Detecting stellar lensing of gravitational waves with ground-based observatories

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We investigate the ability of ground-based gravitational-wave observatories to detect gravitational-wave lensing events caused by stellar mass lenses. We show that LIGO and Virgo possess the sensitivities required to detect lenses with masses as small as $\sim 30 \, M_\odot$ provided that the gravitational wave is observed with a signal-to-noise ratio of ~ 30. Third-generation observatories will allow detection of gravitational-wave lenses with masses of $\sim 1 \, M_\odot$. Finally, we discuss the possibility of lensing by multiple stars, as is the case if the gravitational radiation is passing through a galactic nucleus or a dense star cluster.

I. INTRODUCTION

The recent Laser Interferometer Gravitation-Wave Observatory (LIGO) discoveries of gravitational waves (GWs) from black hole binaries [1–5] opened a new frontier for the study of astrophysical objects using gravitational radiation. Much like electromagnetic (EM) radiation in classical astrophysics, gravitational radiation can be lensed by massive objects. Lensing of gravitational radiation in linearized general relativity can be computed with the same techniques as those employed in the familiar lensing of EM waves.

Much of the previous literature on gravitational lensing of GWs focused on lensing in the geometric optics limit [6,7], where the wavelength of gravitational waves is small compared to the spatial scale of the lenses, and ray optics is sufficient. The lenses responsible for much of the optical depth in this regime are galaxies, which split the gravitational-wave signal into copies separated by a time delay of order a few months.

The stellar lensing events that are considered in this work preempt these strong gravitational lensing by the lens host galaxy. As a beam of gravitational radiation passes through the galaxy, it will first be split into two beams due to it being strongly lensed by the galactic potential. Each of these beams can then be lensed by stars in the galaxy. Because the time delay between the arrival of the two beams is 10–100 days for third-generation (3G) gravitational-wave observatories and $\lesssim 1$ day for LIGO [8], at first only one beam will be detected on Earth. If the first of these beams is found to also be lensed by stars, we can be confident that the beam passes through the core of a galaxy. If this is the case, then it will also be strongly lensed by the galaxy, and thus we can expect that in the close future, the strongly lensed copy of the original beam will arrive on Earth.

In the wave-optics regime, the Laser Interferometer Space Antenna (LISA) [9] possesses the necessary sensitivities to extract the lens' mass from a lensed signal [10]. The goal of our study is to show that such observations can also be achieved by ground-based observatories.

The sensitivities of advanced LIGO [11] and advanced Virgo [12] to lensing by intermediate mass black holes (IMBH) are explored in Ref. [13], which found that current-generation observatories are capable of detecting the lensing IMBH with 98% confidence. Further, Ref. [13] found that LIGO and Virgo can distinguish between a point mass lens and a singular isothermal lens provided that the redshifted lens mass is $200 \, M_\odot$.

In this work, we focus on the capabilities of ground-based GW detectors to detect stellar mass lenses. While stellar mass lenses are more numerous than IMBHs, the amplitude of their lensing signal is much smaller. This requires us to extend our study to include upcoming third-generation gravitational-wave detectors.

II. BACKGROUND AND NOTATIONS

We consider GWs in the perturbed Friedmann-Lemaître-Robertson-Walker (FLRW) metric, written in terms of the conformal time η,

$$ds^2 = a^2[-(1 + 2U)d\eta^2 + (1 - 2U)d\mathbf{r}^2], \quad (1)$$

where $\mathbf{r}$ is the spatial coordinate, U is the gravitational potential of the lenses, and a encodes the universal scale factor. Considering linear perturbation on this metric,

$$g_{\alpha\beta} = g_{\alpha\beta}^B + h_{\mu\nu}, \quad (2)$$

where $h_{\mu\nu}$ is separated into ϕ (its amplitude) and $e_{\mu\nu}$ (its polarization), $h_{\mu\nu} = \phi e_{\mu\nu}$, one can see that the equation of motion for ϕ is simply given by the wave equation,

$$\partial_\mu \left(\sqrt{-g^B} g_B^{\mu\nu} \partial_\nu \phi \right) = 0, \tag{3}$$

where g^B is the determinant of the metric. In Fourier space, $\tilde{\phi}(f, \mathbf{r})$, the equation reads

$$(\nabla^2 + \tilde{\omega}^2)\tilde{\phi} = 4\tilde{\omega}^2 U \tilde{\phi}, \tag{4}$$

where $\tilde{\omega} = 2\pi f$ is the GW frequency. Following [10], we define the amplification factor $F(\omega, \mathbf{r})$ as the ratio between the lensed and unlensed $\tilde{\phi}$.

The setup of our problem consists of three parallel planes, called the source, lens, and observer planes. The angular diameter distances along the normal from the observer plane to the source and lens planes are labeled D_S and D_L, respectively, while the distance between the source and lens planes is labeled as D_{LS}. GWs are emitted by a point in the source plane, and they travel freely to the lens plane, where they are lensed by a gravitational potential U that is assumed to be localized in the thin (width $\ll c/f$) lens plane, before reaching the telescope at the observer plane.

Coordinates can be set up on the three planes. We use the notation of Refs. [10,14], where ξ is the coordinate at the source plane, η is the coordinate at the lens plane, and δ is the coordinate in the observer plane. We also employ the following dimensionless coordinates,

$$\mathbf{x} = \frac{\xi}{\xi_0},$$

$$\mathbf{y} = \frac{D_L}{D_S} \frac{\eta}{\xi_0},$$

$$\mathbf{d} = \left(1 - \frac{D_L}{D_S}\right) \frac{\Delta}{\xi_0},$$

where ξ_0 is some characteristic length scale defined by $\xi_0 = D_L \theta_E$, where θ_E is the Einstein angle for a point mass lens,

$$\theta_E^2 = \frac{4GM_L}{c^2} \frac{D_{LS}}{D_L D_S}, \tag{5}$$

where M_L is the lens mass. Furthermore, we will work with the dimensionless frequency, which for the point mass lens is

$$\omega = \frac{4GM_L(1+z)}{c^3} \tilde{\omega}, \tag{6}$$

where z is the lens redshift. From this point on we will adopt units where $G = c = 1$.

Using this setup, the solution of Eq. (4) is given by the Fresnel-Kirchhoff integral,

$$F(\omega, \mathbf{y}) = \frac{\omega}{2\pi i} \int d^2 x \exp\left[i\omega T(\mathbf{x}, \mathbf{y})\right], \tag{7}$$

where the time delay function $T(\mathbf{x}, \mathbf{y})$ is given by

$$T(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\mathbf{x} - \mathbf{y} - \mathbf{d})^2 - \Psi(\mathbf{x}), \tag{8}$$

with Ψ being the lensing potential.

A. Wave optics lensing

Integrating Eq. (7) with the stationary phase method is valid when the wavelength is much smaller compared to the characteristic scale of the lens. This condition requires

$$\omega \gg 1. \tag{9}$$

A detector operating in the frequency band of LIGO, with a characteristic frequency of $f \sim 100$ Hz, is capable of detecting lenses where $\omega \ll 1$. In this regime, the geometric optics approximation breaks down, and one has to integrate Eq. (7) in full.

For a point mass lens located at $\eta = 0$, $\psi(\mathbf{x}) = \log|\mathbf{x}|$, and Eq. (7) integrates to [10,15]

$$F(\omega) = \exp\left\{\frac{\pi\omega}{4} + i\frac{\omega}{2}\left[\log\left(\frac{\omega}{2}\right) - 2\phi_m(y)\right]\right\} \Gamma\left(1 - \frac{i}{2}\omega\right)$$
$$\times {}_1F_1\left(\frac{i}{2}\omega, 1; \frac{i}{2}\omega y^2\right), \tag{10}$$
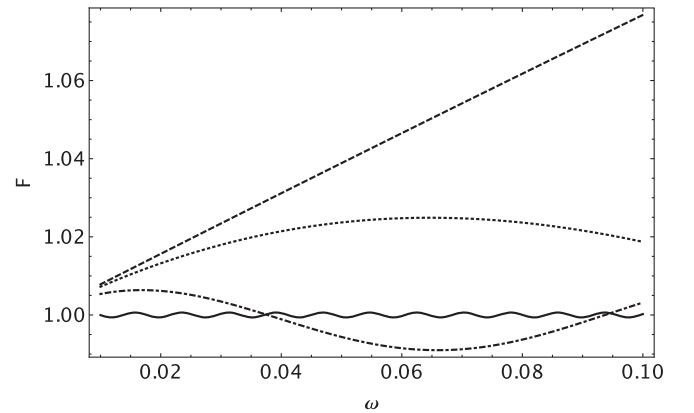

FIG. 1.   The amplification as a function of ω for a point mass lens where $y = 1, 5, 10, 40$ (dashed, dotted, dot-dashed, and solid lines). When the position of the source projected to the lens plane is small ($y \sim 1$), one can obtain amplification that is $\sim$linear in ω. In the LIGO band ($\omega \sim 0.01$–0.1 for a solar mass lens) this results in a deviation from the unlensed signal of a few percent. As the distance increases, the amplitude of the deviation becomes smaller.
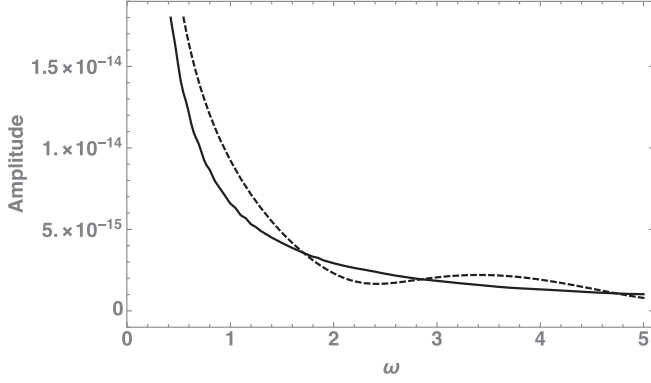
FIG. 2. The amplitude of waveforms from a $30 + 30 \, M_\odot$ binary at 1 parsec lensed by a $30 \, M_\odot$ lens. We exemplify a strongly lensed event with a lens at $y = 1$ (dashed) and a weakly lensed event with a lens at $y = 10$ (solid). An unlensed signal would sit on top of the $y = 10$ line. Note that in addition to the oscillating features at high frequency that are characteristic of wave interference, there is an overall amplification of the signal.

where $y \equiv |\mathbf{y}|$ is the dimensionless impact parameter,

$$\phi_m(y) = (x_m - y)^2/2 - \log x_m, \qquad (11)$$

and

$$x_m = \frac{y + \sqrt{y^2 + 4}}{2}. \qquad (12)$$

The amplification for $\omega \sim 0.01$–0.1 and a variety of y values is plotted in Fig. 1. For a frequency of 100 Hz, this corresponds to lenses of mass 1–10 $M_\odot$.

Two examples of lensed waveforms are shown in Fig. 2. The unlensed waveforms are PhenomA phenomenological models during the inspiral [16,17]. While more complex phenomenological models exist, the PhenomA model suffices for this illustrative purpose.

III. METHOD

We have modified the parameter estimation algorithm currently used by the LIGO and Virgo collaborations [18,19] to allow for the presence of a lens along the line of sight to the source. This is a stochastic sampler that explores the parameter space and produces posterior distributions for the unknown parameters on which the gravitational-wave signal depends. In the absence of a lens, these include masses and spins of the two compact objects, the sky position, distance, orientation, and polarization of the source, as well as the time and phase at coalescence [19]. Throughout this work, we use the effective-precession waveform IMRPhenomPv2 [20].

Our lens model allows for two extra parameters: the mass of the lens, M_L, and the parameter $y' \equiv yM_L$, which are sampled together along the other. Once a waveform corresponding to the unlensed signal is generated, $h_{\text{Unlensed}}(\vec{\theta})$, the

two lens parameters are used to calculate Eq. (10), which yields the lensed signal $h_{\text{Lensed}}(\vec{\theta}, M_{\text{lens}}, y') = F(M_{\text{lens}}, y')h_{\text{Unlensed}}(\vec{\theta})$, where $\vec{\theta}$ are the compact binary coalescence (CBC) parameters in the absence of the lens.

While the physical parameters of the lensing model are M_L and y, we used the parameter y' to smoothly handle the no-lens case without discontinuities or ill-defined parameters. The parameter $y' = yM_L$ is trivially correlated to the lens mass, M_L. However, as they enter different parts of the waveform models and not always in the same combination, they are not degenerate.

Given a GW signal (real or simulated), the algorithm can be run with the lens parameters ("Lens" model) to measure or put an upper bound on the lens mass. After the evidence [21] for both the "Lens" and "No lens" model is calculated, one can compute the odds ratio defined as

$$\text{Odds} = \frac{P(\text{lens}|\text{data})}{P(\text{no lens}|\text{data})}. \qquad (13)$$

IV. RESULTS

A. Current-generation observatories

We ran our code on simulated signals observed by LIGO and Virgo with signal-to-noise (SNR) values of 15, 30, and 60, where we injected lenses of 0, 1, 10, 20, 30, 60, and 100 $M_\odot$ with an impact parameter of an Einstein angle, θ_E. The masses of the simulated CBC signal are compatible with heavy binary sources similar to GW150914.

The results are plotted in Fig. 3. At a SNR ratio of 30, which is moderately high for current-generation observatories, lenses can be detected at $> 3\sigma$ when they possess masses larger than $\sim 30 \, M_\odot$. Higher SNR events allow
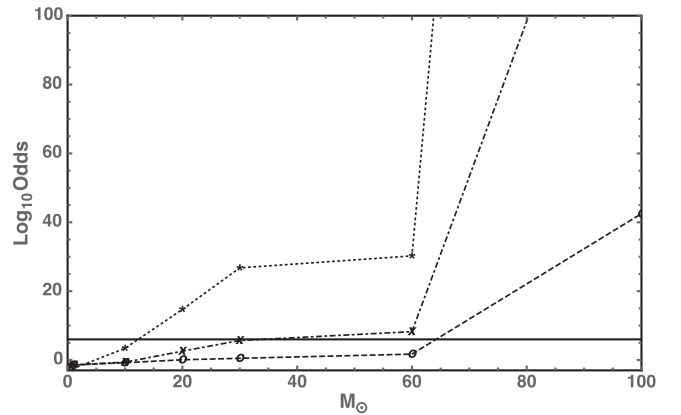

FIG. 3. The odds ratio computed as defined by Eq. (13). The circles, crosses, and stars denote SNR values of 10, 30, and 60, respectively. Dashed, dot-dashed, and dotted lines connecting the points are drawn to guide the eye. Odds values > 1 indicate that the lensed model is preferred over the unlensed model. The solid horizontal line indicates a 3σ detection.

smaller lenses to be detected. At SNR = 60, lenses as small as ~10 $M_\odot$ can be detected. LIGO and Virgo can potentially detect smaller lenses if the impact parameter is significantly smaller than an Einstein radius, but such cases are expected to be rare.

If the mass function of black holes follows the mass function of massive stars, it is reasonable to expect the mass function of binary black holes to be bottom heavy. Assuming that binary black holes are uniformly distributed in space, this bottom-heavy mass function means that SNR > 60 events would be rare. However, there is the possibility that most LIGO black holes are macrolensed by intervening galaxies [22]. Such a macrolensing event would lower the mass requirement for a binary to be the gravitational-wave source of an SNR = 60 event, while allowing said binary to be located at a larger distance. This would enhance the rate of SNR > 60 events and thus open the possibility of LIGO detecting stellar-mass lensing events.

As a straightforward application, we run our algorithm on the stretch of public LIGO data containing the gravitational-wave event GW150914 [1,23]. We find that the waveform detected for GW150914 is consistent with a lens mass of $M = 0$; i.e., GW150914 is most probably not a microlensed event, with an upper bound 90% confidence interval for the lens mass of 50 $M_\odot$.

B. Third-generation observatories

Proposed ground-based 3G GW observatories, such as the Einstein Telescope [24] and the Cosmic Explorer [25], allow detections of BBH events from high redshift [25–27], and will detect nearby events with SNR of hundreds or thousands [28]. Such high SNR events can potentially allow much smaller lenses to be detected. To show this, we ran our algorithm on a simulated GW150914-like source as observed by a third-generation observatory, with an injected lens of 1 $M_\odot$. Figure 4 shows the resulting posterior distribution for the lens mass. As seen in Fig. 4, 3G observatories can detect lenses as small as 1 $M_\odot$. As there are many more lenses with such masses than those with masses of ~30 $M_\odot$, we expect that detection of lensing events by stellar mass lenses will be mostly confined to 3G detectors. If the lenses obey the mass function for stars, the number of lensed events that we can expect from a 3G detector is greater than that of current-generation detectors by a factor q, where

$$q = \frac{R_{3G}}{R_{CG}} \frac{\int_{1\,M_\odot}^\infty m^{-2.3} dm}{\int_{30\,M_\odot}^\infty m^{-2.3} dm} \approx \frac{R_{3G}}{R_{CG}} 100, \qquad (14)$$

where R_{3G} is the overall rate of gravitational-wave detection by 3G observatories, R_{CG} is the overall rate of gravitational-wave detection by current-generation observatories, and we have taken the stellar mass function
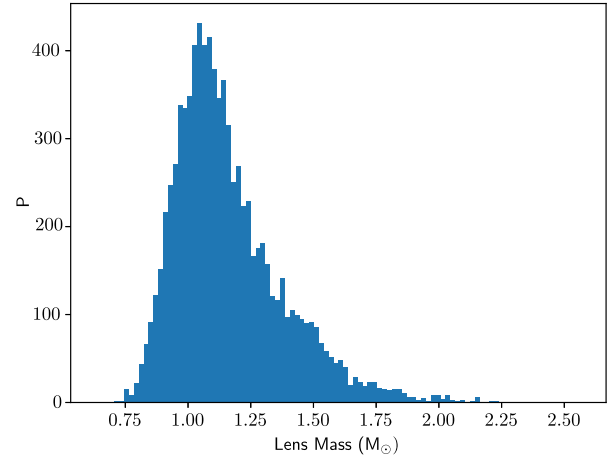

FIG. 4. The resulting posterior distribution of lens mass for an event with an $M = 1\,M_\odot$ lens observed by a third-generation observatory. SNR is set to 3000, and the impact parameter is an Einstein radius. Note that a vanishing lens mass is clearly excluded.

to be of the Salpeter form [29]. Note that we have not fully explored the lower limit of the masses of the lenses that will be detectable by third-generation observatories. It is likely that these observatories will detect lenses with masses even smaller than 1 $M_\odot$.

V. RATE ESTIMATE

The rate of stellar lensing is intrinsically tied to the rate of strong lensing, as the surface number density of stars in the core of galaxies is large enough that if a beam passes within this core, there is an order unity chance that it will pass within the Einstein radius of a star. The number of stellar lensing events per year, N_*, is therefore

$$N_* \sim N_g * \left(\frac{h_c}{h_E}\right)^2, \qquad (15)$$

where h_c is the angular size of the galaxy core, h_E the Einstein angle associated with the galaxy potential, and N_g the number of galaxy (strong) lensing events per year.

The number of galaxy lensing events per year for a 3G telescope has been calculated to be over 100 events per year, where for most of these events the source is a binary black hole [7,30,31]. Plugging in the numbers for a Milky Way equivalent galaxy, as well as putting the source at $z \sim 2$ and the lens halfway to the source, we obtain that $(h_c/h_E)^2 \sim 0.01$. This gives the number of stellar lensing events per year as $N_* \sim$ few. However, this number is extremely conservative. This is because most of the lensing optical depth is provided by galaxies that are much more massive than the Milky Way.

We can scale the factor $(h_c/h_E)^2$ as follows,

$$\left(\frac{h_c}{h_E}\right)^2 \sim \frac{R_c^2}{M_g}, \qquad (16)$$

where R_c is the radius of the galaxy core and M_g the mass of the galaxy. The relation between galactic luminosity and core radius has been found to be $L \sim R_c^{0.84}$ [32]. Therefore,

$$R_c \sim L^{1.19} \sim M_g^{2.38}, \qquad (17)$$

where in the last relation we have used the Faber-Jackson relation. Therefore,

$$\left(\frac{h_c}{h_E}\right)^2 \sim M_g^{3.76}. \qquad (18)$$

Thus, the larger the lensing galaxy, the more probable it is for the gravitational-wave beam to also be stellar lensed. This is pertinent especially because, as mentioned before, most lenses are massive galaxies.

Because most sources are expected to be at $z \sim 2$, the typical detection will not have a SNR of thousands. However, because the gravitational-wave amplitude only goes down as the distance, the difference in SNR between a source at $z \sim 0.1$ and $z \sim 2$ is only ~ 20, leaving a SNR in the hundreds. This is plenty of SNR to detect small lenses with masses of a few solar masses, because as seen in Fig. 3, SNR ~ 60 is enough to provide a $\sim 3\sigma$ detection of ~ 10 $M_\odot$ lenses.

VI. WAVE OPTICS LENSING BY MULTIPLE MASSES

The centers of galaxies are dense enough that the Einstein rings of stars with angular scales of ~ 1 micro-arcseconds (μas) can overlap. Indeed, the probability for there to be another star an Einstein radius, ξ_0, away from a particular star is [33,34]

$$P \approx 1 - \exp\left[-\sigma \pi \xi_0^2\right], \qquad (19)$$

which approaches unity for a stellar mass density, σ, corresponding to ~ 1 g cm^{-2} $\sim 4.8 \times 10^9$ $M_\odot$/kpc^2.

In this regime, it is important to understand the effects of lensing by multiple masses. Assuming that the lensing happened in a thin plane, the lensing potential by N point masses is given by

$$\psi(\mathbf{x}) = \sum_i^N \log |\mathbf{x} - \mathbf{x_i}|. \qquad (20)$$

To simplify our calculation, we employ the fact that distant lenses do not affect the signal by imposing a cutoff on $\psi(\mathbf{x})$. In particular, we will ignore any lenses that are more than an Einstein radius away from the source,

$$\psi(\mathbf{x}) \approx \sum_i^N H(|\mathbf{x} - \mathbf{x_i}|) \log |\mathbf{x} - \mathbf{x_i}|, \qquad (21)$$

where $H(|\mathbf{x} - \mathbf{x_i}|)$ is a top-hat kernel that is unity when $|\mathbf{x} - \mathbf{x_i}|$ is less than an Einstein radius, and zero otherwise. In doing so, we do not need to include all the point masses in the lensing galaxy in $\psi(\mathbf{r})$, but only the lenses whose Einstein rings intersect. Obviously, this number depends on the surface density number of stars in the lensing galaxy.

The upper limit on the stellar surface mass density in a dense system is $\Sigma_{max} \sim 10^{11}$ $M_\odot$/kpc^2 [35]. Assuming that most of the stellar mass is in stars of mass ~ 1 $M_\odot$, this gives a surface number density of $\sim 10^5$ pc^{-2}. Using the fact that at cosmological distances the Einstein angle of such stars are ~ 1 μas, and that stars are randomly distributed in the lensing plane, we create realizations of star fields in the lensing plane. Even for such a dense system, the number of Einstein ring intersections is of order a few. The expected number of overlapping Einstein rings, N_O, can be estimated as follows. If $A_1 = 1/\sigma$ is the area where only one star is expected, then

$$N_O = \frac{\pi(2\xi_0)^2}{A_1} = \pi(2\xi_0)^2 \sigma \qquad (22)$$

$$= 4.3 \times \left(\frac{\sigma}{10^5/\text{pc}^2}\right). \qquad (23)$$

We therefore expect GWs to be significantly lensed by only a few lenses. However, as a Poisson process, N_O is Poisson
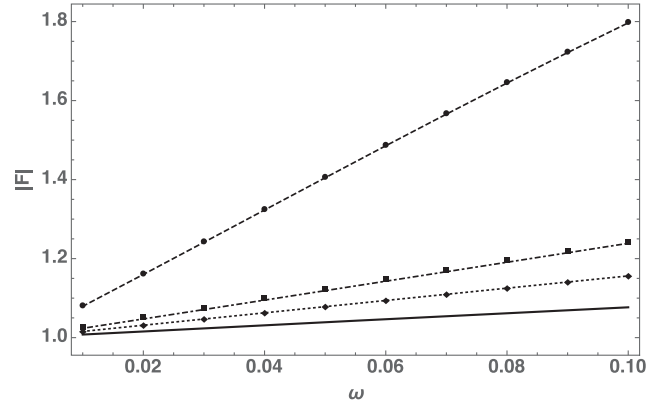

FIG. 5. The amplification factor due to lensing as a function of ω for 1 (analytical solid line), 2, 3, and 10 (dotted, dot-dashed, and dashed, respectively) point mass lenses. For 1 $M_\odot$ lenses, the ω range corresponds to the LIGO frequency range. The lenses are distributed randomly, but consistently in the lens plane, so that the two lens case corresponds to the single lens case plus a randomly distributed second lens, and similarly for the three and ten lens cases. The position of the source in the source plane is (0, 1), and the position of the observer in the observer plane is (0, 0) in Einstein angle units. In this regime where $F(\omega) \propto \omega$, more lenses generally generate a larger lensing effect.

distributed. Therefore, for a $\sigma = 10^5$ pc^{-2} system, there is a $\sim 1\%$ chance for the beam to interact with ~ 10 lenses.

To this end, we calculate the lensing amplitude $F(\omega)$ by numerically integrating Eq. (7) using a Levin method integrator [36]. The resulting magnification amplitudes for 2, 3, and 10 lenses are plotted in Fig. 5. For 1 $M_\odot$ lenses in the LIGO band, $F(\omega)$ scales linearly with ω. The general trend is that more lenses yield a larger deviation in amplitude. This means that if a lensing event is detected at low ω, it might be difficult to distinguish between lensing by a single-point mass or lensing by multiple-point masses. However, if a larger range of ω is observed, these two models will differ significantly.

The computation in this section assumes that the lensing potentials of multiple stars can be superposed in the manner of Eq. (21). In reality, nonlinear effects can produce caustics in the source plane. Further, these caustics can overlap and create large magnifications on scales larger than the Einstein radius. This effect has recently been observed for a star lensed by a galaxy cluster [37], which allowed constraints on compact dark matter to be placed [38]. We leave the computation of caustic curves in the wave optics regime to a future work.

VII. CONCLUSIONS

We have shown that in order for current-generation GW observatories to detect gravitational-wave lensing events, a lens mass of at least ~ 30 $M_\odot$ is required, provided that the gravitational waveform is detected at a signal to noise of ~ 30. If the gravitational-wave source is weaker or is located further away, this number will increase correspondingly. We also note the possibility that the gravitational wave is macrolensed by an intervening galaxy in addition to the stellar lensing event. This could potentially produce an event with large enough SNR for LIGO to detect stellar lenses.

Furthermore, we have shown that 3G detectors can detect lenses of masses as small as 1 $M_\odot$. Since 1 $M_\odot$ lenses are much more numerous than ~ 30 $M_\odot$ lenses, many more lensing events will be detected by third-generation detectors than current generation detectors.

ACKNOWLEDGMENTS

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