

## Critical behavior of Lifshitz dilaton black holes

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(Received 24 August 2018; published 16 November 2018)

Until now, the critical behavior of Lifshitz black holes, in an extended  $P - v$  space, has not been studied, because it is impossible to find an analytical equation of state,  $P = P(v, T)$ , for an arbitrary Lifshitz exponent  $z$ . In this paper, we adopt a new approach toward thermodynamic phase space and successfully explore the critical behavior of  $(n + 1)$ -dimensional Lifshitz dilaton black holes. The most important advantage of this approach is that we keep the cosmological constant as a constant without needing to vary it. For this purpose, we write down the equation of state as  $Q^s = Q^s(T, \Psi)$ , where  $\Psi = (\partial M / \partial Q^s)_{S, P}$  is the conjugate of  $Q^s$ , and construct a Smarr relation based on this new phase space as  $M = M(S, Q^s, P)$ , where  $s = 2p / (2p - 1)$ , with  $p$  the power of the power-law Maxwell Lagrangian. We justify such a choice mathematically and show that with this new phase space, the system admits the critical behavior and resembles the van der Waals fluid system when the cosmological constant (pressure) is treated as a fixed parameter, while the charge of the system varies. We obtain the Gibbs free energy of the system and find a swallowtail shape in Gibbs diagrams, which represents the first-order phase transition. Finally, we calculate the critical exponents and show that although thermodynamic quantities depend on the metric parameters such as  $z$ ,  $p$ , and  $n$ , the critical exponents are the same as the van der Waals fluid-gas system. This alternative viewpoint of the phase space of a Lifshitz dilaton black hole can be understood easily since one can imagine such a change for a given single black hole, i.e., acquiring charge, which induces the phase transition. Our results further support the viewpoint suggested in [A. Dehyadegari, A. Sheykhi, and A. Montakhab, *Phys. Lett. B* **768**, 235 (2017)].

DOI: 10.1103/PhysRevD.98.104026

### I. INTRODUCTION

Historically, Maldacena was the first who suggested, two decades ago, the correspondence between gravity in an anti-de Sitter (AdS) spacetime and the conformal field theory (CFT) living on the boundary of spacetime, known as AdS/CFT correspondence [1]. According to Maldacena's conjecture, the effects of string theory in a  $d$ -dimensional  $\text{AdS}_{n+1} \times S^{d-n-1}$  spacetime can appear in the form of a field theory on an  $n$ -dimensional  $r$ -constant brane which is the boundary of an  $\text{AdS}_{n+1}$  spacetime. This idea has attracted a lot of enthusiasm and has been investigated from various points of view [2]. The metric of the AdS spacetime is given by

$$ds^2 = -\frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 \sum_{i=1}^{n-1} dx_i^2, \quad (1)$$

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which is invariant under an isotropic conformal transformation as follows:

$$t \rightarrow \lambda t, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow \lambda^{-1} r. \quad (2)$$

Here, we further propose the gravitational dual description of a class of critical phenomena exhibiting unconventional scaling of the following form, which is essential for generalized gauge-gravity duality:

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow \lambda^{-1} r, \quad (3)$$

where  $z$  is a dynamical critical exponent and is restricted as  $z > 1$ . This parameter shows the degrees of anisotropy between space and time. The Lifshitz spacetime was first introduced in [3,4] as

$$ds^2 = -\frac{r^{2z}}{l^{2z}} dt^2 + \frac{l^2 dr^2}{r^2} + r^2 \sum_{i=1}^{n-1} dx_i^2. \quad (4)$$

The Lifshitz spacetime is not a vacuum solution of Einstein gravity and so needs a matter source. Usually a massive gauge field plays the role of this matter source,

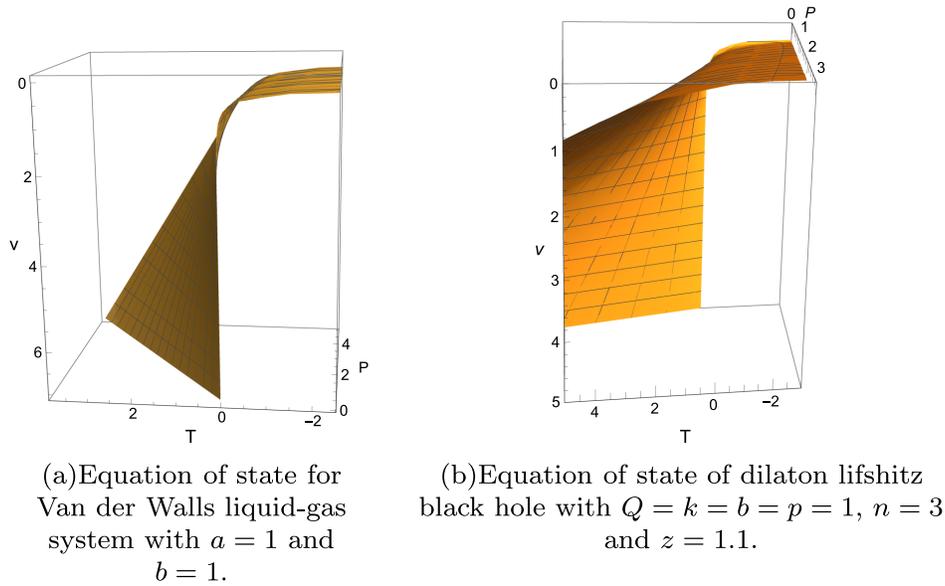


FIG. 1. The 3D diagrams of equations of state for a van der Waals fluid system and Lifshitz black holes. Comparing two diagrams indicates that these systems have a similar phase transition.

but it is nearly impossible to obtain an analytic solution for arbitrary  $z$  in such models. As shown in Ref. [5], considering a dilaton field, instead of a massive gauge field with more than one electromagnetic field, can lead to exact analytical solutions in Lifshitz spacetime (see also [6]). Another motivation is that string theory in its low energy limit reduces to Einstein gravity with a scalar dilaton field coupled to gravity and other fields [7].

On the other hand, studies of the critical behavior of black holes have received a great deal of attention in a wide range of gravity theories. For example, the critical behavior of charged AdS black holes has been studied in [8] and the author completed the analogy between Reissner-Nordström-AdS black holes with the van der Waals liquid-gas system, with the same critical exponents. The key assumption is to enlarge the thermodynamic phase space to include the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume [9–14]. When the gauge field is the Born-Infeld nonlinear electrodynamics, one needs a more extended phase space to introduce a new thermodynamic quantity conjugate to the Born-Infeld parameter which is necessary for the consistency of both the first law of thermodynamics and the corresponding Smarr relation [15]. Treating the cosmological constant as a thermodynamic pressure, thermodynamics and the  $P-v$  criticality of black holes in an extended phase space in the presence of power Maxwell [16] and exponential nonlinear electrodynamics [17] have been explored. The studies were also generalized to other gravity theories. In this regard, the phase structure of asymptotically AdS black holes with higher curvature

corrections such as Gauss-Bonnet [18,19] and Lovelock gravity [20] have also been investigated. The studies were also extended to the rotating black holes, where the phase transition and critical behavior of Myers-Perry black holes have been investigated [21]. Other studies on the critical behavior of black hole spacetimes in an extended phase space have been carried out in [22–26].

The critical behavior of the Einstein-Maxwell dilaton black holes has been studied in [27]. When the gauge field is in the form of a Born-Infeld [28] and power Maxwell [29] field, the critical behavior of  $(n+1)$ -dimensional dilaton black holes in an extended phase space has been investigated. Taking into account the dilaton field in the presence of logarithmic and exponential forms of nonlinear electrodynamics, and considering the cosmological constant and nonlinear parameter as thermodynamic quantities which can vary, it was shown that there is indeed a complete analogy between the nonlinear dilaton black holes with a van der Waals liquid-gas system [30]. In all the studies mentioned above, one assumes the charge of the black hole as an external fixed parameter and treats the cosmological constant as the pressure of the system, which can vary.

In the present work, we would like to investigate the critical behavior of Lifshitz black holes in Einstein-dilaton gravity in the presence of a power-law Maxwell field. It is worthwhile to mention that the  $(n+1)$ -dimensional Lagrangian in power Maxwell theory is conformally invariant provided  $p = (n+1)/4$ , where  $p$  is the power of the Lagrangian. Let us first plot a three-dimensional diagram for the equation of state of a Lifshitz black hole to understand the phase behavior of this system and show its analogy with a van der Waals liquid-gas system

(see Fig. 1). A close look at the temperature expression of Lifshitz dilaton black holes [see [31] and Eq. (27) of the present paper] shows that it is nearly impossible to solve this equation for  $P$  (or more precisely for  $l$ ). Therefore, we cannot have an analytical equation of state,  $P = P(v, T)$ , to investigate the critical behavior or calculate critical quantities of Lifshitz black holes. It is important to note that as one can see from the three-dimensional diagram in Fig. 1, there is indeed a  $P - v$  criticality for the Lifshitz dilaton black holes, similar to the van der Waals liquid-gas system. This implies that one may use the numerical calculations to investigate  $P - v$  criticality of this system and obtain the critical quantities in an extended phase space, at least approximately. However, in this work, we would like to consider a more fascinating and straightforward way to investigate the critical behavior of this general type of black hole, which includes dilaton, power Maxwell field, and Lifshitz effects. In particular, we shall show that the system admits a critical behavior similar to the van der Waals fluid, without needing to extend the phase space.

Another way to investigate the critical behavior of the black holes is to use the method of Refs. [32,33], but as shown in [34], such a view of thermodynamic conjugate variables ( $Q$  and  $\Phi = Q/r_+$ ) which are not mathematically independent can lead to physically irrelevant quantities such as  $(\partial Q/\partial \Phi)_T$ , which is supposed to be a thermodynamic response function, but mathematically ill defined.

To address this problem, an alternative viewpoint toward the thermodynamic phase space of black holes was developed in [34] by treating the cosmological constant as a fixed parameter and considering the charge of the black hole as a thermodynamic variable [35,36]. The cosmological constant is assumed as a constant related to the background of AdS geometry in general relativity or the zero point energy of the background of spacetime in field theory. So, from a physical standpoint, it is difficult to consider the cosmological constant as a pressure of the black hole which can take an arbitrary value. It seems more physical to consider the variation of charge  $Q$  of a black hole instead of the cosmological constant, because the charge of the black hole is really a natural external variable which can vary by absorbing or emitting charged particles. It was argued that, with a fixed cosmological constant, the critical behavior indeed occurs in the  $Q^2 - \Psi$  plane, where  $\Psi = 1/2r_+$  is the conjugate of  $Q^2$ , and thus the equation of state is written as  $Q^2 = Q^2(T, \Psi)$ . We find out that in the case of Lifshitz dilaton black holes, the system admits a critical behavior provided we take the electrodynamics in the form of a power Maxwell field and considering  $Q^s$  as a thermodynamic variable with  $\Psi = (\partial M/\partial Q^s)_{S,P}$  as its conjugate, where  $s = 2p/(2p - 1)$ . In this case we can define a new response function which naturally leads to a physically relevant quantity. Thus, the equation of state is

written in the form of  $Q^s = Q^s(T, \Psi)$  and a Smarr relation based on this new phase space as  $M = M(S, Q^s, P)$ . Clearly, for  $p = 1$ , the power Maxwell field reduces to the standard Maxwell field and  $Q^s \rightarrow Q^2$ . Following [34], in this approach we keep the cosmological constant (pressure) as a fixed quantity, while the charge of the system can vary.

This paper is outlined as follows. In the next section, we present the action and the basic field equations of Lifshitz dilaton black holes, and we review thermodynamic properties of this system. In Sec. III A, we study the phase structure of the solution and present the modified Smarr relation. In Sec. III B, we obtain the equation of state and study the critical behavior of the solutions and compare them with a van der Waals fluid system. We investigate the Gibbs free energy and the critical exponents of the system in Secs. III C and III D, respectively. The last section is devoted to a summary and conclusion.

## II. THERMODYNAMICS OF LIFSHITZ DILATON BLACK HOLES

In this section we are going to review the solutions of charged Lifshitz black holes with a power Maxwell field [31], with an emphasis on their thermodynamic properties. The  $(n + 1)$ -dimensional action of Einstein-dilaton gravity in the presence of a power Maxwell electromagnetic and two linear Maxwell fields can be written as

$$S = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left\{ \mathcal{R} - \frac{4}{n-1} (\nabla\Phi)^2 - 2\Lambda + (-e^{-4/(n-1)\lambda_1\Phi} F)^p - \sum_{i=2}^3 e^{-4/(n-1)\lambda_i\Phi} H_i \right\}, \quad (5)$$

where  $\mathcal{R}$  is the Ricci scalar on manifold  $\mathcal{M}$ ,  $\Phi$  is the dilaton field, and  $\lambda_1$  and  $\lambda_i$  are constants. In Eq. (5)  $F = F_{\mu\nu} F^{\mu\nu}$  and  $H_i$  are the Maxwell invariants of electromagnetic fields, where  $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$  and  $(H_i)_{\mu\nu} = \partial_{[\mu} (B_i)_{\nu]}$ , with  $A_\mu$  and  $(B_i)_\mu$  the electromagnetic potentials. Varying the action (5) with respect to the metric  $g_{\mu\nu}$ , the dilaton field  $\Phi$ , with electromagnetic potentials  $A_\mu$  and  $(B_i)_\mu$ , leads to the following field equations [31]:

$$\begin{aligned} \mathcal{R}_{\mu\nu} = & \frac{g_{\mu\nu}}{n-1} \left\{ 2\Lambda + (2p-1)(-F e^{-4\lambda_1\Phi/(n-1)})^p \right. \\ & \left. - \sum_{i=2}^3 H_i e^{-4\lambda_i\Phi/(n-1)} \right\} + \frac{4}{n-1} \partial_\mu \Phi \partial_\nu \Phi \\ & + 2p e^{-4\lambda_1 p \Phi/(n-1)} (-F)^{p-1} F_{\mu\lambda} F_\nu{}^\lambda \\ & + 2 \sum_{i=2}^3 e^{-4\lambda_i \Phi/(n-1)} (H_i)_{\mu\lambda} (H_i)_{\nu}{}^\lambda, \end{aligned} \quad (6)$$

$$\nabla^2 \Phi - \frac{p\lambda_1}{2} e^{-4\lambda_1 \Phi / (n-1)} (-F)^p + \sum_{i=2}^3 \frac{\lambda_i}{2} e^{-4\lambda_i \Phi / (n-1)} H = 0, \quad (7)$$

$$\nabla_\mu (e^{-4\lambda_1 \Phi / (n-1)} (-F)^{p-1} F^{\mu\nu}) = 0, \quad (8)$$

$$\nabla_\mu (e^{-4\lambda_i \Phi / (n-1)} (H_i)^{\mu\nu}) = 0. \quad (9)$$

We assume the line element of the higher-dimensional asymptotic Lifshitz spacetime has the following form [31]:

$$ds^2 = -\frac{r^{2z} f(r)}{l^{2z}} dt^2 + \frac{l^2 dr^2}{r^2 f(r)} + r^2 d\Omega_{n-1}^2, \quad (10)$$

where  $d\Omega_{n-1}^2$  is an  $(n-1)$ -dimensional hypersurface with constant curvature  $(n-1)(n-2)k$  and volume  $\omega_{n-1}$ . Following the method of [31], one can find the solutions of the field equations (6)–(9) as

$$f(r) = 1 - \frac{m}{r^{n-1+z}} + \frac{kl^2(n-2)^2}{(z+n-3)^2 r^2} + \frac{q^{2p}}{r^{\Gamma+z+n-1}}, \quad (11)$$

$$\Phi(r) = \frac{(n-1)\sqrt{z-1}}{2} \ln\left(\frac{r}{b}\right), \quad (12)$$

$$(A_1)_t = -\frac{q_1 b^{2(z-1)}}{\Gamma r^\Gamma}, \quad (13)$$

$$(A_2)_t = \sqrt{\frac{z-1}{2(n+z-1)}} \frac{r^{n+z-1}}{l^z b^{n-1}}, \quad (14)$$

$$(A_3)_t = \frac{\sqrt{k(n-1)(n-2)(z-1)} r^{z+n-3}}{\sqrt{2}(z+n-3)^{3/2} l^{z-1} b^{n-2}}, \quad (15)$$

where

$$\Gamma = z - 2 + (n-1)/(2p-1), \quad (16)$$

$$q^{2p} = \frac{(2p-1)b^{2(z-1)}}{(n-1)l^{-2p(z-1)-2}\Gamma} (2q_1^2)^p, \quad (17)$$

$$\Lambda = -\frac{(z+n-1)(z+n-2)}{2l^2}. \quad (18)$$

It was argued in [31] that  $p$  and  $z$  are restricted as follows:

$$\begin{aligned} &\text{for } p < 1/2, \quad z-1 > (n-2p)/(1-2p), \\ &\text{for } 1/2 < p \leq n/2, \quad \text{all } z(\geq 1) \text{ values are allowed,} \\ &\text{for } p > n/2, \quad z-1 > (2p-n)/(2p-1). \end{aligned} \quad (19)$$

Using the modified Brown and York formalism [37], one can calculate the mass of the solution per unit volume  $\omega_{n-1}$  as [31]

$$M = \frac{(n-1)m}{16\pi l^{z+1}}, \quad (20)$$

where the mass parameter  $m$  can be written in term of the horizon radius  $r_+$  by using the fact that  $f(r_+) = 0$ . We find

$$m(r_+) = r_+^{z+n-1} + \frac{kl^2(n-2)^2 r_+^{z+n-3}}{(z+n-3)^2} + \frac{q^{2p}}{r_+^\Gamma}. \quad (21)$$

One can also calculate the charge of the black hole by applying the Gauss law

$$Q = \frac{1}{4\pi} \int r^{n-1} e^{-4\lambda_1 \Phi / (n-1)} (-F)^{p-1} F_{\mu\nu} n^\mu u^\nu d\Omega, \quad (22)$$

where  $n^\mu$  and  $u^\nu$  are the unit spacelike and timelike normals to the hypersurface of radius  $r$  given as

$$\begin{aligned} n^\mu &= \frac{1}{\sqrt{-g_{tt}}} dt = \frac{l^z}{r^z \sqrt{f(r)}} dt, \\ u^\nu &= \frac{1}{\sqrt{g_{rr}}} dr = \frac{r \sqrt{f(r)}}{l} dr. \end{aligned}$$

Using (22), we obtain the charge per unit volume  $\omega_{n-1}$  as

$$Q = \frac{2^{p-1} (q_1 l^{z-1})^{2p-1}}{4\pi}. \quad (23)$$

The electric potential  $U$ , measured at infinity with respect to the horizon, is defined by

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+}, \quad (24)$$

where  $\chi = p\partial_t$  is the null generator of the horizon. Using (13), we can obtain the electric potential

$$U = \frac{pq_1 b^{2(z-1)}}{\Gamma r_+^\Gamma}. \quad (25)$$

The entropy of the black holes can be calculated by using the area law of the entropy which is applied to almost all kinds of black holes in Einstein gravity including dilaton black holes. Thus, the entropy of our solutions per unit volume  $\omega_{n-1}$  is

$$S = \frac{r_+^{n-1}}{4}. \quad (26)$$

The Hawking temperature can also be obtained as

$$\begin{aligned} T_+ &= \frac{r_+^{z+1} f'(r_+)}{4\pi l^{z+1}} \\ &= \frac{1}{4\pi} \left\{ \frac{(n-1+z)r_+^z}{l^{z+1}} + \frac{k(n-2)^2 r_+^{z-2}}{l^{z-1}(z+n-3)} - \frac{\Gamma q^{2p}}{l^{z+1} r_+^{\Gamma+n-1}} \right\}. \end{aligned} \quad (27)$$

As one can see from expression (27), it is nearly impossible to solve this equation for  $P$  (or more precisely for  $l$ ) and write an analytical equation of state,  $P = P(v, T)$ , for an arbitrary Lifshitz exponent  $z$ . This implies that, for the Lifshitz dilaton black holes, one cannot investigate the critical behavior of the system through an extended  $P - v$  phase space by treating the cosmological constant (pressure) as a thermodynamic variable. However, as we shall see in the next section, it is quite possible to investigate the critical behavior of this system through a new  $Q^s - \Psi$  phase space and show its similarity with a van der Waals fluid system.

### III. CRITICAL BEHAVIOR OF LIFSHITZ DILATON BLACK HOLES

#### A. Phase structure

It is now generally accepted that charged black holes in AdS spaces allow critical behavior similar to the van der Waals fluid system, provided one treats the cosmological constant as a thermodynamic variable (pressure) in an extended phase space [8]. Also it has been shown in [34] that there is a deeper connection between charged AdS black holes and a van der Waals fluid system. Indeed, it was argued that similar behavior can be found without extending the phase space [34] by keeping the cosmological constant as a fixed parameter. The key assumption in this picture is to treat the square of the charge of the black hole,  $Q^2$ , as a thermodynamic variable instead of charge  $Q$  [34]. Also, the equation of state has been written as  $Q^2 = Q^2(T, \Psi)$ , where  $\Psi = 1/v$  (conjugate of  $Q^2$ ) is the inverse of the specific volume. With this new picture, the authors completed the analogy of charged AdS black holes as a van der Waals fluid system with exactly the same critical exponents. In this section, we would like to consider Lifshitz dilaton black holes with a power-law Maxwell field and investigate their critical behavior, as well as an analogy with a van der Waals fluid for this system.

The usual first law of thermodynamics in an extended phase space is in the form of

$$dM = TdS + VdP + UdQ. \quad (28)$$

From this point of view, the usual Smarr relation which was obtained from thermodynamic variables (25)–(27) and mass (20) can be written as

$$M = \frac{n-1}{z+n-3} TS + \frac{-2}{z+n-3} VP + \frac{2p-1}{2p} \left(1 + \frac{\Gamma}{z+n-3}\right) UQ, \quad (29)$$

where

$$P = \frac{n(n-1)r_+^{z-1}}{16\pi l^{z+1}}, \quad V = \int 4Sdr_+ = \frac{r_+^n \omega_{n-1}}{n}. \quad (30)$$

The definition of pressure and volume are chosen such that they can satisfy the Smarr formula and reduce to the definitions in [8,27–30] in limiting cases. One may define  $P = -\Lambda/8\pi$  such as [38,39], but in this way the volume should be redefined, too. It was shown in [34] that by replacing the term  $UdQ$  in the first law with the term  $\Psi dQ^2$ , the system allows critical behavior similar to the van der Waals fluid system. First of all, let us review the motivation of this selection. The well-known thermodynamic quantities of AdS black holes are given as [8,34]

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} + \frac{r_+^3}{2l^2}, \quad (31)$$

$$T = \frac{1}{4\pi r_+} \left(1 + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2}\right), \quad (32)$$

$$U = \frac{Q}{r_+} \Rightarrow QU = \frac{Q^2}{r_+}, \quad (33)$$

and the usual Smarr formula is [8,34]

$$M = 2(TS - VP) + QU. \quad (34)$$

It is clear that  $M$ ,  $T$ , and the term  $(QU)$  in the Smarr formula include terms proportional to the square of the charge of a black hole  $Q^2$  (not the black hole charge  $Q$ ). Therefore, when pressure ( $\Lambda$ ) is a fixed parameter,  $Q^2$  is the best choice as a new variable [34] and one may replace  $UdQ$  with  $\Psi d^2Q$ , where  $\Psi = 1/2r_+$  is the conjugate of  $Q^2$  [34]. Therefore, inspired by the expression (32), one may write the equation of state as [34]

$$Q^2(T, \Psi) = r_+^2 + \frac{3r_+^4}{l^2} - 4\pi r_+^3 T. \quad (35)$$

As mentioned before, in the case of constant  $l$  (or  $\Lambda$ ), the equation of state (35) leads to a critical behavior similar to a van der Waals fluid system [34]. Now, we are going to employ this approach for charged Lifshitz black holes with a power Maxwell field. Let us write thermodynamic quantities in terms of  $Q$  (the charge of a black hole) instead of  $q$  (the charge parameter). It is a matter of calculation to show by using Eqs. (17) and (23) that the mass (20), temperature (27), and electric potential (25) are the same as

$$M = \frac{(n-1)\omega_{n-1}}{16\pi l^{z+1}} \left\{ r_+^{z+n-1} + \frac{kl^2(n-2)^2 r_+^{z+n-3}}{(z+n-3)^2} + \frac{(2p-1)b^{2z-2} l^2 \pi^s Q^s}{(n-1)\Gamma 2^{\frac{zs}{2}} \omega_{n-1} r_+^\Gamma} \right\} \quad (36)$$

$$T_+ = \frac{1}{4\pi} \left\{ \frac{(n-1+z)r_+^z}{l^{z+1}} + \frac{k(n-2)^2 r_+^{z-2}}{l^{z-1}(z+n-3)} - \frac{(2p-1)b^{2z-2} 2^{\frac{5s}{2}} \pi^s Q^s}{(n-1)l^{z-1} \omega_{n-1}^s r_+^{\Gamma_+^{n-1}}} \right\} \quad (37)$$

$$U = \frac{pb^{2(z-1)}}{\Gamma l^{z-1} r_+^\Gamma} \left( \frac{\pi Q}{2^{3-p} w} \right)^{(s-1)} \\ \Rightarrow QU = \frac{pb^{2(z-1)}}{\Gamma l^{z-1} r_+^\Gamma} \left( \frac{\pi}{2^{3-p} w} \right)^{(s-1)} Q^s, \quad (38)$$

where

$$s = 2p/(2p-1). \quad (39)$$

It is clear that the charge of a black hole appears as  $Q^s$  in the above equations. This motivates us to choose the new thermodynamic variable as  $Q^s$ , which can simplify all the calculations. Also, from Eq. (37) we see that any other choice except  $Q^s$  makes the equation of state so complicated that it cannot be solved analytically to investigate its critical behavior and critical quantities, while selecting  $Q^s$  as a new variable creates a simple and solvable equation of state. Fortunately, in the limit of  $p = z = 1$  and  $n = 3$ ,  $Q^s \rightarrow Q^2$ ; i.e., our result reduces to that of [34]. As we will show in the next sections, the system allows critical behavior similar to the van der Waals fluid with a fixed cosmological constant by replacing  $UdQ$  with  $\Psi dQ^s$ . Thus, we write down the first law in the form

$$dM = TdS + VdP + \Psi dQ^s, \quad (40)$$

where  $T = (\partial M / \partial S)_{P, Q^s}$ ,  $V = (\partial M / \partial P)_{S, Q^s}$ , and the conjugate of  $Q^s$  is

$$\Psi = \left( \frac{\partial M}{\partial Q^s} \right)_{S, P} = \frac{(2p-1)\omega_{2p-1}^{\frac{1}{2}} b^{(2z-2)} \pi^{\frac{1}{2p-1}}}{16\Gamma l^{z-1} r_+^{\frac{\eta}{2p-1}}}, \quad (41)$$

while  $\eta = 2pz + n - 4p - z + 1$ . When  $p = z = 1$  and in three dimensions, the above definition for  $\Psi$  reduces to  $\Psi = 1/(2r_+)$  [34]. In this new picture, we can write the Smarr formula for the charged Lifshitz dilaton black hole as

$$M = \frac{n-1}{z+n-3} TS + \frac{-2}{z+n-3} VP \\ + \frac{2p-1}{2p} \left( 1 + \frac{\Gamma}{z+n-3} \right) \Psi Q^s. \quad (42)$$

It is worth noting that we have replaced the usual  $\Phi dQ$  term in the first law with  $\Psi dQ^s$ . The extended phase space associated with  $P = -\Lambda/(8\pi)$  is still the same.

Using (38) and (41), straightforward calculations show  $QU = \Psi Q^s$  and so the Smarr equations (29) and (42) are the same.

## B. Equation of state

Using Eq. (41) and treating pressure or more precisely  $l$  as a fixed parameter, Eq. (27) can be written as

$$Q^s = \frac{(z+n-1)Y^{2\beta}}{Xl^{z+1}} \Psi^{-\frac{2(2p-1)\beta}{\eta}} + \frac{k(n-2)^2 Y^{2\delta}}{(z+n-3)Xl^{z-1}} \Psi^{-\frac{2(2p-1)\delta}{\eta}} \\ - \frac{4\pi T}{X} Y^{2p-1} \Psi^{-\frac{\alpha}{\eta}}, \quad (43)$$

where

$$\alpha = 2np + 2pz - 6p - z + 2, \\ \beta = \frac{np + 2pz - 3p - z + 1}{2p-1}, \\ \delta = \frac{np + 2pz - 5p - z + 2}{2p-1}, \\ \eta = 2pz + n - 4p - z + 1, \quad (44)$$

and

$$X = \frac{(2p-1)b^{2z-2} \pi^{\frac{5s}{2}}}{l^{z-1} \omega^s (n-1)}, \\ Y = \left( \frac{(2p-1)b^{2(z-1)} 2^{\frac{5s}{2}} \pi^{\frac{s}{2p}}}{16l^{z-1} \Gamma \omega^{2p}} \right)^{\frac{2p-1}{\eta}}. \quad (45)$$

In order to compare the critical behavior of the system with a van der Waals gas, one may plot isotherm diagrams  $Q^s - \Psi$ , which are displayed in Figs. 2–4. As we know, a second-order phase transition occurs in the point with the following conditions:

$$\left. \frac{\partial Q^s}{\partial \Psi} \right|_{T_c} = 0, \quad \left. \frac{\partial^2 Q^s}{\partial \Psi^2} \right|_{T_c} = 0. \quad (46)$$

Solving Eqs. (46) yields the coordinates of the critical point as

$$\Psi_c = \left( \frac{k(n-2)^2 \delta l^2 (-z+2)}{Y^2 (z+n-3) \beta (z+n-1) z} \right)^{\frac{-\eta}{2(2p-1)}}, \quad (47)$$

$$T_c = \frac{(2p-1)(n-2)^z}{l\pi\alpha} \left( \frac{(z+n-1)\beta}{-z+2} \right)^{1-\frac{z}{2}} \left( \frac{k\delta}{z(z+n-3)} \right)^{\frac{z}{2}}, \quad (48)$$

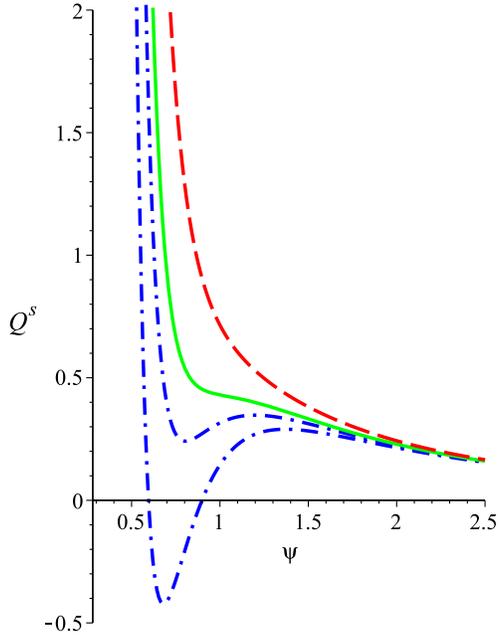


FIG. 2.  $Q^s - \Psi$  diagram of Lifshitz black holes for  $b = 1$ ,  $n = 3$ ,  $q = 1$ ,  $l = 1$ ,  $p = 1$ , and  $z = 0.6$ .

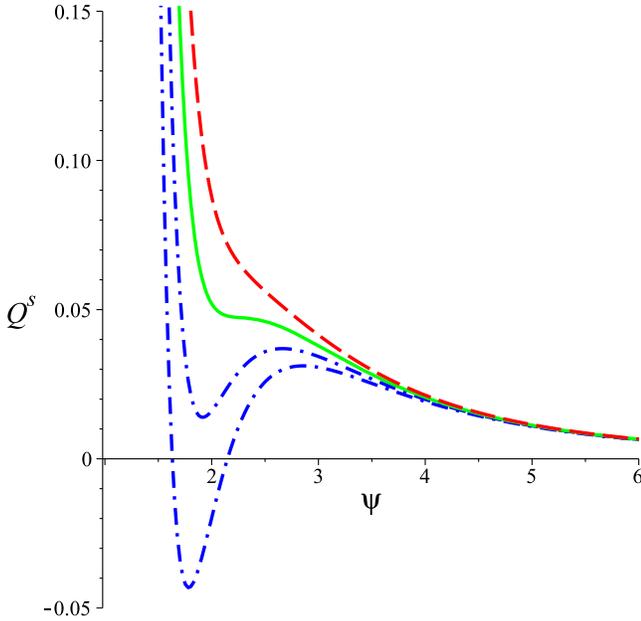


FIG. 3.  $Q^s - \Psi$  diagram of Lifshitz black holes for  $b = 1$ ,  $n = 3$ ,  $q = 1$ ,  $l = 1$ ,  $p = 1.2$ , and  $z = 1$ .

$$Q_c^s = \frac{w^s(-z+2)(2p-1)b^{-2z+2}(n-1)}{2^{\frac{5s}{2}}\alpha\pi^s} \times \left( \frac{l^2(-z+2)(2p-1)\delta}{z(z+n-1)} \right)^\delta \times \left( \frac{k(n-2)^2}{(z+n-3)(2p-1)\beta} \right)^\beta. \quad (49)$$

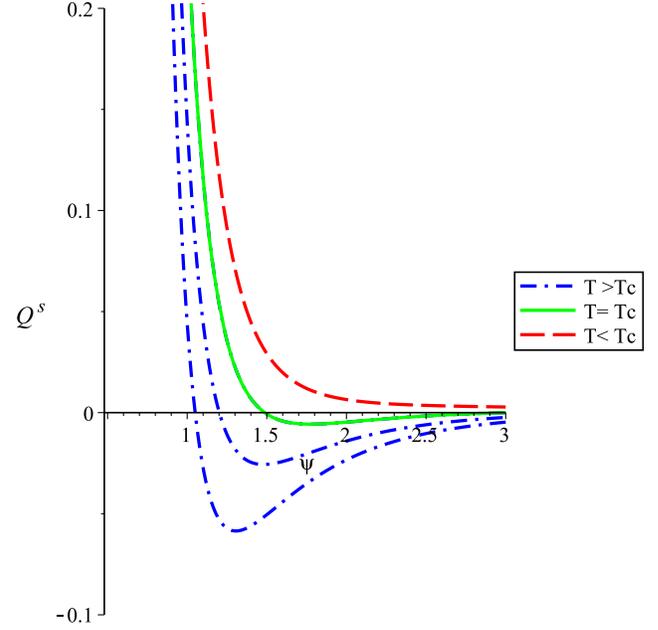


FIG. 4.  $Q^s - \Psi$  diagram of Lifshitz black holes for  $b = 1$ ,  $n = 3$ ,  $q = 1$ ,  $l = 1$ ,  $p = 1.2$ , and  $z = 1.4$ .

Following the new definition  $\rho_c = Q^s T_c \Psi_c$  [34], the energy density of a Lifshitz black hole at the critical point is

$$\rho_c = \frac{(2p-1)^2 w(n-1)(n-2)^{n+2z-1}}{l^{3-n} \pi^2 \Gamma \alpha^2} (z+n-1)^{\frac{5-n+2z}{2}} \times \left( \frac{(-z+2)\delta}{z\beta} \right)^{\frac{n-3+2z}{2}} \left( \frac{k}{z+n-3} \right)^{\frac{n-1+2z}{2}}. \quad (50)$$

Since  $\rho_c$  should be a positive quantity, we can determine the range of the parameters which satisfy this condition. In the general case it is difficult to calculate it, but using diagram (Fig. 5), we show that  $\rho_c$  is positive provided Eq. (19) is satisfied. For  $n = 3$ ,  $\rho_c$  is independent of the value of  $l$ . As we expect when  $n = 3$ ,  $z = 1 = p$ , our results reduce to those of Reissner-Nordström-AdS black holes [34]:

$$\Psi_c = \sqrt{\frac{3}{2l^2}}, \quad Q_c^2 = \frac{l^2}{36}, \\ T_c = \frac{1}{\pi l} \sqrt{\frac{2}{3}}, \quad \rho = \frac{1}{36\pi}. \quad (51)$$

To see how the critical quantities change with  $p$  and  $z$ , one may plot Figs. 6–8. It is clear that in the limit of  $z = 2$ ,  $\rho_c$  and  $Q_c^2$  are equal to zero, while  $\Psi_c$  goes to infinity. Indeed,  $T_c$  has a minimum value at  $z = 2$ , while for  $z > 2$ ,  $T_c$  may have an imaginary value.

### C. Gibbs free energy

Gibbs free energy is one of the most important items which can help us to study the phase transition of a

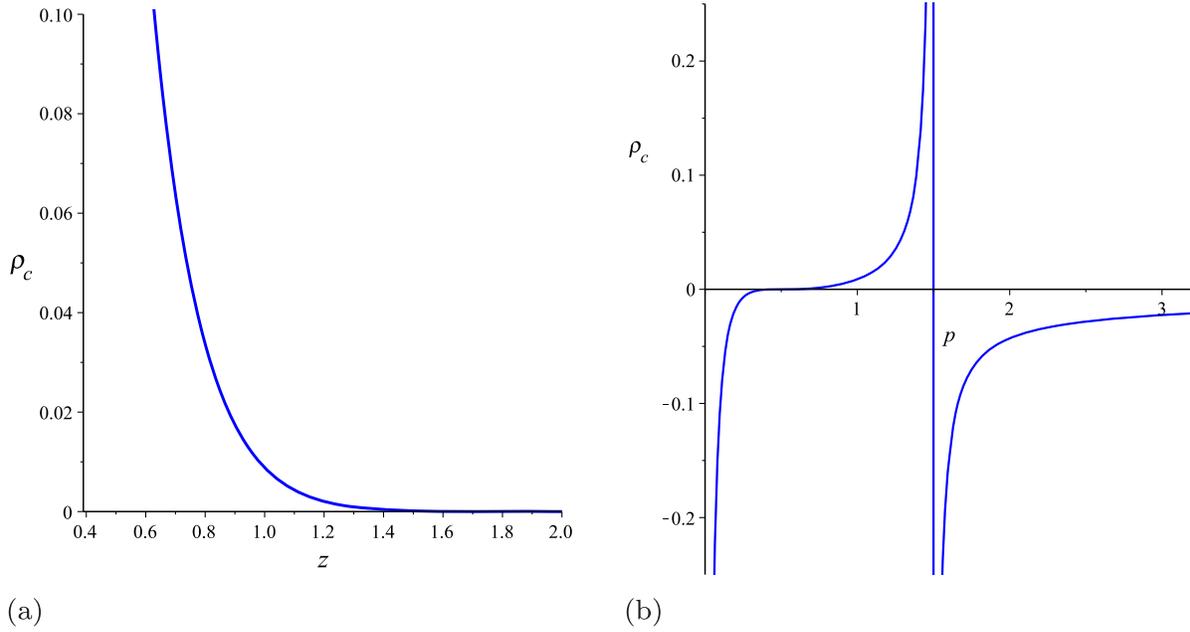


FIG. 5.  $\rho_c$  diagram of Lifshitz black holes. (a)  $\rho_c - z$  for  $b = 1, n = 3, q = 1$  and  $p = 1$ , (b)  $\rho_c - p$  for  $b = 1, n = 3, q = 1$  and  $z = 1$ . As we see  $\rho_c$  is positive in the range  $1/2 < \rho < 3/2$ .

thermodynamical system. As we know, there is no phase transition when Gibbs free energy is a continuous function. Any discontinuity in Gibbs free energy is known as a zero-order phase transition. Also, a first-order phase transition occurs when the Gibbs free energy is continuous, but its first derivative with respect to the temperature and pressure

is discontinuous. At first, we calculate the Gibbs free energy of a Lifshitz dilaton black hole. Then, we try to plot Gibbs diagrams to find out more details about the phase transition of the system. We associate the energy of the system with the Gibbs free energy  $G = M - TS$  [9]. The Gibbs free energy can be obtained as

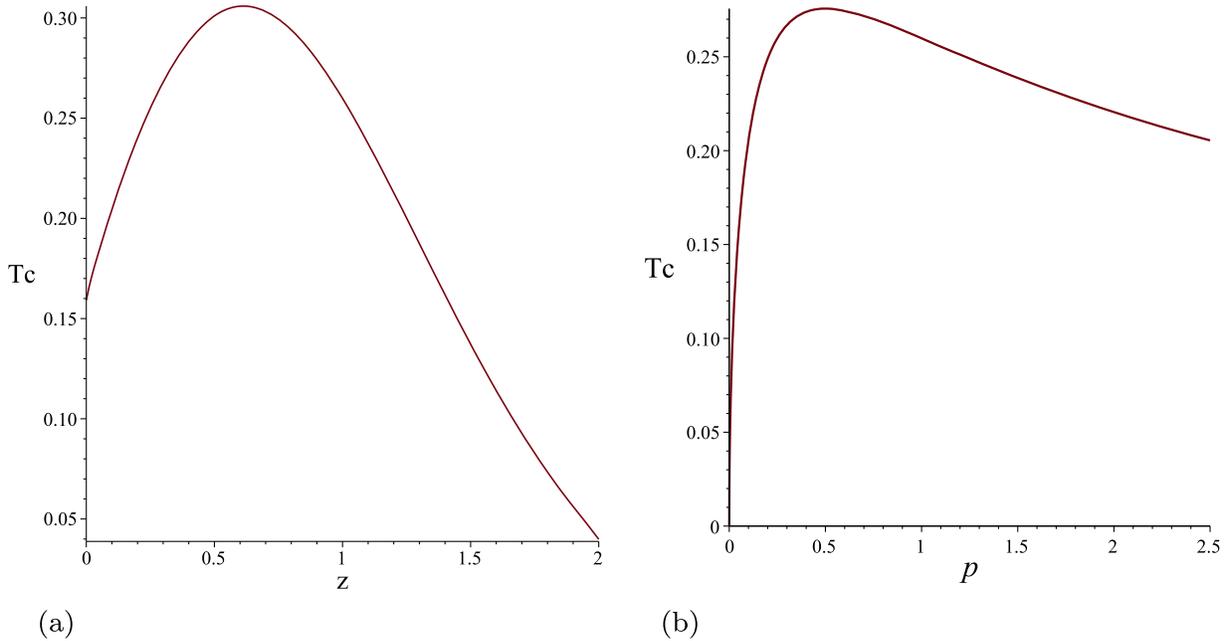


FIG. 6.  $T_c$  diagram of Lifshitz black holes. (a)  $T_c$  versus  $z$  for  $b = 1, n = 3, q = 1$  and  $p = 1$ , (b)  $T_c$  versus  $p$  for  $b = 1, n = 3, q = 1$  and  $z = 1$ .

$$\begin{aligned}
 G = G(Q^s, T) = & -\frac{k(n-2)^2 w(z-2) l^{1-z} r^{n+z-3}}{16\pi(n+z-3)^2} - \frac{w l^{-z-1} (n+2z-1) r^{n+z-1}}{16\pi} \\
 & + \frac{2^{\frac{4-3p}{2p-1}} (2p-1) \pi^{\frac{1}{2p-1}} b^{2z-2} l^{1-z} Q^s w^{\frac{1}{1-2p}} (2np+n+2p(2z-5)-2z+3) r^{\frac{n+2p(z-2)-z+1}{1-2p}}}{(n-1)\eta}.
 \end{aligned} \tag{52}$$

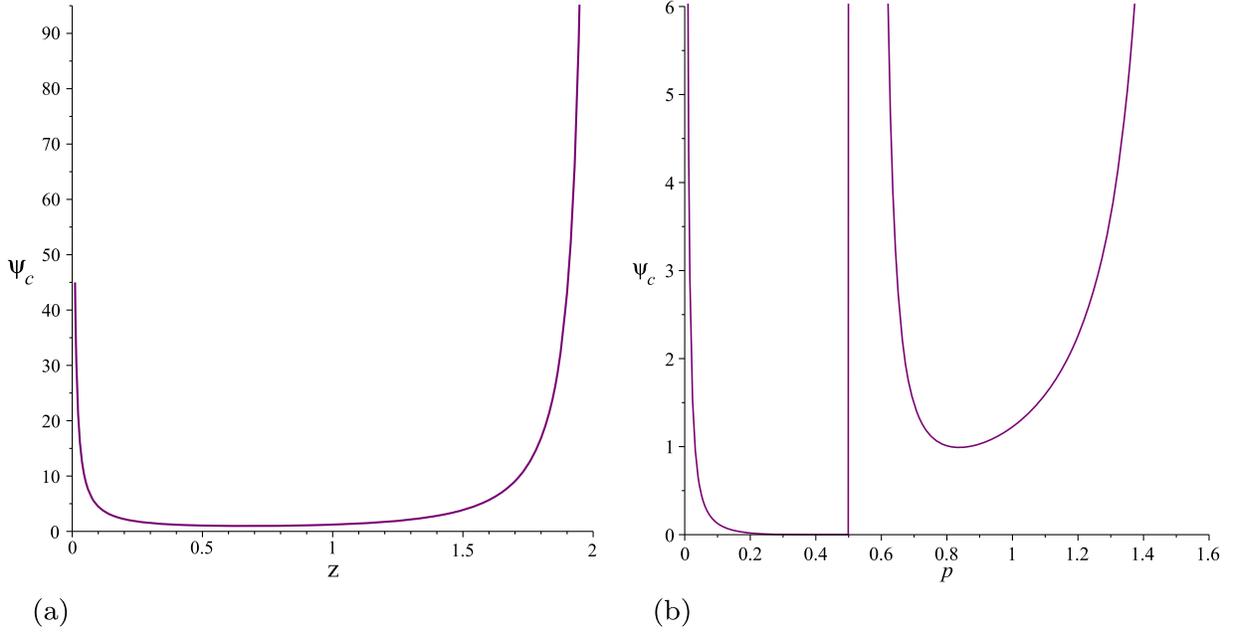


FIG. 7.  $\psi_c$  diagram of Lifshitz black holes. (a)  $\Psi_c$  versus  $z$  for  $b = 1$ ,  $n = 3$ ,  $q = 1$  and  $p = 1$ , (b)  $\Psi_c$  versus  $p$  for  $b = 1$ ,  $n = 3$ ,  $q = 1$  and  $z = 1$ .

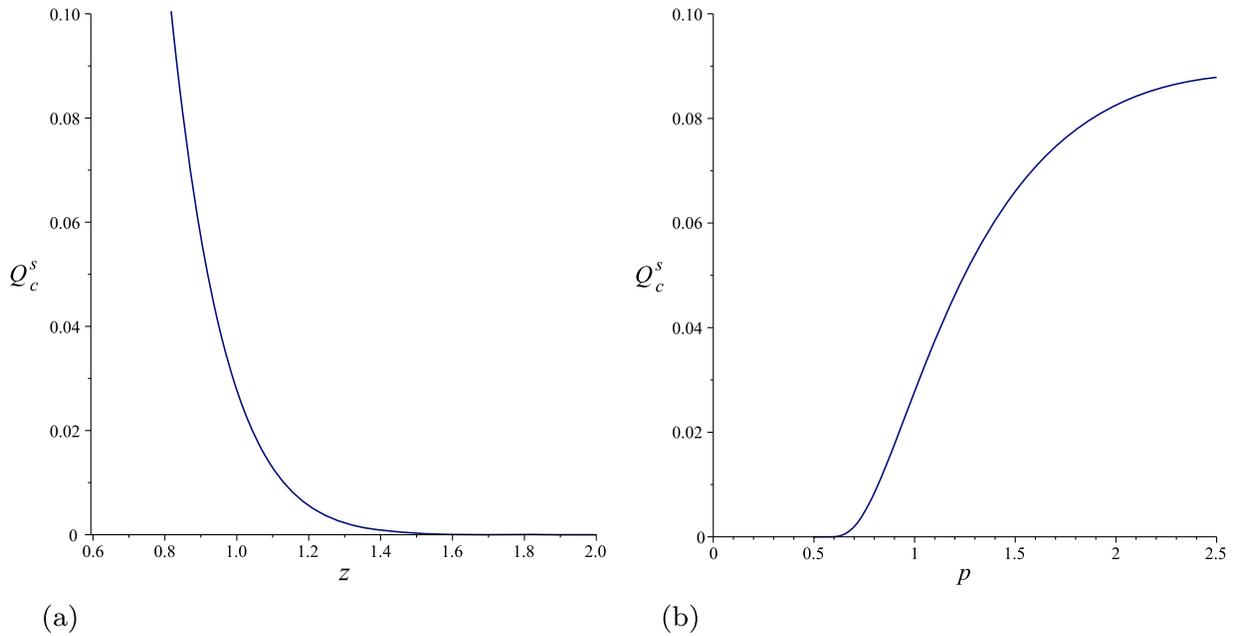


FIG. 8.  $Q_c^s$  diagram of Lifshitz black holes. (a)  $Q_c^s$  versus  $z$  for  $b = 1$ ,  $n = 3$ ,  $q = 1$  and  $p = 1$ , (b)  $Q_c^s$  versus  $p$  for  $b = 1$ ,  $n = 3$ ,  $q = 1$  and  $z = 1$ .

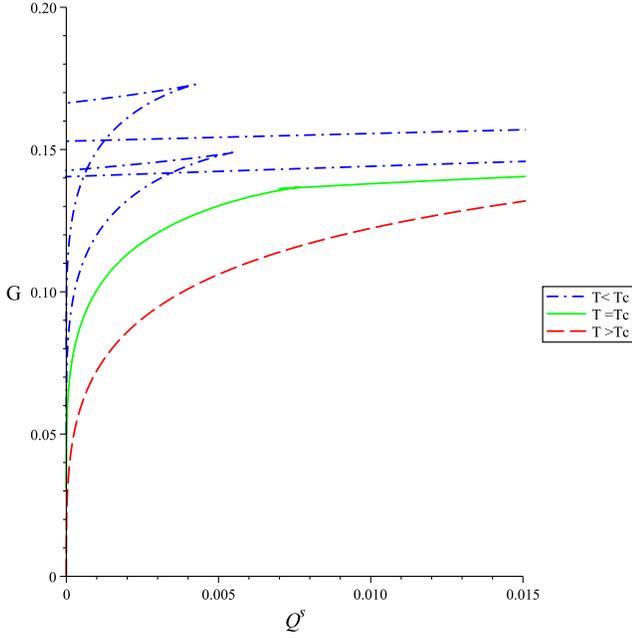


FIG. 9.  $G - Q^s$  diagram of Lifshitz black holes for  $b = 1$ ,  $n = 3$ ,  $q = 1$ ,  $l = 1$ ,  $p = 0.8$ , and  $z = 1.4$ .

In the limiting case where  $p = z = 1$  and  $n = 3$ , the Gibbs free energy reduces to [34]

$$G = G(T, Q^2) = \frac{r_+}{4} + \frac{3Q^2}{4r_+} - \frac{r_+^3}{4l^2}, \quad (53)$$

where  $r_+ = r_+(T, Q^2)$ . We have plotted the Gibbs energy diagrams in Figs. 9–11. These diagrams have been shifted

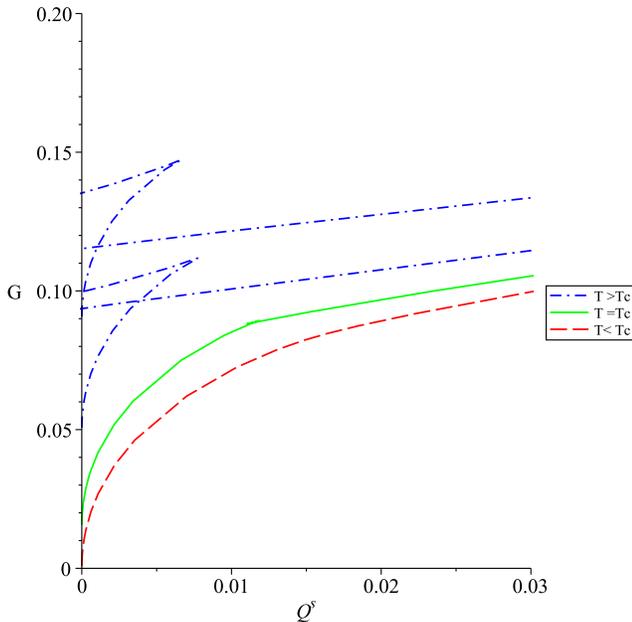


FIG. 10.  $G - Q^s$  diagram of Lifshitz black holes for  $b = 1$ ,  $n = 3$ ,  $q = 1$ ,  $l = 1$ ,  $p = 1$ , and  $z = 1.1$ .

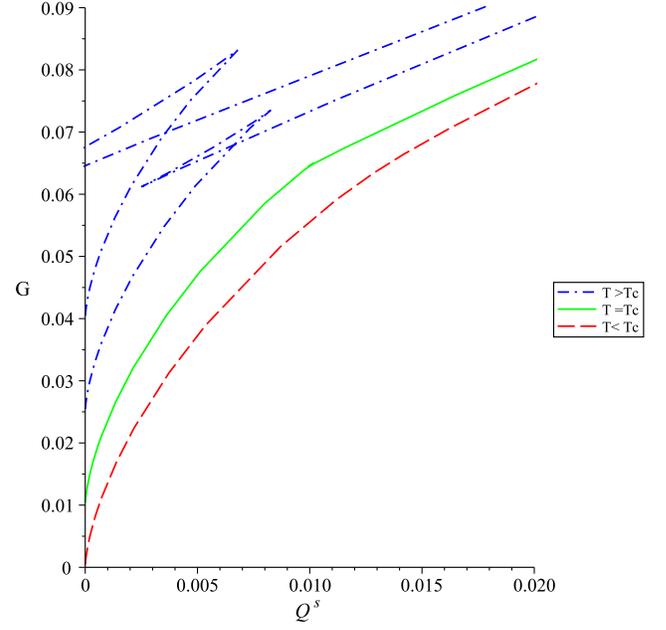


FIG. 11.  $G - Q^s$  diagram of Lifshitz black holes for  $b = 1$ ,  $n = 3$ ,  $q = 1$ ,  $l = 1$ ,  $p = 1.2$ , and  $z = 1.2$ .

for more clarity. The swallowtail behavior of Figs. 9–11 shows that a first-order phase transition occurs in the system.

#### D. Critical exponents

The behavior of the physical quantities in the vicinity of a critical point can be characterized by the critical exponents. Following the approach of [15], one can calculate the critical exponents  $\alpha'$ ,  $\beta'$ ,  $\gamma'$ , and  $\delta'$  for the phase transition of charged Lifshitz black holes in the presence of a power Maxwell field. To obtain the critical exponents, we define the reduced thermodynamic variables as

$$T_r = \frac{T}{T_c}, \quad \psi_r = \frac{\psi}{\psi_c}, \quad Q_r^s = \frac{Q^s}{Q_c^s}. \quad (54)$$

Since the critical exponents should be studied near the critical point, we write the reduced variables in the form  $T_r = 1 + t$  and  $\psi_r = 1 + \phi$ , indicating deviation from the critical point. One may expand Eq. (43) near the critical point as

$$Q_r^s = 1 + At - Bt\phi - C\phi^3 + O(t\phi^2, \phi^4), \quad (55)$$

where

$$A = \frac{-4\delta\beta}{z(-z+2)}, \quad B = \frac{-4\alpha\delta\beta}{z(-z+2)\eta}, \quad C = \frac{2(2p-1)^2\alpha\beta\delta}{3\eta^3}. \quad (56)$$

To calculate the critical exponent  $\alpha'$ , we consider the entropy  $S$  given in Eq. (26) as a function of  $T$  and  $\psi$ . Using Eq. (41) we have

$$S = S(T, \Psi) = \frac{Y^{n-1}}{4\pi} \Psi^{(-2p+1)/\eta}.$$

Obviously, this is independent of  $T$  and therefore the specific heat vanishes,  $C_\psi = T(\partial S/\partial T)_\psi = 0$ . Since the exponent  $\alpha'$  governs the behavior of the specific heat at fixed  $\psi$ ,  $C_\psi \propto |t|^{\alpha'}$ , hence the exponent  $\alpha' = 0$ .

Differentiating Eq. (55) at a fixed  $t < 0$  with respect to  $\phi$ , we get

$$dQ_r^s = -(Bt + 3C\phi^2)d\phi. \quad (57)$$

Now, we apply the Maxwell's equal area law [40]. Denoting the variable  $\phi$  for small and large black holes with  $\phi_s$  and  $\phi_l$ , respectively, we obtain

$$\begin{aligned} Q_r^s &= 1 + At - Bt\phi_l - C\phi_l^3 = 1 + At - Bt\phi_s - C\phi_s^3, \\ 0 &= \int_{\phi_l}^{\phi_s} \phi dQ_r^s. \end{aligned} \quad (58)$$

Equation (58) leads to the unique nontrivial solution

$$\phi_l = -\phi_s = \sqrt{-\frac{Bt}{C}}, \quad (59)$$

which gives the order parameter as

$$|\phi_s - \phi_l| = 2\phi_s = 2\sqrt{-\frac{B}{C}}t^{1/2}. \quad (60)$$

Thus, the exponent  $\beta'$  which describes the behavior of the order parameter near the critical point is  $\beta' = 1/2$ . To calculate the exponent  $\gamma'$ , one may determine the behavior of the following function near the critical point:

$$\chi_T = \left. \frac{\partial \psi}{\partial Q^s} \right|_T.$$

Differentiating Eq. (55) with respect to  $\phi$  near the critical point may be written as

$$\chi_T \propto -\frac{\psi_c}{BQ_c^s t} \implies \gamma' = 1. \quad (61)$$

Finally, the shape of the critical isotherm  $t = 0$  is given by Eq. (55). We find

$$Q_r^s - 1 = -C\phi^3 \implies \delta' = 3. \quad (62)$$

#### IV. SUMMARY AND CONCLUSION

The critical behavior of the Lifshitz dilaton black hole in an extended phase space, where the cosmological constant is treated as the thermodynamical pressure, cannot be studied due to the complicated form of the solution. Indeed, it can be seen from Eq. (27) that it is almost impossible to solve this equation for  $P$  (or more precisely for  $l$ ). Therefore, we cannot analytically investigate the  $P - v$  critically of the black hole (although one may use the numerical calculations or plotting diagrams to do that). Also, investigating the phase space of the system in the  $U - Q$  plan leads to physically irrelevant quantities which are mathematically ill defined [34]. Here, we address, for the first time, the critical behavior of an  $(n + 1)$ -dimensional dilaton Lifshitz black hole in the presence of a power-law Maxwell field via an alternative phase space developed in [34]. We have treated the cosmological constant as a fixed parameter and the charge of the system as a thermodynamical variable. It was argued in [34] that without extension of the phase space and by keeping the cosmological constant (pressure) as a fixed quantity instead of the charge of the system, it is quite possible to have critical behavior similar to that of a van der Waals system provided one takes the equation of state of the form  $Q^2 = Q^2(T, \psi)$ , where  $\Psi = 1/2r_+$  is a conjugate of  $Q^2$ .

In this work, we disclosed that in order to investigate the critical behavior of Lifshitz black holes with a power Maxwell field, we should modify the method developed in [34] by considering  $Q^s$  as a thermodynamic variable and write down the equation of state in the form of  $Q^s = Q^s(T, \psi)$ , where  $s = 2p/(2p - 1)$ , with  $p$  the power of the power Maxwell Lagrangian. In this approach we keep the cosmological constant (pressure) as a fixed quantity and treat the charge of the black hole as a thermodynamic variable. This is in contrast to the extended phase space of [8], where the charge is fixed and the cosmological constant is treated as a thermodynamic variable. The isotherm diagrams  $Q^s - \Psi$  show the complete analogy between our system and a van der Waals liquid-gas system. Also, the swallowtail behavior of the Gibbs free energy represents a first-order phase transition occurring in the system. Furthermore, we calculated the critical quantities such as  $T_c$ ,  $\rho_c$ ,  $P_c$ , and  $\Psi_c$  at the critical point, which depends on the metric parameters. Finally, we obtained the critical exponents of the system and found that they are universal and exactly the same as those of a van der Waals fluid system. The result obtained here for a general Lifshitz dilaton black hole coupled to a power Maxwell field with an arbitrary dilaton coupling  $\alpha$ , Lifshitz exponent  $z$ , and power Maxwell parameter  $p$  further supports the arguments given in Ref. [34]. Clearly, for  $z = 1$ ,  $\alpha = 0$ , and  $p = 1$ , our general Lifshitz dilaton black hole reduces to a charged AdS black hole and our obtained results recover those obtained in [34]. This confirms that the approach presented here is independent

of the power Maxwell, Lifshitz exponent, and dilaton coupling constant. Indeed, our study shows that the approach here is powerful enough to study the critical behavior of any kind of black hole and the system resembles the van der Waals fluid without needing to consider the cosmological constant (pressure) as a thermodynamic variable. It could help us to extract critical exponents of the system without extending the phase space, which is useful in studying the thermodynamical properties of black holes. We expect to confirm

that this approach is viable and can be applied to other gravity theories such as Gauss-Bonnet and Lovelock gravity.

### ACKNOWLEDGMENTS

We thank the Shiraz University Research Council. The work of A.S. has been supported financially by the Research Institute for Astronomy and Astrophysics of Maragha, Iran.

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