Dynamics of cosmological perturbations at first and second order

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(Received 10 August 2018; published 30 November 2018)

In this paper we give five gauge-invariant systems of governing equations for first and second order scalar perturbations of flat Friedmann-Lemaître universes that are minimal in the sense that they contain no redundant equations or variables. We normalize the variables so that they are dimensionless, which leads to systems of equations that are simple and ready-to-use. We compare the properties and utility of the different systems. For example, they serve as a starting point for finding explicit solutions for two benchmark problems in cosmological perturbation theory at second order: adiabatic perturbations in the superhorizon regime (the long wavelength limit) and perturbations of ΛCDM universes. However, our framework has much wider applicability and serves as a reference for future work in the field.

DOI: 10.1103/PhysRevD.98.103534

I. INTRODUCTION

Perturbations of Friedmann-Lemaître (FL) cosmologies play an essential role in confronting theoretical models with observations of the anisotropy of the cosmic microwave background (CMB) and the inhomogeneity of the large scale structure (LSS) of the Universe. Initially linear perturbations were adequate but now the increasing accuracy of the observations necessitates the use of second order (nonlinear) perturbations to analyze, for example, the presence of non-Gaussianity in the CMB and the LSS.[1]

In this paper we consider first and second order scalar perturbations of FL universes subject to the following assumptions:

(i) the spatial background is flat;

(ii) the stress-energy tensor can be written in the form $T^a{}_b = (\rho + p)u^a u_b + p\delta^a{}_b$, thereby describing perfect fluids and scalar fields;

(iii) the linear perturbation is purely scalar.

The dynamics of perturbations of FL universes are governed by the perturbed Einstein equations and the perturbed matter equations. For *scalar perturbations* the perturbed Einstein equations give four equations (linear combinations of the components of the perturbed Einstein tensor) which include evolution equations for the metric perturbations. The perturbed conservation equations provide evolution equations for two primary matter perturbations, the density perturbation and the scalar velocity perturbation. Only four of these six equations are needed to fully describe the perturbations, but in order to obtain a well-defined system the gauge freedom has to be eliminated by fixing the gauge.

Since 2004 much work aimed at confronting theoretical models with observations has been done using second order perturbations. In one respect second order perturbations are analogous to first order perturbations: the *leading order terms* in the equations have exactly the same form. The greater complexity at second order arises from the fact that each equation is augmented by so-called *source terms* that depend quadratically on the first order perturbations. There are various ways of formulating the governing equations for second order perturbations, depending on the choice of variables and gauge. These choices are influenced by various factors such as the problem to be investigated, for example, long wavelength perturbations or perturbations of the ΛCDM universe, or in the case of numerical work, by the availability of numerical packages. A number of detailed formulations of the governing equations have been given,[2] but mainly due to the complexity of the source terms no standard systems have emerged: it is as though the necessary technical infrastructure for analyzing second order perturbations has not been sufficiently well developed.

With this as motivation, our goal in this paper is to present five systems of equations that are suitable for analyzing the dynamics of both first and second order scalar perturbations of FL universes. To accomplish this we begin by imposing the so-called C-gauge of Hwang and Noh [3] up to second order which fixes the spatial gauge,

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[1]See, for example, Bartolo *et al.* (2010) [1] and Tram *et al.* (2016) [2].

[2]See, for example, Noh and Hwang (2004) [3], and Nakamura (2007) [4].

but we initially keep an arbitrary temporal gauge. Within this framework we construct a set of leading order and quadratic source terms for the perturbed Einstein field equations (a set of four scalar equations) and the perturbed energy-momentum conservation equations (a set of two scalar equations). Finally we construct five specific systems of gauge invariant equations by also fixing the temporal gauge. First, specializing the perturbed Einstein field equations to the Poisson (longitudinal, zero shear) gauge and the uniform (flat) curvature gauge yields two systems of governing equations. Second, specializing three of the perturbed Einstein field equations together with the perturbed momentum conservation equation to the Poisson gauge and the total matter gauge results in two more systems. Finally we create a fifth system by using the perturbed energy-momentum equations to describe the evolution of the density perturbation in the total matter gauge, and the velocity perturbation in the Poisson gauge, with two of the perturbed Einstein equations acting as constraints to determine the metric perturbations. We regard these five systems of equations as *ready-to-use* since they are gauge invariant, contain no redundant equations or variables, and do not require that any further simplifications be made before use.

It is important to note that in general the above systems of equations are not closed (not fully determined) since the nonadiabatic pressure perturbation has to be specified. However, for a barotropic perfect fluid and a minimally coupled scalar field, the systems are fully determined, once an equation of state and a scalar field potential, respectively, has been given. Moreover, we present the systems in a manner that makes it possible to apply them to more general matter models such as multifluids and multiple scalar fields.

The present paper is the second of four related papers by the authors. The first paper [5], hereafter referred to as UW1, gives a unified and simplified formulation of gauge change formulas at second order, while the third paper [6], called UW3, uses the present paper, in conjunction with UW1, to give new conserved quantities and derive the general explicit solution at second order for adiabatic perturbations in the long wavelength limit, results that are subsequently adapted to inflationary universes with a single scalar field in [7], which we refer to as UW4.

The outline of the paper is as follows. In Sec. II we introduce the metric and matter perturbation variables. In Sec. III we present leading order and quadratic source terms for the perturbed Einstein field equations, and in Sec. IV we present the leading order and quadratic source terms for the perturbed conservation equations. In both cases the details of the source terms are deferred to an appendix. The central goal of the paper is reached in Sec. V where we derive the five ready-to-use systems of governing equations. Finally in Sec. VI we comment on specific applications of the five systems and on their relative merits.

II. METRIC AND MATTER PERTURBATION VARIABLES

A. Background geometrical and matter scalars

The background Robertson-Walker (RW) metric has the form

$$ds^2 = a^2(-d\eta^2 + \gamma_{ij}dx^i dx^j), \tag{1}$$

where a is the background scale factor, η is conformal time, and γ_{ij} is the flat spatial 3-metric. The evolution of the background geometry is governed by the scalars

$$\mathcal{H} = \frac{a'}{a}, \qquad q = -\frac{\mathcal{H}'}{\mathcal{H}^2}, \tag{2a}$$

where $'$ denotes differentiation with respect to η, and $\mathcal{H} = aH$, with H being the background Hubble parameter and q the background deceleration parameter. We associate the following scalars with the background stress-energy tensor:

$$w = \frac{p_0}{\rho_0}, \qquad c_s^2 = \frac{p_0'}{\rho_0'}, \tag{2b}$$

where ρ_0 and p_0 are the background energy density and pressure, respectively. The density parameter is defined as usual by

$$\Omega = \frac{\rho_0}{3H^2}, \tag{2c}$$

where we have set $c = 1$ and $8\pi G = 1$, where c is the speed of light and G the gravitational constant.

The Einstein equations for a spatially flat background can be written as

$$3\mathcal{H}^2 = a^2\rho_0, \qquad 2(-\mathcal{H}' + \mathcal{H}^2) = a^2(\rho_0 + p_0), \tag{3}$$

or equivalently, using Eqs. (2), in the following form:

$$\Omega = 1, \qquad 2(1+q) = 3(1+w). \tag{4}$$

One can use the second equation to switch between $1+q$ and $1+w$ and in what follows we will use either expression depending on the context.

As regards dimensions, we make the choice that the scale factor a is dimensionless, which implies via (1) that the conformal time η has dimensions of *length*. It follows that H and $\mathcal{H}$ have dimension $(\text{length})^{-1}$ and ρ_0 and p_0 have dimension $(\text{length})^{-2}$ while q, w, c_s^2, and Ω are dimensionless.

We now introduce the background *e-fold time variable* N, defined by

$$N = \ln(a/a_0). \tag{5}$$

This variable describes the number of background e-foldings with respect to some reference epoch $a = a_0$. Although conformal time η is arguably the most commonly used background time variable in cosmological perturbation theory, the e-fold time variable N is used in inflationary cosmology and also when doing numerical simulations.[3] In this paper we will primarily use e-fold time N but on occasion we will make the transition to conformal time η. In changing time variables, note that

$$\partial_\eta = \mathcal{H}\partial_N, \qquad \partial_\eta^2 = \mathcal{H}^2(\partial_N^2 - q\partial_N), \tag{6}$$

and that the deceleration parameter q can be written in either of the following forms:

$$q = -\frac{\partial_N \mathcal{H}}{\mathcal{H}} = -\frac{\partial_N H}{H} - 1. \tag{7}$$

In order to write simple expressions for the perturbed Einstein tensor it is helpful to introduce an additional background geometrical scalar $\mathcal{C}^2$, which is defined in terms of the background Einstein tensor according to

$$\mathcal{C}^2 = -\frac{1}{3}\partial_N {}^{(0)}G^i{}_i / \partial_N {}^{(0)}G^\eta{}_\eta. \tag{8}$$

On noting that ${}^{(0)}T^i{}_i = 3p_0$, ${}^{(0)}T^\eta{}_\eta = -\rho_0$ it follows from (2b) that $\mathcal{C}^2$ is the geometrical analogue of c_s^2 and that the Einstein equations in the background imply

$$\mathcal{C}^2 = c_s^2. \tag{9}$$

For future use we note that definition (8) leads to the following derivative[4]:

$$\partial_N q = -(1+q)(1+3\mathcal{C}^2 - 2q). \tag{10}$$

B. Metric perturbation variables

To perturb a flat RW background geometry we write the metric in the form

$$ds^2 = a^2(-(1+2\phi)d\eta^2 + f_{\eta i} d\eta dx^i + f_{ij} dx^i dx^j), \tag{11}$$

where we assume that the metric components can be expanded as a Taylor series in a perturbation parameter ϵ, e.g.,

$$\phi = \epsilon^{(1)}\phi + \frac{1}{2}\epsilon^{2\,(2)}\phi + \cdots. \tag{12}$$

We also assume that the metric can be decomposed into scalar, vector, and tensor perturbations according to

$$f_{\eta i} = \mathbf{D}_i B + B_i, \tag{13a}$$

$$f_{ij} = (1-2\psi)\gamma_{ij} + 2\mathbf{D}_i\mathbf{D}_j C + 2\mathbf{D}_{(i}C_{j)} + 2C_{ij}, \tag{13b}$$

where $\mathbf{D}^i B_i = 0$, $\mathbf{D}^i C_i = 0$, $C^i{}_i = 0$, $\mathbf{D}^i C_{ij} = 0$, which ensures that ψ, B, and C describe scalar perturbations. Here $\mathbf{D}_i$ is the spatial covariant derivative corresponding to the flat metric γ_{ij}. Use of Cartesian background coordinates yields $\gamma_{ij} = \delta_{ij}$ and $\mathbf{D}_i = \partial/\partial x^i$. As regards dimensions, since we have made the choice that the scale factor a is dimensionless, it follows that the coordinates η and x^i have dimensions of length since ds^2 has dimension length^2. Therefore, due to the structure of equations (11) and (13), ϕ and ψ are dimensionless while B has dimension length, since $\mathbf{D}_i$ has dimension $(\text{length})^{-1}$.

From now on we completely fix the spatial gauge freedom by setting the metric functions C and C_i in (13) to be zero order by order,[5] which up to second order gives

$${}^{(r)}C = 0, \qquad {}^{(r)}C_i = 0, \qquad r = 1, 2, \tag{14}$$

where the transformation laws for C and C_i were given in Eqs. (B10e) and (B10f) in [10]. Furthermore, in this paper we are restricting our considerations to perturbations that are *purely scalar at linear order*, and hence the metric perturbations B_i and C_{ij} satisfy[6]

$${}^{(1)}B_i = 0, \qquad {}^{(1)}C_{ij} = 0. \tag{15}$$

Since purely scalar perturbations at linear order will generate vector and tensor perturbations at second order, it follows that these perturbations will have ${}^{(2)}B_i \neq 0$, ${}^{(2)}C_{ij} \neq 0$. However, our interest in this paper is discussing the scalar perturbations at first and second order. Thus the metric perturbations we consider are given by

$${}^{(r)}f = ({}^{(r)}\phi, \mathcal{H}{}^{(r)}B, {}^{(r)}\psi), \quad r = 1, 2, \tag{16}$$

where we have scaled ${}^{(r)}B$ with a factor of $\mathcal{H}$. We introduced this scaling in our earlier paper UW1 [5], motivated by the transformation properties of the metric perturbations under a change of gauge [see Eqs. (23) and (24) in that paper], and by the fact that $\mathcal{H}{}^{(r)}B$ is dimensionless, as we have confirmed here. By inspection it follows from (11) and (13b) that ϕ and ψ are dimensionless.

[3] See, for example, Huston and Malik (2009) [8].

[4] Write the background Einstein tensor in the form ${}^{(0)}G^\eta{}_\eta = -3H^2$, ${}^{(0)}G^i{}_i = 3H^2(1-2q)$.

[5] This excludes the synchronous gauge (for a recent work using the synchronous gauge, see e.g., [9]), but apart from this gauge most commonly used gauges are included in this class.

[6] This assumption is often made, but see, for example, Carrilho and Malik (2016) [11].

For future reference when scaling variables with $\mathcal{H}$ we use

$$\mathcal{H}\partial_N f = (\partial_N + q)(\mathcal{H}f), \tag{17}$$

as follows from (7).

C. Matter perturbation variables

We consider a stress-energy tensor which can be written in the form

$$T^a{}_b = (\rho + p)u^a u_b + p\delta^a{}_b, \tag{18}$$

which describes both perfect fluid models and models with a minimally coupled scalar field. In addition we assume that it can be expanded in a Taylor series in ϵ, e.g.,

$$\rho(\epsilon) = \rho_0 + \epsilon\,{}^{(1)}\rho + \frac{1}{2}\epsilon^2\,{}^{(2)}\rho + \cdots, \tag{19}$$

for the energy density and similarly for the pressure $p(\epsilon)$. We then normalize the perturbations of ρ and p with $\rho_0 + p_0$ and define

$$ {}^{(r)}\boldsymbol{\delta} = \frac{{}^{(r)}\rho}{\rho_0 + p_0}, \qquad {}^{(r)}P = \frac{{}^{(r)}p}{\rho_0 + p_0}, \tag{20}$$

which are dimensionless.

To define the scalar velocity perturbations we find it convenient to work with the *covariant* 4-velocity u_b, which we normalize with a conformal factor a according to $u_b = aV_b$, in analogy with the conformal factor a^2 in the metric (11). We then expand and decompose the spatial components of V_b according to

$$V_i = \epsilon\,{}^{(1)}V_i + \frac{1}{2}\epsilon^2\,{}^{(2)}V_i + \cdots, \tag{21a}$$

$${}^{(r)}V_i = \mathbf{D}_j\,{}^{(r)}V + {}^{(r)}\tilde{V}_i, \quad r = 1, 2, \ldots, \tag{21b}$$

with $\mathbf{D}^i\,{}^{(r)}\tilde{V}_i = 0$, so that ${}^{(r)}V$ represents the scalar perturbations. Since we are focusing on scalar perturbations in this paper we set the first order vector term to zero (${}^{(1)}\tilde{V}_i = 0$). Since the V_b are dimensionless and the x^i have dimension length it follows from Eq. (21) that ${}^{(r)}V$ has dimension length. As in the case of ${}^{(r)}B$ we normalize ${}^{(r)}V$ with $\mathcal{H}$ and thereby consider $\mathcal{H}\,{}^{(r)}V$.

Next we introduce the nonadiabatic pressure perturbations, which we denote by ${}^{(r)}\Gamma$, $r = 1, 2$. Following Bartolo *et al.* (2004) [12], but using the normalized pressure perturbation (20), we define[7]

$${}^{(r)}\Gamma = {}^{(r)}P_\rho, \quad r = 1, 2, \tag{22}$$

i.e., *the nonadiabatic pressure perturbations equal the pressure perturbations in the uniform density gauge* (defined by ${}^{(r)}\boldsymbol{\delta} = 0$, $r = 1, 2$). This definition ensures that the ${}^{(r)}\Gamma$, $r = 1, 2$ are dimensionless gauge invariants and that if $p = p(\rho)$, then ${}^{(r)}\Gamma = 0$. Since we will need to express ${}^{(r)}\Gamma$, $r = 1, 2$, in terms of other spatially fixed gauges we use a change of gauge formula to express ${}^{(r)}P_\rho$, $r = 1, 2$, in terms of the normalized pressure and density perturbations in a temporally arbitrary but spatially fixed gauge.[8] On introducing the scaled density perturbation (20) and the e-fold time variable N we obtain the following expressions:

$${}^{(1)}\Gamma = {}^{(1)}P - c_s^2\,{}^{(1)}\boldsymbol{\delta}, \tag{23a}$$

$$\begin{aligned}{}^{(2)}\Gamma = {}^{(2)}P - c_s^2\,{}^{(2)}\boldsymbol{\delta} + \frac{1}{3}(\partial_N c_s^2)\,{}^{(1)}\boldsymbol{\delta}^2 \\ + \frac{2}{3}\,{}^{(1)}\boldsymbol{\delta}[\partial_N - 3(1 + c_s^2)]\,{}^{(1)}\Gamma.\end{aligned} \tag{23b}$$

In what follows we will replace the pressure perturbations ${}^{(r)}P$ with the gauge invariants ${}^{(r)}\Gamma$, $r = 1, 2$, which means that the basic matter perturbations that we use are the dimensionless quantities

$${}^{(r)}M = ({}^{(r)}\boldsymbol{\delta}, {}^{(r)}\Gamma, \mathcal{H}\,{}^{(r)}V), \quad r = 1, 2. \tag{24}$$

III. THE PERTURBED EINSTEIN EQUATIONS IN A GENERAL TEMPORAL GAUGE

A. Perturbed Einstein and stress energy tensors

We assume that the Einstein tensor for the metric (11) has a Taylor expansion of the form

$$G^a{}_b(\epsilon) = {}^{(0)}G^a{}_b + \epsilon\,{}^{(1)}G^a{}_b + \frac{1}{2}\epsilon^2\,{}^{(2)}G^a{}_b + \cdots. \tag{25}$$

The first and second order perturbations of the Einstein tensor have the following general structure:

$$H^{-2}\,{}^{(1)}G^a{}_b = \mathsf{G}^a{}_b({}^{(1)}f), \tag{26a}$$

$$H^{-2}\,{}^{(2)}G^a{}_b = \mathsf{G}^a{}_b({}^{(2)}f) + \mathbb{G}^a{}_b({}^{(1)}f), \tag{26b}$$

where we have normalized with the background quantity H^{-2} to ensure that $\mathsf{G}^a{}_b(f)$ and $\mathbb{G}^a{}_b({}^{(1)}f)$ are dimensionless. Observe that the first and second order perturbations have

[7]See in particular [12] Eqs. (136), (137), and (145) for a general discussion and the desired expressions. The Bartolo expression $\delta P_{\rm nad}$ is related to our Γ by $\delta P_{\rm nad} = (\rho_0 + p_0)\Gamma$. Previously, Kodama and Sasaki (1984) [13] used the symbol Γ in this context: their expression is related to ours according to $w\Gamma_{\rm KS} = (1 + w)\Gamma$. See Eq. (II.3.38).

[8]See Malik and Wands (2004) [14], Eqs. (4.19) and (4.20), and [12], Eqs. (137) and (145).

a common *leading order term* of the form $\mathsf{G}^a{}_b(f)$, where f denotes ${}^{(1)}f \equiv ({}^{(1)}\phi, \mathcal{H}^{(1)}B, {}^{(1)}\psi)$ or ${}^{(2)}f \equiv ({}^{(2)}\phi, \mathcal{H}^{(2)}B, {}^{(2)}\psi)$, while $H^{-2(2)}G^a{}_b$ also has a *source term* $\mathbb{G}^a{}_b({}^{(1)}f)$ which depends quadratically on ${}^{(1)}f$. We use the fonts $\mathcal{G}$, G, $\mathbb{G}$ as notational conventions for background (zeroth order), first/leading order, and second order source terms, respectively, although we will deviate from these conventions when they clash with notation that has become fairly standard in the literature.

The perturbations of the stress-energy tensor have a similar general structure:

$$(\rho_0 + p_0)^{-1(1)}T^a{}_b = \mathsf{T}^a{}_b({}^{(1)}M), \tag{27a}$$

$$(\rho_0 + p_0)^{-1(2)}T^a{}_b = \mathsf{T}^a{}_b({}^{(2)}M) + \mathbb{T}^a{}_b({}^{(1)}f, {}^{(1)}M), \tag{27b}$$

where ${}^{(r)}M = ({}^{(r)}\boldsymbol{\delta}, {}^{(r)}\Gamma, \mathcal{H}^{(r)}V)$, $r = 1, 2$, are the matter perturbation variables. We have chosen the normalization factor $(\rho_0 + p_0)^{-1}$ to be compatible with (20), resulting in dimensionless variables.

We express the perturbed Einstein equations ${}^{(r)}G^a{}_b = {}^{(r)}T^a{}_b$, $r = 1, 2$, in leading order terms and source terms using Eqs. (26) and (27). Since $(\rho_0 + p_0)/H^2 = 3(1 + w)\Omega = 3(1 + w)$, the first and second order perturbed Einstein equations take the form

$$\mathsf{G}^a{}_b({}^{(1)}f) = 3(1 + w)\mathsf{T}^a{}_b({}^{(1)}M), \tag{28a}$$

$$\mathsf{G}^a{}_b({}^{(2)}f) + \mathbb{G}^a{}_b({}^{(1)}f) = 3(1 + w)(\mathsf{T}^a{}_b({}^{(2)}M) + \mathbb{T}^a{}_b({}^{(1)}f, {}^{(1)}M)). \tag{28b}$$

B. The scalar mode

The scalar mode of the leading order tensor $\mathsf{G}^a{}_b(f)$ in (26) is described by the following linear combinations of $\mathsf{G}^a{}_b$[9]:

$$\mathsf{G}^i{}_i, \qquad \mathcal{H}^2\mathcal{S}^{ij}\mathsf{G}_{ij}, \qquad \mathcal{H}\mathcal{S}^i\mathsf{G}^\eta{}_i, \qquad \mathsf{G}^\eta{}_\eta, \tag{29}$$

and similarly for the leading order tensor $\mathsf{T}^a{}_b(M)$ in (27) and the source terms $\mathbb{G}^a{}_b$ and $\mathbb{T}^a{}_b$. Here the *scalar mode extraction operators* $\mathcal{S}^i$ and $\mathcal{S}^{ij}$ [see Uggla and Wainwright (2013) [15]] are defined as follows:

$$\mathcal{S}^i = \mathbf{D}^{-2}\mathbf{D}^i, \tag{30a}$$

$$\mathcal{S}^{ij} = \frac{3}{2}(\mathbf{D}^{-2})^2\mathbf{D}^{ij}, \tag{30b}$$

where $\mathbf{D}_{ij} = \mathbf{D}_{(i}\mathbf{D}_{j)} - \frac{1}{3}\gamma_{ij}\mathbf{D}^2$, $\mathbf{D}^2$ is the spatial Laplacian, and $\mathbf{D}^{-2}$ is the inverse Laplacian. To ensure that the expressions (29) are dimensionless we have scaled $\mathcal{S}^i$ and $\mathcal{S}^{ij}$ appropriately with $\mathcal{H}$ [see also (32) below and Appendix A].

We have found that significant simplifications occur in the perturbed Einstein equations if one replaces $\mathsf{G}^i{}_i$ and $\mathsf{T}^i{}_i$ by the following combinations:

$$\frac{1}{3}\mathsf{G}^i{}_i + \mathcal{C}^2\mathsf{G}^\eta{}_\eta, \qquad \frac{1}{3}\mathsf{T}^i{}_i + c_s^2\mathsf{T}^\eta{}_\eta, \tag{31}$$

where $\mathcal{C}^2$ and c_s^2 are the background scalars defined by Eqs. (2b) and (8), with $\mathcal{C}^2 = c_s^2$ when the background Einstein equations are satisfied. The motivation for this choice is clear in the case of the stress-energy tensor, since it follows that $\frac{1}{3}\mathsf{T}^i{}_i(M) + c_s^2\mathsf{T}^\eta{}_\eta(M) = \Gamma$, the nonadiabatic pressure perturbation. Consistency then requires that we use $\frac{1}{3}\mathsf{G}^i{}_i + \mathcal{C}^2\mathsf{G}^\eta{}_\eta$ for the Einstein tensor.

With the above as motivation we now define the following linear combinations of the components of the leading order tensors $\mathsf{G}^a{}_b(f)$ and $\mathsf{T}^a{}_b(M)$ for scalar perturbations:

$$\mathsf{G}^\Gamma(f) \coloneqq \frac{1}{3}\mathsf{G}^i{}_i + \mathcal{C}^2\mathsf{G}^\eta{}_\eta, \qquad \mathsf{T}^\Gamma(M) \coloneqq \frac{1}{3}\mathsf{T}^i{}_i + c_s^2\mathsf{T}^\eta{}_\eta = \Gamma, \tag{32a}$$

$$\mathsf{G}^\pi(f) \coloneqq \mathcal{H}^2\mathcal{S}^{ij}\mathsf{G}_{ij}, \qquad \mathsf{T}^\pi(M) \coloneqq \mathcal{H}^2\mathcal{S}^{ij}\mathsf{T}_{ij} = 0, \tag{32b}$$

$$\mathsf{G}^q(f) \coloneqq \mathcal{H}\mathcal{S}^i\mathsf{G}^\eta{}_i, \qquad \mathsf{T}^q(M) \coloneqq \mathcal{H}\mathcal{S}^i\mathsf{T}^\eta{}_i = \mathcal{H}V, \tag{32c}$$

$$\mathsf{G}^\rho(f) \coloneqq -\mathsf{G}^\eta{}_\eta, \qquad \mathsf{T}^\rho(M) \coloneqq -\mathsf{T}^\eta{}_\eta = \boldsymbol{\delta}, \tag{32d}$$

where $f = {}^{(1)}f$ or ${}^{(2)}f$ and $M = {}^{(1)}M$ or ${}^{(2)}M$, as given by (16) and (24), respectively. We will use the same linear combinations for the source terms $\mathbb{G}^a{}_b({}^{(1)}f)$ and $\mathbb{T}^a{}_b({}^{(1)}f, {}^{(1)}M)$, which are given by the above equations with G and T replaced by $\mathbb{G}$ and $\mathbb{T}$, respectively.

C. The leading order Einstein tensor terms

The expressions for the leading order Einstein terms in (32) can be obtained by specializing equations (19) in Uggla and Wainwright (2013) [15], which yields[10]

$$\frac{1}{2}\mathsf{G}^\Gamma(f) + \frac{1}{3}\mathcal{H}^{-2}\mathbf{D}^2\mathsf{G}^\pi(f) = (\mathcal{L}_\mathrm{B} - \mathcal{C}^2\mathcal{H}^{-2}\mathbf{D}^2)\psi + \mathcal{L}_1(\phi - \psi) + \mathcal{C}^2\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}B), \tag{33a}$$

$$\mathsf{G}^\pi(f) = -(\mathcal{L}_2 + q)(\mathcal{H}B) - \phi + \psi, \tag{33b}$$

[9] The components G_{ij} contain a vector mode and a tensor mode in addition to the scalar mode. The operator $\mathcal{S}^{ij}$ is a concise way of extracting the scalar mode. Similarly the operator $\mathcal{S}^i$ extracts the scalar mode from $\mathsf{G}^\eta{}_i$.

[10] The notation in [15] is related to the notation in the present paper as follows: $\mathsf{G} = \mathcal{H}^2\mathsf{G}^\Gamma$, $\mathcal{S}^{ij}\hat{\mathsf{G}}_{ij} = \mathsf{G}^\pi$, $\mathcal{S}^i\mathsf{G}_i = \mathcal{H}^2(\mathsf{G}^\rho - 3\mathsf{G}^q)$, and $\mathcal{S}^i\mathsf{G}^0{}_i = \mathcal{H}\mathsf{G}^q$. The differential operators have been scaled and relabeled as $\mathcal{L}_A \equiv \mathcal{H}\mathcal{L}_1$, $\mathcal{L}_B \equiv \mathcal{H}\mathcal{L}_2$, and $\mathcal{L} \equiv \mathcal{H}^2\mathcal{L}_B$. Specialize f_{ab} according to $f_{00} = -2\phi$, $f_{0i} = \mathbf{D}_iB$, $f_{ij} = -2\psi\gamma_{ij}$, and set $K = 0$, since we are considering a flat background.

$$\mathsf{G}^q(f) = -2(\partial_N\psi + \phi), \tag{33c}$$

$$\mathsf{G}^\rho(f) - 3\mathsf{G}^q(f) = 2\mathcal{H}^{-2}\mathbf{D}^2(\psi - \mathcal{H}B), \tag{33d}$$

where $f = {}^{(1)}f$ or $f = {}^{(2)}f$. The temporal differential operators $\mathcal{L}_1$, $\mathcal{L}_2$, which are first order in time, are defined by

$$\mathcal{L}_1 = \partial_N + 1 + 3\mathcal{C}^2 - 2q,$$
$$\mathcal{L}_1 f = (1+q)\partial_N((1+q)^{-1}f), \tag{34a}$$

$$\mathcal{L}_2 = \partial_N + 2,$$
$$\mathcal{L}_2 f = a^{-2}\partial_N(a^2 f), \tag{34b}$$

where the compact expressions in the second equation in each pair are derived using $\partial_N a = a$ and the derivative (10) of q. The Bardeen operator $\mathcal{L}_\mathrm{B}$, which is of second order, is defined as

$$\mathcal{L}_\mathrm{B}(f) = \mathcal{L}_1(\mathcal{H}\mathcal{L}_2(\mathcal{H}^{-1}f)), \tag{35}$$

where f is an arbitrary function. Expanding the product form (35) using the compact expressions in (34), and using the definition (2a) of q and the derivative (10) of q, leads to

$$\mathcal{L}_\mathrm{B} = \partial_N^2 + (3(1+\mathcal{C}^2) - q)\partial_N + 1 + 3\mathcal{C}^2 - 2q. \tag{36}$$

The operators $\mathcal{L}_B$, $\mathcal{L}_1$, and $\mathcal{L}_2$ play a central role in determining the evolution of scalar perturbations at first and second order. Note that they are purely kinematical in nature, and therefore relevant for any metric theory that involves the Einstein tensor. The operator $\mathcal{L}_B$ is associated with the Poisson gauge, and it gained prominence through the seminal paper of Bardeen (1980) [16], while the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are associated with the uniform curvature gauge and the work of Kodama and Sasaki [13] [see Eqs. (4.6a,b)], but have been less used. The scalars q and $\mathcal{C}^2$ are determined by the background Einstein equations once the stress-energy tensor has been specified, for example a perfect fluid with barotropic equation of state, pressure-free matter (cold dark matter (CDM)) with a cosmological constant, or a minimally coupled scalar field. In this way the Bardeen operator has appeared in the literature in a variety of different forms, usually using conformal time as the time variable,[11]

$$\mathcal{L}_\mathrm{B} = \mathcal{H}^{-2}(\partial_\eta^2 + 3(1+\mathcal{C}^2)\mathcal{H}\partial_\eta + \mathcal{H}^2(1+3\mathcal{C}^2-2q)). \tag{37}$$

[11]The third term on the right side appears in different forms, for example $\mathcal{H}^2(1+3\mathcal{C}^2-2q) = 2\mathcal{H}' + \mathcal{H}^2(1+3c_s^2) = 3\mathcal{H}^2(c_s^2 - w)$, for a perfect fluid universe [e.g., Mukhanov *et al.* (1992) [17], Eq. (5.22); Nakamura (2007) [4], Eq. (6.65); and Malik and Wands (2009) [18], Eq. (8.31)].

D. The perturbed Einstein equations: Scalar mode

We are now in a position to specialize the perturbed Einstein field equations (28) to the case of scalar perturbations using the linear combinations (32),

$$\mathsf{G}^\Gamma({}^{(1)}f) = 3(1+w)^{(1)}\Gamma;$$
$$\mathsf{G}^\Gamma({}^{(2)}f) = 3(1+w)^{(2)}\Gamma - \mathbb{S}^\Gamma, \tag{38a}$$

$$\mathsf{G}^\pi({}^{(1)}f) = 0;$$
$$\mathsf{G}^\pi({}^{(2)}f) = -\mathbb{S}^\pi, \tag{38b}$$

$$\mathsf{G}^q({}^{(1)}f) = 3(1+w)\mathcal{H}^{(1)}V;$$
$$\mathsf{G}^q({}^{(2)}f) = 3(1+w)\mathcal{H}^{(2)}V - \mathbb{S}^q, \tag{38c}$$

$$\mathsf{G}^\rho({}^{(1)}f) = 3(1+w)^{(1)}\boldsymbol{\delta};$$
$$\mathsf{G}^\rho({}^{(2)}f) = 3(1+w)^{(2)}\boldsymbol{\delta} - \mathbb{S}^\rho, \tag{38d}$$

where the complete source terms have the following form:

$$\mathbb{S} = \mathbb{G}({}^{(1)}f) - 3(1+w)\mathbb{T}({}^{(1)}f, {}^{(1)}M), \tag{38e}$$

for the superscripts Γ, π, q, ρ. In these equations the leading order terms $\mathsf{G}({}^{(1)}f)$ and $\mathsf{G}({}^{(2)}f)$ are given by (33) and the source terms $\mathbb{G}({}^{(1)}f)$ and $\mathbb{T}({}^{(1)}f, {}^{(1)}M)$ are given by (A4) and (A8) in Appendix A.

In Sec. V we will specialize Eqs. (38) to the Poisson gauge ($B = 0$); label the remaining variables with a subscript $_\mathrm{p}$, the uniform curvature gauge ($\psi = 0$); label the remaining variables with a subscript $_\mathrm{c}$ and the total matter gauge ($V = 0$); and label the remaining variables with a subscript $_\mathrm{v}$. When specifying a gauge we will use the following shorthand notation for the source terms, for example in the Poisson gauge:

$$\mathbb{G}({}^{(1)}f_\mathrm{p}) = \mathbb{G}_\mathrm{p}, \qquad \mathbb{T}({}^{(1)}f_\mathrm{p}, {}^{(1)}M_\mathrm{p}) = \mathbb{T}_\mathrm{p}, \tag{39}$$

for each of the superscripts Γ, π, q, ρ, and similarly in the other gauges.

The role played by each of the four equations in the set (38) depends on the choice of gauge, as can be seen by referring to the expressions (33) for the leading order terms (in what follows we identify the four equations in (38) by using the symbol for the leading order Einstein tensor terms):

(i) The G^Γ equation gives
 (a) a second order evolution equation for ψ_p (the Bardeen equation) in the Poisson gauge,
 (b) a first order evolution equation for ϕ_c in the uniform curvature gauge.

(ii) The G^π equation gives
 (a) a constraint equation for ϕ_p in the Poisson gauge,
 (b) a first order evolution equation for B_c in the uniform curvature gauge,

(c) a first order evolution equation for B_v in the total matter gauge.

(iii) The G^q equation gives
(a) a constraint equation for V_p in the Poisson gauge,
(b) a constraint equation for V_c in the uniform curvature gauge,
(c) a first order evolution equation for ψ_v in the total matter gauge.

(iv) The G^ρ equation gives a constraint equation for $\boldsymbol{\delta}$ in all three gauges.

Finally, before specializing the gauge, we present the general expression for ${}^{(r)}\boldsymbol{\delta}$, $r = 1, 2$, valid in any temporal gauge, that arises from the constraint referred to in (iv) above, and that will be useful later. Forming the linear combination (38d)-3(38c) (i.e., the $\mathsf{G}^\rho - 3\mathsf{G}^q$ equation) and using the leading order term (33d) we obtain

$$ {}^{(1)}\boldsymbol{\delta} = 3\mathcal{H}{}^{(1)}V + \frac{2}{3}(1+w)^{-1}\mathcal{H}^{-2}\mathbf{D}^2({}^{(1)}\psi - \mathcal{H}{}^{(1)}B), \tag{40a} $$

$$ {}^{(2)}\boldsymbol{\delta} = 3\mathcal{H}{}^{(2)}V + \frac{1}{3}(1+w)^{-1}(2\mathcal{H}^{-2}\mathbf{D}^2({}^{(2)}\psi - \mathcal{H}{}^{(2)}B) + \mathbb{S}^\rho - 3\mathbb{S}^q), \tag{40b} $$

where

$$ \mathbb{S}^\rho = \mathbb{G}^\rho - 3(1+w)\mathbb{T}^\rho, \qquad \mathbb{S}^q = \mathbb{G}^q - 3(1+w)\mathbb{T}^q, \tag{40c} $$

using the notation in (38e).

IV. THE PERTURBED CONSERVATION EQUATIONS

When using the perturbed Einstein equations, the metric perturbations are determined by the evolution equations, while the matter perturbations are determined by the constraint equations. As an alternative approach one can use the perturbed conservation equations to determine the evolution of the density and velocity perturbations and use two of the perturbed Einstein equations acting as constraints to determine the metric perturbations.

In order to determine the perturbed conservation equations we associate an energy term $E(\epsilon)$ and a scalar momentum term $M(\epsilon) = \mathcal{S}^i E_i(\epsilon)$ with the divergence $\boldsymbol{\nabla}_b(\epsilon)T^b{}_a(\epsilon)$ of the stress-energy tensor $T^a{}_b(\epsilon)$, where

$$ E(\epsilon) = H^{-1}(\rho(\epsilon) + p(\epsilon))^{-1}u^a(\epsilon)\boldsymbol{\nabla}_b(\epsilon)T^b{}_a(\epsilon), \tag{41a} $$

$$ E_i(\epsilon) = (\rho(\epsilon) + p(\epsilon))^{-1}\boldsymbol{\nabla}_b(\epsilon)T^b{}_i(\epsilon). \tag{41b} $$

The Taylor expansion for $T^b{}_a(\epsilon)$ leads to a Taylor series expansion for $E(\epsilon)$,

$$ E(\epsilon) = {}^{(0)}E + \epsilon{}^{(1)}E + \frac{1}{2}\epsilon^2{}^{(2)}E + \cdots, \tag{42} $$

and similarly for $M(\epsilon)$. The coefficients in this expansion have a structure analogous to the perturbations of the Einstein tensor and stress-energy tensor in (26) and (27),

$$ {}^{(1)}E = \mathsf{E}({}^{(1)}F), \qquad {}^{(1)}M = \mathsf{M}({}^{(1)}F), \tag{43a} $$

$$ {}^{(2)}E = \mathsf{E}({}^{(2)}F) + \mathbb{E}({}^{(1)}F), \qquad {}^{(2)}M = \mathsf{M}({}^{(2)}F) + \mathbb{M}({}^{(1)}F), \tag{43b} $$

where

$$ F = (\phi, \mathcal{H}B, \psi, \boldsymbol{\delta}, \Gamma, \mathcal{H}V). \tag{43c} $$

The first and second order perturbations have a common *leading order term* of the form $\mathsf{E}(F)$ or $\mathsf{M}(F)$, where $F = {}^{(1)}F$ or $F = {}^{(2)}F$ while ${}^{(2)}E$ and ${}^{(2)}M$ also have a *source term* $\mathbb{E}({}^{(1)}F)$ and $\mathbb{M}({}^{(1)}F)$ which depends quadratically on ${}^{(1)}F$.

Performing the perturbation expansion in (41) to first order gives the following expressions for the leading order terms:

$$ \mathsf{E}(F) = \partial_N(\boldsymbol{\delta} - 3\psi) + \mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}V - \mathcal{H}B) + 3\Gamma, \tag{44a} $$

$$ \mathsf{M}(F) = (\partial_N + 1 + q)(\mathcal{H}V) + \phi + c_s^2(\boldsymbol{\delta} - 3\mathcal{H}V) + \Gamma, \tag{44b} $$

where we have used (17), which introduces q into the equation and suggests that we use $1+q$ instead of $\frac{3}{2}(1+w)$. A similar but more lengthy calculation to second order leads to the expressions for the quadratic source terms $\mathbb{E}({}^{(1)}F)$ and $\mathbb{M}({}^{(1)}F)$ that are given by Eqs. (A9) in Appendix A.

Referring to Eq. (43) the perturbed conservation equations at first and second order are given by ${}^{(i)}E = 0$, $i = 1, 2$ (conservation of energy) and ${}^{(i)}M = 0$, $i = 1, 2$ (conservation of momentum). From Eqs. (43a) and (43b) we obtain

$$ \mathsf{E}({}^{(1)}F) = 0, \qquad \mathsf{M}({}^{(1)}F) = 0, \tag{45a} $$

$$ \mathsf{E}({}^{(2)}F) + \mathbb{E}({}^{(1)}F) = 0, \qquad \mathsf{M}({}^{(2)}F) + \mathbb{M}({}^{(1)}F) = 0, \tag{45b} $$

where the leading order terms are given by (44) and the source terms are given by (A9).

The perturbed energy conservation equation at second order has been given by Malik and Wands (2004) [14] in the long wavelength limit [see Eq. (5.33) where they use ${}^{(2)}\rho$ and ${}^{(2)}p$ rather than ${}^{(2)}\boldsymbol{\delta}$ and ${}^{(2)}\Gamma$ as perturbation variables]. We are aware of two general formulations of the perturbed conserved equations to second order, namely, Hwang and Noh (2007) [19] and Nakamura (2009) [20]. For purposes of comparison we refer to their equations when specialized to the case of perfect fluid and scalar perturbations: in [19] see Eq. (100) with (95) for conservation of energy and (101) for conservation of momentum, and in [20] see Eqs. (4.8)–(4.10) for conservation of energy and (4.14), (4.18), and (4.19) for conservation of momentum. In contrast to our approach these authors use the unscaled density and pressure perturbations as matter variables and do not introduce the nonadiabatic pressure perturbation Γ, which means that an immediate comparison cannot be made. As regards gauge choice, Hwang and Noh give their equations for an arbitrary choice of temporal gauge, while Nakamura effectively uses the Poisson gauge.

V. READY-TO-USE SYSTEMS OF GOVERNING EQUATIONS

In this section, by specializing the perturbed Einstein equations (38) and conservation equations (44) and (45) to various gauges we derive the ready-to-use systems of governing equations described in the Introduction.

A. The Poisson gauge

The Poisson gauge is defined by the condition $B = 0$. The scalar metric and matter perturbations are denoted by $\phi_{\rm p}, \psi_{\rm p}$, with $B_{\rm p} = 0$, and $V_{\rm p}$, $\boldsymbol{\delta}_{\rm p}$ with the subscript $_{\rm p}$ indicating the Poisson gauge while a superscript indicates the order of the perturbation, e.g., ${}^{(r)}\psi_{\rm p}$, $r = 1, 2$ (since ${}^{(1)}\Gamma$ and ${}^{(2)}\Gamma$ are gauge invariants, they will not have a subscript in any gauge).

We insert $B = 0$ into the leading order terms (33), and label the remaining variables with a subscript $_{\rm p}$. These leading order terms (first and second order), when inserted into Eqs. (38), give the perturbed Einstein equations in the Poisson gauge. It is convenient, however, to obtain $\boldsymbol{\delta}_{\rm p}$ directly by choosing the Poisson gauge in Eq. (40). In addition, in order to obtain the Bardeen equation (48a) below in a direct way we form the linear combination (38a) $+2(\mathcal{L}_1 + \frac{1}{3}\mathcal{H}^{-2}\mathbf{D}^2)$ (38b) of the perturbed Einstein equations and use the following relation for the leading order Einstein terms:

$$\mathsf{G}^{\Gamma}_{\rm p} + 2\left(\mathcal{L}_1 + \frac{1}{3}\mathcal{H}^{-2}\mathbf{D}^2\right)\mathsf{G}^{\pi}_{\rm p} = 2(\mathcal{L}_{\rm B} - \mathcal{C}^2\mathcal{H}^{-2}\mathbf{D}^2)\psi_{\rm p}, \tag{46}$$

where we have used the linear combination 2 (33a) $+2\mathcal{L}_1$ (33b) in the Poisson gauge.

1. The Bardeen equation for $\boldsymbol{\psi}_{\rm p}$

At first order the above procedure leads to the following system:

$$(\mathcal{L}_B - c_s^2\mathcal{H}^{-2}\mathbf{D}^2){}^{(1)}\psi_{\rm p} = \frac{3}{2}(1+w){}^{(1)}\Gamma, \tag{47a}$$

$${}^{(1)}\phi_{\rm p} = {}^{(1)}\psi_{\rm p}, \tag{47b}$$

$$\mathcal{H}{}^{(1)}V_{\rm p} = -\frac{2}{3}(1+w)^{-1}(\partial_N{}^{(1)}\psi_{\rm p} + {}^{(1)}\phi_{\rm p}), \tag{47c}$$

$${}^{(1)}\boldsymbol{\delta}_{\rm p} = 3\mathcal{H}{}^{(1)}V_{\rm p} + \frac{2}{3}(1+w)^{-1}\mathcal{H}^{-2}\mathbf{D}^{2\,(1)}\psi_{\rm p}, \tag{47d}$$

where $\mathcal{L}_B$ is given by (36), although the product form (35) of the operator is useful when solving the equation. Observe that ${}^{(1)}\psi_{\rm p}$ is the primary dynamical variable and is determined by the Bardeen equation (47a).

The second order perturbation equations have the following form:

$$(\mathcal{L}_B - c_s^2\mathcal{H}^{-2}\mathbf{D}^2){}^{(2)}\psi_{\rm p} = \frac{3}{2}(1+w){}^{(2)}\Gamma - \frac{1}{2}\mathbb{S}^{\Gamma}_{\rm p} - \left(\mathcal{L}_1 + \frac{1}{3}\mathcal{H}^{-2}\mathbf{D}^2\right)\mathbb{S}^{\pi}_{\rm p}, \tag{48a}$$

$${}^{(2)}\phi_{\rm p} = {}^{(2)}\psi_{\rm p} + \mathbb{S}^{\pi}_{\rm p}, \tag{48b}$$

$$\mathcal{H}{}^{(2)}V_{\rm p} = -\frac{2}{3}(1+w)^{-1}\left(\partial_N{}^{(2)}\psi_{\rm p} + {}^{(2)}\phi_{\rm p} - \frac{1}{2}\mathbb{S}^{q}_{\rm p}\right), \tag{48c}$$

$${}^{(2)}\boldsymbol{\delta}_{\rm p} = 3\mathcal{H}{}^{(2)}V_{\rm p} + \frac{2}{3}(1+w)^{-1}\left(\mathcal{H}^{-2}\mathbf{D}^{2\,(2)}\psi_{\rm p} + \frac{1}{2}\mathbb{S}^{\rho}_{\rm p} - \frac{3}{2}\mathbb{S}^{q}_{\rm p}\right), \tag{48d}$$

where the source terms $\mathbb{S}_{\rm p}$, for the superscripts Γ, π, q, ρ, are given by

$$\mathbb{S}_{\rm p} = \mathbb{G}_{\rm p} - 3(1+w)\mathbb{T}_{\rm p}, \tag{49}$$

using the notation (38e) and (39). To complete the specification of the equations we need to give the explicit form of the source terms $\mathbb{G}_{\rm p}$ and $\mathbb{T}_{\rm p}$, which are obtained by specializing Eqs. (A4) and (A8) in Appendix A to the Poisson gauge ($B = 0$) and inserting the relations ${}^{(1)}\phi_{\rm p} = {}^{(1)}\psi_{\rm p}$ [Eq. (47b)]. Equations (A4) yield[12]

$$\mathbb{G}^{\Gamma}_{\rm p} = -8\mathcal{L}_1(\psi_{\rm p}^2) + \frac{2}{3}(1+3c_s^2)\mathbb{X}_{\rm p} - \frac{8}{3}\mathcal{H}^{-2}(\mathbf{D}\psi_{\rm p})^2, \tag{50a}$$

$$\mathbb{G}^{\pi}_{\rm p} = 4(\psi_{\rm p}^2 - \mathbb{D}_0(\psi_{\rm p})), \tag{50b}$$

$$\mathbb{G}^{q}_{\rm p} = 4(2\psi_{\rm p}^2 - \mathcal{S}^i[(\partial_N\psi_{\rm p})\mathbf{D}_i\psi_{\rm p}]), \tag{50c}$$

$$\mathbb{G}^{\rho}_{\rm p} = 24\psi_{\rm p}^2 - 2\mathbb{X}_{\rm p}, \tag{50d}$$

[12]These expressions have been given by Uggla and Wainwright (2013) [15]; see Eq. (35). Here and elsewhere, in order to simplify the notation we omit the superscript ${}^{(1)}$ on the linear perturbations in the source terms.

where the mode extraction operator $\mathcal{S}^i$ was given in (30) while the spatial differential operator $\mathbb{D}_0$ is defined in Eq. (A1b) in Appendix A. In addition,

$$\mathbb{X}_{\rm p} = -3(\partial_N \psi_{\rm p})^2 + 5\mathcal{H}^{-2}(\mathbf{D}\psi_{\rm p})^2 - 4\mathcal{H}^{-2}\mathbf{D}^2\psi_{\rm p}^2. \qquad (50e)$$

Equations (A8) yield

$$\mathbb{T}_{\rm p}^{\Gamma} = \frac{2}{3}(1-3c_s^2)(\mathbf{D}V_{\rm p})^2 - \frac{1}{3}(\partial_N c_s^2)(\boldsymbol{\delta}_{\rm p})^2 - \frac{2}{3}{}^{(1)}\boldsymbol{\delta}_{\rm p}(\partial_N - 3(1+c_s^2))\Gamma, \qquad (51a)$$

$$\mathbb{T}_{\rm p}^{\pi} = 2\mathbb{D}_0(\mathcal{H}V_{\rm p}), \qquad (51b)$$

$$\mathbb{T}_{\rm p}^{q} = \mathcal{S}^i[2((1+c_s^2)\boldsymbol{\delta}_{\rm p} - \phi_{\rm p} + \Gamma)\mathbf{D}_i(\mathcal{H}V_{\rm p})], \qquad (51c)$$

$$\mathbb{T}_{\rm p}^{\rho} = 2(\mathbf{D}V_{\rm p})^2. \qquad (51d)$$

The perturbed Einstein equations at second order in the Poisson gauge have been given in different forms by various authors.[13]

2. Coupled evolution equations for $\psi_{\rm p}$ and $V_{\rm p}$

An alternative approach to analyzing the dynamics in the Poisson gauge is to use $\psi_{\rm p}$ and $V_{\rm p}$ as primary dynamical variables, with the perturbed Einstein equation $\mathsf{G}_{\rm p}^q$ as evolution equation for $\psi_{\rm p}$ and the perturbed conservation of momentum equation $\mathsf{M}_{\rm p}$ as evolution equation for $V_{\rm p}$.

To obtain the first equation we use the perturbed Einstein equation (48c), with ${}^{(2)}\phi_{\rm p}$ eliminated using (48b). To obtain the second equation we use the perturbed conservation of momentum equation (45b) at second order in the Poisson gauge which reads

$$(\partial_N + 1 + q)(\mathcal{H}{}^{(2)}V_{\rm p}) + {}^{(2)}\phi_{\rm p} + c_s^2({}^{(2)}\boldsymbol{\delta} - 3\mathcal{H}{}^{(2)}V_{\rm p}) + {}^{(2)}\Gamma + \mathbb{M}_{\rm p} = 0, \qquad (52)$$

and use (48b) to eliminate ${}^{(2)}\phi_{\rm p}$ and (48d) to eliminate ${}^{(2)}\boldsymbol{\delta}_{\rm p} - 3\mathcal{H}{}^{(2)}V_{\rm p}$. The resulting equations are as follows:

$$(\partial_N + 1){}^{(2)}\psi_{\rm p} + (1+q)\mathcal{H}{}^{(2)}V_{\rm p} - \frac{1}{2}\mathbb{S}_{\rm p}^q + \mathbb{S}_{\rm p}^{\pi} = 0, \qquad (53a)$$

$$(\partial_N + 1 + q)(\mathcal{H}{}^{(2)}V_{\rm p}) + (1 + (1+q)^{-1}c_s^2\mathcal{H}^{-2}\mathbf{D}^2){}^{(2)}\psi_{\rm p} + {}^{(2)}\Gamma + \mathbb{S} = 0, \qquad (53b)$$

where

$$\mathbb{S} = \mathbb{M}_{\rm p} + \mathbb{S}_{\rm p}^{\pi} + \frac{1}{2}(1+q)^{-1}c_s^2(\mathbb{S}_{\rm p}^{\rho} - 3\mathbb{S}_{\rm p}^{q}). \qquad (53c)$$

The source terms with kernel $\mathbb{S}_{\rm p}$ are given by (49), and $\mathbb{M}_{\rm p}$ is given by (A9b). Equations (53) form a coupled system of evolution equations for ${}^{(2)}\psi_{\rm p}$ and ${}^{(2)}V_{\rm p}$. The corresponding system for ${}^{(1)}\psi_{\rm p}$ and ${}^{(1)}V_{\rm p}$ is obtained by dropping the source terms and changing ${}^{(2)}$ to ${}^{(1)}$.

The system of Eqs. (53) has the same dynamical content as the second order Bardeen equation (48a) for ${}^{(2)}\psi_{\rm p}$. One can derive the Bardeen equation from (53) by solving the first equation algebraically for $\mathcal{H}{}^{(2)}V_{\rm p}$ and substituting it into the second equation. The difference is that the source term obtained in this way has a different form from the source term in (48a).

B. The uniform curvature gauge

The uniform curvature gauge is defined by the condition $\psi = 0$. The scalar metric perturbations are denoted by $\phi_{\rm c}$, $B_{\rm c}$, with $\psi_{\rm c} = 0$, and the matter variables by $\boldsymbol{\delta}_{\rm c}$, $V_{\rm c}$, and Γ, with a superscript indicating the order of the perturbation, e.g., ${}^{(r)}\phi_{\rm c}$, $r = 1, 2$. We insert $\psi = 0$ into the leading order terms (33), and label the remaining variables with a subscript ${}_{\rm c}$. These leading order terms (first and second order), when inserted into Eqs. (38), give the perturbed Einstein equations in the uniform curvature gauge. It is convenient, however, to obtain $\boldsymbol{\delta}_{\rm c}$ directly by choosing the uniform curvature gauge in Eq. (40).

1. Coupled evolution equations for $\phi_{\rm c}$ and $B_{\rm c}$

At first order the above procedure leads to the following system:

$$(1+q)\partial_N((1+q)^{-1\,(1)}\phi_{\rm c}) = -c_s^2\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}{}^{(1)}B_{\rm c}) + (1+q){}^{(1)}\Gamma, \qquad (54a)$$

$$\partial_N(a^{2\,(1)}B_{\rm c}) = -a^2\mathcal{H}^{-1\,(1)}\phi_{\rm c}, \qquad (54b)$$

$$\mathcal{H}{}^{(1)}V_{\rm c} = -(1+q)^{-1\,(1)}\phi_{\rm c}, \qquad (54c)$$

$${}^{(1)}\boldsymbol{\delta}_{\rm c} = 3\mathcal{H}{}^{(1)}V_{\rm c} - (1+q)^{-1}\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}{}^{(1)}B_{\rm c}), \qquad (54d)$$

where we have used the expression $\mathcal{L}_1 f = (1+q)\partial_N((1+q)^{-1}f)$, given in (34a), which introduces q into the equations. We have chosen not to replace $1+q$ by $\frac{3}{2}(1+w)$. At second order we obtain

$$(1+q)\partial_N((1+q)^{-1\,(2)}\phi_{\rm c}) = -c_s^2\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}{}^{(2)}B_{\rm c}) + (1+q){}^{(2)}\Gamma - \frac{1}{2}\mathbb{S}_{\rm c}^{\Gamma} - \frac{1}{3}\mathcal{H}^{-2}\mathbf{D}^2\mathbb{S}_{\rm c}^{\pi}, \qquad (55a)$$

$$\partial_N(a^{2\,(2)}B_{\rm c}) = -a^2\mathcal{H}^{-1}({}^{(2)}\phi_{\rm c} - \mathbb{S}_{\rm c}^{\pi}), \qquad (55b)$$

$$\mathcal{H}{}^{(2)}V_{\rm c} = -(1+q)^{-1}\left({}^{(2)}\phi_{\rm c} - \frac{1}{2}\mathbb{S}_{\rm c}^{q}\right), \qquad (55c)$$

[13] See, for example, Noh and Hwang (2004) [3], Eq. (303), and Nakamura (2007) [4], Eqs. (6.38), (6.41), (6.42), and (6.44).

$$ {}^{(2)}\boldsymbol{\delta}_{\rm c} = 3\mathcal{H}^{(2)}V_{\rm c} + (1+q)^{-1}\bigg(-\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}^{(2)}B_{\rm c}) + \frac{1}{2}(\mathbb{S}_{\rm c}^{\rho} - 3\mathbb{S}_{\rm c}^{q})\bigg). \tag{55d} $$

The source terms with kernel $\mathbb{S}_{\rm c}$ are given by

$$ \mathbb{S}_{\rm c} = \mathbb{G}_{\rm c} - 3(1+w)\mathbb{T}_{\rm c}, \tag{56} $$

using the notation (38e) and (39). The source terms for the Einstein tensor, with kernel $\mathbb{G}_{\rm c}$, are given by Eqs. (A4) in Appendix A with $\psi_{\rm c} = 0$,[14]

$$ \mathbb{G}_{\rm c}^{\Gamma} = -2\mathcal{L}_1(4\phi_{\rm c}^2 - (\mathbf{D}B_{\rm c})^2) - \frac{4}{3}\mathcal{H}^{-2}[(\partial_N\phi_{\rm c} - 4\phi_{\rm c}) \times \mathbf{D}^2(\mathcal{H}B_{\rm c}) + (\mathbf{D}\phi_{\rm c})^2] - \frac{1}{3}(2+3c_s^2)\mathbb{W}_{\rm c} - \frac{2}{3}(1+3c_s^2)\mathcal{H}^{-2}\mathbf{D}^2\mathbb{D}_2(B_{\rm c}), \tag{57a} $$

$$ \mathbb{G}_{\rm c}^{\pi} = 2\mathbb{D}_0(\phi_{\rm c}) + 2\mathcal{S}^{ij}[(2(\mathbf{D}_i\phi_{\rm c})\mathbf{D}_j(\mathcal{H}B_{\rm c}) + (\partial_N\phi_{\rm c})\mathbf{D}_{ij}(\mathcal{H}B_{\rm c})] + \mathbb{D}_2(B_{\rm c}), \tag{57b} $$

$$ \mathbb{G}_{\rm c}^{q} = 2(4\phi_{\rm c}^2 - (\mathbf{D}B_{\rm c})^2) - 2\mathcal{H}^{-2}\mathcal{S}^i \times \bigg[(\mathbf{D}_j\phi_{\rm c})\bigg(\mathbf{D}^j{}_i - \frac{2}{3}\delta^j{}_i\mathbf{D}^2\bigg)(\mathcal{H}B_{\rm c})\bigg], \tag{57c} $$

$$ \mathbb{G}_{\rm c}^{\rho} = 6(4\phi_{\rm c}^2 - (\mathbf{D}B_{\rm c})^2) + \mathbb{W}_{\rm c} + 2\mathcal{H}^{-2}\mathbf{D}^2\mathbb{D}_2(B_{\rm c}), \tag{57d} $$

where

$$ \mathbb{W}_{\rm c} = \mathcal{H}^{-2}[8\phi_{\rm c}\mathbf{D}^2(\mathcal{H}B_{\rm c}) + 4(\mathbf{D}^k\phi_{\rm c})\mathbf{D}_k(\mathcal{H}B_{\rm c})]. \tag{57e} $$

The spatial differential operators $\mathbb{D}_0$ and $\mathbb{D}_2$ are defined in Eqs. (A1b) and (A1c) in Appendix A.

The source terms for the stress-energy tensor, with kernel $\mathbb{T}$, are given by Eqs. (A8) in Appendix A, specialized to the uniform curvature gauge,

$$ \mathbb{T}_{\rm c}^{\rho} = \gamma^{ij}(\mathbb{V}_{2,{\rm c}})_{ij}, \tag{58a} $$

$$ \mathbb{T}_{\rm c}^{\Gamma} = \frac{1}{3}(1-3c_s^2)\gamma^{ij}(\mathbb{V}_{2,{\rm c}})_{ij} - \frac{1}{3}(\partial_N c_s^2)\boldsymbol{\delta}_{\rm c}^2 - \frac{2}{3}\boldsymbol{\delta}_{\rm c}(\partial_N - 3(1+c_s^2))\Gamma, \tag{58b} $$

$$ \mathbb{T}_{\rm c}^{q} = 2\mathcal{S}^i[((1+c_s^2)\boldsymbol{\delta}_{\rm c} - \phi_{\rm c} + \Gamma)\mathbf{D}_i(\mathcal{H}V_{\rm c})], \tag{58c} $$

$$ \mathbb{T}_{\rm c}^{\pi} = \mathcal{H}^2\mathcal{S}^{ij}(\mathbb{V}_{2,{\rm c}})_{ij}, \tag{58d} $$

where

$$ (\mathbb{V}_{2,{\rm c}})_{ij} = 2\mathcal{H}^{-2}(\mathbf{D}_i\mathcal{H}V_{\rm c})\mathbf{D}_j(\mathcal{H}V_{\rm c} - \mathcal{H}B_{\rm c}). \tag{58e} $$

To the best of our knowledge the system of Eqs. (55) and the associated source terms are new. We comment on the utility of these equations in the discussion in Sec. VI.

C. The total matter gauge

The total matter gauge is defined by the condition $V = 0$. There are thus three metric perturbation variables, $\phi_{\rm v}$, $\psi_{\rm v}$, and $B_{\rm v}$, but only two matter perturbation variables $\boldsymbol{\delta}_{\rm v}$ and Γ. The perturbations of the stress-energy tensor thereby simplify. It follows from (32) and (A8) that the leading order terms and the source terms, respectively, satisfy

$$ \mathsf{T}_{\rm v}^{q} = 0, \quad \mathsf{T}_{\rm v}^{\pi} = 0; \quad \mathbb{T}_{\rm v}^{\rho} = 0, \quad \mathbb{T}_{\rm v}^{q} = 0, \quad \mathbb{T}_{\rm v}^{\pi} = 0. \tag{59} $$

1. Coupled evolution equations for $\psi_{\rm v}$ and $B_{\rm v}$

When working in the total matter gauge it is convenient to replace the perturbed Einstein equation associated with G^{Γ} by the perturbed conservation of momentum equation, since this equation determines $\phi_{\rm v}$ algebraically. Specifically, in the total matter gauge the perturbed conservation of momentum equations (44b) and (A9b) lead to

$$ {}^{(1)}\phi_{\rm v} = -c_s^2\,{}^{(1)}\boldsymbol{\delta}_{\rm v} - {}^{(1)}\Gamma, \tag{60a} $$

$$ {}^{(2)}\phi_{\rm v} = -c_s^2\,{}^{(2)}\boldsymbol{\delta}_{\rm v} - {}^{(2)}\Gamma - \mathbb{M}_{\rm v}, \tag{60b} $$

where

$$ \mathbb{M}_{\rm v} = -2\phi_{\rm v}^2 + (\mathbf{D}B_{\rm v})^2 - \bigg[c_s^2(1+c_s^2) + \frac{1}{3}\partial_N c_s^2\bigg]\boldsymbol{\delta}_{\rm v}^2 - \Gamma^2 - \frac{2}{3}\boldsymbol{\delta}_{\rm v}\partial_N\Gamma + 2\mathcal{S}^i[\Gamma\mathbf{D}_i\boldsymbol{\delta}_{\rm v}]. \tag{60c} $$

On substituting (59) into the perturbed Einstein equations (38) the first and second order equations, excluding the G^{Γ} equation, become

$$ \mathsf{G}^{\pi}({}^{(1)}f_{\rm v}) = 0, \qquad \mathsf{G}^{\pi}({}^{(2)}f_{\rm v}) + \mathbb{G}_{\rm v}^{\pi} = 0, \tag{61a} $$

$$ \mathsf{G}^{q}({}^{(1)}f_{\rm v}) = 0, \qquad \mathsf{G}^{q}({}^{(2)}f_{\rm v}) + \mathbb{G}_{\rm v}^{q} = 0, \tag{61b} $$

$$ \mathsf{G}^{\rho}({}^{(1)}f_{\rm v}) = 3(1+w)\,{}^{(1)}\boldsymbol{\delta}_{\rm v}, \quad \mathsf{G}^{\rho}({}^{(2)}f_{\rm v}) + \mathbb{G}_{\rm v}^{\rho} = 3(1+w)\,{}^{(2)}\boldsymbol{\delta}_{\rm v}. \tag{61c} $$

As governing equations we use Eqs. (60) in conjunction with the perturbed Einstein equations (61). To obtain the detailed form of the equations we substitute the expressions

[14] The source terms for the perturbed Einstein tensor in the uniform curvature gauge have been given by Uggla and Wainwright (2013) [15] [see Eqs. (96)].

for the leading terms of the Einstein tensor from Eqs. (33). At first order we obtain

$$\partial_N {}^{(1)}\psi_{\rm v} = c_s^2 {}^{(1)}\boldsymbol{\delta}_{\rm v} + {}^{(1)}\Gamma, \tag{62a}$$

$$\mathcal{H}a^{-2}\partial_N(a^2 {}^{(1)}B_{\rm v}) = {}^{(1)}\psi_{\rm v} + c_s^2 {}^{(1)}\boldsymbol{\delta}_{\rm v} + {}^{(1)}\Gamma, \tag{62b}$$

$${}^{(1)}\boldsymbol{\delta}_{\rm v} = \frac{2}{3}(1+w)^{-1}\mathcal{H}^{-2}\mathbf{D}^2({}^{(1)}\psi_{\rm v} - \mathcal{H}{}^{(1)}B_{\rm v}), \tag{62c}$$

while the second order equations can be written as

$$\partial_N {}^{(2)}\psi_{\rm v} = c_s^2 {}^{(2)}\boldsymbol{\delta}_{\rm v} + {}^{(2)}\Gamma + \mathbb{M}_{\rm v} + \frac{1}{2}\mathbb{G}_{\rm v}^q, \tag{63a}$$

$$\mathcal{H}a^{-2}\partial_N(a^2 {}^{(2)}B_{\rm v}) = {}^{(2)}\psi_{\rm v} + c_s^2 {}^{(2)}\boldsymbol{\delta}_{\rm v} + {}^{(2)}\Gamma + \mathbb{M}_{\rm v} + \mathbb{G}_{\rm v}^\pi, \tag{63b}$$

$${}^{(2)}\boldsymbol{\delta}_{\rm v} = \frac{2}{3}(1+w)^{-1}\left(\mathcal{H}^{-2}\mathbf{D}^2({}^{(2)}\psi_{\rm v} - \mathcal{H}{}^{(2)}B_{\rm v}) + \frac{1}{2}(\mathbb{G}_{\rm v}^\rho - 3\mathbb{G}_{\rm v}^q)\right), \tag{63c}$$

where $\mathbb{M}_{\rm v}$ is given by (60c). The Einstein source terms, labeled $\mathbb{G}_{\rm v}$, are obtained by evaluating Eq. (A4) in Appendix A in the total matter gauge. Since the three metric perturbations are in general nonzero in this gauge the Einstein source terms do not simplify in general. However, as we will explain in the Discussion, in the two benchmark problems mentioned in the Introduction, additional restrictions arise which simplify the leading order Einstein terms and the Einstein source terms significantly, making the total matter gauge an ideal choice for these problems.

D. The perturbed conservation equations

In this subsection we use the matter variables $\boldsymbol{\delta}$ and V as the primary dynamical variables and we use both perturbed conservation equations (energy and momentum) to obtain the evolution equations.

The perturbed conservation equations at second order are given by Eqs. (44) and (45b), which we repeat here:

$${}^{(2)}E = \partial_N({}^{(2)}\boldsymbol{\delta} - 3{}^{(2)}\psi) + \mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}{}^{(2)}V - \mathcal{H}{}^{(2)}B) + 3{}^{(2)}\Gamma + \mathbb{E} = 0, \tag{64a}$$

$${}^{(2)}M = (\partial_N + 1 + q)(\mathcal{H}{}^{(2)}V) + {}^{(2)}\phi + c_s^2({}^{(2)}\boldsymbol{\delta} - 3\mathcal{H}{}^{(2)}V) + {}^{(2)}\Gamma + \mathbb{M} = 0. \tag{64b}$$

These equations provide evolution equations for $\boldsymbol{\delta}$ and V, but they do not form a closed evolution system since they are coupled to the metric perturbations. However, we can circumvent this difficulty by an appropriate use of two gauges: in the following section we consider $\boldsymbol{\delta}_{\rm v}$ (total matter gauge) and $V_{\rm p}$ (Poisson gauge).

1. Coupled evolution equations for $\boldsymbol{\delta}_{\rm v}$ and $\mathbf{D}^2 V_{\rm p}$

To obtain the first equation we calculate ${}^{(2)}E_{\rm v} - 3{}^{(2)}M_{\rm v}$ starting with (64). We eliminate $\partial_N {}^{(2)}\psi_{\rm v}$ using

$$\partial_N {}^{(2)}\psi_{\rm v} + {}^{(2)}\phi_{\rm v} = \frac{1}{2}\mathbb{G}_{\rm v}^q, \tag{65}$$

which follows from Eqs. (60b) and (63a), and we replace ${}^{(2)}B_{\rm v}$ by $-{}^{(2)}V_{\rm p}$ plus source terms using the change of gauge formula

$$\mathcal{H}{}^{(2)}B_{\rm v} + \mathcal{H}{}^{(2)}V_{\rm p} = \mathbb{S}[B_{\rm v} + V_{\rm p}], \tag{66}$$

where the source term is given by (A13a). Here we have introduced the notation $\mathbb{S}[\cdots]$ for the source terms associated with a change of gauge formula at second order. To obtain the second equation we evaluate $\mathbf{D}^{2(1)}M_{\rm p}$ starting with (64). We replace ${}^{(2)}\phi_{\rm p}$ by ${}^{(2)}\psi_{\rm p}$ plus source terms using (48b). We next use the general relativity (GR) version of Poisson's equation at second order,

$$(1+q){}^{(2)}\boldsymbol{\delta}_{\rm v} - \mathcal{H}^{-2}\mathbf{D}^{2(2)}\psi_{\rm p} = \mathbb{S}_{\rm Poisson}, \tag{67}$$

where the source term is given by (A10), to express $\mathbf{D}^{2(2)}\psi_{\rm p}$ in terms of ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ plus source terms. We finally express ${}^{(2)}\boldsymbol{\delta}_{\rm p} - 3{}^{(2)}\mathcal{H}V_{\rm p}$ in terms of ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ and source terms using the change of gauge formula

$${}^{(2)}\boldsymbol{\delta}_{\rm v} - {}^{(2)}\boldsymbol{\delta}_{\rm p} + 3{}^{(2)}\mathcal{H}V_{\rm p} = \mathbb{S}[\boldsymbol{\delta}_{\rm v} - \boldsymbol{\delta}_{\rm p} + \mathcal{H}V_{\rm p}], \tag{68}$$

where the source term is given by (A13b). This procedure leads to the following system of evolution equations:

$$(\partial_N - 3c_s^2){}^{(2)}\boldsymbol{\delta}_{\rm v} + \mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}{}^{(2)}V_{\rm p}) + \mathbb{S}_\rho = 0, \tag{69a}$$

$$(\partial_N + 1 - q)[\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}{}^{(2)}V_{\rm p})] + (1 + q + c_s^2\mathcal{H}^{-2}\mathbf{D}^2){}^{(2)}\boldsymbol{\delta}_{\rm v} + \mathcal{H}^{-2}\mathbf{D}^{2(2)}\Gamma + \mathbb{S}_V = 0, \tag{69b}$$

where the source terms are

$$\mathbb{S}_\rho = \mathbb{E}_{\rm v} - 3\mathbb{M}_{\rm v} - \frac{3}{2}\mathbb{G}_{\rm v}^q - \mathcal{H}^{-2}\mathbf{D}^2(\mathbb{S}[B_{\rm v} + V_{\rm p}]), \tag{70a}$$

$$\mathbb{S}_V = \mathcal{H}^{-2}\mathbf{D}^2(\mathbb{M}_{\rm p} + \mathbb{S}_{\rm p}^\pi + c_s^2\mathbb{S}[\boldsymbol{\delta}_{\rm p} - 3\mathcal{H}V_{\rm p} - \boldsymbol{\delta}_{\rm v}]) - \mathbb{S}_{\rm Poisson}. \tag{70b}$$

The source terms $\mathbb{E}_{\rm v}$, $\mathbb{M}_{\rm v}$, and $\mathbb{M}_{\rm p}$ are obtained by choosing the total matter gauge and the Poisson gauge in the general formulas (A9) in Appendix A, and the source terms associated with a change of gauge formulas are given by (A13).

The source term $\mathbb{S}_{\text{Poisson}}$, which is the source term in the GR version of Poisson's equation (67) at second order, requires explanation. This equation is derived by choosing the Poisson gauge in Eq. (40b) and then using (68) to express $\boldsymbol{\delta}_{\text{p}}$ in terms of $\boldsymbol{\delta}_{\text{v}}$. Collecting all the source terms gives the expression for $\mathbb{S}_{\text{Poisson}}$ in Eq. (A10).

2. Second order evolution equation for $\boldsymbol{\delta}_{\text{v}}$

In this section the first order evolution equations for $\boldsymbol{\delta}_{\text{v}}$ and V_{p} are combined to give a second order evolution equation for $\boldsymbol{\delta}_{\text{v}}$. This evolution equation for the density perturbation is obtained by eliminating $\mathbf{D}^2({}^{(2)}V_{\text{p}})$ in Eqs. (69). After expanding the derivatives we obtain

$$(\mathcal{L}_D - c_s^2\mathcal{H}^{-2}\mathbf{D}^2)^{(2)}\boldsymbol{\delta}_{\text{v}} + \mathbb{S}_{\mathcal{L}_D} = \mathcal{H}^{-2}\mathbf{D}^{2(2)}\Gamma, \tag{71a}$$

where the differential operator $\mathcal{L}_D$ is given by

$$\begin{aligned}\mathcal{L}_D = \partial_N^2 + (1-q-3c_s^2)\partial_N - (1-3c_s^2)(1+q) \\ - 6c_s^2 - 3\partial_N c_s^2,\end{aligned} \tag{71b}$$

and the source term $\mathbb{S}_{\mathcal{L}_D}$ is given by

$$\mathbb{S}_{\mathcal{L}_D} = (\partial_N + 1 + q)\mathbb{S}_\rho - \mathbb{S}_V. \tag{71c}$$

We note in passing that the source term in the evolution equation (71) simplifies significantly in the case of a perturbed ΛCDM universe, which permits one to quickly find the second order density perturbation ${}^{(2)}\boldsymbol{\delta}_{\text{v}}$ by solving the evolution equation.

3. Alternative choices of variables

The perturbed conservation equations (64) at second order have been used to derive evolution equations that differ from those in Sec. V D 1, but have the unsatisfactory feature of not forming a closed system. First, Fitzpatrick *et al.* (2010) [21] have used the conservation equations in the Poisson gauge, first specialized to the case of pressure-free matter and then to the case of radiation [see Eqs. (26)–(29)]. The resulting equations are first order evolution equations for ${}^{(2)}\boldsymbol{\delta}_{\text{p}}$ and ${}^{(2)}V_{\text{p}}$, but are less simple than Eqs. (69) in that they are coupled to the metric perturbation ${}^{(2)}\psi_{\text{p}}$. As a second example Doran *et al.* (2003) [22] apply the perturbed conservation equations at linear order in a different way, using as variables the density perturbation in the uniform curvature gauge ${}^{(1)}\boldsymbol{\delta}_{\text{c}}$ and the velocity perturbation in the Poisson gauge [see Eqs. (A.29) and (A.30)]. Again these equations are coupled to the metric perturbation ${}^{(1)}\psi_{\text{p}}$.

VI. DISCUSSION

In this paper we have given five ready-to-use systems of governing equations for second order scalar perturbations, subject to the assumption that at first order the perturbations are purely scalar. Here we summarize their identifying features and give their active dynamical variables:

(i) Equations (48) using the Poisson gauge (variable ψ_{p}),
(ii) Equations (53) using the Poisson gauge (variables ψ_{p}, V_{p}),
(iii) Equations (55) using the uniform curvature gauge (variables ϕ_{c}, B_{c}),
(iv) Equations (63) using the total matter gauge (variables ψ_{v}, B_{v}),
(v) Equations (69) using the conservation equations (variables $\boldsymbol{\delta}_{\text{v}}$, $\mathbf{D}^2V_{\text{p}}$).

Other systems of equations that are more general than ours, as regards matter content and gauge choices, have been developed in the extensive series of papers by Hwang and Noh (see for example [3,19]) and Nakamura [4,23]. We regard our less general but more focused framework, which comprises the above five ready-to-use systems of equations, as complementing the more general systems in the above references. Each of our systems is minimal in the sense that there are no redundant equations or variables, and the matter content is restricted so that the systems are closed once the nonadiabatic pressure perturbation Γ is specified. Although we are primarily motivated by the needs of second order perturbation theory, we note that our framework can be specialized to linear perturbations by simply dropping the source terms. Because of this we hope that our framework will form a useful reference for both linear and second order perturbations.

We now make some remarks concerning the utility of the five systems of governing equations as regards applications. The unified nature of our formulation of these systems of equations enables one to easily compare their relative merits as regards a chosen application. We begin by noting that in cosmological perturbation theory the evolution of the perturbations is described in general by partial differential equations. Usually, in order to obtain explicit, approximate, or numerical solutions in a particular physical context, one applies the Fourier transform to the partial differential equations which converts them to ordinary differential equations for the Fourier coefficients of the perturbation variables, with the wave number k as a parameter, together with algebraic constraints relating the Fourier coefficients. For first order perturbations the spatial derivatives appear only via the spatial Laplacian $\mathbf{D}^2$, and one can implement the transition by simply making the replacement $\mathbf{D}^2 \to -k^2$. At second order, however, the process is more complicated since one has to use the convolution theorem to take the Fourier transform of products of the first order perturbations that appear in the source terms.[15]

There are, however, two important applications of cosmological perturbation theory, namely, adiabatic perturbations in the superhorizon regime (the long wavelength limit)

[15]See, for example, Tram *et al.* (2016) [2], Eqs. (1.1)–(1.3) and Vretblad (2005) [24] for details.

and perturbations of ΛCDM universes, in which it is not necessary to make the transition to Fourier space since the evolution equations automatically simplify to ordinary differential equations. We regard these applications as elementary but important benchmark problems in cosmological perturbation theory.

First we note that long wavelength adiabatic perturbations are defined by the requirement that terms of order 2 in the scaled dimensionless spatial differential operator $\mathcal{H}^{-1}\mathbf{D}^i$ can be neglected, and that the nonadiabatic pressure perturbation is negligible (${}^{(r)}\Gamma \approx 0$, $r = 1, 2$). However, the background matter scalars w and c_s^2 are unrestricted. Second, as shown in Appendix B, when the background model is the ΛCDM universe we have $w_m = 0$, and hence the background matter scalars are given by

$$c_s^2 = 0, \qquad 1 + w = \Omega_m, \tag{72}$$

which implies that the perturbations are adiabatic (${}^{(r)}\Gamma = 0$, $r = 1, 2$). In both cases the term $c_s^2\mathbf{D}^2$, which appears in the leading order terms in the evolution equations, is negligible, and it is this property that reduces the evolution equations to ordinary differential equations.

It turns out that in these two benchmark problems it is possible to explicitly solve the ordinary differential equations and obtain the general time dependence of the perturbations at first and second order, including both growing and decaying modes. The spatial dependence is described by arbitrary spatial functions that arise as constants of integration. In order to achieve this goal it is necessary to make an appropriate choice from among the five ready-to-use systems. Observe that in both systems (iii) (uniform curvature gauge) and (iv) (total matter gauge) the two evolution equations decouple, and can thus be solved successively for the two active variables, first at linear order and then, after using the linear solution to calculate the source terms, at second order. However, on evaluating the source terms one finds that *system (iv), using the total matter gauge, provides the simplest method of solution for the two benchmark problems.* Details concerning the derivation of the solution in the case of long wavelength perturbations are given in UW3 [6].

We conclude with some brief remarks on the relative merits of the five systems for problems other than the two benchmark problems. An immediate conclusion is that system (iv) is no longer the simplest system since the presence of a nonzero c_s^2 complicates the evolution equations considerably, since the term $c_s^2\boldsymbol{\delta}_{\rm v}$, which depends on $\psi_{\rm v} - \mathcal{H}B_{\rm v}$, appears on the right side of both evolution equations in (62) and (63). In addition the presence of this term makes the source terms more complicated. Instead it appears that *system (iii), based on the uniform curvature gauge, is the simplest system*, which makes it a natural choice for numerical experiments or qualitative analysis using dynamical systems methods. This system has not been given before.[16] Our analysis suggests that it is worthy of further study.

APPENDIX A: THE GENERAL SOURCE TERMS

The major technical problem in second order perturbation theory is managing the quadratic source terms. Our strategy is to use a consistent and easy to identify notation. We use the same letter for the kernel in the symbol for the source terms as we do for the leading order terms but with a different font: G and $\mathbb{G}$ for the leading order and source terms of the Einstein tensor and T and $\mathbb{T}$ for the stress-energy tensor with superscripts Γ, π, q, ρ indicating the components and subscripts p, c, v indicating the gauge, as in Sec. III D. For the conserved energy and momentum equations we use E and $\mathbb{E}$ and M and $\mathbb{M}$, respectively, with the usual subscripts indicating the gauge. We also introduce a notation for the source terms associated with the change of gauge formulas at second order, $\mathbb{S}(\cdots)$, where $(\cdots)$ identifies the formula [see Eqs. (A11) and (A12)]. Other source terms are defined as linear combinations of the above basic expressions (see Secs. V D 1 and V D 2).

In writing the source terms it is convenient to follow Appendix B in UW1 [5] and define the following spatial differential operators:

$$(\mathbf{D}C)^2 = \gamma^{ij}\mathbf{D}_iC\mathbf{D}_jC, \tag{A1a}$$

$$\mathbb{D}_0(C) = \mathcal{S}^{ij}\mathbf{D}_iC\mathbf{D}_jC, \tag{A1b}$$

$$\mathbb{D}_2(C) = \frac{1}{3}(\mathbf{D}^2\mathbb{D}_0(C) - (\mathbf{D}C)^2), \tag{A1c}$$

where the scalar mode extraction operators $\mathcal{S}^i$ and $\mathcal{S}^{ij}$ are given by (30).

By inspection of the expressions for the source terms one finds that $\mathcal{H}$ appears explicitly only in the variables $\mathcal{H}B$ and $\mathcal{H}V$ and as a coefficient of the spatial differential operator, in the form $\mathcal{H}^{-1}\mathbf{D}_i$. We can thus absorb all multiplicative factors of $\mathcal{H}$ by introducing the following overbar notation:

$$\bar{B} = \mathcal{H}B, \qquad \bar{V} = \mathcal{H}V, \qquad \bar{\mathbf{D}}_i = \mathcal{H}^{-1}\mathbf{D}_i, \tag{A2}$$

thereby making the expressions for the source terms simpler. The definition of $\bar{\mathbf{D}}_i$ leads to barred expressions for the associated spatial differential operators:

[16] We mention, however, that Malik and co-workers have used the uniform curvature gauge to study second order perturbations of inflationary universes with single and multiple scalar fields [see, for example, Malik (2007) [25], Huston and Malik (2009) [8], and Christopherson *et al.* (2015) [26]]. The structure of the governing equations in these references is specifically adapted to the scalar fields, and as a result they do not have much in common with our governing equations.

$$\bar{\mathbf{D}}^2 = \mathcal{H}^{-2}\mathbf{D}^2, \qquad \bar{\mathbf{D}}_{ij} = \mathcal{H}^{-2}\mathbf{D}_{ij}, \tag{A3a}$$

$$(\bar{\mathbf{D}}C)^2 = \mathcal{H}^{-2}(\mathbf{D}C)^2, \qquad \bar{\mathbb{D}}_2(C) = \mathcal{H}^{-2}\mathbb{D}_2(C), \tag{A3b}$$

$$\bar{\mathcal{S}}^i = \mathcal{H}\mathcal{S}^i, \qquad \bar{\mathcal{S}}^{ij} = \mathcal{H}^2\mathcal{S}^{ij}. \tag{A3c}$$

We could also use the barred dimensionless expressions to simplify the terms in the ready-to-use systems of the governing equation in Sec. V but have decided to give the more familiar forms in which $\mathcal{H}$ is visible.

1. The Einstein tensor source terms

The expressions for the Einstein source terms in (38) can be obtained by specializing Eqs. (75)–(78) in Uggla and Wainwright (2013) [15].[17] With the above notation the source terms can be written in the following form:

$$\mathbb{G}^\rho({}^{(1)}f) = 6[4\phi^2 - (\bar{\mathbf{D}}\,\bar{B})^2] + \mathbb{W} - 2\mathbb{X} + 2\bar{\mathbf{D}}^2\bar{\mathbb{D}}_2(\bar{B}), \tag{A4a}$$

$$\mathbb{G}^\Gamma({}^{(1)}f) = -2\mathcal{L}_1[4\phi^2 - (\bar{\mathbf{D}}\,\bar{B})^2] - \frac{1}{3}(2+3\mathcal{C}^2)\mathbb{W} + \frac{2}{3}(1+3\mathcal{C}^2)\mathbb{X} - \frac{2}{3}\mathbb{R} - \frac{2}{3}(1+3\mathcal{C}^2)\bar{\mathbf{D}}^2\bar{\mathbb{D}}_2(\bar{B}), \tag{A4b}$$

$$\mathbb{G}^q({}^{(1)}f) = 2[4\phi^2 - (\bar{\mathbf{D}}\,\bar{B})^2] + 8(\phi-\psi)\partial_N\psi + \bar{\mathcal{S}}^i\mathbb{R}_i, \tag{A4c}$$

$$\begin{aligned}\mathbb{G}^\pi({}^{(1)}f) = {}& 4\psi^2 + 2\mathbb{D}_0(\phi) - 6\mathbb{D}_0(\psi) + 2\bar{\mathcal{S}}^{ij}[2(\phi-\psi)\bar{\mathbf{D}}_{ij}\psi \\ & + 2(\bar{\mathbf{D}}_i\phi)\bar{\mathbf{D}}_j\bar{B} + (\partial_N(\phi+\psi))\bar{\mathbf{D}}_{ij}\bar{B}] \\ & + 4\bar{\mathcal{S}}^{ij}[(\psi-\phi)\bar{\mathbf{D}}_{ij}\mathsf{G}^\pi({}^{(1)}f) + (\bar{\mathbf{D}}_i\psi)\bar{\mathbf{D}}_j\mathsf{G}^\pi({}^{(1)}f)] \\ & + \bar{\mathbb{D}}_2(\bar{B}),\end{aligned} \tag{A4d}$$

where

$$\mathbb{W} = 24(\phi-\psi)\partial_N\psi + 8(\phi-\psi)\bar{\mathbf{D}}^2\bar{B} + 4(\bar{\mathbf{D}}_i\bar{B})\bar{\mathbf{D}}^i(\phi+\psi), \tag{A5a}$$

$$\mathbb{X} = -3(\partial_N\psi)^2 + 5(\bar{\mathbf{D}}\psi)^2 - 4\bar{\mathbf{D}}^2\psi^2 - 2\bar{\mathbf{D}}^i((\bar{\mathbf{D}}_i\bar{B})\partial_N\psi), \tag{A5b}$$

$$\begin{aligned}\mathbb{R} = {}& 12(\phi-\psi)(\partial_N^2 - q\partial_N)\psi + 6(\partial_N\psi)\partial_N(\phi-\psi) \\ & + 4(\phi-\psi)\bar{\mathbf{D}}^2\psi + 2(\bar{\mathbf{D}}\phi)^2 + 2(\bar{\mathbf{D}}\psi)^2 \\ & + 4(\bar{\mathbf{D}}_i\bar{B})\bar{\mathbf{D}}^i\partial_N\psi + 2(\partial_N\phi)\mathbf{D}^2\bar{B} \\ & - 2[2(\phi-\psi)\bar{\mathbf{D}}^2 + (\bar{\mathbf{D}}_i\psi)\bar{\mathbf{D}}^i][2\bar{B} + \mathsf{G}^\pi(f)],\end{aligned} \tag{A5c}$$

$$\begin{aligned}\mathbb{R}_i = {}& -4(\partial_N\psi)\bar{\mathbf{D}}_i\phi + 2(\bar{\mathbf{D}}_j\bar{B})\left(\bar{\mathbf{D}}^j{}_i + \frac{4}{3}\delta^j{}_i\bar{\mathbf{D}}^2\right)\psi \\ & - 2(\bar{\mathbf{D}}_j\phi)\left(\bar{\mathbf{D}}^j{}_i - \frac{2}{3}\delta^j{}_i\bar{\mathbf{D}}^2\right)\bar{B}.\end{aligned} \tag{A5d}$$

Here and elsewhere in this appendix, in order to simplify the notation we have omitted the superscript ${}^{(1)}$ on the linear perturbations in the source terms. We have simplified the term (A4d) in an important way, as follows. This term initially contains the expression $\mathbb{D}_2^*(B)$, where

$$\mathbb{D}_2^*(B) \coloneqq 2\mathcal{S}^{ij}\left(\frac{1}{3}(\mathbf{D}^2B)\mathbf{D}_{ij}B - (\mathbf{D}_{k\langle i}B)\mathbf{D}^k{}_{j\rangle}B\right), \tag{A6}$$

but in the flat case it can be shown using the commutation identities for $\mathbf{D}_i$ that

$$\mathbb{D}_2^*(B) = \mathbb{D}_2(B). \tag{A7}$$

2. The stress-energy tensor source terms

The components of the source term, identified by a kernel $\mathbb{T}$, are given by

$$\mathbb{T}^\rho = \gamma^{ij}(\mathbb{V}_2)_{ij}, \tag{A8a}$$

$$\mathbb{T}^\Gamma = \frac{1}{3}(1-3c_s^2)\gamma^{ij}(\mathbb{V}_2)_{ij} - \frac{1}{3}(\partial_N c_s^2)\boldsymbol{\delta}^2 - \frac{2}{3}\boldsymbol{\delta}(\partial_N - 3(1+c_s^2))\Gamma, \tag{A8b}$$

$$\mathbb{T}^q = \bar{\mathcal{S}}^i[2((1+c_s^2)\boldsymbol{\delta} + \Gamma - \phi)\bar{\mathbf{D}}_i\bar{V}], \tag{A8c}$$

$$\mathbb{T}^\pi = \bar{\mathcal{S}}^{ij}(\mathbb{V}_2)_{ij}, \tag{A8d}$$

where

$$(\mathbb{V}_2)_{ij} \coloneqq 2(\mathbf{D}_iV)\mathbf{D}_j(V-B) = 2(\bar{\mathbf{D}}_i\bar{V})\bar{\mathbf{D}}_j(\bar{V}-\bar{B}), \tag{A8e}$$

where $(\mathbb{V}_2)_{ij}$ has weight 2 in $\mathbf{D}_i$, motivating the subscript ${}_2$.

3. The source terms for the conservation equations

The quadratic source terms are given by

$$\begin{aligned}\mathbb{E}({}^{(1)}F) = {}& -\partial_N[6\psi^2 - (\bar{\mathbf{D}}\,\bar{V})^2 + (1+c_s^2)\boldsymbol{\delta}^2 + 2\boldsymbol{\delta}\Gamma] - 6\Gamma^2 \\ & + 2\bar{\mathbf{D}}^k[\phi\bar{\mathbf{D}}_k\bar{V} + 2\psi\bar{\mathbf{D}}_k(\bar{V}-\bar{B})] \\ & + 2[\bar{\mathbf{D}}^k(\boldsymbol{\delta} - 3\psi)]\bar{\mathbf{D}}_k(\bar{V}-\bar{B}),\end{aligned} \tag{A9a}$$

$$\begin{aligned}\mathbb{M}({}^{(1)}F) = {}& -2\phi^2 + [\bar{\mathbf{D}}(\bar{V}-\bar{B})]^2 - \left[c_s^2(1+c_s^2) + \frac{1}{3}(\partial_N c_s^2)\right]\boldsymbol{\delta}^2 \\ & -\Gamma^2 - \frac{2}{3}\boldsymbol{\delta}\partial_N\Gamma - 2\bar{\mathcal{S}}^i\{[\phi(\partial_N + 1 - 3c_s^2) \\ & - \partial_N(c_s^2\boldsymbol{\delta} + \Gamma) + 3\Gamma]\bar{\mathbf{D}}_i\bar{V} - \Gamma\bar{\mathbf{D}}_i\boldsymbol{\delta}\},\end{aligned} \tag{A9b}$$

[17] See footnote 10 for the relation between the notation in [15] and in the present paper.

where we have used the first order equation $\mathsf{E}(^{(1)}F) = 0$ in deriving the first equation.

The source term $\mathbb{S}_{\text{Poisson}}$ in the GR version of Poisson's equation at second order (67), that is used in the source term (70b), has the following form:

$$\mathbb{S}_{\text{Poisson}} = \bar{\mathbf{D}}^2(4\psi_{\text{p}}^2 - (1+q)(\bar{V}_{\text{p}})^2) - 5(\bar{\mathbf{D}}\psi_{\text{p}})^2 - 6\bar{\mathcal{S}}^i[\bar{V}_{\text{p}}\bar{\mathbf{D}}_i(\bar{\mathbf{D}}^2\psi_{\text{p}})]. \tag{A10}$$

4. The source terms for the change of gauge formulas

We have recently given in UW1 [5] a general formalism for relating gauge invariants associated with different gauges at second order. In this paper we need the gauge change formula that relates $\bar{B}_{\text{v}}$ to $\bar{V}_{\text{p}}$ which we write in the following form using the bar notation:

$$^{(2)}\bar{B}_{\text{v}} + {}^{(2)}\bar{V}_{\text{p}} = \mathbb{S}(\bar{B}_{\text{v}} + \bar{V}_{\text{p}}), \tag{A11}$$

and also the one that relates $^{(2)}\boldsymbol{\delta}_{\text{v}}$ to $^{(2)}\boldsymbol{\delta}_{\text{p}}$:

$$^{(2)}\boldsymbol{\delta}_{\text{p}} - 3{}^{(2)}\bar{V}_{\text{p}} - {}^{(2)}\boldsymbol{\delta}_{\text{v}} = \mathbb{S}(\boldsymbol{\delta}_{\text{p}} - 3\bar{V}_{\text{p}} - \boldsymbol{\delta}_{\text{v}}). \tag{A12}$$

The source terms are given by

$$\mathbb{S}(\bar{B}_{\text{v}} + \bar{V}_{\text{p}}) = (\partial_N + 2q)[\mathbb{D}_0(\bar{V}_{\text{p}}) - \bar{V}_{\text{p}}^2] - 2\bar{V}_{\text{p}}^2 - 2\bar{\mathcal{S}}^i[\phi_{\text{p}}\bar{\mathbf{D}}_i\bar{V}_{\text{p}}], \tag{A13a}$$

$$\mathbb{S}(\boldsymbol{\delta}_{\text{p}} - 3\bar{V}_{\text{p}} - \boldsymbol{\delta}_{\text{v}}) = 3(1+q+3(1+c_s^2))\bar{V}_{\text{p}}^2 - 6\bar{\mathcal{S}}^i[\phi_{\text{v}}\bar{\mathbf{D}}_i\bar{V}_{\text{p}}] + 2(3\boldsymbol{\delta}_{\text{v}} + \bar{\mathbf{D}}^2\bar{V}_{\text{p}})\bar{V}_{\text{p}}. \tag{A13b}$$

For the first equation we used Eq. (42a) in [5] with $\square = \bar{V} = \mathcal{H}V$ and the total matter gauge on the right side, and for the second equation we used Eq. (42b) with $\square = \frac{1}{3}\boldsymbol{\delta}$ and the Poisson gauge on the right side. It is then necessary to use the definitions (36) of the hatted variables.

APPENDIX B: THE FRACTIONAL DENSITY PERTURBATION

We emphasize that $\boldsymbol{\delta}$ is defined by normalizing the density perturbation with $\rho_0 + p_0$ as in Eq. (20), while the commonly used fractional density perturbation δ is defined by normalizing the density perturbation with the background matter density, which we denote by $^{(0)}\rho_m$, while ρ_0 denotes the total matter/energy density. If there is a cosmological constant, then

$$^{(0)}\rho = \rho_0 = {}^{(0)}\rho_m + \Lambda, \qquad {}^{(0)}p = p_0 = {}^{(0)}p_m - \Lambda, \tag{B1}$$

and we also introduce

$$w_m = \frac{^{(0)}p_m}{^{(0)}\rho_m}, \qquad \Omega_m = \frac{^{(0)}\rho_m}{3H^2}, \tag{B2}$$

while c_s^2 is unaffected. It follows that

$$1 + w = \Omega_m(1 + w_m). \tag{B3}$$

The fractional density perturbation is defined by

$$^{(r)}\delta = \frac{^{(r)}\rho}{^{(0)}\rho_m}, \quad r = 1, 2. \tag{B4}$$

It follows that $\delta = (1 + w_m)\boldsymbol{\delta}$, which simplifies to $\delta = (1 + w)\boldsymbol{\delta}$ if $\Lambda = 0$. In particular for a ΛCDM universe we have $w_m = 0$, $c_s^2 = 0$ and (B3) reduces to

$$1 + w = \Omega_m. \tag{B5}$$

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