Test of special relativity using comparisons of clock frequencies

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We use the frequency comparisons of two kinds of clocks to reanalyze the test of special relativity in the Robertson-Mansouri-Sexl kinematical framework, which involves the Michelson-Morley experiment and the optical atomic clock-comparison experiment. The light clock involved in the Michelson-Morley experiment is composed of two mirrors and a photon propagating back and forth between them, which can be regarded as a kind of structure different from the point-particle clock. Similarly, the inner structure of the atom in optical atomic clock-frequency comparisons should also be considered, which has not been mentioned before. Because of the structure effect, our result shows that the measurable parameter in the optical atomic clock-comparison experiment should be $\alpha - (\beta + 2\delta)/3$ instead of the widely recognized α, where β and δ are defined in the text.

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I. INTRODUCTION

Special relativity (SR) was proposed more than a hundred years ago and has been one of the cornerstones of modern physics. It is derived from two fundamental postulates: the principle of relativity and the constancy of the speed of light [1]. As the principle of relativity means the laws of physics are invariant under the Lorentz transformation, it is usually termed as Lorentz invariance (LI). Since gravity still has not been merged with quantum mechanics, a more fundamental theory is needed to provide a unified description of all interactions, and most of the unified theories [2–5] imply tiny violations of SR. Searching for the LI violation is motivated by uncovering a possible violation of SR and exploring the unification theories, which can help us to understand the Universe surrounding us and find new physics.

There are various theoretical frameworks to study the possibility of LI violation [2,6–12]; here we only discuss two of them: the kinematical framework developed by Robertson, Mansouri, and Sexl (RMS framework), and the Lorentz violating extension of the standard model developed by Kostelecky and coworkers over the last two decades (SME framework). In the former frameworks, three parameters are related to describe the deviation from SR, and the corresponding experimental tests of LI are mainly Michelson-Morley (MM) [13–15], Kennedy-Thorndike (KT) [14,16,17], Ives-Stilwell (IS) [18–20] experiments and the comparisons of two same optical clocks in different locations [21]. These experiments restrict the three parameters by probing the experimental dependence on orientation of the speed of light, or getting the time dilation via the precise measurement of the Doppler effect. In the SME framework, the parameter space describing LI violation is vast, and the associated experimental tests are various, such as the gravitational experiments in a short range [22–24], the clock-comparison experiments [25–33], and so on. All of them probe the dependence of the observable on the orientation and boost of the laboratory reference frame, and finally limit the combinations of SME violating parameters.

In this paper, we reanalyze the light-clock comparison (MM experiment) and the optical atomic clock-comparison experiments in the RMS framework. The light clock consists of two mirrors at a distance L_0 and a photon propagating back and forth between the two mirrors [34,35], and the flight time of the photon from one mirror to the other reflects the clock's frequency. While the optical atomic clock is based on trapped single ion or many neutral atoms, the frequency is presented by the ions or atoms jumping between two energy levels. For the light clock, the distance between two mirrors and the propagating direction of the light in the laboratory reference frame can be regarded as the structure of the light clock, which influences the clocks' frequency due to the anisotropy of the speed of light implied by the LI violation. Similarly, the inner structure of optical atomic clocks (that is, the inner structure of the atoms) should also be taken into account. Here, we review the test of SR by the light-clock comparison (MM experiment), and develop a method to calculate the influence of the structure of the light clock. With a similar method, the change of the atomic clock's

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frequency depending on the atomic structure has been first analyzed in the RMS framework. For the experiment of frequency comparison between two optical atomic clocks linked by a fiber network [21], our result shows that the measurable parameter should be $\alpha - 2(\beta + 2\delta)/3$ instead of α, where β and δ are defined in Eq. (2) below.

The paper is organized as follows. In Sec. II, we review the RMS kinematical framework, in which the LI violation is embodied by three parameters, and also introduce the current stringent limits on these parameters. In Sec. III, we develop a method to reanalyze the influence of structure effect on light-clock comparison (MM experiment) and the optical atomic clock comparison. Finally, the paper is concluded in Sec. IV.

II. ROBERTSON-MANSOURI-SEXL FRAMEWORK

The RMS framework, embodying the LI violation with a simpler form, has been pioneered by Robertson [6] and further refined by Mansouri and Sexl [7] and others. In this framework, it is postulated that there are two reference frames: a preferred frame $\Sigma(T, X, Y, Z)$ and a moving frame $S(t, x, y, z)$. The X and x axes for the two frames are parallel. The Σ frame is usually considered as the cosmic microwave background (CMB) frame [16,36], in which there is no distinguished direction and the speed of light is constant. The S frame, usually considered as the laboratory reference frame, has a relative constant speed to the Σ frame along the x-axis direction.

In SR, the speed of light is assumed to be isotropic, and two inertial coordinate frames are linked by Lorentz transformation. Robertson assumed that the two-way speed of light is anisotropic, and found a transformation (Robertson transformation) deviating from the Lorentz transformation, which describes the LI violation with three parameters. Then, Mansouri and Sexl further postulated that the one-way speeds of light are unequal, which is dependent on the synchronization transport, and finally gave a more general form of the transformation between the S and Σ frame (MS transformation). This transformation is in fact equal to the Robertson transformation, since the one-way speed of light is physically unobservable. Similarly, Edwards transformation can return to Lorentz transformation, if one changes the arbitrary synchronization convention to the Einstein synchronization convention. The relations of the four transformations mentioned above are shown in Fig. 1 [37].

In RMS framework, assuming the S frame has a relative velocity v to the Σ frame along the positive direction of the x axis, the MS transformation between the two frames can be expressed as [7]

$$t = aT + \vec{\varepsilon} \cdot \vec{x}, \qquad x = b(X - vT),$$
$$y = dY, \qquad z = dZ. \tag{1}$$

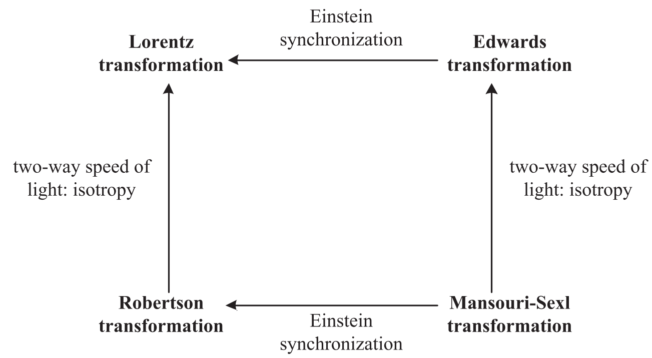


FIG. 1. Schematic diagram of the relations among the four transformations [37]: Lorentz, Edwards, Robertson and Mansouri-Sexl transformations.

Here, $\vec{\varepsilon}$ reflects the transport synchronization of clocks. As the one-way speed of light is unobservable, all of the synchronization conventions are physically equivalent. For simplification, the Einstein synchronization convention of $\varepsilon = -va(v)/[(1 - v^2)b(v)]$ [7] can be adopted, resulting in the Robertson transformation. The parameters a, b, d in Eq. (1), respectively describing the LI violation along the time and space axes of the laboratory reference frame, can be expanded with v^2/c^2 to the first order as [38,39]

$$a(v) = 1 + \left(\alpha - \frac{1}{2}\right)\frac{v^2}{c^2},$$
$$b(v) = 1 + \left(\beta + \frac{1}{2}\right)\frac{v^2}{c^2},$$
$$d(v) = 1 + \delta\frac{v^2}{c^2}, \tag{2}$$

where c is the velocity of light in vacuum of the CMB reference frame. If $\alpha = \beta = \delta = 0$, Eq. (1) reduces to the Lorentz transformation. According to Eqs. (1) and (2), the light cone $\vec{X}^2 - c^2T^2 = 0$ in Σ frame can be rewritten in S frame as $\vec{x}^2 - c^2(\theta)t^2 = 0$ with the velocity of light in laboratory reference frame [7]

$$c(\theta) = c \cdot \left[1 + (\delta - \beta)\frac{v^2}{c^2}\sin^2\theta + (\beta - \alpha)\frac{v^2}{c^2}\right], \tag{3}$$

where θ is the angle between the directions of the light ray propagation and the velocity v.

Currently, the tests of SR in RMS framework can be classified as the MM, KT, IS experiments and the recent comparison of two same optical clocks in different locations linked by a fiber network. With the orthogonal standing-wave optical cavities interrogated by a laser, MM experiment can be used to probe the LI violation due to anisotropy of the resonance frequencies of electromagnetic cavities, and gives the best constrain on $\delta - \beta = (-1.6 \pm 6.0 \pm 1.2) \times 10^{-12}$ [15]. By comparing

TABLE I. Present limits for the violating parameters in the RMS framework.

Reference	$\delta - \beta$	$\beta - \alpha$	α
P. Wolf *et al.* (2003) [14]	$(1.2 \pm 2.2) \times 10^{-9}$	$(1.6 \pm 3.0) \times 10^{-7}$	⋯
S. Reinhardt *et al.* (2007) [19]	⋯	⋯	$\leq 8.4 \times 10^{-8}$
C. Eisele *et al.* (2009) [15]	$(-1.6 \pm 6.0 \pm 1.2) \times 10^{-12}$	⋯	⋯
M. E. Tobar *et al.* (2010) [16]	⋯	$(-1.7 \pm 4.0) \times 10^{-8}$	⋯
B. Botermann *et al.* (2014) [20]	⋯	⋯	$\leq 2.0 \times 10^{-8}$
P. Delva *et al.* (2017) [21]	⋯	⋯	$\leq 1.1 \times 10^{-8}$

a cryogenic sapphire oscillator and a hydrogen maser, the KT experiment showed no violating signal and limited $\beta - \alpha = (-1.7 \pm 4.0) \times 10^{-8}$ [16]. For the violating parameter α, the best results for IS experiment and the frequency comparison between two optical atomic clocks are $|\alpha| \leq 2.0 \times 10^{-8}$ [20] and $|\alpha| \leq 1.1 \times 10^{-8}$ [21], respectively. Table I gives the present limits on the three violating parameters for the above four experiments.

III. INFLUENCE OF THE CLOCKS' STRUCTURES ON THE CLOCK COMPARISONS

The basic idea of testing the LI violation with the clock comparison is exploring the dependence of a clock's frequency on its orientation, since the LI violation implies the anisotropy of the spacetime. For the clock-comparison experiment, it can be described as a two-way frequency transfer between two observers I and II [21,40–43]. the observer I emits an electromagnetic signal with the proper frequency ν_1, and this signal is received by the observer II with the frequency ν_2. As the first order of the Doppler effect is rather large in the direct comparison $(\nu_2 - \nu_1)/\nu_1$, the two-way frequency transfer is usually adopted to eliminate it, in which the signal is again back to I, and observed with the frequency ν_3. Finally, the relative differential signal of the clock comparison can be written as

$$\Delta \equiv \frac{\nu_2 - \nu_1}{\nu_1} - \frac{\nu_3 - \nu_1}{2\nu_1}, \tag{4}$$

where the orientation-dependence LI violation is included in $(\nu_2 - \nu_1)/\nu_1$. The second term in Eq. (4) is unnecessary when the two clocks approximately locate on the same places.

This signal includes the general relativistic effect, the special relativistic effect, and also the deviations from them. Since we only focus on the violation of the SR here, Eq. (4) can be further split into four parts,

$$\Delta = \Delta_{gr} + \Delta_{\mathrm{SR}} + \Delta_\alpha + \Delta_\xi. \tag{5}$$

Here, Δ_{gr} stands for the gravitational redshift effect, Δ_{SR} represents the SR effect (the second order of the Doppler effect), and the other two terms are contributed by the deviations of SR, in which Δ_α and Δ_ξ are respectively the deviations from the Lorentz transformation in the time and space axes. For Δ_α, it has been studied extensively, and obtained as [21]

$$\Delta_\alpha = \alpha c^{-2}[2\vec{w} \cdot (\vec{v}_I - \vec{v}_{II}) + (\vec{v}^2{}_I - \vec{v}^2{}_{II})] + o(c^{-3}), \tag{6}$$

where $\vec{w}$ is the velocity of the Earth with respect to the CMB frame; $\vec{v}_I$ and $\vec{v}_{II}$ are the velocities of clocks I and II in the nonrotating geocentric celestial reference system, respectively. For Δ_ξ, it can be regarded as a kind of structure effect, since it depends on the clock's orientation. This effect has not been analyzed before, which is focused on by this paper. As the structure effects in the light-clock comparison and the optical atomic clock comparison are different, we calculate both of them below.

A. Structure effect in the light-clock comparison (MM experiment)

The apparatus of the MM experiment is similar to a Michelson interferometer, which contains two interferometer arms [see Fig. 2(a)]. Through the half-silvered mirror A, the transmitted ray is reflected by the mirror B and then A, and finally reaches the detector D, forming the interference arm I. Similarly, the reflected ray propagates to D and forms the interference arm II. In the MM experiment, the difference between the flight time of photons propagating along the two interference arms is measured by interferometry. Then, the apparatus is rotated to produce the modulated signal. As the period reflects the frequency, each interference arm in the MM experiment can be considered as a light clock [see Fig. 2(b)] [34,35]. Therefore, the MM experiment equivalently makes a frequency comparison between two light clocks to test SR (testing the dependence of the speed of light on the interference-arm orientation). In the following, we analyze the influence of the clocks' structure on this comparison.

Let us consider the light clock I at rest in the S frame, shown in Fig. 3(a). It consists of two mirrors with a distance L_0 and a photon propagating back and forth between them. Any observer at rest in the S frame can measure the proper length L_0 between the two mirrors A and B. Since the S frame moves with a constant velocity relative to the Σ frame, the length measured by an observer at rest in Σ frame is not L_0, and should introduce the additional contraction,

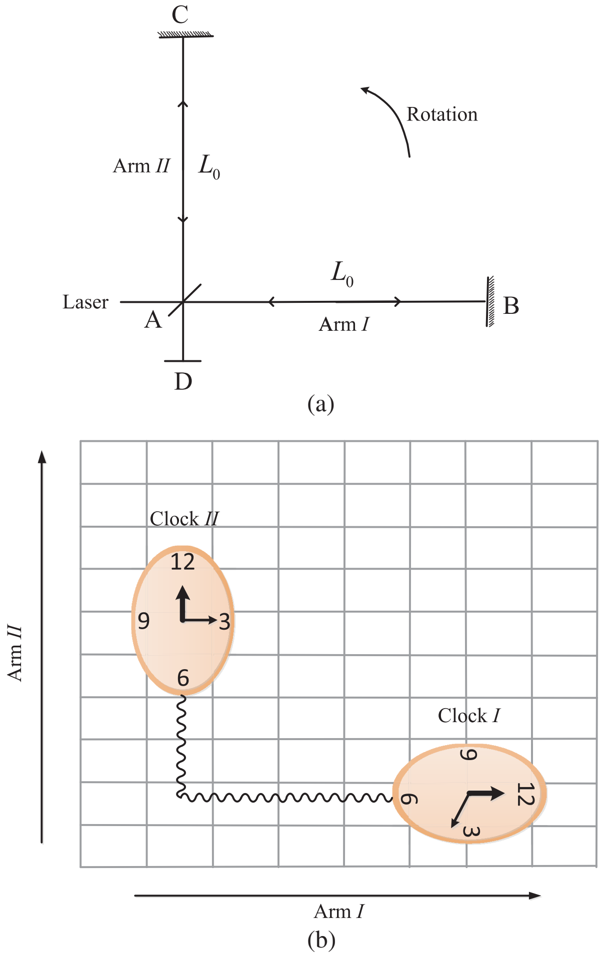


FIG. 2. (a) Schematic diagram of the MM experiment. (b) Equivalent counterpart of the MM experiment—frequency comparison between the two light clocks.

which can be calculated based on the MS transformation. To make the analysis simpler, an imaginary inertial frame $\tilde{\Sigma}(\tilde{T},\tilde{X},\tilde{Y},\tilde{Z})$ can be introduced to link the reference frame Σ and the laboratory reference frame S, which is equivalent to decomposing MS transformation to a Lorentz transformation and a scalar modification [see Fig. 3(b)]. The laws of physics are invariant for the Σ and $\tilde{\Sigma}$ frames with respect to Lorentz transformation,

$$\tilde{T}=\frac{T-vX/c^2}{\sqrt{1-v^2/c^2}},\qquad \tilde{X}=\frac{X-vT}{\sqrt{1-v^2/c^2}},$$
$$\tilde{Y}=Y,\qquad \tilde{Z}=Z. \tag{7}$$

The four coordinates of $\tilde{\Sigma}(\tilde{T},\tilde{X},\tilde{Y},\tilde{Z})$ are physically insignificant and cannot be measured directly. The scalar modification between the $\tilde{\Sigma}$ and S frames represents the deviation from SR. Combining Eqs. (1) and (7), we can obtain the scalar modification as

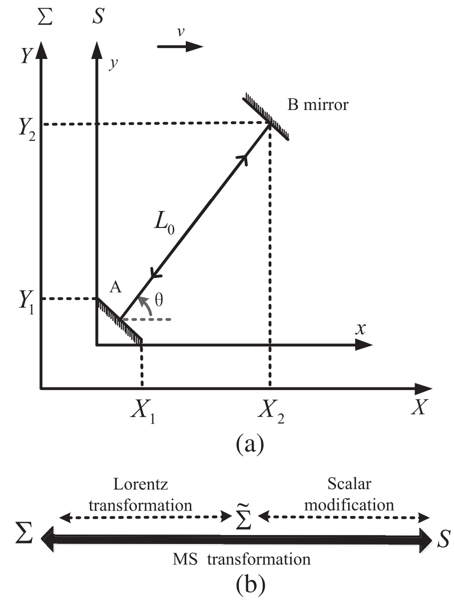


FIG. 3. (a) Schematic diagram of the Σ and S frame. The S frame has the velocity v in the positive direction of X axis of the frame Σ. (b) Relations among the three frames: Σ, $\tilde{\Sigma}$, and S frames.

$$\tilde{T}=\frac{t\sqrt{1-v^2/c^2}}{a},\qquad \tilde{X}=\frac{x}{(b\sqrt{1-v^2/c^2})},$$
$$\tilde{Y}=\frac{y}{d},\qquad \tilde{Z}=\frac{z}{d}. \tag{8}$$

From Eqs. (4) and (5), the $\tilde{\Sigma}$ frame is the bridge to connect the Σ and S frames.

According to Eq. (8), the $\tilde{X}$ and $\tilde{Y}$ components of the contracted length for the separation between mirrors A and B can be obtained as $\tilde{L}_{\tilde{X}}=L_0\cos\theta/(b\sqrt{1-v^2/c^2})$ and $\tilde{L}_{\tilde{Y}}=L_0\sin\theta/d$. Therefore, the contracted length in the $\tilde{\Sigma}$ frame can be derived as

$$\tilde{L}=\sqrt{\tilde{L}_{\tilde{X}}^2+\tilde{L}_{\tilde{Y}}^2}$$
$$=\sqrt{\left(\frac{L_0\cos\theta}{b\sqrt{1-v^2/c^2}}\right)^2+\left(\frac{L_0\sin\theta}{d}\right)^2}. \tag{9}$$

As the speed of light in $\tilde{\Sigma}$ frame, associated with the Σ frame by the Lorentz transformation, is the constant c, the travel time for the photon propagating from A to B, and then back to A, can be denoted by

$$\Delta\tilde{T}_{AB}=\frac{2\tilde{L}}{c}. \tag{10}$$

If the flight time is ΔT_{AB} for an observer at rest in Σ frame, we can get

$$\Delta T_{AB} \equiv \frac{2L}{c} = \frac{\Delta \tilde{T}_{AB}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2L_0}{c}\cdot\left[1-\left(\beta-\frac{1}{2}\right)\frac{v^2}{c^2}-(\delta-\beta)\sin^2\theta\frac{v^2}{c^2}\right] \quad (11)$$

based on the time dilation in SR, which has been kept to the first order of $(v/c)^2$. Here, L is the measured result of the distance between mirrors A and B for an observer at rest in Σ frame. In the S frame, the travel time can be expressed as

$$\Delta t_{AB} = a\Delta T_{AB} \simeq \frac{2L_0}{c(\theta)}, \quad (12)$$

where the first equality arises from the MS transformation and the second one is the most general definition of time with $c(\theta)$ embodying the anisotropy of the speed of light. Combining Eqs. (11) and (12), one can derive the same result for $c(\theta)$ with Eq. (3).

Now we analyze the MM experiment, which can be equivalently regarded as the frequency comparison of two light clocks (clocks I and II). We determine the clock I frequency ν_1,

$$\frac{\nu_1}{\nu_0} = \frac{2L_0/c}{\Delta T_{AB}} = a + \xi(\theta). \quad (13)$$

Here, $\nu_0 \equiv c/2L_0$, and the parameter a stands for the time dilation for a moving clock. $\xi(\theta)$ reflects the correction related to the structure of the light clock, which can be written as

$$\xi(\theta) = \frac{c(\theta)}{c} - 1 = (\beta-\alpha)\frac{v^2}{c^2} + (\delta-\beta)\sin^2\theta\frac{v^2}{c^2}. \quad (14)$$

Finally, the relative frequency difference of clocks II and I in MM experiment can be derived as

$$\Delta \equiv \frac{\nu_2-\nu_1}{\nu_0} = \xi\left(\theta+\frac{\pi}{2}\right) - \xi(\theta) \equiv \Delta_\xi. \quad (15)$$

As the locations and velocities of the two light clocks I and II are approximately the same, the gravitational redshift effect, SR effect, and Δ_α can be neglected, and only Δ_ξ remains. Based on Eq. (15), the influence of the structure effect on testing SR with the light-clock comparison can be calculated. However, the tests involving the optical atomic clocks (such as KT, IS, and the optical atomic clock-comparison experiments) have not considered the influence of the atomic inner structure on the experimental results. We take the optical atomic clock-comparison experiment as an example to calculate this effect in the next section.

B. Structure effect in the optical atomic clock comparison

The atomic clocks, based on the atoms emitting or absorbing the photons, have very high stability and accuracy, which can offer some of the most powerful tests of Lorentz violations. Analogous to the calculation of the structure effect in the light clock-comparison experiment, this similar effect in the optical atomic clock comparison is analyzed with an imaginary frame $\tilde{\Sigma}$ in the following.

In discussing the light clock-comparison experiment, there involves three coordinate frames: the preferred frame Σ, the imaginary inertial frame $\tilde{\Sigma}$, and the laboratory frame S. The speed of light in them is respectively c, c, and $c(\theta)$. The observed lengths of a light clock at rest in the S frame are respectively L, $\tilde{L}$, and L_0 from the view of three observers staying statically in these three frames. According to Eqs. (1), (7), and (8), L and $\tilde{L}$ are linked by the Lorentz contracted factor, $\tilde{L}$ and L_0 are connected by a scalar modification, and the relationship between L and L_0 is related by the MS transformation. In the atomic clock-comparison experiment, here we take a hydrogenlike atom for an example to analyze, in which the electrons form a closed shell. From the perspective of three observers at rest in three frames, the measured frequencies of an atomic clock at rest in the S frame are different, which can be regarded as the difference of the observed radial distances between nucleus and electron. Similar to the lengths of the light clock, the three distances respectively denoted with R, $\tilde{R}$, and r satisfy the same contracted relation. The above description has been simply shown in Fig. 4. We focus on calculating the change of the atomic energy level due to the LI violation in this paper, in which the imaginary frame $\tilde{\Sigma}$ is adopted to make solving of the Dirac equation simpler.

Make the convention: the wave functions, momenta, electromagnetic four-dimension potential, and radial distances of the atoms in the three frames Σ, $\tilde{\Sigma}$, and S are respectively $(\psi, \tilde{\psi}, \phi)$, $(P_i = -i\hbar\nabla_{X^i}, \tilde{P}_i = -i\hbar\nabla_{\tilde{X}^i}, p_i = -i\hbar\nabla_{x^i})$, $(\mathcal{A}_\mu = \{\mathcal{A}_0, \mathcal{A}_i\}, \tilde{A}_\mu = \{\tilde{A}_0, \tilde{A}_i\}, A_\mu = \{A_0, A_i\})$, and $(R = \sqrt{X^2+Y^2+Z^2}, \tilde{R} = \sqrt{\tilde{X}^2+\tilde{Y}^2+\tilde{Z}^2}, r = \sqrt{x^2+y^2+z^2})$, where the spacetime coordinates 0, 1, 2, 3 are denoted by greek indices μ and the space coordinates 1, 2, 3 are denoted by latin indices i. The free Dirac equation (without potentials) in the Σ frame [44] is

$$i\hbar\frac{\partial\psi}{\partial T} = (c\gamma^0\gamma^i P_i + \gamma^0 m_e c^2)\psi, \quad (16)$$

with $H_f = c\gamma^0\gamma^i P_i + \gamma^0 m_e c^2$ being the relativistic Hamiltonian of the Dirac particle,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^{\rm i} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

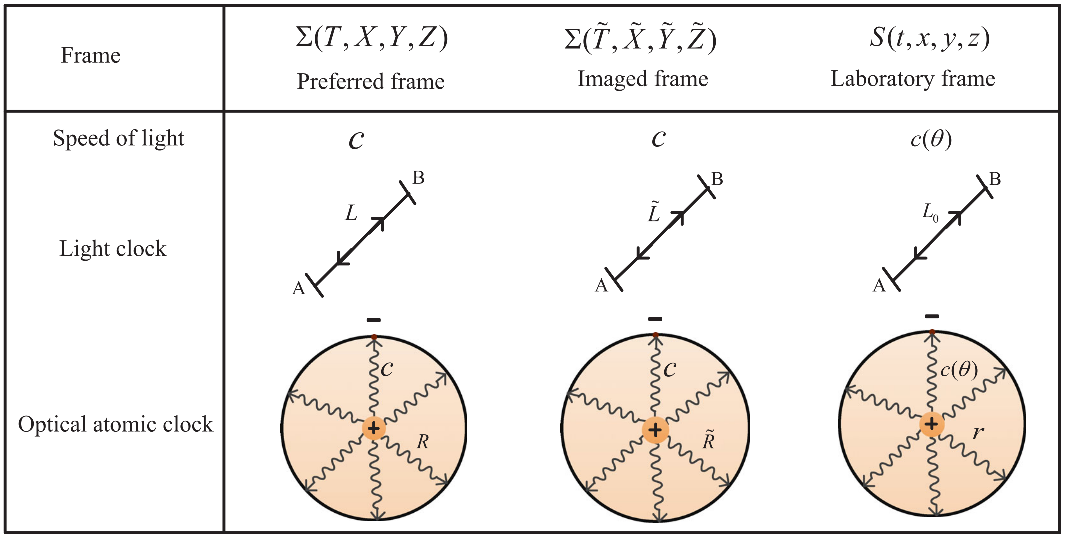


FIG. 4. The lengths of the light clock and the optical atomic clock in the Σ, $\tilde{\Sigma}$, and S frames.

the 4×4 gamma matrices in the Dirac representation. When the Dirac particle moves in an electromagnetic field, the relativistic Hamiltonian can be further expressed as

$$H = c\gamma^0\gamma^i\left(P_i - \frac{e}{c}A_i\right) + \gamma^0 m_e c^2 + eV_\Sigma, \tag{17}$$

with V_Σ being the value of A_0 in the Σ frame. As the Σ and $\tilde{\Sigma}$ frames are linked by Lorentz transformation, one can obtain the relativistic Hamiltonian in the $\tilde{\Sigma}$ frame as

$$\tilde{H} = c\gamma^0\gamma^i\left(\tilde{P}_i - \frac{e}{c}\tilde{A}_i\right) + \gamma^0 m_e c^2 + eV_{\tilde{\Sigma}}. \tag{18}$$

Because $\tilde{H}$ always contains parts coupling together the free positive and negative energy solutions, the Foldy-Wouthuysen (FW) transformation [45–47] provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum mechanics. The relativistic Hamiltonian in Eq. (18) can be simply written as $\tilde{H} \equiv \gamma^0 m_e c^2 + \varepsilon + O$ [46], where the odd operator $O \equiv c\gamma^0\gamma^i(\tilde{P}_i - \frac{e}{c}\tilde{A}_i)$ and the even operator $\varepsilon \equiv eV_{\tilde{\Sigma}}$ are off diagonal and diagonal, respectively. Then, based on the FW transformation, Eq. (18) can be further written as [47]

$$\begin{aligned}\tilde{H}_{FW} = {}& \gamma^0\left(m_e c^2 + \frac{1}{2m_e}(\tilde{P}_i - \frac{e}{c}\tilde{A}_i)^2 - \frac{1}{8m_e^3c^6}\tilde{P}_i{}^4\right) + eV_{\tilde{\Sigma}} \\ & - \frac{1}{2m_e c}\gamma^0 e\hbar(\Xi\cdot\tilde{B}) - \frac{ie\hbar^2c^2}{8m_e^2c^4}\Xi\cdot(\tilde{\nabla}\times\tilde{E}) \\ & - \frac{e\hbar}{4m_e^2c^4}\Xi\cdot(\tilde{E}\times\tilde{P}_i) - \frac{e\hbar^2c^2}{8m_e^2c^4}\tilde{\nabla}\cdot\tilde{E},\end{aligned} \tag{19}$$

with the spin related matrix

$$\Xi = \begin{pmatrix}\sigma^i & 0 \\ 0 & \sigma^i\end{pmatrix}.$$

Here, Eq. (19) has been kept up to the order of $1/m_e^3c^6$. In the nonrelativistic limit, the Dirac equation can be transformed into the Schrodinger equation.

Based on Eq. (8), since the difference of the coordinates between the $\tilde{\Sigma}$ and S frames is just the respective coordinate scaling, the quantities $V_{\tilde{\Sigma}}$, $\tilde{A}_i$, $\tilde{B}$, $\tilde{E}$ in the $\tilde{\Sigma}$ frame can be contracted to V_S, A_i, B, E in the S frame. Thus, the evolution equation of the Dirac particle in the S frame can be expressed as

$$\begin{aligned}\frac{a}{\sqrt{1-v^2/c^2}}i\hbar\frac{\partial\phi}{\partial t} = {}& \left\{\gamma^0 m_e c^2 + \frac{\gamma^0}{2m_e}\left[-i\hbar(b\sqrt{1-v^2/c^2}\nabla_x + d\nabla_y + d\nabla_z) - \frac{e}{c}A_i\right]^2 - \frac{\gamma^0}{8m_e^3c^6}\tilde{P}_i{}^4 + eV_S\right. \\ & \left. - \frac{1}{2m_e c}\gamma^0 e\hbar(\Xi\cdot B) - \frac{ie\hbar^2c^2}{8m_e^2c^4}\Xi\cdot(\nabla\times E) - \frac{e\hbar}{4m_e^2c^4}\Xi\cdot(E\times\tilde{P}_i) - \frac{e\hbar^2c^2}{8m_e^2c^4}\nabla\cdot E\right\}\phi.\end{aligned} \tag{20}$$

In Eq. (20), the left-hand side represents the time part of Lorentz violation; for the right-hand side, the first three terms in the brace describe the relativistic mass increase. Especially, the second term describes the space part of Lorentz violation, which is the atomic structure effect we focus on. The following two terms describe the electrostatic energy and magnetic dipole energy. The next two terms stand for the spin-orbit interaction, and the last one is the Darwin term. Here, the influence of the Lorentz-violation perturbations on the relativistic terms is negligible.

It is worth noting that the Coulomb potential V_S in laboratory reference (S frame) should be given by the form of $-Ze^2/r$. To demonstrate it, we consider an analogy between the MM experiment and optical atomic clock-comparison experiment. For the MM experiment, the arm between mirrors may be replaced by an infinite potential well

$$V_{\mathrm{MM}}(x) = \begin{cases} 0, & 0 \leq x \leq L_0 \\ \infty, & x < 0, x > L_0 \end{cases}. \tag{21}$$

The photon's motion in the infinite potential well characterizes time, which corresponds to the clock conception in the RMS framework [Fig. 5(a)]. The infinite potential well has a proper length L_0 in space, which corresponds to "rod" conception in the RMS framework. A similar analysis may be implemented for the structure of the atom, which implies that the electron-state transition and Coulomb potential also correspond to clock and rod conceptions in the RMS framework, respectively. Therefore, the Coulomb potential V_S should be $-Ze^2/r$ [Fig. 5(b)].

For the application in the atomic clock-comparison experiment, only nonrelativistic terms are needed. The last four terms of Eq. (20) can be neglected, since we do not focus on the spin contributions and Darwin term. Considering the atom is stationary relative to the S frame, the magnetic field vanishes. Furthermore, as the antiparticle part in the wave function is not needed, we just need to calculate the particle-part contribution. Thus, Eq. (20) can be rewritten as

$$\frac{a}{\sqrt{1-v^2/c^2}} i\hbar \frac{\partial \phi}{\partial t} = \left\{ m_e c^2 + \frac{1}{2m_e}\left[-i\hbar\left(b\sqrt{1-v^2/c^2}\nabla_x + d\nabla_y + d\nabla_z\right)\right]^2 - \frac{Ze^2}{r} \right\}\phi, \tag{22}$$

where ϕ represents the particle's wave function, and the constant term $m_e c^2$ can be neglected in the following.

For the Coulomb potential, it can be expanded as

$$-\frac{Ze^2}{r} = -\frac{Ze^2}{\tilde{R}} + \bar{V}, \tag{23}$$

with

$$\bar{V} = \frac{Ze^2}{\tilde{R}} \cdot \frac{\beta + 2\delta}{3} \cdot \frac{v^2}{c^2} + \frac{Ze^2}{\tilde{R}} \cdot \frac{\beta - \delta}{3} \cdot \frac{2\tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2}{\tilde{R}^2} \cdot \frac{v^2}{c^2}, \tag{24}$$

which can be regarded as a perturbation, and vanishes when SR is not violated. Therefore, Eq. (22) can be further expressed as

$$i\hbar \partial_{\tilde{T}} \phi = \left(\frac{\tilde{P}^2}{2m_e} - \frac{Ze^2}{\tilde{R}} + \bar{V} \right)\phi. \tag{25}$$

In the following, we calculate the influence of LI violation on the atomic energy level with the perturbation method.

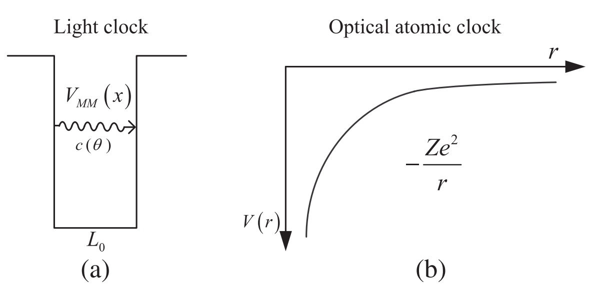
In the laboratory reference frame S, the exact energy eigenfunctions for the exact energy eigenvalues $E_n^{(0)}$ can be written as

$$\phi(\vec{\tilde{R}}, \tilde{T}) = \langle \vec{\tilde{R}} | n, l, m \rangle = \exp\left(-\frac{i}{\hbar} \tilde{E}_n^{(0)} \tilde{T} \right) \phi(\vec{\tilde{R}}), \tag{26}$$

with $|n, l, m\rangle$ being the exact energy eigenkets in frame S and

$$\tilde{E}_n^{(0)} = \left\langle -\frac{Ze^2}{\tilde{R}} \right\rangle_{nl} = -\frac{m_e Z^2 e^4}{2n^2\hbar^2} \tag{27}$$

being the exact energy eigenvalue in the imaginary frame $\tilde{\Sigma}$. According to Eq. (8), the eigenfunctions can be expressed by the laboratory coordinates (t, x, y, z) as



	Light clock	Optical atomic clock
Clock (time)	Photon motion	Electron-state transition
Rod (space)	$V_{MM}(x)$	$-Ze^2/r$

(c)

FIG. 5. (a) The infinite potential well for the light clock with the length L_0. (b) The Coulomb potential $-Ze^2/r$ for the electron in S frame. (c) The analogy between the light clock and optical atomic clock.

$$\phi(\vec{r},t) = \langle \vec{r}|n,l,m\rangle = \exp\left(-\frac{i}{\hbar}\frac{a}{\sqrt{1-v^2/c^2}}\tilde{E}_n^{(0)}t\right)\phi(\vec{r}). \tag{28}$$

Combining Eqs. (2) and (28), we can obtain the energy eigenvalues in the S frame

$$E_n^{(0)} = \frac{a}{\sqrt{1-v^2/c^2}}\tilde{E}_n^{(0)} \simeq (1+\alpha v^2/c^2)\cdot \tilde{E}_n^{(0)}. \tag{29}$$

When the Lorentz transformation violates as indicated by Eq. (8), the energy level E_n of an atom in the frame S not only has a contraction compared with its proper values $\tilde{E}_n^{(0)}$ in $\tilde{\Sigma}$ frame [show in Eq. (29), but also an increment ξ_n arising from the structure effect, namely the perturbation potential $\bar{V}$. Simplified by the spherical harmonic $Y_2^0 = \sqrt{5/16\pi}\cdot(3\tilde{z}^2 - \tilde{R}^2)/\tilde{R}^2$, Eq. (24) can be written as

$$\bar{V} = \frac{Ze^2}{\tilde{R}}\cdot\frac{v^2}{c^2}\left(\frac{\beta+2\delta}{3} + \sqrt{\frac{16\pi}{5}}\cdot\frac{\beta-\delta}{3}\cdot Y_2^0\right). \tag{30}$$

Then, the shift of energy level can be obtained as

$$\xi_n = \langle n,l,m|\bar{V}|n,l,m\rangle = -\tilde{E}_n^{(0)}\frac{v^2}{c^2}\left(\frac{\beta+2\delta}{3} + \sqrt{\frac{16\pi}{5}}\frac{\beta-\delta}{3}\langle l,m|Y_2^0|l,m\rangle\right) \tag{31}$$

with the matrix element of operator Y_2^0 between states $\langle l,m|$ and $|l,m\rangle$ given by [48]

$$\langle l,m|T_0^{(2)}|l,m\rangle = \left(\frac{C_{20lm}^{lm}}{C_{20ll}^{ll}}\right)\langle l,l|T_0^{(0)}|l,l\rangle, \tag{32}$$

where C_{20lm}^{lm} is the usual Clebsch-Gordan coefficient and the ratio of the Clebsch-Gordan coefficients $(C_{20lm}^{lm})/(C_{20ll}^{ll})$ is $[3m^2 - l(l+1)]/[3l^2 - l(l+1)]$. Since $\delta - \beta$ has been limited by the MM experiment with a high precision $(-1.6\pm 6\pm 1.2)\times 10^{-12}$ [15], here we only consider the constraints of the atomic clock comparison on $\beta + 2\delta$. Thus, the structure effect approximately shifts the energy level of the atoms by

$$\xi_n \simeq -\frac{\beta+2\delta}{3}\cdot\frac{v^2}{c^2}\cdot\tilde{E}_n^{(0)}. \tag{33}$$

Finally, based on Eqs. (29) and (33), the energy level of an atom in the frame S is

$$E_n \equiv E_n^{(0)} + \xi_n = \left[1 + \left(\alpha - \frac{\beta+2\delta}{3}\right)\frac{v^2}{c^2}\right]\cdot\tilde{E}_n^{(0)}. \tag{34}$$

For the atomic clock comparison [34], as the velocities of clocks I and II are different due to the different locations, both of the LI violations in the time and space axes exist. Based on Eqs. (29) and (33), we can derive

$$\Delta_\alpha \equiv \frac{E_{n(I)}^{(0)} - E_{n(II)}^{(0)}}{\tilde{E}_n^{(0)}} = \alpha c^{-2}[2\vec{w}(\vec{v}_I - \vec{v}_{II}) + (\vec{v}^2{}_I - \vec{v}^2{}_{II})] + o(c^{-3}) \tag{35}$$

with $E_{n(I)}^{(0)}$ and $E_{n(II)}^{(0)}$, respectively, the energy eigenvalues of clocks I and II, and

$$\Delta_\xi \equiv \frac{\xi_{n(I)} - \xi_{n(II)}}{\tilde{E}_n^{(0)}} = -\frac{(\beta+2\delta)}{3}c^{-2}[2\vec{w}(\vec{v}_I - \vec{v}_{II}) + (\vec{v}^2{}_I - \vec{v}^2{}_{II})] + o(c^{-3}), \tag{36}$$

which oscillates with a sidereal period. Therefore, the deviating effect of SR in the atomic clock comparison can be written as

$$\Delta \equiv \Delta_\alpha + \Delta_\xi = \left(\alpha - \frac{\beta+2\delta}{3}\right)c^{-2}[2\vec{w}(\vec{v}_I - \vec{v}_{II}) + (\vec{v}^2{}_I - \vec{v}^2{}_{II})] + o(c^{-3}). \tag{37}$$

That is, for the test of SR with the frequency comparison between two atomic clocks linked by a fiber network, the modifying factor, introduced by the SR violation, of the measurable parameter should be $\alpha - (\beta+2\delta)/3$ instead of the widely recognized α.

IV. SUMMARY

In this paper, we reviewed the Robertson-Mansouri-Sexl framework, based on which the deviating effects of SR in the light-clock comparison and the atomic clock comparison have been calculated. For simplification, an imaginary inertial frame is introduced, which is connected with the laboratory frame by a scalar modification, representing the deviation from the LI.

The LI violation can be divided into the violations along the time axis and space axis (structure effect). Adopting the imaginary frame to analyze these violating effects, we find the following: for the light-clock comparison (MM experiment), as the locations of the two clocks are approximately same, only the structure effect remains, and our calculated result is the same as the previous work; for the atomic clock comparison, we give a more complete expression for the violating effect, which shows that the modifying factor of the measurable parameter should be $\alpha - (\beta+2\delta)/3$ instead of the widely recognized α.

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