made in [6] and obtain the SN equation as the nonrelativistic limit of the Einstein-Dirac equations. Since the spin of the Dirac field couples naturally to torsion, we also study the Einstein-Cartan-Dirac equations, and obtain its nonrelativistic limit. These equations are a special case of the Einstein-Cartan-Sciama-Kibble theory [7–14], which we henceforth refer to as the Einstein-Cartan theory.

The plan of the paper is as follows. In Sec. II we describe the Einstein-Cartan-Dirac equations. Section III is the central part of the paper—the nonrelativistic limit of the Einstein-Dirac equations is derived here. We first describe the ansatz for the Dirac state and for the metric, which is used to derive the nonrelativistic limit. We then describe in detail the nonrelativistic expansion for the Dirac equation, and for the energy-momentum tensor. It is then shown that the nonrelativistic limit of the Eistein-Dirac equations is the Schrödinger-Newton equation, as expected. In Sec. IV, the nonrelativistic limit of the Einstein-Cartan-Dirac equations—which include the torsion of the Dirac field—is derived. It is shown that torsion does not contribute in the nonrelativistic limit, and once again we obtain the Schrödinger-Newton equation. Conclusions are presented in the next section, while the detailed Appendix gives calculations of the geometric variables such as metric, connection, and curvature, as well as the energy-momentum tensor, for the ansatz used in this paper.

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The present paper is part of a series of our works [15–21] that investigate the role of torsion in microscopic physics, and the motivation for including torsion in the Einstein-Dirac equations. The fundamental motivation comes from noting that given a relativistic point mass m, Einstein equations as well as the Dirac equation both claim to hold for it, irrespective of the numerical value of the mass. This is because there is no mass scale in either of the systems of equations, but of course both cannot hold for all masses. Only from experiments we know that Einstein equations hold for macroscopic masses, and the Dirac equation for small masses. But how large is large, and how small is small? There has to be an underlying dynamics with an in-built mass scale, to which the Dirac equation and Einstein equation are small mass and large mass approximations, respectively. The search for this underlying dynamics is aided by the fact that general relativity has Schwarzschild radius as a fundamental length (depending linearly on mass) and the Dirac equation has Compton wavelength as fundamental length (depending inversely on mass). This strongly suggests that the underlying theory should have one unified length, and also that it should include torsion, which dominates over curvature for small masses, because in this domain spin dominates mass. We have developed such curvature-torsion models, and investigated what physical role torsion might play in the modified Dirac equation. It is in this spirit that in this paper we are studying the nonrelativistic limit of the Eistein-Cartan-Dirac equations, to look for signatures of torsion.

II. PRELIMINARIES: THE EINSTEIN-CARTAN-DIRAC EQUATIONS

The antisymmetric part of the affine connection,

$$Q_{\alpha\beta}{}^{\mu} = \Gamma_{[\alpha\beta]}{}^{\mu} = \frac{1}{2}(\Gamma_{\alpha\beta}{}^{\mu} - \Gamma_{\beta\alpha}{}^{\mu}), \tag{4}$$

is called torsion. The affine connection is related to the Christoffel symbols by

$$\Gamma_{\alpha\beta}{}^{\mu} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} - K_{\alpha\beta}{}^{\mu}, \tag{5}$$

where $K_{\alpha\beta}{}^{\mu}$ is the contorsion tensor, and is given by $K_{\alpha\beta}{}^{\mu} = -Q_{\alpha\beta}{}^{\mu} - Q^{\mu}{}_{\alpha\beta} + Q_{\beta}{}^{\mu}{}_{\alpha}$.

For a matter field ψ, which is minimally coupled to gravity and torsion, the action is given by [8]

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_m(\psi, \nabla\psi, g) - \frac{1}{2k}R(g, \partial g, Q)\right], \tag{6}$$

where $k = 8\pi G/c^4$. The first and the second term on the right-hand side correspond to contribution from matter and gravity, respectively. Varying the action with respect to ψ (matter field), $g_{\mu\nu}$ (metric), and $K_{\alpha\beta\mu}$ (contorsion), the following field equations are obtained:

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi} = 0, \tag{7}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \quad \text{and} \tag{8}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta K_{\alpha\beta\mu}}. \tag{9}$$

Equation (7) yields the matter field equation on a space-time with torsion. The right-hand side of Eq. (8) is related to the metric energy-momentum tensor $T_{\mu\nu}$, while the right-hand side of Eq. (9) is associated with the spin density tensor $S^{\mu\beta\alpha}$. Equations (8) and (9) together give the Einstein-Cartan field equations,

$$G^{\mu\nu} = k\ \Sigma^{\mu\nu}, \tag{10}$$

$$T^{\mu\beta\alpha} = k\ \tau^{\mu\beta\alpha}. \tag{11}$$

$G^{\mu\nu}$ is the asymmetric Einstein tensor constructed from the asymmetric connection. $\Sigma^{\mu\nu}$ is the canonical energy-momentum tensor (asymmetric) constructed from the metric energy-momentum tensor (symmetric) and the spin density tensor. In Eq. (11), $T^{\mu\beta\alpha}$ is the modified torsion (traceless part of the torsion tensor); it is algebraically related to $S^{\mu\beta\alpha}$ on the right-hand side. On setting the torsion to 0, the field equations of general relativity are recovered.

For a Dirac field (ψ), the matter Lagrangian density is given by

$$\mathcal{L}_m = \frac{i\hbar c}{2}(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi) - mc^2\bar{\psi}\psi. \tag{12}$$

We denote a Riemannian space-time by V_4 and a space-time with torsion by U_4. Minimally coupling a Dirac field on U_4 leads to the Einstein-Cartan-Dirac (ECD) theory. The spinors are defined on V_4 and U_4 using tetrads. We use $\hat{e}^{\mu} = \partial^{\mu}$ as the coordinate basis, which is covariant under general coordinate transformations. Spinors (defined on a Minkowski space-time) on the other hand are associated with basis vectors that are covariant under local Lorentz transformations. To this aim, we define at each point on the manifold four orthonormal basis fields (tetrad fields) $\hat{e}^{i}(x)$, one for each i value. The tetrad fields satisfy the relation $\hat{e}^{i}(x) = e^{i}_{\mu}(x)\hat{e}^{\mu}$, where the transformation matrix e^{i}_{μ} is such that

$$e^{(i)}_{\mu}e^{(k)}_{\nu}\eta_{(i)(k)} = g_{\mu\nu}. \tag{13}$$

The transformation matrix $e^{(i)}_{\mu}$ facilitates the conversion of the components of any world tensor (which transform according to general coordinate transformations) to the corresponding components in a local Minkowski space (these latter components being covariant under local Lorentz transformations). Greek indices are raised and lowered using the metric $g_{\mu\nu}$, while the latin indices are raised and lowered using $\eta_{(i)(k)}$. Parentheses around indices is a matter of convention.

We adopt the following conventions for the remainder of the paper:

(i) Objects with greek indices (world indices), e.g., α, ζ, δ, transform according to general coordinate transformations and are raised and lowered using the metric $g_{\mu\nu}$.

(ii) Objects with latin indices within parentheses (tetrad indices), e.g., (a) or (i), transform according to local Lorentz transformations and are raised and lowered using $\eta_{(i)(k)}$.

(iii) Latin indices without parentheses, e.g., i, j, b, c, refer to objects in Minkowski space, which transform according to global Lorentz transformations.

(iv) In general 0, 1, 2, 3 refer to world indices while (0), (1), (2), (3) refer to tetrad indices.

(v) $\nabla^{\{\}}$ represents the covariant derivative with the Christoffel connections ($\{\}$), while ∇ denotes the total covariant derivative.

(vi) Commas (,) refer to partial derivatives and semicolons (;) to the Riemannian covariant derivative, which implies (;) and $\nabla^{\{\}}$ are the same for tensors. For spinors, (;) involves a partial derivative and the Riemannian part of the spin connection.

Just as the affine connection Γ facilitates parallel transport of geometrical objects with world (greek) indices, the spin connection (γ) does so for anholonomic objects (those with latin indices). The affine connection Γ has two parts—Riemannian ($\{\}$) and torsional (constructed from the contorsion tensor ${K_\mu}^{(k)(i)}$); similarly, the spin connection (${\gamma_\mu}^{(i)(k)}$) has two parts—Riemannian (denoted by $\gamma_\mu^{o\ (i)(k)}$) and torsional (constructed from the contorsion tensor ${K_\mu}^{(k)(i)}$). These quantities are interrelated by

$$ {\gamma_\mu}^{(i)(k)} = \gamma_\mu^{o\ (i)(k)} - {K_\mu}^{(k)(i)} \quad \text{and} \tag{14} $$

$$ \begin{aligned} {\gamma_\mu}^{(i)(k)} &= e^{(i)}_\alpha e^{\nu(k)} {\Gamma_{\mu\nu}}^\alpha - e^{\nu(k)}\partial_\mu e^{(i)}_\nu \\ &= e^{(i)}_\alpha e^{\nu(k)} \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - {K_\mu}^{(k)(i)} - e^{\nu(k)}\partial_\mu e^{(i)}_\nu . \end{aligned} \tag{15} $$

Using Eqs. (14) and (15), the Riemannian part of the spin connection can be expressed entirely in terms of the Christoffel symbols and the tetrads as [13]

$$ \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = e^\alpha_{(i)} e_{\nu(k)} {\gamma_\mu}^{o(k)(i)} + e^\alpha_{(i)}\partial_\mu e^{(i)}_\nu . \tag{16} $$

One can thus define the covariant derivative for spinors as

$$ \psi_{;\mu} = \partial_\mu\psi + \frac{1}{4}\gamma^o_{\mu(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi \quad (\text{on } V_4) \tag{17} $$

and
$$ \begin{aligned} \nabla_\mu\psi = \partial_\mu\psi &+ \frac{1}{4}\gamma^0_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi \\ &- \frac{1}{4}K_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi. \quad (\text{on } U_4) \end{aligned} \tag{18} $$

The explicit form of the matter Lagrangian density is obtained by substituting Eqs. (17) and (18) in Eq. (12). The Dirac equation is then given by Eq. (7),

$$ i\gamma^\mu\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 \qquad (\text{on } V_4) \tag{19} $$

and
$$ i\gamma^\mu\psi_{;\mu} + \frac{i}{4}K_{(a)(b)(c)}\gamma^{[(a)}\gamma^{(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 \quad (\text{on } U_4). \tag{20} $$

The gravitational field equations are obtained using Eqs. (8) and (12),

$$ G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad (\text{on } V_4) \tag{21} $$

and
$$ G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \quad (\text{on } U_4). \tag{22} $$

The metric energy-momentum (EM) tensor (symmetric) is defined by

$$ \begin{aligned} T_{\mu\nu} &= \Sigma_{(\mu\nu)}(\{\}) \\ &= \frac{i\hbar c}{4}[\bar\psi\gamma_\mu\psi_{;\nu} + \bar\psi\gamma_\nu\psi_{;\mu} - \bar\psi_{;\mu}\gamma_\nu\psi - \bar\psi_{;\nu}\gamma_\mu\psi]. \end{aligned} \tag{23} $$

Equations (19) and (21) are the governing equations for the Einstein-Dirac theory. The spin density tensor is obtained from the matter Lagrangian density (12),

$$ S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar\psi\gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}\psi. \tag{24} $$

Using Eqs. (24) and (9), Eq. (20) simplifies to the Hehl-Datta equation [8,10], which together with Eq. (22) and the relation between the modified torsion tensor and the spin density tensor constitutes the field equations for the Einstein-Cartan-Dirac theory,

$$ G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}, \tag{25} $$

$$ T_{\mu\nu\alpha} = -K_{\mu\nu\alpha} = \frac{8\pi G}{c^4}S_{\mu\nu\alpha}, \quad \text{and} \tag{26} $$

$$ i\gamma^\mu\psi_{;\mu} = +\frac{3}{8}L^2_{\rm Pl}\bar\psi\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{mc}{\hbar}\psi. \tag{27} $$

Here, $L_{\rm Pl}$ is the Planck length. The Lorentz signature, diag$(+,-,-,-)$ is used throughout the paper. The gamma matrices are represented in the Dirac basis, which happens to be the matrix representation of Clifford algebra $Cl_{1,3}[\mathbb{R}]$,

$$\gamma^0=\beta=\begin{pmatrix}\mathbb{I}_2 & 0\\ 0 & -\mathbb{I}_2\end{pmatrix},\quad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},$$
$$\gamma^5=\frac{i}{4!}\epsilon_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l=\begin{pmatrix}0 & \mathbb{I}_2\\ \mathbb{I}_2 & 0\end{pmatrix}\ \text{and}\ \alpha^i=\beta\gamma^i=\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}. \tag{28}$$

III. NONRELATIVISTIC LIMIT OF THE EINSTEIN-DIRAC EQUATIONS

A. Ansatz for the spinor and the metric

Ansatz for the Dirac spinor: We expand $\psi(x,t)$ as $\psi(x,t)=e^{iS(x,t)\hbar}$ (which can be done for any complex function of x and t). S can be expressed as a perturbative power series in $\sqrt{\hbar}$ or $(1/c)$, to obtain the semiclassical and the nonrelativistic limit, respectively. The scheme for obtaining the nonrelativistic limit has been employed by Kiefer and Singh [22]. Giulini and Großardt in their work [6] construct a new ansatz with the parameter $\sqrt{\hbar}/c$ as follows:

$$\psi(\mathbf{r},t)=e^{\frac{ic^2}{\hbar}S(\mathbf{r},t)}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n(\mathbf{r},t), \tag{29}$$

where $S(\mathbf{r},t)$ is a scalar function and $a_n(\mathbf{r},t)$ is a spinor field. We use this ansatz in our present work.

Ansatz for the metric: We express a general metric as a perturbative power series in the parameter $\sqrt{\hbar}/c$, similar to the expansion for the spinor,

$$g_{\mu\nu}(\mathbf{r},t)=\eta_{\mu\nu}+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n g_{\mu\nu}^{[n]}(\mathbf{r},t), \tag{30}$$

where $g_{\mu\nu}^{[n]}(x)$ are metric functions indexed by n. In the nonrelativistic scheme, gravitational potentials are weak and cannot produce velocities comparable to c. Hence, we assume the leading order function to be the Minkowski metric, $g_{\mu\nu}^{[0]}(x)=\eta_{\mu\nu}$. The generic power series for the tetrads, spin coefficients, and Einstein tensor are then given by

$$e^{\mu}_{(i)}=\delta^{\mu}_{(i)}+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(i)},$$
$$\gamma_{(a)(b)(c)}=\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{[n]}_{(a)(b)(c)}, \tag{31}$$

$$e^{(i)}_{\mu}=\delta^{(i)}_{\mu}+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{(i)[n]}_{\mu}\quad\text{and}\quad G_{\mu\nu}=\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n G^{[n]}_{\mu\nu}, \tag{32}$$

where $e^{\mu[n]}_{(i)}$, $e^{(i)[n]}_{\mu}$, $\gamma^{[n]}_{(a)(b)(c)}$, and $G^{[n]}_{\mu\nu}$ are functions of the metric $g^{[n]}_{\mu\nu}$ and its derivatives.

B. Analyzing the Dirac equation with the above ansatz

We first separate the spatial and the temporal part of the Dirac equation on V_4 [Eq. (19)]. (Note that $\gamma^{(a)}\psi_{;(a)}=e^{(a)}_{\mu}e^{\nu}_{(a)}\gamma^{\mu}\psi_{;\nu}=\delta^{\nu}_{\mu}\gamma^{\mu}\psi_{;\nu}=\gamma^{\mu}\psi_{;\mu}$.)

$$i\gamma^{\mu}\psi_{;\mu}-\frac{mc}{\hbar}\psi=0 \tag{33}$$

$$\Rightarrow i\gamma^0\partial_0\psi+\frac{i}{4}\gamma^{(0)}\gamma^{o}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi+i\gamma^{\alpha}\partial_{\alpha}\psi+\frac{i}{4}\gamma^{(j)}\gamma^{o}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi-\frac{mc}{\hbar}\psi=0. \tag{34}$$

Multiplying both sides by $e^{(0)}_0\gamma^{(0)}c$, we get

$$i\partial_t\psi+\frac{ic}{4}\gamma^{o}_{0(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi+ice^{(0)}_0e^{\alpha}_{(a)}\alpha^{(a)}\partial_{\alpha}\psi+\frac{ic}{4}e^{(0)}_0\alpha^{(j)}\gamma^{o}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi-e^{(0)}_0\beta\frac{mc^2}{\hbar}\psi=0. \tag{35}$$

Using the series expansion for the tetrads and the Riemannian part of the spin connection [Eqs. (31) and (32)], we keep terms of the order c^2, c and 1, and neglect terms of the order $(\frac{1}{c^n})$ with $n\geq 1$. This is sufficient for obtaining the equation obeyed by the leading order spinor term, a_0. We thus obtain

$$i\partial_t\psi+\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{0(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi+ic\alpha.\nabla\psi+i\sqrt{\hbar}\vec{E}.\nabla\psi+\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi-\beta\frac{mc^2}{\hbar}\psi-\beta\frac{mc}{\sqrt{\hbar}}e^{(0)[1]}_0\psi-\beta me^{(0)[2]}_0\psi=0, \tag{36}$$

where $\vec{E}=([e^{(0)[1]}_0\alpha^{(\mathbf{1})}+e^{\mathbf{1}[1]}_{(a)}\alpha^{(a)}],[e^{(0)[1]}_0\alpha^{(\mathbf{2})}+e^{\mathbf{2}[1]}_{(a)}\alpha^{(a)}],[e^{(0)[1]}_0\alpha^{(\mathbf{3})}+e^{\mathbf{3}[1]}_{(a)}\alpha^{(a)}])$. We now evaluate each term of Eq. (36) by substituting the spinor ansatz (29).

Term 1

$$+i\partial_t\psi = i\partial_t\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]$$
$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n[-\dot{S}a_{n-1}+i\dot{a}_{n-3}]. \quad (37)$$

Term 2

$$+\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi$$
$$=+\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]$$
$$=e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}a_{n-3}\right]. \quad (38)$$

Term 3

$$ic\alpha.\nabla\psi = ic\vec{\alpha}\cdot\vec{\nabla}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]$$
$$= ic\vec{\alpha}\cdot\left[e^{\frac{ic^2S}{\hbar}}\frac{c^2}{\hbar}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n(i\vec{\nabla}Sa_n+\vec{\nabla}a_{n-2})\right]$$
$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n$$
$$\times[-\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}Sa_n+i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}a_{n-2}]. \quad (39)$$

Term 4

$$+i\sqrt{\hbar}\vec{E}.\vec{\nabla}\psi = i\sqrt{\hbar}\vec{E}.\vec{\nabla}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]$$
$$= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}[-\sqrt{\hbar}\vec{E}.\vec{\nabla}Sa_{n-1}+i\sqrt{\hbar}\vec{E}.\vec{\nabla}a_{n-3}]. \quad (40)$$

Term 5

$$+\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi$$
$$=+\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]$$
$$=e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}a_{n-3}\right]. \quad (41)$$

Term 6

$$-\beta\frac{mc^2}{\hbar}\psi = -\beta\frac{mc^2}{\hbar}e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n$$
$$= -e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n(\beta ma_{n-1}). \quad (42)$$

Term 7

$$-\beta\frac{mc}{\sqrt{\hbar}}e_0^{(0)[1]}\psi = -\beta\frac{mc}{\sqrt{\hbar}}e_0^{(0)[1]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]$$
$$= -e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n[\beta me_0^{(0)[1]}a_{n-2}]. \quad (43)$$

Term 8

$$-\beta me_0^{(0)[2]}\psi = -\beta m\left[e_0^{(0)[2]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right]\right.$$
$$= -e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\beta me_0^{(0)[2]}a_{n-3}. \quad (44)$$

We thus obtain

$$e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[(-\sqrt{\hbar}\vec{\alpha}.\vec{\nabla}S)a_n\right.$$
$$-(\dot{S}+\beta m+\sqrt{\hbar}\vec{E}.\vec{\nabla}S)a_{n-1}+(i\sqrt{\hbar}\vec{\alpha}.\vec{\nabla}-\beta me_0^{(0)[1]})a_{n-2}$$
$$+\left(i\partial_t+i\sqrt{\hbar}\vec{E}.\vec{\nabla}+\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\right.$$
$$\left.\left.+\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}-\beta me_0^{(0)[2]}\right)a_{n-3}\right]=0. \quad (45)$$

At the leading order ($n=0$), we get

$$\mathbf{\nabla}S=0, \quad (46)$$

which implies that the scalar S is a function of time only, i.e., $S=S(t)$. The Dirac spinor is a four-component object that can be written as $a_n=(a_{n,1},a_{n,2},a_{n,3},a_{n,4})$. We split it into two-component spinors, $a_n^{>}=(a_{n,1},a_{n,2})$ and $a_n^{<}=(a_{n,3},a_{n,4})$. At order $n=1$, we get $\dot{S}+\beta m+\sqrt{\hbar}\vec{E}.\vec{\nabla}S=0$. Since $\vec{\nabla}S=0$, this implies

$$(m+\dot{S})a_0^{>}=0, \quad (47a)$$

$$\text{and}\quad (m-\dot{S})a_0^{<}=0, \quad (47b)$$

which satisfies either $S = -mt$ with $a_0^< = 0$ or $S = +mt$ with $a_0^> = 0$. The wave function at this order is $\psi = e^{\frac{\pm imc^2 t}{\hbar}} a_0$, which corresponds to particles of positive energy (lower sign) and negative energy (upper sign), at rest. We restrict ourselves to $S = -mt$ and $a_0^< = 0$, i.e., the positive energy solutions. It is implicitly assumed that the two cases (positive and negative energies) can be studied separately. We digress at this point and analyze the energy-momentum tensor.

C. Analyzing the energy-momentum tensor $T_{\mu\nu}$ with the above ansatz

The dynamical energy-momentum tensor is given by Eq. (23). We analyze all the sixteen components of $T_{\mu\nu}$.

(1) kT_{00} (with the indices of the gamma matrices raised),

$$kT_{00} = \frac{4i\pi G\hbar}{c^4}\left[\bar{\psi}\gamma^0\left(\partial_t\psi + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\psi\right) - \left(\partial_t\bar{\psi} + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\bar{\psi}\right)\gamma^0\psi\right] \tag{48}$$

$$= \frac{4i\pi G\hbar}{c^4}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right) \times \left[\bar{\psi}\gamma^{(0)}\left(\partial_t\psi + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\psi\right) - \left(\partial_t\bar{\psi} + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\bar{\psi}\right)\gamma^{(0)}\psi\right]. \tag{49}$$

Substituting the spinor ansatz [Eq. (29)] in Eq. (48), we obtain a series expansion for kT_{00}. At the leading order we get

$$kT_{00} = \frac{4i\pi G}{c^2}\left\{\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n^\dagger\right) \times \left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m [i\dot{S}a_m + \dot{a}_{m-2}]\right) + \left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n [i\dot{S}a_n^\dagger - \dot{a}^\dagger_{n-2}]\right) \times \left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m a_m\right)\right\} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right), \tag{50}$$

with $(n + m = 0)$, i.e.,

$$kT_{00} = \frac{4\pi Gi}{c^2}\{i(-m)a_0^{>\dagger}a_0^> + i(-m)a_0^{>\dagger}a_0^>\} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{51}$$

$$= \frac{8\pi Gm|a_0^>|^2}{c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{52}$$

(2) $kT_{0\mu}$ ($\mu = 1, 2, 3$),

$$kT_{0\mu} = \frac{2i\pi G\hbar}{c^4}\left[c\bar{\psi}\gamma_0\left(\partial_\mu\psi + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\right) + c\bar{\psi}\gamma_\mu\left(\partial_0\psi + \frac{1}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\right) - c\left(\partial_\mu\bar{\psi} + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\right)\gamma_0\psi - c\left(\partial_0\bar{\psi} + \frac{1}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\right)\gamma_\mu\psi\right]. \tag{53}$$

Terms containing the spin coefficients ($\gamma_{\mu(i)(j)}$) are of the order $\frac{1}{c^3}$ or higher and hence do not contribute at the order $\frac{1}{c^2}$. The rest of the terms are analyzed in Appendix G 1, and are shown to have no contribution at the order $\frac{1}{c^2}$. Hence,

$$kT_{0\mu} = \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{54}$$

(3) $kT_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$),

$$kT_{\mu\nu} = \frac{2i\pi G\hbar}{c^3}\left[+\bar{\psi}\gamma_\mu\left(\partial_\nu\psi + \frac{1}{4}[\gamma^o_{\nu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\right) + \bar{\psi}\gamma_\nu\left(\partial_\mu\psi + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\right) - \left(\partial_\nu\bar{\psi} + \frac{1}{4}[\gamma^o_{\nu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\right)\gamma_\mu\psi - \left(\partial_\mu\bar{\psi} + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\right)\gamma_\nu\psi\right]. \tag{55}$$

Once again, terms containing the spin coefficients ($\gamma_{\mu(i)(j)}$) are of the order $\frac{1}{c^3}$ or higher and hence do not contribute at the order $\frac{1}{c^2}$. The rest of the terms are analyzed in Appendix G 2, and are shown to have no contribution at the order $\frac{1}{c^2}$. Hence,

$$kT_{\mu\nu} = \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{56}$$

Results of the order analysis of the EM tensor, summarized by Eqs. (52), (54), and (56) imply

$$\frac{|T_{00}|}{|T_{0i}|} \ll 1, \qquad \frac{|T_{00}|}{|T_{ij}|} \ll 1, \quad \text{and} \quad k|T_{00}| \sim O\left(\frac{1}{c^2}\right) \quad \forall\ i, j \in (1, 2, 3). \tag{57}$$

Owing to Einstein's equations, the same relations then hold for the components of the Einstein tensor, i.e.,

$$\frac{|G_{00}|}{|G_{0i}|} \ll 1, \qquad \frac{|G_{00}|}{|G_{ij}|} \ll 1, \quad \text{and}$$
$$|G_{00}| \sim O\left(\frac{1}{c^2}\right) \quad \forall\ i,j \in (1,2,3). \tag{58}$$

D. Constraints on the metric

In Sec. III C we showed that $|G_{00}| \sim O(\frac{1}{c^2})$ while all the other components of $G_{\mu\nu}$ are of higher order. For a generic metric ansatz, $G_{\mu\nu}$ has been calculated in Appendix A. At this point we make an important assumption—the metric field is asymptotically flat. This leads to the following constraints on the metric components (proved in Appendix B):

(1) $G^{[1]}_{\mu\nu} = 0$ $(\forall\, \mu, \nu)$ together with the condition of asymptotic flatness of the metric, leads to the following results (proved in Appendix B 1):

$$g^{[1]}_{\mu\nu} = 0, \qquad e^{\mu[1]}_{(i)} = 0, \qquad e^{(i)[1]}_{\mu} = 0, \quad \text{and}$$
$$\gamma^{[1]}_{(i)(j)(k)} = 0 \quad \forall\ i,j,k,\mu,\nu \in (0,1,2,3). \tag{59}$$

(2) $G^{[2]}_{\mu\nu} = 0$ (except for $\mu = \nu = 0$) leads to the following constraint: $g^{[2]}_{\mu\nu} = F(\mathbf{r},t)\delta_{\mu\nu}$ for some field $F(\mathbf{r},t)$ (proved in Appendix B 2).

The full metric is then given by

$$g_{\mu\nu}(\mathbf{r},t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \left(\frac{\hbar}{c^2}\right) \begin{bmatrix} F & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} (\mathbf{r},t) + \sum_{n=3}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n \begin{bmatrix} g^{[n]}_{00} & g^{[n]}_{01} & g^{[n]}_{02} & g^{[n]}_{03} \\ g^{[n]}_{10} & g^{[n]}_{11} & g^{[n]}_{12} & g^{[n]}_{13} \\ g^{[n]}_{20} & g^{[n]}_{21} & g^{[n]}_{22} & g^{[n]}_{23} \\ g^{[n]}_{30} & g^{[n]}_{31} & g^{[n]}_{32} & g^{[n]}_{33} \end{bmatrix} (\mathbf{r},t), \tag{60}$$

where $g^{[2]}_{00} = g^{[2]}_{11} = g^{[2]}_{22} = g^{[2]}_{33} = F(\mathbf{r},t)$. The above metric has been employed to calculate other objects (tetrads, spin coefficients, etc.) in Appendixes C, E, D, and F.

E. NR limit of the Einstein-Dirac equations

1. Dirac equation

In Sec. III B we analyzed Eq. (45) for $n = 0$ and $n = 1$. Using the results of Appendixes C, D, and E, Eq. (45) can be further simplified to

$$e^{\frac{ic^2 S}{\hbar}} \frac{c^3}{\hbar^{3/2}} \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n \Big[-(\dot{S} + \beta m)a_{n-1} + i\dot{a}_{n-3} + i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla} a_{n-2} - \beta \frac{mF(\mathbf{r},t)}{2} a_{n-3}\Big] = 0. \tag{61}$$

At order $n = 2$, Eq. (61) gives us

$$\begin{pmatrix} \dot{S}+m & 0 \\ 0 & \dot{S}-m \end{pmatrix} \begin{pmatrix} a_1^{>} \\ a_1^{<} \end{pmatrix} - i\sqrt{\hbar} \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\nabla} \\ \vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix} \begin{pmatrix} a_0^{>} \\ a_0^{<} \end{pmatrix} = 0. \tag{62}$$

The first of these two equations is trivially satisfied. The second equation yields a relation between $a_1^{<}$ and $a_0^{>}$,

$$a_1^{<} = \frac{-i\sqrt{\hbar}\vec{\sigma}\cdot\vec{\nabla}}{2m} a_0^{>}. \tag{63}$$

At order $n = 3$, we get

$$\begin{pmatrix} \dot{S}+m & 0 \\ 0 & \dot{S}-m \end{pmatrix} \begin{pmatrix} a_2^{>} \\ a_2^{<} \end{pmatrix} - i\sqrt{\hbar} \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\nabla} \\ \vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix} \begin{pmatrix} a_1^{>} \\ a_1^{<} \end{pmatrix} - \begin{pmatrix} i\partial_t - \frac{mF(\mathbf{r},t)}{2} & 0 \\ 0 & i\partial_t + \frac{mF(\mathbf{r},t)}{2} \end{pmatrix} \begin{pmatrix} a_0^{>} \\ a_0^{<} \end{pmatrix} = 0, \tag{64}$$

which comprises two equations. Using Eq. (63), the first of these two equations gives us

$$i\hbar \frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + \frac{m\hbar F(\mathbf{r},t)}{2} a_0^{>}. \tag{65}$$

2. Einstein's equations

The Einstein tensor has been evaluated in Appendix F. Equating G_{00} to kT_{00} we get

$$\frac{\hbar\nabla^2 F(\mathbf{r},t)}{c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) = \frac{8\pi G m |a_0^{>}|^2}{c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{66}$$

At the leading order, this gives us

$$\nabla^2 F(\mathbf{r},t) = \frac{8\pi G m |a_0^{>}|^2}{\hbar}. \tag{67}$$

Recognizing the quantity $\frac{\hbar F(\mathbf{r},t)}{2}$ as the Newtonian potential ϕ, we obtain the Schrödinger-Newton system of equations ($m\phi \rightarrow$ gravitational potential energy and $m|a_0^{>}|^2 \rightarrow$ mass density) as the NR limit of the Einstein-Dirac equations,

$$i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + m\phi(\mathbf{r},t)a_0^{>} \quad \text{and} \tag{68}$$

$$\nabla^2\phi(\mathbf{r},t) = 4\pi Gm|a_0^{>}|^2 = 4\pi G\rho(\mathbf{r},t) \tag{69}$$

$$\Rightarrow i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} - Gm^2\int\frac{|a_0^{>}(\mathbf{r}\prime,t)|^2}{|\mathbf{r}-\mathbf{r}\prime|}d^3\mathbf{r}\prime a_0^{>}, \tag{70}$$

the physical picture for which has already been discussed in Sec. I. This completes the derivation of the Schrödinger-Newton equation as the nonrelativistic limit of the Einstein-Dirac equations.

IV. NONRELATIVISTIC LIMIT OF THE EINSTEIN-CARTAN-DIRAC EQUATIONS

We now employ the Wentzel-Kramers-Brillouin (WKB) type ansatz of the previous section to the case when torsion is included. It is to be noted that the torsion of the Dirac field can be expressed directly in terms of the Dirac spinors. Once the substitution of the torsion tensor has been done in terms of the Dirac spinors, the nonlinear Dirac equation no longer makes any reference to torsion. Similarly, in the Einstein-Cartan field equations, the contribution coming from torsion can be expressed in terms of the Dirac state. Thus the Einstein-Cartan-Dirac system is a coupled differential system for the metric and the Dirac state—just like the Einstein-Dirac system is—only, the nonlinear terms are now different. Thus the WKB ansatz used earlier can be directly used in the presence of torsion as well.

The Dirac equation on U_4 (also known as the Hehl-Datta equation) is given by [Eq. (27)]

$$i\gamma^\mu\psi_{;\mu} - \frac{3}{8}L_{\mathrm{Pl}}^2\bar{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi - \frac{mc}{\hbar}\psi = 0. \tag{71}$$

We have already analyzed the first and the last term on the left-hand side using the ansatz for the spinor (29) and the metric (60). The second term arises due to torsion and makes the equation nonlinear. We evaluate this term similar to the other two (Sec. III B). Multiplying the middle term by $e_0^{(0)}\gamma^{(0)}c$ [as was done in Sec. III B to obtain (35) from (34)], we get

$$\begin{aligned}&-e_0^{(0)}\gamma^{(0)}\frac{3c}{8}L_{\mathrm{Pl}}^2\bar{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi\\&\quad=-\frac{3c}{8}l_{\mathrm{Pl}}^2e^{\frac{ic^2S}{\hbar}}\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]\\&\quad\quad\times\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n^\dagger\right)\gamma^5\gamma_{(a)}\left(\sum_{l=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_l\right)\\&\quad\quad\times\gamma^5\gamma^{(a)}\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_m\right),\end{aligned} \tag{72}$$

which simplifies to

$$\begin{aligned}&e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]\\&\times\frac{3G}{8}\left(\sum_{n_1,n_2,n_3=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_{n_1-i}^\dagger\gamma^5\gamma_{(a)}a_{n_2-j}\gamma^5\gamma^{(a)}a_{n_3-k}\right),\end{aligned} \tag{73}$$

where $n = n_1 + n_2 + n_3$. This term modifies Eq. (61) as follows:

$$\begin{aligned}&e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[ma_{n-1}+i\dot{a}_{n-3}+i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}a_{n-2}\right.\\&-\beta ma_{n-1}-\beta\frac{mF(\mathbf{r},t)}{2}a_{n-3}\\&\left.+\frac{3G}{8}\left(\sum_{n_1,n_2,n_3=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_{n_1-i}^\dagger\gamma^5\gamma_{(a)}a_{n_2-j}\gamma^5\gamma^{(a)}a_{n_3-k}\right)\right]=0,\end{aligned} \tag{74}$$

where $n=n_1+n_2+n_3$ and $i+j+k=5$, with $i \le n_1$, $j \le n_2$ and $k \le n_3$. Further, $i, j, k, n_1, n_2, n_3 \in (0,1,2,3,4,5)$. The nonlinear term contributes only at order $n = 5$ and higher. As a result, the analysis for $n = 0$, 1, 2, and 3 (considered in Sec. III E) holds good. Thus $a_0^{>}$ satisfies the Schrödinger equation, i.e., $i\hbar\frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + \frac{m\hbar F(\mathbf{r},t)}{2}a_0^{>}$.

Einstein's equations on U_4 read $G_{\mu\nu}(\{\}) = kT_{\mu\nu} - \frac{1}{2}k^2g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}$. $G_{\mu\nu}(\{\})$ and $T_{\mu\nu}$ have already been analyzed in Sec. III E. The second term on the right-hand side, i.e., $-\frac{1}{2}k^2g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}$, involves a contraction of the spin density tensor (24). We consider only the first term in the series expansion of the metric, because the other terms together with the coupling constant are of orders not relevant for the NR limit. We thus obtain

$$\begin{aligned}&\frac{-1}{2}k^2g_{00}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}\\&\quad=-g_{00}\frac{2\pi^2G^2\hbar^2}{c^6}\sum_{N=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_k^\dagger\gamma^0\gamma^{[c}\gamma^a\gamma^{b]}\right)\\&\quad\quad\times\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_m^\dagger\gamma^0\gamma_{[c}\gamma_a\gamma_{b]}n_m\right)=\sum_{n=6}^{\infty}O\left(\frac{1}{c^n}\right),\end{aligned} \tag{75}$$

which implies that this additional term does not contribute at the order $(\frac{1}{c^2})$ on the right-hand side of Eq. (25). Hence, we once again recover Poisson's equation. Thus the Schrödinger-Newton equation also happens to be the NR limit of the ECD theory, which implies that torsion does not contribute at the leading order.

V. CONCLUSIONS

While the nonrelativitic limit of the Einstein-Dirac equations for a self-gravitating Dirac field has been calculated by Giulini and Großardt [6], we relax the assumption of a spherically symmetric metric in our present work. The Schrödinger-Newton equation is obtained as the nonrelativistic limit for a general metric, by considering a perturbative series in the parameter $(\frac{\sqrt{\hbar}}{c})$, for the spinor, the metric, and other relevant quantities. This scheme for obtaining the nonrelativistic limit follows the WKB-like expansion given by Giulini and Großardt [6].

The Einstein-Cartan-Dirac equations provide an elegant system for coupling matter to the geometry of space-time, where torsion arises due to the spin of the Dirac field. The nonrelativistic limit of this system of equations (derived in Sec. IV) yields the Schrödinger-Newton equation, at the leading order of the parameter $(\frac{1}{c})$. This suggests that torsion does not manifest itself at this order.

The effect of torsion in the higher order corrections to the Schrödinger-Newton equation can be obtained from the Einstein-Cartan-Dirac equations, by considering a WKB-type expansion for the spinor and other relevant quantities, as was done in the present work. However, in this paper, we have restricted ourselves to the analysis at the leading order, which gives us the nonrelativistic limit. A similar prescription may also be employed to obtain the higher order corrections to the Schrödinger equation (starting from Dirac's equation) and Newton's equation for gravitation (starting from Einstein's equations).

The Einstein-Cartan-Dirac equations with the unified new length scale [19] provide for the possibility of a solitonic solution that interpolates between a black hole and a Dirac fermion. This is one of the primary motivations for us to study this system of field equations. The search for such solutions has been attempted in [21] and further work is in progress. One could well ask if Derrick's theorem [23] could compel such solitonic solutions to be unstable. The theorem suggests that stationary localized solutions to nonlinear wave equations such as those considered here are unstable. In the present situation, however, the inclusion of torsion (which has a dispersive effect) makes it more plausible to achieve a stable balancing solution where the dispersive aspect due to torsion balances the collapse aspect due to gravity. Moreover, a way out of Derrick's no-go theorem is that the sought-for solitonic solutions are periodic in time, rather than time independent. Such solutions were actually reported by us in [21]. Rigorously speaking, the so-called Vakhitov-Kolokolov stability criterion [24] provides a precise condition for the linear stability of a periodic solitary wave solution. This requirement continues to hold for the Einstein-Cartan-Dirac equations as well.

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APPENDIX A: FORM OF THE EINSTEIN TENSOR EVALUATED USING THE GENERIC METRIC UP TO SECOND ORDER

The ansatz for the metric is given by [Eq. (30)]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n g_{\mu\nu}^{[n]}(x).$$

To the second order, the metric and its inverse is then given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \left(\frac{\sqrt{\hbar}}{c}\right) g_{\mu\nu}^{[1]} + \left(\frac{\hbar}{c^2}\right) g_{\mu\nu}^{[2]} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \quad \text{and} \tag{A1}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \left(\frac{\sqrt{\hbar}}{c}\right) g^{\mu\nu[1]} - \left(\frac{\hbar}{c^2}\right)[g_{\beta}^{\mu[1]} g^{\beta\nu[1]} + g^{\mu\nu[2]}] + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{A2}$$

We evaluate Christoffel symbols, Riemann curvature tensor, Ricci tensor, and scalar curvature to obtain the Einstein tensor $G_{\mu\nu}$ up to the second order as follows:

$$G_{\mu\nu}(\{\}) = \left(\frac{\sqrt{\hbar}}{c}\right) G_{\mu\nu}^{[1]}(\{\}) + \left(\frac{\hbar}{c^2}\right) G_{\mu\nu}^{[2]}(\{\}), \tag{A3}$$

where

$$G_{\mu\nu}^{[1]}(\{\}) = -\frac{1}{2}\Box \bar{g}_{\mu\nu}^{[1]}, \quad \bar{g}_{ij}^{[1]} = g_{\mu\nu}^{[1]} - \frac{1}{2}\eta_{\mu\nu} g^{[1]}, \quad g^{[1]} = (\eta^{\mu\nu} g_{\mu\nu}^{[1]}), \tag{A4}$$

$$G_{\mu\nu}^{[2]}(\{\}) = -\frac{1}{2}\Box \bar{g}_{\mu\nu}^{(2)} + f(g_{\mu\nu}^{[1]}), \qquad \bar{g}_{ij}^{[2]} = g_{\mu\nu}^{[2]} - \frac{1}{2}\eta_{\mu\nu} g^{[2]} \quad \text{and} \quad g^{[2]} = (\eta^{\mu\nu} g_{\mu\nu}^{[2]}). \tag{A5}$$

In Eq. (A5), f is a function of $g^{[1]}_{\mu\nu}$, which is given by

$$f(g^{[1]}_{\mu\nu}) = -\frac{1}{4}[2\partial^{\lambda}g^{[1]}\partial_{\nu}g^{[1]}_{\lambda\mu} - 2\partial^{\lambda}g^{[1]}\partial_{\lambda}g^{[1]}_{\mu\nu} - \partial_{\rho}g^{\lambda[1]}_{\nu}\partial_{\mu}g^{\rho[1]}_{\lambda} - \partial_{\rho}g^{\lambda[1]}_{\nu}\partial_{\lambda}g^{\rho[1]}_{\mu} + \partial_{\rho}g^{\lambda[1]}_{\nu}\partial^{\rho}g^{[1]}_{\lambda\mu} + \partial_{\nu}g^{\lambda[1]}_{\rho}\partial_{\mu}g^{\rho[1]}_{\lambda} + \partial_{\nu}g^{\lambda[1]}_{\rho}\partial_{\lambda}g^{\rho[1]}_{\mu} - \partial_{\nu}g^{\lambda[1]}_{\rho}\partial^{\rho}g^{[1]}_{\lambda\mu}] - \frac{1}{8}[2\partial^{\lambda}g^{[1]}\partial_{\nu}g^{[1]}_{\lambda\mu} - 2\eta_{\mu\nu}\partial^{\lambda}g^{[1]}\partial_{\lambda}g^{[1]} - \partial_{\rho}g^{\lambda[1]}_{\nu}\partial_{\mu}g^{\rho[1]}_{\lambda} - \partial_{\rho}g^{\lambda[1]}_{\mu}\partial_{\lambda}g^{\rho[1]}_{\nu} + \partial_{\rho}g^{\lambda[1]}_{\mu}\partial^{\rho}g^{[1]}_{\lambda\nu} + \partial_{\mu}g^{\lambda[1]}_{\rho}\partial_{\nu}g^{\rho[1]}_{\lambda} + \partial_{\mu}g^{\lambda[1]}_{\rho}\partial_{\lambda}g^{\rho[1]}_{\nu} - \partial_{\nu}g^{\lambda[1]}_{\rho}\partial^{\rho}g_{\lambda\mu[1]}].$$

$G_{\mu\nu}$ happens to be the same as $G_{\mu\nu}(\{\})$ for V_4. For U_4 on the other hand, $G_{\mu\nu}(\{\})$ is the Riemannian part of $G_{\mu\nu}$ (a symmetric tensor constructed from the Christoffel symbols).

APPENDIX B: CONSTRAINTS ON THE METRIC DUE TO THE ASYMPTOTIC FLATNESS CONDITION

1. Constraint on $g^{[1]}_{\mu\nu}$

From the analysis of Sec. III C one can argue that all the components of $G^{[1]}_{\mu\nu}$ are 0, which implies $\Box\bar{g}^{[1]}_{\mu\nu} = \Box g^{[1]}_{\mu\nu} = 0$, from Eq. (A4) for $\mu \neq \nu$ (off-diagonal terms). Gravitational waves are the nontrivial solutions to this equation. However, they do not respect asymptotic flatness. We are therefore obliged to consider the trivial solution, i.e., $g^{[1]}_{\mu\nu} = 0$ for the off-diagonal terms. In order to evaluate the diagonal terms, we consider the following general form of the metric:

$$g^{[1]}_{\mu\nu} = \begin{pmatrix} f^{[1]}_1 & 0 & 0 & 0 \\ 0 & f^{[1]}_2 & 0 & 0 \\ 0 & 0 & f^{[1]}_3 & 0 \\ 0 & 0 & 0 & f^{[1]}_4 \end{pmatrix}. \tag{B1}$$

Hence,

$$\bar{g}^{[1]}_{00} = \frac{f^{[1]}_1 + f^{[1]}_2 + f^{[1]}_3 + f^{[1]}_4}{2}, \tag{B2}$$

$$\bar{g}^{[1]}_{11} = \frac{f^{[1]}_1 + f^{[1]}_2 - f^{[1]}_3 - f^{[1]}_4}{2}, \tag{B3}$$

$$\bar{g}^{[1]}_{22} = \frac{f^{[1]}_1 + f^{[1]}_3 - f^{[1]}_2 - f^{[1]}_4}{2} \quad \text{and} \tag{B4}$$

$$\bar{g}^{[1]}_{33} = \frac{f^{[1]}_1 + f^{[1]}_4 - f^{[1]}_2 - f^{[1]}_3}{2}. \tag{B5}$$

Using the above equations, we get

$$\Box\bar{g}^{[1]}_{00} = \Box\frac{f^{[1]}_1 + f^{[1]}_2 + f^{[1]}_3 + f^{[1]}_4}{2} = 0$$
$$\Rightarrow \Box f^{[1]}_1 + \Box f^{[1]}_2 + \Box f^{[1]}_3 + \Box f^{[1]}_4 = 0, \tag{B6}$$

$$\Box\bar{g}^{[1]}_{11} = \Box\frac{f^{[1]}_1 + f^{[1]}_2 - f^{[1]}_3 - f^{[1]}_4}{2} = 0$$
$$\Rightarrow \Box f^{[1]}_1 + \Box f^{[1]}_2 = \Box f^{[1]}_3 + \Box f^{[1]}_4, \tag{B7}$$

$$\Box\bar{g}^{[1]}_{22} = \Box\frac{f^{[1]}_1 + f^{[1]}_3 - f^{[1]}_2 - f^{[1]}_4}{2} = 0$$
$$\Rightarrow \Box f^{[1]}_1 + \Box f^{[1]}_3 = \Box f^{[1]}_2 + \Box f^{[1]}_4 \quad \text{and} \tag{B8}$$

$$\Box\bar{g}^{[1]}_{33} = \Box\frac{f^{[1]}_1 + f^{[1]}_4 - f^{[1]}_2 - f^{[1]}_3}{2} = 0$$
$$\Rightarrow \Box f^{[1]}_1 + \Box f^{[1]}_4 = \Box f^{[1]}_2 + \Box f^{[1]}_3. \tag{B9}$$

Equations (B7)–(B9) imply

$$\Box f^{[1]}_2 = \Box f^{[1]}_1 \Rightarrow f^{[1]}_2 = f^{[1]}_1 + c_1, \tag{B10}$$

$$\Box f^{[1]}_3 = \Box f^{[1]}_1 \Rightarrow f^{[1]}_3 = f^{[1]}_1 + c_2 \quad \text{and} \tag{B11}$$

$$\Box f^{[1]}_4 = \Box f^{[1]}_1 \Rightarrow f^{[1]}_4 = f^{[1]}_1 + c_3. \tag{B12}$$

The constants c_1, c_2, and c_3 must be 0, as any one of them not being equal to 0 would violate the condition of asymptotic flatness. Hence, Eq. (B6) implies $4\Box f^{[1]}_1 = 0 \Rightarrow f^{[1]}_1 = 0$ (wave solutions and nonzero constants also satisfy the equation, but do not respect asymptotic flatness). Hence, $f^{[1]}_i = 0 \ \forall \ i$, which in turn implies

$$g^{[1]}_{\mu\nu} = 0 \ \forall \ \mu, \nu. \tag{B13}$$

2. Constraint on $g^{[2]}_{\mu\nu}$

From the analysis of Sec. III C one can argue that the off-diagonal components of $G^{[2]}_{\mu\nu}$ are 0. This implies $\Box\bar{g}^{[2]}_{\mu\nu} = \Box g^{[2]}_{\mu\nu} = 0$ $(\mu \neq \nu)$, from Eq. (A4). Following the same arguments of Appendix B 1, $g^{[2]}_{\mu\nu} = 0$ $(\mu \neq \nu)$ is the only allowed solution to the above equation. Once again, we consider the following general form of the metric in order to evaluate the diagonal terms:

$$g^{[2]}_{\mu\nu} = \begin{pmatrix} f^{[2]}_1 & 0 & 0 & 0 \\ 0 & f^{[2]}_2 & 0 & 0 \\ 0 & 0 & f^{[2]}_3 & 0 \\ 0 & 0 & 0 & f^{[2]}_4 \end{pmatrix}. \tag{B14}$$

Hence,

$$\bar{g}^{[2]}_{00} = \frac{f^{[2]}_1 + f^{[2]}_2 + f^{[2]}_3 + f^{[2]}_4}{2}, \tag{B15}$$

$$\bar{g}^{[2]}_{11} = \frac{f^{[2]}_1 + f^{[2]}_2 - f^{[2]}_3 - f^{[2]}_4}{2}, \tag{B16}$$

$$\bar{g}^{[2]}_{22} = \frac{f^{[2]}_1 + f^{[2]}_3 - f^{[2]}_2 - f^{[2]}_4}{2} \quad \text{and} \tag{B17}$$

$$\bar{g}^{[2]}_{33} = \frac{f^{[2]}_1 + f^{[2]}_4 - f^{[2]}_2 - f^{[2]}_3}{2}. \tag{B18}$$

At the second order, all the components of the Einstein tensor are 0, except for the 00 component. This implies

$$\Box\bar{g}^{[2]}_{00} = \Box\frac{f^{[2]}_1 + f^{[2]}_2 + f^{[2]}_3 + f^{[2]}_4}{2} \Rightarrow \Box f^{[2]}_1 + \Box f^{[2]}_2 + \Box f^{[2]}_3 + \Box f^{[2]}_4 \neq 0, \tag{B19}$$

$$\Box\bar{g}^{[2]}_{11} = \Box\frac{f^{[2]}_1 + f^{[2]}_2 - f^{[2]}_3 - f^{[2]}_4}{2} \Rightarrow \Box f^{[2]}_1 + \Box f^{[2]}_2 = \Box f^{[2]}_3 + \Box f^{[2]}_4, \tag{B20}$$

$$\Box\bar{g}^{[2]}_{22} = \Box\frac{f^{[2]}_1 + f^{[2]}_3 - f^{[2]}_2 - f^{[2]}_4}{2} \Rightarrow \Box f^{[2]}_1 + \Box f^{[2]}_3 = \Box f^{[2]}_2 + \Box f^{[2]}_4 \quad \text{and} \tag{B21}$$

$$\Box\bar{g}^{[2]}_{33} = \Box\frac{f^{[2]}_1 + f^{[2]}_4 - f^{[2]}_2 - f^{[2]}_3}{2} \Rightarrow \Box f^{[2]}_1 + \Box f^{[2]}_4 = \Box f^{[2]}_2 + \Box f^{[2]}_3. \tag{B22}$$

Equations (B20)–(B22) imply

$$\Box f^{[2]}_2 = \Box f^{[2]}_1 \Rightarrow f^{[2]}_2 = f^{[2]}_1, \tag{B23}$$

$$\Box f^{[2]}_3 = \Box f^{[2]}_1 \Rightarrow f^{[2]}_3 = f^{[2]}_1 \quad \text{and} \tag{B24}$$

$$\Box f^{[2]}_4 = \Box f^{[2]}_1 \Rightarrow f^{[2]}_4 = f^{[2]}_1. \tag{B25}$$

The absence of constants in the above equations follows from the arguments of Appendix B 1. Using Eqs. (B23)–(B25), we get $f^{[2]}_1 = f^{[2]}_2 = f^{[2]}_3 = f^{[2]}_4 = F(\mathbf{r}, t)$, hence,

$$g^{[2]}_{\mu\nu} = \begin{pmatrix} F(\mathbf{r},t) & 0 & 0 & 0 \\ 0 & F(\mathbf{r},t) & 0 & 0 \\ 0 & 0 & F(\mathbf{r},t) & 0 \\ 0 & 0 & 0 & F(\mathbf{r},t) \end{pmatrix}. \tag{B26}$$

APPENDIX C: METRIC AND CHRISTOFFEL SYMBOLS

The metric defined in Eq. (60) is of the form

$$g_{\mu\nu} = \begin{pmatrix} 1 + \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{C1}$$

and

$$g^{\mu\nu} = \begin{pmatrix} 1 - \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{C2}$$

Christoffel connections: For the above metric the nonzero Christoffel connections are

$$\Gamma^0_{0\mu} = \frac{\hbar\partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\Gamma^\mu_{00} = \frac{\hbar\partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \quad \text{and}$$
$$\Gamma^\mu_{\mu\mu} = \frac{-\hbar\partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right), \tag{C3}$$

where μ runs from 1 to 3 (spatial coordinates only). It is worth noting that the zeroth and first order terms in $(\frac{1}{c})$ are absent in Eq. (C3). The other nonzero Christoffel connections are of order 3 and higher in $(\frac{1}{c})$, which we do not mention here.

APPENDIX D: TETRADS

Tetrads were introduced in Sec. II. For the metric defined by

$$dS^2 = \left[1 + \frac{\hbar F(\mathbf{r}, t)}{c^2}\right] c^2 dt^2 - \left[1 - \frac{\hbar F(\mathbf{r}, t)}{c^2}\right] d\mathbf{r}^2, \tag{D1}$$

the tetrad fields over the entire manifold are given by

$$\hat{e}_{(0)} = \frac{1}{c}\left(1 + \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_t, \qquad \hat{e}_{(1)} = \left(1 - \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_x,$$
$$\hat{e}_{(2)} = \left(1 - \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_y \quad \text{and} \quad \hat{e}_{(3)} = \left(1 - \frac{\hbar F}{c^2}\right)^{\frac{1}{2}} \partial_z. \tag{D2}$$

The transformation matrices [defined in Eq. (13)] that relate the world components with the anholonomic components are given by

$$e_\mu^{(i)} = \begin{pmatrix} 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right), \tag{D3}$$

$$e^\mu_{(i)} = \begin{pmatrix} 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right), \tag{D4}$$

$$e_{\nu(k)} = \begin{pmatrix} 1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & -1 + \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \quad \text{and} \tag{D5}$$

$$e^{\nu(k)} = \begin{pmatrix} 1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & -1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{D6}$$

APPENDIX E: RIEMANNIAN PART OF THE SPIN CONNECTIONS ($\gamma^o_{(a)(b)(c)}$)

Using the relation between Christoffel connections and tetrad transformation matrices [Eq. (16)], the Riemannian part of the spin connections [defined by Eq. (14)] is obtained as follows:

$$\gamma^o_{(0)(0)(0)} = \frac{-\hbar\partial_0 F}{2c^2} \frac{(1 + \frac{\hbar F}{2c^2})}{(1 - \frac{\hbar F}{2c^2})} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(i)(0)(0)} = \left(\frac{-\hbar\partial_i F}{2c^2}\right) \frac{\hbar F/2c^2}{(1 + \frac{\hbar F}{2c^2})} + \sum_{n=5}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(0)(i)(0)} = \frac{-\hbar\partial_i F}{2c^2} \frac{(1 + \frac{\hbar F}{2c^2})}{(1 - \frac{\hbar F}{2c^2})} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(0)(0)(i)} = \frac{\hbar\partial_i F}{2c^2} \frac{1}{(1 + \frac{\hbar F}{2c^2})},$$
$$\gamma^o_{(i)(i)(i)} = \frac{\hbar\partial_i F}{2c^2} \frac{\hbar F/2c^2}{(1 + \frac{\hbar F}{2c^2})} + \sum_{n=5}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(i)(i)(0)} = \gamma^o_{(i)(0)(i)} = +\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(0)(i)(i)} = \frac{-\hbar\partial_0 F}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(0)(i)(j)} = \gamma^o_{i0j} = \gamma^o_{ij0} = +\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right),$$
$$\gamma^o_{(i)(j)(j)} = \frac{-\hbar\partial_0 F}{2c^2} \frac{(1 - \frac{\hbar F}{2c^2})}{(1 + \frac{\hbar F}{2c^2})} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)$$
$$\text{and} \quad \gamma^o_{(i)(j)(k)} = \gamma^o_{(i)(j)(i)} = \gamma^o_{(j)(j)(i)} = +\sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{E1}$$

The torsional part of the spin connections [defined by Eq. (14)] manifests itself as a nonlinear term in the Hehl-Datta equation. This term, being completely expressible in terms of the Dirac spinor, is evaluated using the spinor ansatz while deriving the nonrelativistic limit of the ECD system of equations.

APPENDIX F: EINSTEIN TENSOR

In this section, we aim to evaluate the Einstein tensor. $G^{[1]}_{\mu\nu}$ has already been shown to be 0. Since $g^{[1]}_{\mu\nu}$ is 0, $\mathrm{f}[g^{[1]}_{\mu\nu}]$ [defined in Eq. (A5)] is also 0. Using $g^{[2]}_{\mu\nu}$ (defined in Appendix B 2), $G^{[2]}_{\mu\nu}$ [Eq. (A5)] is evaluated as follows:

$$G^{[2]}_{\mu\nu} = -\frac{1}{2}\Box\bar{g}^{[2]}_{\mu\nu} \quad \text{where}$$
$$\bar{g}^{[2]}_{\mu\nu} = g^{[2]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}h_{\alpha\beta}), \text{ now} \tag{F1}$$

$$\eta^{\mu\nu}h_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix}$$
$$= \frac{-2\hbar F(\mathbf{r},t)}{c^2}; \tag{F2}$$

thus $G_{\mu\nu} = 0$ for $\mu \neq \nu$. The diagonal components are given by

$$G_{00} = -\frac{1}{2}\Box\bar{g}^{[2]}_{00} = -\frac{\hbar}{c^2}\Box F(\mathbf{r},t)$$
$$= \left[-\frac{\hbar\partial_t^2 F(\mathbf{r},t)}{c^4} + \frac{\hbar\nabla^2 F(\mathbf{r},t)}{c^2}\right] \tag{F3}$$

$$\text{and} \quad G_{\alpha\alpha} = 0 \quad \text{because } \bar{g}^{[2]}_{\alpha\alpha} = 0 \quad \text{for } \alpha \in (1,2,3). \tag{F4}$$

Thus,

$$G_{\mu\nu} = \frac{\hbar}{c^2}\begin{pmatrix} \nabla^2 F(\mathbf{r},t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{F5}$$

APPENDIX G: ANALYSIS OF THE COMPONENTS OF THE METRIC EM TENSOR

This section contains the calculations and proofs for some of the results used in Sec. III C.

1. Analysis of $kT_{0\mu}$

After excluding the terms containing the spin coefficients $\gamma_{\mu(i)(j)}$, $kT_{0\mu}$ [Eq. (53)] is given by

$$kT_{0\mu} = \frac{2i\pi G\hbar}{c^4}[c\bar{\psi}\gamma^0\partial_\mu\psi - c\bar{\psi}\gamma^\mu\partial_0\psi - c\partial_\mu\bar{\psi}\gamma^0\psi + c\partial_0\bar{\psi}\gamma^\mu\psi] \tag{G1}$$

$$= \frac{-2i\pi G\hbar}{c^3}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right) \times [\bar{\psi}\gamma^{(0)}\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^{(0)}\psi] + \frac{2i\pi G\hbar}{c^4}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(a)}\right) \times [\partial_t\bar{\psi}\gamma^{(a)}\psi - \bar{\psi}\gamma^{(a)}\partial_t\psi]. \tag{G2}$$

The first term on the right-hand side of Eq. (G2) is

$$= \frac{2i\pi G\hbar}{c^3}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n (a^\dagger_{n_1}\partial_\mu a_{n_2} - \partial_\mu a^\dagger_{n_1}a_{n_2}) \quad (n = n_1 + n_2)$$
$$= \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right), \tag{G3}$$

while the second term is

$$= \frac{2i\pi G}{c^2}\left\{\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a^\dagger_n\right)\alpha^{(a)}\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m [iSa_m + \dot{a}_{m-2}]\right) + \left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n [iSa^\dagger_n - \dot{a}^\dagger_{n-2}]\right)\alpha^{(a)}\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m a_m\right)\right\} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)$$
$$= \frac{4\pi Gm}{c^2}(a^\dagger_0\alpha^{(a)}a_0) + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)$$
$$= \frac{4\pi Gm}{c^2}\left[\begin{pmatrix} a^>_0 & 0 \end{pmatrix}^\dagger \begin{pmatrix} 0 & \sigma^{(a)} \\ \sigma^{(a)} & 0 \end{pmatrix}\begin{pmatrix} a^>_0 \\ 0 \end{pmatrix}\right] + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)$$
$$= \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \tag{G4}$$

H ence, there is no contribution at the second order.

2. Analysis of $kT_{\mu\nu}$

After excluding the terms containing the spin coefficients $\gamma_{\mu(i)(j)}$, $kT_{\mu\nu}$ [Eq. (55)] is given by

$$
\begin{aligned}
kT_{\mu\nu} &= \frac{2i\pi G\hbar}{c^3}[-\bar{\psi}\gamma^{\mu}\partial_{\nu}\psi - \bar{\psi}\gamma^{\nu}\partial_{\mu}\psi + \partial_{\nu}\bar{\psi}\gamma^{\mu}\psi + \partial_{\mu}\bar{\psi}\gamma^{\nu}\psi] \\
&= \frac{2i\pi G\hbar}{c^3}\left(1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(a)}\right)[\psi^{\dagger}\alpha^{(a)}\partial_{\nu}\psi - \partial_{\nu}\psi^{\dagger}\alpha^{(a)}\psi] + \frac{2i\pi G\hbar}{c^3}\left(1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\nu[n]}_{(b)}\right)[\partial_{\mu}\psi^{\dagger}\alpha^{(b)}\psi - \psi^{\dagger}\alpha^{(b)}\partial_{\mu}\psi] \\
&= \frac{2i\pi G\hbar}{c^3}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n (e^{\mu}_{(a)}a^{\dagger}_{n_1}\alpha^{(a)}\partial_{\nu}a_{n_2} - e^{\mu}_{(a)}\partial_{\nu}a^{\dagger}_{n_1}\alpha^{(a)} + e^{\nu}_{(b)}a^{\dagger}_{n_1}\alpha^{(b)}\partial_{\mu}a_{n_2} - e^{\nu}_{(b)}\partial_{\mu}a^{\dagger}_{n_1}\alpha^{(b)}a_{n_2}) \\
&= \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right). \qquad \text{(G5)}
\end{aligned}
$$

Hence, there is no contribution at the second order.

APPENDIX H: GENERIC COMPONENTS OF $T_{\mu\nu}$

Using the spin connections of Appendix E, we analyze the metric energy-momentum tensor [Eq. (23)], whose components are given below.

$$
T_{\mu\nu} = \frac{i\hbar c}{4}\begin{pmatrix}
\begin{array}{c} 2\bar{\psi}\gamma_0(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi}\Big)2\gamma_0\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_0\partial_1\psi+\bar{\psi}\gamma_1(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\partial_1\bar{\psi}\gamma_0\psi-\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi}\Big)\gamma_1\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_0\partial_2\psi+\bar{\psi}\gamma_2(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\partial_2\bar{\psi}\gamma_0\psi-\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi}\Big)\gamma_2\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_0\partial_3\psi+\bar{\psi}\gamma_3(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\partial_3\bar{\psi}\gamma_0\psi-\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi}\Big)\gamma_3\psi \end{array} \\
\begin{array}{c} \bar{\psi}\gamma_1(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ +\bar{\psi}\gamma_0\partial_1\psi-\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\ +\gamma_{0\alpha 0}\gamma^i\gamma^0]\bar{\psi}\Big)\gamma_1\psi-\partial_1\bar{\psi}\gamma_0\psi \end{array} &
2(\bar{\psi}\gamma_1\partial_1\psi-\partial_1\psi\bar{\gamma}_1\psi) &
\begin{array}{c} \bar{\psi}\gamma_1\partial_2\psi+\bar{\psi}\gamma_2\partial_1\psi \\ -\partial_2\bar{\psi}\gamma_1\psi-\partial_1\bar{\psi}\gamma_2\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_1\partial_3\psi+\bar{\psi}\gamma_3\partial_1\psi \\ -\partial_3\bar{\psi}\gamma_1\psi-\partial_1\bar{\psi}\gamma_3\psi \end{array} \\
\begin{array}{c} \bar{\psi}\gamma_2(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ +\bar{\psi}\gamma_0\partial_2\psi-\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\ +\gamma_{0\alpha 0}\gamma^i\gamma^0]\bar{\psi}\Big)\gamma_2\psi-\partial_2\bar{\psi}\gamma_0\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_2\partial_1\psi+\bar{\psi}\gamma_1\partial_2\psi \\ -\partial_1\bar{\psi}\gamma_2\psi-\partial_2\bar{\psi}\gamma_1\psi \end{array} &
2(\bar{\psi}\gamma_2\partial_2\psi-\partial_2\psi\bar{\gamma}_2\psi) &
\begin{array}{c} \bar{\psi}\gamma_2\partial_3\psi+\bar{\psi}\gamma_3\partial_2\psi \\ -\partial_3\bar{\psi}\gamma_2\psi-\partial_2\bar{\psi}\gamma_3\psi \end{array} \\
\begin{array}{c} \bar{\psi}\gamma_3(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha+\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ +\bar{\psi}\gamma_0\partial_3\psi-\Big(\partial_0\bar{\psi}+\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\ +\gamma_{0\alpha 0}\gamma^i\gamma^0]\bar{\psi}\Big)\gamma_3\psi-\partial_3\bar{\psi}\gamma_0\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_3\partial_1\psi+\bar{\psi}\gamma_1\partial_3\psi \\ -\partial_1\bar{\psi}\gamma_3\psi-\partial_3\bar{\psi}\gamma_1\psi \end{array} &
\begin{array}{c} \bar{\psi}\gamma_3\partial_2\psi+\bar{\psi}\gamma_2\partial_3\psi \\ -\partial_2\bar{\psi}\gamma_3\psi-\partial_3\bar{\psi}\gamma_2\psi \end{array} &
2(\bar{\psi}\gamma_3\partial_3\psi-\partial_3\psi\bar{\gamma}_3\psi)
\end{pmatrix} \qquad \text{(H1)}
$$

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