Tensor $O(N)$ model near six dimensions: Fixed points and conformal windows from four loops

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In search of nontrivial field theories in high dimensions, we study further the tensor representation of the $O(N)$-symmetric ϕ^4 field theory introduced by Herbut and Janssen [Phys. Rev. D **93**, 085005 (2016)] by using four-loop perturbation theory in two cubic interaction coupling constants near six dimensions. For infinitesimal values of the parameter $\epsilon = (6-d)/2$ we find an infrared-stable fixed point with two relevant quadratic operators for N within the conformal windows $1 < N < 2.653$ and $2.999 < N < 4$, and compute critical exponents at this fixed point to the order ϵ^4. Taking the four-loop beta functions at their face value we determine the higher-order corrections to the edges of the above conformal windows at finite ϵ, to find both intervals to shrink to zero above $\epsilon \approx 0.15$. The disappearance of the conformal windows with the increase of ϵ is due to the collision of the Wilson-Fisher $\mathcal{O}(\epsilon)$ infrared fixed point with the $\mathcal{O}(1)$ mixed-stable fixed point that appears at two and persists at higher loops. The latter may be understood as a Banks-Zaks type fixed point that becomes weakly coupled near the right edge of either conformal window. The consequences and issues raised by such an evolution of the flow with dimension are discussed. It is also shown both within the perturbation theory and exactly that the tensor representation at $N = 3$ and right at the $\mathcal{O}(\epsilon)$ infrared-stable fixed point exhibits an emergent $U(3)$ symmetry. A role of this enlarged symmetry in possible protection of the infrared fixed point at $N = 3$ is noted.

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I. INTRODUCTION

The question of the existence of interacting conformally invariant field theories in space-time dimensions $d \geq 4$ has been long standing. Using the random walk representation, it has been, for example, rigorously proven that repulsive $O(N)\phi^4$ field theories are trivial for $N = 1$ [1] and $N = 2$ [2] i.e. that the only fixed point of the scale transformation in such a field theory is the Gaussian one. On the other hand, there is a bicritical fixed point for the $O(N)\phi^4$ theory at $d > 4$ and any N, but at a negative (attractive) self-interaction. One may then ask if there is a way to represent this fixed point of the ϕ^4 theory as an infrared (IR)-stable fixed point of another field theory with the same symmetry, which would serve as its ultraviolet (UV) completion. Indeed, there are at least two instances where this is possible: (1) the ϕ^4 theory at the IR Wilson-Fisher fixed point is believed to be equivalent to the nonlinear sigma model at the UV fixed point, (2) the Gross-Neveu model at the UV fixed point is equivalent to the Gross-Neveu-Yukawa field theory at its IR critical point [3].

For the $O(N)$-invariant self-attractive ϕ^4 theory this question was raised by Fei, Giombi, and Klebanov [4], who Hubbard-Stratonovich decoupled the interaction term, to eventually replace the original theory with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu z_a)^2 + \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{g_1}{2} z_a \phi_i \Lambda^a_{ij} \phi_j + \frac{g_2}{6} d^{abc} z_a z_b z_c \tag{1}$$

with the indices $i, j = 1, 2, \ldots N$, the index $a = 0$, the matrix $\Lambda^0_{ij} = \delta_{ij}$, and the symbol $d^{000} = 1$. ϕ_i are the original N real fields that transform as a vector, and z_0 is the real Hubbard-Stratonovich field, scalar under $O(N)$. The $O(N)$ symmetry also allows for two quadratic terms, $m_z^2 z_a z_a$ and $m_\phi^2 \phi_i \phi_i$ which are IR-relevant, and which in (1) have been and hereafter will be tuned to zero. To understand the origin of the field theory (1) notice that when $g_2 = 0$ and $m_z \neq 0$ the field z_0 can be integrated out, to recover the original interacting term $(\phi_i\phi_i)^2$ to the leading order in gradient expansion. The interaction g_2 and the kinetic energy term for z_a can be thought of, in turn, as being generated by integration over high-energy modes of the fields ϕ_i, in close analogy to the passage from the Gross-Neveu to the Gross-Neveu-Yukawa field theories, for example. The same theory was also considered as a toy model of hadron dynamics by Ma. [5] We will call it the "scalar representation."

What is gained by this reformulation of the ϕ^4 theory is a new possibility for a systematic search for nontrivial fixed points using $(6-d)$ expansion, since both cubic interaction couplings g_1 and g_2 become marginal in the same dimension $d = 6$. It was found that for large N one-loop, beta functions for the two interaction coupling constants do have a Wilson-Fisher $\mathcal{O}(6-d)$ IR-stable fixed point. (For the purposes of this paper an IR-stable fixed point will be defined as a fixed point with only two relevant couplings, which are the above two masses m_z and m_ϕ.) As $N \to 1038.266$, however, the Wilson-Fisher fixed point collides with another $\mathcal{O}(6-d)$ fixed point with mixed stability, which has one additional relevant direction in the IR. Both fixed points then become complex when $N < 1038.266$. Higher-loop computations [6,7] conformed to the same scenario, but significantly reduced the value of the critical number of components N above which the real IR-stable fixed point exists at finite $d < 6$. Nevertheless, the question of the existence of the IR-stable fixed point of the theory (1) in scalar representation for single-digit values of N remained.

It was further realized by Herbut and Janssen [8] that it is equally possible to decouple the interaction term of the ϕ^4 theory in the tensor channel, so that one ends up with the theory in the form of (1), but with the index $a = 1, \ldots M_N$ with $M_N = (N-1)(N+2)/2$ as the number of components of the irreducible second-rank tensor under $O(N)$, and the M_N matrices Λ^a as a basis in the space of traceless, real, symmetric, N-dimensional matrices. The real fields z_a transform then under $O(N)$ as components of a second-rank tensor, and the symbol

$$d^{abc} = \mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c). \tag{2}$$

In particular, for $N = 2$ the two Λ^a matrices are nothing but the two real Pauli matrices, and for $N = 3$ the five Λ^a matrices are the familiar real Gell-Mann matrices [9]. We will call this realization of the $O(N)$ symmetry the "tensor representation." The combination of the scalar and the tensor representations of the $O(N)$ model was also considered and studied near six dimensions [10,11]. The pure tensor representation of the $O(N)$ model defined as above will be the main subject of the present study.

One-loop renormalization group (RG) analysis of the tensor representation [8] revealed, quite surprisingly and in contrast to the scalar representation [4], that the IR-stable fixed point exists in the intervals $1 < N < 2.653$ and $2.999 < N < 4$, which include physically interesting low values of $N = 2$ and $N = 3$. This suggests that there could be examples of $O(N)$-symmetric field theories, albeit in higher (second-rank tensor) representation, that are nontrivial in say five dimensions. This conclusion would follow, of course, only if the obtained IR-stable fixed point at a given N within the above conformal windows survives the increase of the parameter $6-d$, from an infinitesimal to a physical integer value of one or two. This however is not guaranteed. To examine this issue, Roscher and Herbut [12] have performed perturbative two-loop and functional renormalization group calculations for $N = 2$, which is particularly simple because the second cubic term identically vanishes, i.e. $d^{abc} \equiv 0$. They found that the second-loop terms introduce an additional nontrivial mixed-stable fixed point of the beta function, which is $\mathcal{O}(1)$, and which collides with the $\mathcal{O}(\epsilon)$ IR-stable fixed point as the parameter $\epsilon = (6-d)/2$ is increased. At $\epsilon > \epsilon_c$, with $\epsilon_c \ll 1$, this way the single beta function at $N = 2$ for the coupling g_1 had no real zeros left. If correct beyond the particular approximations which were employed, this result would conform to one's expectation of triviality of the $O(N)$ theories in $d = 5$ or $d = 4$, but only at the expense of having a new mixed-stable, in this case actually UV-stable finite-coupling fixed point already at the upper critical dimension of $d = 6$. In other words, the triviality of the tensor representation of the $O(2)$ model in $d = 5$ this way is obtained at the price of nontriviality in $d = 6$.

In this work we further study the theory (1) in tensor representation, at general N, and by using state-of-the-art perturbation theory in the couplings g_1 and g_2 to four loops. This first allows us to compute Taylor expansions for the critical exponents at the IR-stable fixed point to the order ϵ^4. Setting $N = 2$ we recover the previous two-loop result [12], and for all $1 < N < 2.653$ we find the same scenario of fixed-point collision and annihilation determining the critical line $N_c(\epsilon)$. A good fit to the numerically computed edge of this conformal window to three loops for example is

$$N_c(\epsilon) = 2.65 - 4\epsilon^{1/2} + 0.75\epsilon + \mathcal{O}(\epsilon^{3/2}). \tag{3}$$

The mixed-stable fixed point that annihilated the Wilson-Fisher fixed point, while inevitably becoming $\mathcal{O}(1)$ right at $N = 2$, is actually weakly coupled near to and left of the right edge of the conformal window at $N = 2.653$

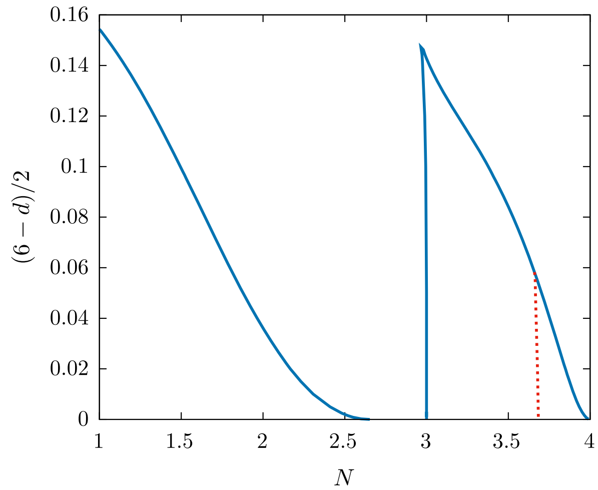

FIG. 1. Conformal regions (below the full lines) at which the theory (1) in tensor representation for $N > 1$ has an IR-stable fixed point, computed using two-loop beta functions for g_1 and g_2. At the boundaries of both regions the IR-stable fixed point collides with another fixed point and becomes complex. At the nearly vertical dashed line which terminates at $N = 3.684$ and $\epsilon = (6-d)/2 = 0$ the IR-stable and mixed-stable fixed points exchange stability. (For further discussion see the text.)

(Fig. 1). It is another example of a Banks-Zaks fixed point [13], and the above expansion may be understood as an extrapolation out of the strictly perturbative regime near the end point $\epsilon = 0$ and $N = 2.653$, where the line $N_c(\epsilon)$ is determined by the collision of two weakly coupled fixed points: one being $\mathcal{O}(\epsilon)$ (IR-stable Wilson-Fisher) and the other being $\mathcal{O}(2.653 - N)$ (mixed-stable Banks-Zaks).

The second conformal window for $2.999 < N < 4$ in Fig. 1 also reduces to zero width with the increase of ϵ, but in a more elaborate way, as there are several exchanges of stability between the existing fixed points before the IR-stable fixed point finally disappears above a certain value of ϵ. These intricacies notwithstanding, the final disappearance, or rather complexification of the IR fixed point, is again due to a collision with an $\mathcal{O}(1)$ fixed point of mixed stability introduced by higher order terms. The physically relevant case of $N = 3$ is particularly involved. We first find that at $N = 3$ both the vector components ϕ_i and the tensor components z_a acquire exactly the same anomalous dimensions, up to the computed fourth order in ϵ. This indicates an emergence of a larger symmetry between these two representations of $O(3)$, which we then show to be $U(3)$: the fixed point values of the couplings g_1 and g_2 are precisely such that the two cubic interaction terms in the theory (1) at the fixed point can be rewritten compactly as a single trace of the third power of a traceless, Hermitean, three-dimensional matrix [Eq. (26)]. The IR-stable $\mathcal{O}(\epsilon)$ fixed point at $N = 3$ possesses therefore a larger emergent $U(3)$ symmetry, and the critical exponents reduce to those already computed in [7] for the matrix $U(N)$ model. Increasing the value of ϵ, however, we find this fixed point first to exchange stability with another nearby $\mathcal{O}(\epsilon)$ fixed point with mixed stability in the g_1-g_2 plane, before that other fixed point, now IR-stable, collides with an $\mathcal{O}(1)$ fixed point at some critical value of ϵ and annihilates. Again, near the right corner of the conformal window we find that the collision that determines the boundary is between the Wilson-Fisher $\mathcal{O}(\epsilon)$ fixed point and the $\mathcal{O}(4 - N)$ Banks-Zaks-like fixed point. This perturbative region, however, is not smoothly connected to the region that contains $N = 3$, since there is an exchange of stability between a pair of fixed points around $N = 3.684$. Nevertheless, the mixed-stable fixed point that annihilates the Wilson-Fisher $\mathcal{O}(\epsilon)$ fixed point for $2.999 < N < 3.684$ is also a Banks-Zaks, albeit weakly coupled at a slightly larger value of N of 4.0057.

The paper is further organized as follows. In the next section we describe the four-loop renormalization group calculation and give the values of the critical exponents, with the emphasis on $N = 3$. In Sec. III we demonstrate the emergence of the $U(3)$ symmetry at the IR-stable $\mathcal{O}(\epsilon)$ fixed point. In Sec. IV we discuss the evolution of conformal windows with parameters ϵ and N, and show how it leads to finite conformal regions in the N-ϵ plane. We summarize and comment further on our findings in the concluding section. Full four-loop expressions for the beta functions and the exponents at a general N are given in the appendices.

II. RENORMALIZATION

We turn now to the technical details of the evaluation of the renormalization group functions for (1). The procedure we followed was similar to that used for the scalar channel of the $O(N)$-symmetric case at four loops given in [7] and summarize it in this context. As there are two couplings and fields there are a sizeable number of Feynman diagrams to evaluate for the 2- and 3-point functions which could only be handled by an automatic computation. All the graphs to four loops for the two 2-point functions are generated using the FORTRAN based package QGRAF [14], which is then mapped into the syntax of the symbolic manipulation language FORM [15,16], which is the working language for the automatic evaluation. At this stage $O(N)$ group indices are added and the group relation

$$\Lambda^a_{ij}\Lambda^a_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl} \tag{4}$$

is implemented within a FORM module. The divergent part of each individual graph in dimensional regularization in $d = 6 - 2\epsilon$ dimensions is determined by using the Laporta algorithm [17]. We used both versions of the

REDUZE implementation [18,19] of the algorithm for the present computation. In general this systematically uses integration by parts to reduce all Feynman integrals of a Green's function down to a small base or master set of integrals. The values of these are determined directly by nonintegration by parts methods. For (1) the set of four loop master integrals for six-dimensional 2-point functions were given in [7] having been constructed using the Tarasov method [20,21] from the corresponding four-dimensional masters given in [22]. In terms of graphs computed for the ϕ_i 2-point function, we evaluated 1, 5, 48, and 637 graphs at the respective loop orders from one to four. For the z_a field the corresponding numbers were 2, 7, 60, and 723.

This summarizes the process for the wave function renormalization. Clearly there is a large number of graphs to compute but this is substantially smaller than the respective number for the coupling constant renormalization. However, to determine the two beta functions to four loops we exploited a novel feature of renormalizing this scalar field theory in six dimensions. As there are no 4-point and higher vertices in (1) all the graphs contributing to the 3-point functions can be simply generated by making an insertion on each propagator in the 2-point function graphs [7]. In parallel to this the divergent part of each vertex function of (1) in dimensional regularization can be extracted by nullifying the external momentum of one of the legs. As the critical dimension of (1) is six then a propagator $1/(k^2)^2$ in a 3-point function, where k is a loop momentum, does *not* introduce any infrared divergences unlike the case if the critical dimension was four. This means that we do not have to separately generate a substantial number of graphs using QGRAF for the vertex functions. Instead in the computation of the 2-point functions we merely make the replacements [7]

$$\frac{\delta_{ij}}{k^2} \to \frac{\delta_{ij}}{k^2} + \frac{g_1 \Lambda^{a_e}_{ij}}{(k^2)^2}$$
$$\frac{\Lambda^a_{ik}\Lambda^b_{kj}}{k^2} \to \frac{\Lambda^a_{ik}\Lambda^b_{kj}}{k^2} + \frac{g_2}{(k^2)^2}\Lambda^a_{ik}\Lambda^{b_e}_{kl}\Lambda^b_{lj} \tag{5}$$

respectively for the ϕ_i and z_a propagators where a_e and b_e are the indices of the external leg. In the second equation, the Λ matrices correspond to the matrices from the vertices at either end of the propagator. They are included so that internally one can distinguish which vertex is being inserted in the FORM identification. This algebraic expansion is straightforward to implement within the program in the 2-point function. It is truncated in such a way that no more than one term deriving from the $1/(k^2)^2$ correction is retained. This ensures that only the graphs corresponding to the vertex functions themselves are selected. Both replacements are implemented in each 2-point function with the g_1 coupling constant being deduced from the ϕ_i 2-point function. The other coupling is renormalized via the z_a 2-point function. The advantage of proceeding in this fashion, aside from not having to have a separate vertex function computation, is that the Laporta algorithm can be applied to both wave function and coupling constant renormalization in the *same* full automatic computation.

Moreover as we are interested in the renormalization of the masses of the two fields we can simply implement a replacement similar to (5) for this which is

$$\frac{1}{k^2} \to \frac{1}{k^2} + \frac{\mu_\phi^2}{(k^2)^2}$$
$$\frac{1}{k^2} \to \frac{1}{k^2} + \frac{\mu_z^2}{(k^2)^2} \tag{6}$$

for the ϕ_i and z_a propagators respectively [7]. The replacement is truncated at $O(\mu_i^2)$ for $i = \phi$ or z. Here the quantities μ_ϕ^2 and μ_z^2 are not masses but merely symbols which tag the origin of the additional term. These are necessary in order to extract the renormalization constants of the 2×2 mass mixing matrix of (1). A similar procedure was used in [7]. The final part of the automatic evaluation of the n-point functions is the extraction of the renormalization constants. This is also achieved automatically by the method discussed in [23]. For each of the Green's functions leading to the wave function, coupling constant and mass renormalization one uses the Lagrangian with bare parameters such as g_i and the masses. The respective counterterms are introduced by replacing the bare ones with the corresponding renormalized variables. The overall divergence remaining at each loop order after the lower loop counterterm values have been determined is absorbed into the respective counterterm for that particular n-point function. As we have implemented a zero momentum insertion for the coupling constant renormalization, we determine the renormalization group functions in the $\overline{\rm MS}$ scheme.

Given the number of Feynman graphs we had to compute for (1) together with the presence of an $O(N)$ symmetry the full expression for each of the renormalization group functions is cumbersome and are given in the Appendix, and provided electronically in the Supplemental Material [24]. To give a flavor of this, we record the expressions for the specific case of $N = 3$. First, the two anomalous dimensions are

$$
\begin{aligned}
\gamma_\phi(g_i) = {} & \frac{5}{9}g_1^2 + 35[-17g_1^2 + 24g_1g_2 - 11g_2^2]\frac{g_1^2}{972} \\
& - 5[29808\zeta_3 g_1^4 - 66638g_1^4 + 31542g_1^3g_2 - 9072\zeta_3 g_1^2g_2^2 - 12383g_1^2g_2^2 + 15288g_1g_2^3 - 13587g_2^4]\frac{g_1^2}{52488} \\
& + 5[216635472\zeta_3 g_1^6 + 12060576\zeta_4 g_1^6 - 479623680\zeta_5 g_1^6 + 185271707g_1^6 - 192961440\zeta_3 g_1^5g_2 \\
& \quad + 47029248\zeta_4 g_1^5g_2 + 333931878g_1^5g_2 + 152962992\zeta_3 g_1^4g_2^2 + 84832272\zeta_4 g_1^4g_2^2 - 209018880\zeta_5 g_1^4g_2^2 \\
& \quad - 106561007g_1^4g_2^2 + 166779648\zeta_3 g_1^3g_2^3 + 82791072\zeta_4 g_1^3g_2^3 - 179951100g_1^3g_2^3 + 20112624\zeta_3 g_1^2g_2^4 \\
& \quad + 34128864\zeta_4 g_1^2g_2^4 - 300464640\zeta_5 g_1^2g_2^4 + 238593145g_1^2g_2^4 - 37612512\zeta_3 g_1g_2^5 - 20085408\zeta_4 g_1g_2^5 \\
& \quad + 108293094g_1g_2^5 + 12183696\zeta_3 g_2^6 + 3347568\zeta_4 g_2^6 - 68577061g_2^6]\frac{g_1^2}{34012224} + \mathcal{O}(g_i^{10}) \qquad (7)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_z(g_i) = {} & [3g_1^2 + 7g_2^2]\frac{1}{18} + 7[-42g_1^4 + 72g_1^3g_2 - 33g_1^2g_2^2 - 113g_2^4]\frac{1}{972} \\
& - [178848\zeta_3 g_1^6 - 473475g_1^6 + 404712g_1^5g_2 - 136080\zeta_3 g_1^4g_2^2 - 130473g_1^4g_2^2 + 386568g_1^3g_2^3 - 176001g_1^2g_2^4 \\
& \quad + 371952\zeta_3 g_2^6 - 1217531g_2^6]\frac{1}{209952} \\
& + [119362896\zeta_3 g_1^8 + 6030288\zeta_4 g_1^8 - 239811840\zeta_5 g_1^8 + 60083890g_1^8 - 175624848\zeta_3 g_1^7g_2 + 1469664\zeta_4 g_1^7g_2 \\
& \quad + 325111584g_1^7g_2 + 135218160\zeta_3 g_1^6g_2^2 + 99610560\zeta_4 g_1^6g_2^2 - 209018880\zeta_5 g_1^6g_2^2 - 97863857g_1^6g_2^2 \\
& \quad + 320005728\zeta_3 g_1^5g_2^3 + 120022560\zeta_4 g_1^5g_2^3 - 327324144g_1^5g_2^3 + 105561792\zeta_3 g_1^4g_2^4 + 55275696\zeta_4 g_1^4g_2^4 \\
& \quad - 600929280\zeta_5 g_1^4g_2^4 + 389081833g_1^4g_2^4 - 102359376\zeta_3 g_1^3g_2^5 + 229171488g_1^3g_2^5 + 39018672\zeta_3 g_1^2g_2^6 \\
& \quad - 13390272\zeta_4 g_1^2g_2^6 - 91542339g_1^2g_2^6 + 419008464\zeta_3 g_2^8 + 15023232\zeta_4 g_2^8 - 598752000\zeta_5 g_2^8 \\
& \quad - 55379079g_2^8]\frac{1}{11337408} + \mathcal{O}(g_i^{10}) \qquad (8)
\end{aligned}
$$

where ζ_z is the Riemann zeta function and g_i in the order symbol represents either coupling constant. The two (UV) beta functions are

$$
\begin{aligned}
\beta_1(g_i) = {} & -\frac{1}{2}\epsilon g_1 - [-17g_1^2 + 42g_1g_2 - 7g_2^2]\frac{g_1}{36} + [-6617g_1^4 + 3591g_1^3g_2 - 1988g_1^2g_2^2 + 2625g_1g_2^3 - 791g_2^4]\frac{g_1}{1944} \\
& - [4380480\zeta_3 g_1^6 + 2469685g_1^6 - 1959552\zeta_3 g_1^5g_2 + 9953370g_1^5g_2 + 9897552\zeta_3 g_1^4g_2^2 - 3206105g_1^4g_2^2 \\
& \quad + 10723104\zeta_3 g_1^3g_2^3 - 3644172g_1^3g_2^3 + 2830464\zeta_3 g_1^2g_2^4 + 4790919g_1^2g_2^4 - 2231712\zeta_3 g_1g_2^5 \\
& \quad + 4461618g_1g_2^5 + 371952\zeta_3 g_2^6 - 1217531g_2^6]\frac{g_1}{419904} \\
& + [-13032226080\zeta_3 g_1^8 - 571011120\zeta_4 g_1^8 + 5888920320\zeta_5 g_1^8 - 2545129585g_1^8 - 6517823760\zeta_3 g_1^7g_2 \\
& \quad + 2609143488\zeta_4 g_1^7g_2 + 18674530560\zeta_5 g_1^7g_2 - 14014594272g_1^7g_2 - 41976125856\zeta_3 g_1^6g_2^2 - 1654270128\zeta_4 g_1^6g_2^2 \\
& \quad + 41686202880\zeta_5 g_1^6g_2^2 + 1287653416g_1^6g_2^2 + 55393105824\zeta_3 g_1^5g_2^3 + 2510431056\zeta_4 g_1^5g_2^3 - 53417387520\zeta_5 g_1^5g_2^3 \\
& \quad - 6038544078g_1^5g_2^3 - 62879166000\zeta_3 g_1^4g_2^4 - 739159344\zeta_4 g_1^4g_2^4 + 63110638080\zeta_5 g_1^4g_2^4 + 8724733738g_1^4g_2^4 \\
& \quad + 26658589104\zeta_3 g_1^3g_2^5 - 2891318976\zeta_4 g_1^3g_2^5 - 19379969280\zeta_5 g_1^3g_2^5 - 8431548300g_1^3g_2^5 - 32862267648\zeta_3 g_1^2g_2^6 \\
& \quad - 1099390320\zeta_4 g_1^2g_2^6 + 30451438080\zeta_5 g_1^2g_2^6 + 3554373284g_1^2g_2^6 - 7103920320\zeta_3 g_1g_2^7 + 256456368\zeta_4 g_1g_2^7 \\
& \quad + 10777536000\zeta_5 g_1g_2^7 - 252392070g_1g_2^7 + 1257025392\zeta_3 g_2^8 + 45069696\zeta_4 g_2^8 - 1796256000\zeta_5 g_2^8 \\
& \quad - 166137237g_2^8]\frac{g_1}{68024448} + \mathcal{O}(g_i^{11}) \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_2(g_i) = & -\frac{1}{2}\epsilon g_2 - [6g_1^3 - 3g_1^2 g_2 - 13g_2^3]\frac{1}{12} + [549g_1^5 - 537g_1^4 g_2 + 747g_1^3 g_2^2 - 420g_1^2 g_2^3 - 2951g_2^5]\frac{1}{648} \\
& - [-629856\zeta_3 g_1^7 + 1501722g_1^7 + 1648512\zeta_3 g_1^6 g_2 - 469209g_1^6 g_2 + 2822688\zeta_3 g_1^5 g_2^2 - 1281780g_1^5 g_2^2 \\
& \quad + 1006992\zeta_3 g_1^4 g_2^3 + 1066437g_1^4 g_2^3 - 637632\zeta_3 g_1^3 g_2^4 + 1008810g_1^3 g_2^4 - 563883g_1^2 g_2^5 + 2549232\zeta_3 g_2^7 \\
& \quad - 1449209g_2^7]\frac{1}{139968} \\
& + [-297429408\zeta_3 g_1^9 + 15081552\zeta_4 g_1^9 + 876199680\zeta_5 g_1^9 - 478212348g_1^9 - 1099291824\zeta_3 g_1^8 g_2 \\
& \quad - 142615728\zeta_4 g_1^8 g_2 + 1323630720\zeta_5 g_1^8 g_2 - 59500523g_1^8 g_2 + 3065116464\zeta_3 g_1^7 g_2^2 - 21205152\zeta_4 g_1^7 g_2^2 \\
& \quad - 3532792320\zeta_5 g_1^7 g_2^2 + 488212362g_1^7 g_2^2 - 6242637600\zeta_3 g_1^6 g_2^3 + 289477152\zeta_4 g_1^6 g_2^3 + 6083942400\zeta_5 g_1^6 g_2^3 \\
& \quad + 780950008g_1^6 g_2^3 + 4325835456\zeta_3 g_1^5 g_2^4 + 182693232\zeta_4 g_1^5 g_2^4 - 2768567040\zeta_5 g_1^5 g_2^4 - 2207956584g_1^5 g_2^4 \\
& \quad - 6776964144\zeta_3 g_1^4 g_2^5 - 38537856\zeta_4 g_1^4 g_2^5 + 5538067200\zeta_5 g_1^4 g_2^5 + 569390506g_1^4 g_2^5 - 1272647376\zeta_3 g_1^3 g_2^6 \\
& \quad + 254531808\zeta_4 g_1^3 g_2^6 + 1539648000\zeta_5 g_1^3 g_2^6 - 261985350g_1^3 g_2^6 + 273150144\zeta_3 g_1^2 g_2^7 - 101570112\zeta_4 g_1^2 g_2^7 \\
& \quad - 62132436g_1^2 g_2^7 - 4547016000\zeta_3 g_2^9 - 298260144\zeta_4 g_2^9 + 3467318400\zeta_5 g_2^9 + 710525229g_2^9]\frac{1}{7558272} + \mathcal{O}(g_i^{11}).
\end{aligned}
\tag{10}
$$

For the renormalization of the masses we define the mixing matrix of anomalous dimensions by $\gamma_{ij}(g_k)$ where the masses of ϕ_i and z_a are labeled respectively by 1 and 2. The presence of the two tags μ_i^2 allows one to extract the four respective renormalization constants by the same method as [7] which results in

$$
\begin{aligned}
-\gamma_{11}(g_i) = & -\frac{10}{9}g_1^2 + 25[-23g_1^2 - 21g_1 g_2 + 7g_2^2]\frac{g_1^2}{486} \\
& - 35[51840\zeta_3 g_1^4 + 55841g_1^4 - 15552\zeta_3 g_1^3 g_2 - 28374g_1^3 g_2 + 5184\zeta_3 g_1^2 g_2^2 + 42923g_1^2 g_2^2 - 2418g_1 g_2^3 + 3672g_2^4]\frac{g_1^2}{104976} \\
& + 5[-260546544\zeta_3 g_1^6 - 263046528\zeta_4 g_1^6 + 345487680\zeta_5 g_1^6 - 1105616335g_1^6 - 1164844800\zeta_3 g_1^5 g_2 \\
& \quad + 459025056\zeta_4 g_1^5 g_2 - 88179840\zeta_5 g_1^5 g_2 - 258947892g_1^5 g_2 - 1012943232\zeta_3 g_1^4 g_2^2 - 332797248\zeta_4 g_1^4 g_2^2 \\
& \quad + 1422308160\zeta_5 g_1^4 g_2^2 - 74040764g_1^4 g_2^2 - 2119745376\zeta_3 g_1^3 g_2^3 - 54377568\zeta_4 g_1^3 g_2^3 + 1685214720\zeta_5 g_1^3 g_2^3 \\
& \quad - 117454092g_1^3 g_2^3 - 1103962608\zeta_3 g_1^2 g_2^4 - 55030752\zeta_4 g_1^2 g_2^4 + 1913829120\zeta_5 g_1^2 g_2^4 - 776455043g_1^2 g_2^4 \\
& \quad + 43382304\zeta_3 g_1 g_2^5 + 40170816\zeta_4 g_1 g_2^5 - 119868840g_1 g_2^5 - 10977120\zeta_3 g_2^6 - 6695136\zeta_4 g_2^6 \\
& \quad + 68062526g_2^6]\frac{g_1^2}{34012224} + \mathcal{O}(g_i^{10})
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
-\gamma_{21}(g_i) = & -g_1^2 + [17g_1^2 - 63g_1 g_2 - 21g_2^2]\frac{g_1^2}{18} \\
& - [-23328\zeta_3 g_1^4 + 83771g_1^4 - 52318g_1^3 g_2 - 6048\zeta_3 g_1^2 g_2^2 + 8813g_1^2 g_2^2 - 60480\zeta_3 g_1 g_2^3 + 7462g_1 g_2^3 \\
& \quad + 55776g_2^4]\frac{g_1^2}{3888} \\
& + [-73183392\zeta_3 g_1^6 - 7231680\zeta_4 g_1^6 + 124688160\zeta_5 g_1^6 - 34030172g_1^6 + 120331008\zeta_3 g_1^5 g_2 \\
& \quad - 10859184\zeta_4 g_1^5 g_2 - 158124960\zeta_5 g_1^5 g_2 - 87191811g_1^5 g_2 - 3123792\zeta_3 g_1^4 g_2^2 - 12845952\zeta_4 g_1^4 g_2^2 \\
& \quad + 10886400\zeta_5 g_1^4 g_2^2 - 47057234g_1^4 g_2^2 - 14230944\zeta_3 g_1^3 g_2^3 - 13281408\zeta_4 g_1^3 g_2^3 - 126554400\zeta_5 g_1^3 g_2^3 \\
& \quad + 64341480g_1^3 g_2^3 + 11941776\zeta_3 g_1^2 g_2^4 - 18996768\zeta_4 g_1^2 g_2^4 - 25855200\zeta_5 g_1^2 g_2^4 - 60530316g_1^2 g_2^4 \\
& \quad - 17720640\zeta_3 g_1 g_2^5 + 31652208\zeta_4 g_1 g_2^5 - 67495680\zeta_5 g_1 g_2^5 - 58803297g_1 g_2^5 + 3592512\zeta_3 g_2^6 \\
& \quad + 2449440\zeta_4 g_2^6 - 1753038g_2^6]\frac{g_1^2}{629856} + \mathcal{O}(g_i^{10})
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
-\gamma_{12}(g_i) = {} & -\frac{5}{3}g_1^2 + 5[2g_1^2 - 189g_1g_2 - 14g_2^2]\frac{g_1^2}{162} \\
& - 5[-88128\zeta_3 g_1^4 + 244250g_1^4 - 81648\zeta_3 g_1^3 g_2 - 43974g_1^3g_2 + 45360\zeta_3 g_1^2g_2^2 + 74669g_1^2g_2^2 - 27216\zeta_3 g_1 g_2^3 \\
& \quad -139482g_1g_2^3 + 27216\zeta_3 g_2^4 + 77273g_2^4]\frac{g_1^2}{34992} \\
& + 5[-209158200\zeta_3 g_1^6 + 17128584\zeta_4 g_1^6 + 336506400\zeta_5 g_1^6 - 204713141g_1^6 + 309799728\zeta_3 g_1^5 g_2 \\
& \quad -11512368\zeta_4 g_1^5g_2 - 430284960\zeta_5 g_1^5 g_2 - 263394642g_1^5g_2 - 27025488\zeta_3 g_1^4g_2^2 - 114225552\zeta_4 g_1^4 g_2^2 \\
& \quad +6531840\zeta_5 g_1^4g_2^2 + 14773990g_1^4g_2^2 - 23024736\zeta_3 g_1^3g_2^3 - 20575296\zeta_4 g_1^3g_2^3 - 507034080\zeta_5 g_1^3g_2^3 \\
& \quad +220078110g_1^3g_2^3 + 86388120\zeta_3 g_1^2g_2^4 - 31230360\zeta_4 g_1^2g_2^4 - 43273440\zeta_5 g_1^2g_2^4 - 186362841g_1^2g_2^4 \\
& \quad +52390800\zeta_3 g_1g_2^5 + 19350576\zeta_4 g_1 g_2^5 - 119206080\zeta_5 g_1g_2^5 - 166599216g_1g_2^5 + 41504400\zeta_3 g_2^6 \\
& \quad -16819488\zeta_4 g_2^6 + 29393280\zeta_5 g_2^6 - 18765152g_2^6]\frac{g_1^2}{5668704} + \mathcal{O}(g_i^{10})
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
-\gamma_{22}(g_i) = {} & -[-3g_1^2 + 35g_2^2]\frac{1}{18} + [-633g_1^4 - 315g_1^3g_2 + 546g_1^2g_2^2 + 959g_2^4]\frac{1}{486} \\
& - [1228608\zeta_3 g_1^6 - 82353g_1^6 - 1143072\zeta_3 g_1^5g_2 - 69300g_1^5g_2 + 2313360\zeta_3 g_1^4g_2^2 + 3746673g_1^4g_2^2 \\
& \quad +489888\zeta_3 g_1^3g_2^3 - 2310084g_1^3g_2^3 + 313131g_1^2g_2^4 + 2004912\zeta_3 g_2^6 + 9924901g_2^6]\frac{1}{209952} \\
& + [186594192\zeta_3 g_1^8 - 75011184\zeta_4 g_1^8 - 359134560\zeta_5 g_1^8 - 323438987g_1^8 - 724653216\zeta_3 g_1^7 g_2 \\
& \quad +531773424\zeta_4 g_1^7g_2 - 401708160\zeta_5 g_1^7g_2 - 594763890g_1^7g_2 - 367325280\zeta_3 g_1^6g_2^2 - 614646144\zeta_4 g_1^6g_2^2 \\
& \quad +736464960\zeta_5 g_1^6g_2^2 + 427325500g_1^6g_2^2 - 3795924384\zeta_3 g_1^5g_2^3 - 194975424\zeta_4 g_1^5g_2^3 + 2478833280\zeta_5 g_1^5g_2^3 \\
& \quad +234595536g_1^5g_2^3 - 2130450336\zeta_3 g_1^4g_2^4 - 305526816\zeta_4 g_1^4g_2^4 + 4778040960\zeta_5 g_1^4g_2^4 - 3088527995g_1^4g_2^4 \\
& \quad -328769280\zeta_3 g_1^3g_2^5 + 202078800\zeta_4 g_1^3g_2^5 + 264539520\zeta_5 g_1^3g_2^5 - 108887142g_1^3g_2^5 + 351975456\zeta_3 g_1^2g_2^6 \\
& \quad -79525152\zeta_4 g_1^2g_2^6 + 546998382g_1^2g_2^6 - 3258671472\zeta_3 g_2^8 - 753366096\zeta_4 g_2^8 + 6203342880\zeta_5 g_2^8 \\
& \quad -1091649300g_2^8]\frac{1}{11337408} + \mathcal{O}(g_i^{10})
\end{aligned}
\tag{14}
$$

for $N = 3$. In accordance with our definition of $\epsilon = (6 - d)/2$, further definitions of anomalous dimensions, beta functions, and the mass matrix in Eqs. (7)–(14) also differ from standard definitions [3] by an overall factor of two, to agree with the ones used in [7]. We will provide the standard values of the exponents below.

One aspect of a high loop order computation such as the one undertaken here is ensuring that it is correct. Given the way we enacted the renormalization automatically already provides one internal check. This is because the nonsimple poles in ϵ of the renormalization constants are not independent. Instead they are precisely determined by all the poles in the renormalization constant from the lower loop order. We note therefore that this internal renormalization group consistency check was satisfied in all the renormalization constants computed which gives us confidence in the correctness of our calculation. Another hidden check emerges from an observation to do with the fixed-point structure of the $N = 3$ case. By solving $\beta_i(g_j) = 0$ for the critical couplings it transpires that there is a solution where $g_1^* = -g_2^*$. This was observed originally in the one-loop calculation [8] and as we show in the next section corresponds to an emergent $U(3)$ symmetry, at which the anomalous dimensions of both fields become identical. In [7] the renormalization group functions for a six-dimensional cubic scalar field theory with $U(N_c)$ symmetry were computed directly. Specifying those results to $U(3)$ and evaluating the corresponding critical exponents we find exact agreement with those of (1) at the emergent $U(3)$ fixed point, which provides a solid check on our $O(N)$ tensor theory. In particular the (standard) anomalous dimensions $\eta_\phi = 2\gamma_\phi(g_i^*)$ and $\eta_z = 2\gamma_z(g_i^*)$ at the $U(3)$-symmetric fixed point are

$$\eta_\phi = \eta_z = \frac{10}{33}\epsilon + \frac{1000}{11979}\epsilon^2 + 10[104544\zeta_3 + 220057]\frac{\epsilon^3}{4348377} + 10[5936914368\zeta_3 + 170772624\zeta_4 - 5025952800\zeta_5 - 192710239]\frac{\epsilon^4}{4735382553} + \mathcal{O}(\epsilon^5) \quad (15)$$

where the one-loop term is in agreement with [8] and the two-loop term is in agreement with [25]. The fixed-point values of the couplings are

$$(g_1^*)^2 = (g_2^*)^2 = \frac{3}{11}\epsilon + \frac{1301}{3993}\epsilon^2 + [1672704\zeta_3 + 3301487]\frac{\epsilon^3}{5797836} + [107879214960\zeta_3 + 2732361984\zeta_4 - 76753591200\zeta_5 - 6440858957]\frac{\epsilon^4}{9470765106} + \mathcal{O}(\epsilon^5). \quad (16)$$

Using these values then the higher order corrections to the universal exponents corresponding to the two IR relevant directions out of the critical surface are deduced from the eigenvalues of the mass mixing matrix γ_{ij} at the fixed point. We find

$$\theta_1 = 2 + \frac{2}{33}\epsilon + \frac{6250}{11979}\epsilon^2 + 5[209088\zeta_3 + 285869]\frac{\epsilon^3}{1449459} + 5[3466069200\zeta_3 + 2049271488\zeta_4 + 5535604800\zeta_5 + 5263987741]\frac{\epsilon^4}{9470765106} + \mathcal{O}(\epsilon^5),$$

$$\theta_2 = 2 + \frac{50}{33}\epsilon + \frac{3830}{3993}\epsilon^2 + 5[2718144\zeta_3 + 5472707]\frac{\epsilon^3}{4348377} + 5[132436914192\zeta_3 + 8880176448\zeta_4 - 117320322240\zeta_5 + 26569626097]\frac{\epsilon^4}{9470765106} + \mathcal{O}(\epsilon^5), \quad (17)$$

where θ_i are the two eigenvalues of the matrix $2(\delta_{ij} + \gamma_{ij}(g_i^*))$. Recall that our $\epsilon = (6-d)/2$ yields the one-loop term in agreement with [8]. The eigenvector corresponding to the larger eigenvalue (θ_2) is also at every order a symmetric combination of the two masses, just as one would expect for the $U(3)$-symmetric theory. This is then yet another check on our calculation.

III. EMERGENCE OF $U(3)$ SYMMETRY

Let us next show that on the line $g_1 + g_2 = 0$ the last two terms in (1) for $N = 3$ can be rewritten in a manifestly $U(3)$ invariant way. Consider $\mathrm{Tr}(M^3)$ where M is a three-dimensional, traceless, Hermitean matrix. It can be expanded in terms of Gell-Mann matrices as

$$M = z_a\Lambda^a + \phi_i S^i, \quad (18)$$

where Λ^a are the five real, and S^i the three imaginary Gell-Mann matrices, with matrix elements

$$(S^i)_{jk} = i\epsilon_{ijk}, \quad (19)$$

with ϵ_{ijk} as the fully antisymmetric tensor. We make no distinction between upper and lower indices, and use them both solely for notational clarity. Evidently, the three S^i are the generators of $SO(3)$ in the adjoint representation, and the five Λ^a are a second-rank tensorial representation of the same group. Taken together the eight matrices close the algebra of $SU(3)$, as is well known.

Then,

$$\mathrm{Tr}(M^3) = z_a z_b z_c \mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c) + 3z_a\phi_i\phi_j\mathrm{Tr}(\Lambda^a S^i S^j) + 3z_a z_b\phi_k\mathrm{Tr}(\Lambda^a\Lambda^b S^k) + \phi_i\phi_j\phi_k\mathrm{Tr}(S^i S^j S^k). \quad (20)$$

The last two terms are identically zero. Consider first the last term:

$$\phi_i\phi_j\phi_k\mathrm{Tr}(S^i S^j S^k) = -i\phi_i\phi_j\phi_k\epsilon_{inm}\epsilon_{jml}\epsilon_{kln}. \quad (21)$$

Since $\epsilon_{inm}\epsilon_{jml} = \delta_{il}\delta_{nj} - \delta_{ij}\delta_{ln}$ the last line reduces to

$$-i\phi_i\phi_j\phi_k(\epsilon_{kij} - \delta_{ij}\epsilon_{kll}) = 0. \quad (22)$$

The third term in Eq. (20) is proportional to

$$z_a z_b\phi_k\mathrm{Tr}(\Lambda^a\Lambda^b S^k) = \frac{i}{2}z_a z_b\phi_k(\Lambda^a_{ij}\Lambda^b_{jl} + \Lambda^a_{lj}\Lambda^b_{ji})\epsilon_{kli}, \quad (23)$$

where we used the symmetry property of the Λ matrices in the second term. By exchanging the indices i and l in the second term we can further rewrite this as

$$\frac{i}{2} z_a z_b \phi_k \Lambda^a_{ij} \Lambda^b_{jl} (\epsilon_{kli} + \epsilon_{kil}) = 0. \tag{24}$$

Finally the second term in Eq. (20) can be simplified as

$$3 z_a \phi_i \phi_j \mathrm{Tr}(\Lambda^a S^i S^j) = -3 z_a \phi_i \phi_j \Lambda^a_{ij}, \tag{25}$$

using the tracelessness of matrices Λ^a.

Altogether, the field theory (1) for $N = 3$ and when $g_1 = -g_2$ can be written simply as

$$\mathcal{L} = \frac{1}{4} \mathrm{Tr}(\partial_\mu M)^2 + \frac{g_2}{6} \mathrm{Tr}(M^3), \tag{26}$$

which is invariant under a global transformation

$$M \to U M U^\dagger, \tag{27}$$

where U is an arbitrary three-dimensional unitary matrix.

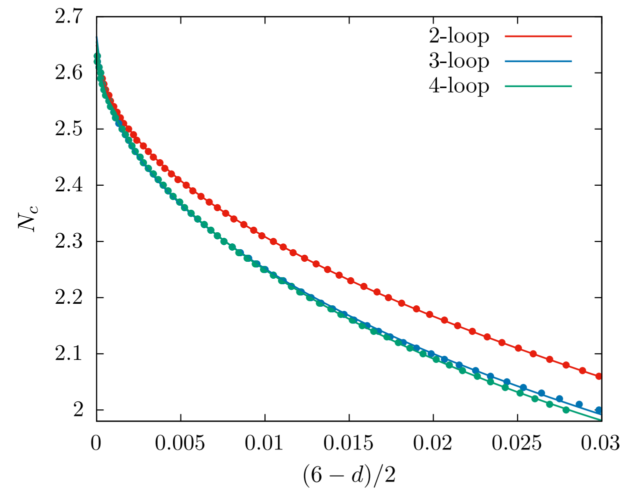

FIG. 3. Comparison of $N_c(\epsilon)$ extracted from two-loop, three-loop, and four-loop β functions, for $2 < N < 2.6535$.

IV. EVOLUTION OF CONFORMAL WINDOWS

One-loop beta functions display IR-stable fixed points when N falls within certain intervals, as discussed in [8]. Let us briefly recapitulate these results, as they will be now confirmed and extended. First, besides the Gaussian fixed point at $g_1 = g_2 = 0$, for any $N < 4$ there is a fixed point at $g_1 = 0$ and $g_2 \sim \epsilon$ at which the fields ϕ_i and z_a decouple, first identified by Priest and Lubensky [26]. For $1 < N < 2.653$ there is an IR-stable fixed point at finite g_1 and g_2 (the fixed point "B" in Fig. 2 in [8]). When $N \to 2.653$ the latter fixed point runs away to infinity, in this way terminating the first conformal window within the one-loop approximation. Increasing N further, at $N = 2.999$ two new fixed points at finite g_1 and g_2 appear, one of which is IR-stable, and the other one is of mixed stability, just like the still present Priest-Lubensky fixed point (Fig. 3 in [8]). For $N = 3$ the IR-stable fixed point is at the $U(3)$-invariant line $g_1 + g_2 = 0$, at which the fields ϕ_i and z_a acquire identical anomalous dimensions. As $N \to 3.684$ the IR-stable fixed point at finite g_1 exchanges stability with the Priest-Lubensky fixed point at $g_1 = 0$, which now becomes IR-stable. Finally, at $N = 4$ the Priest-Lubensky fixed point itself runs away to infinity as well [26].

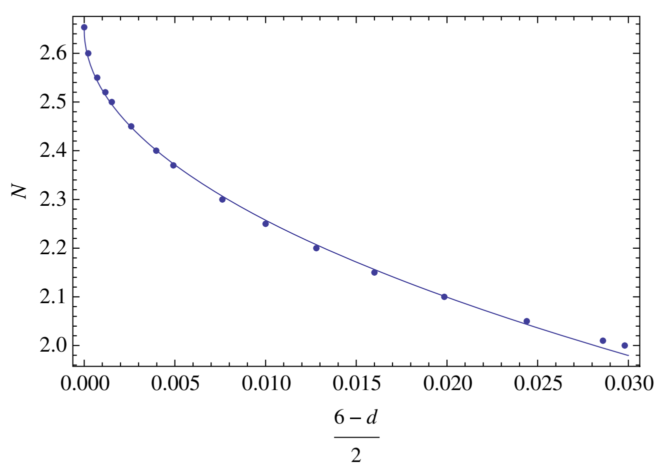

FIG. 2. $N_c(\epsilon)$ extracted from three-loop β functions, for $2 < N < 2.6535$. The fit to points is a Taylor expansion in $\epsilon^{1/2}$, given by Eq. (3) in the Introduction. The Wilson-Fisher IR fixed point exists below the curve.

In sum, to the leading order the beta functions of the theory (1) in tensor representation display two conformal windows: (1) $1 < N < 2.653$, which surrounds the point $N = 2$, and (2) $2.999 < N < 4$, which contains $N = 3$ very close to the left edge of the window, and also exhibits the exchange of stability between two fixed points at $N = 3.684$.

The values of the boundaries of the conformal windows quoted above may be understood as the leading ($\sim\epsilon^0$) approximation to some unknown functions $N_c(\epsilon)$ [6,27,28], which may be expressed as Taylor series in $\epsilon^{1/2}$. Tracking the evolution of the fixed-point structure of the two-loop beta functions, for example, we determine the conformal regions in the N-ϵ plane, within which an IR-stable $\mathcal{O}(\epsilon)$ fixed point in the theory (1) exists (Fig. 1). Adding the third and fourth loop terms turns out to alter this structure only quantitatively, and only slightly (Fig. 2 and Fig. 3). The principal new feature brought by two loops and higher is the appearance of the $\mathcal{O}(1)$ fixed points already at $d = 6$, with mixed stability, which collide with and annihilate the IR-stable $\mathcal{O}(\epsilon)$ fixed points at the upper edges of both conformal regions. This scenario replaces the runaway to infinity of the IR-stable fixed points at $N = 2.653$ and $N = 4$ observed in the one-loop calculation. In fact, near these boundary values the mixed-stable fixed point in question at $d = 6$ becomes weakly coupled, since it is proportional to $\sim(2.653 - N)$ for $N < 2.653$, and to $\sim(4 - N)$ for $N < 4$, so that at small epsilon and near and left of both

$N = 2.653$ and $N = 4$ the boundaries of both conformal regions result from a collision of two weakly coupled fixed points. The forms of both lines at $\epsilon = 0$ are this way determined solely by the universal coefficients of the two-loop beta functions. While for the left conformal window the exact form seems difficult to obtain analytically due to both beta functions being involved, the right conformal window is determined only by the second beta function at $g_1 = 0$, and as $N \to 4$ is given by

$$\epsilon = \frac{441}{320}(4 - N)^2 + \mathcal{O}((4 - N)^3). \tag{28}$$

The present calculation also agrees well with what was previously found for $N = 2$ at two loops [12], which is now a point on the line that goes from (1,0.154) to (2.653,0) in the (N, ϵ) plane. Focusing on this line alone for a moment, we show in Fig. 2 the result of the three-loop calculation, together with a fit to the points provided by the Eq. (3). The two leading terms in this expansion should be universal.

One also notices nearly vertical lines on the left edge and inside the right conformal region in Fig. 1. The first corresponds to the collision of the two $\mathcal{O}(\epsilon)$ fixed points as N approaches it from the right, and the second to the exchange of stability with the Priest-Lubensky fixed point. Both lines are in fact slightly bent towards the left for $\epsilon > 0$. Finally, we should mention that precisely at $N = 3$ with increase of ϵ two things happen in succession: first, at a very low value of ϵ the $U(3)$-symmetric IR-stable fixed point and the second mixed-stable $\mathcal{O}(\epsilon)$ fixed point very close to it exchange stability, so that the $U(3)$-symmetric fixed point acquires mixed stability in the g_1-g_2 plane. Increasing ϵ further at fixed $N = 3$ then ultimately brings the new IR-stable [but $U(3)$-asymmetric] fixed point in collision with another $\mathcal{O}(1)$ fixed point, just as at any other N. The $U(3)$-symmetric mixed-stable fixed point is still present at the tip of the right conformal region, and disappears only at a higher value of ϵ, via collision with another IR-unstable fixed point that lies at the $U(3)$-symmetry line.

V. SUMMARY AND DISCUSSION

In conclusion, we considered a higher-loop perturbative renormalization group for the tensorial representation of the $O(N)\phi^4$ model close to six dimensions. In general, the Wilson-Fisher IR-stable $\mathcal{O}(\epsilon)$ fixed point which was previously found in the one-loop calculation with increase of the parameter $\epsilon = (6 - d)/2$ becomes complex at some value of ϵ which is low enough to render the theory trivial in $d = 5$. This, however, happens only because higher loops in the beta functions generally introduce a nontrivial $\mathcal{O}(1)$ mixed-stable fixed point which exists even in $d = 6$, and which collides with and annihilates [29] the IR fixed points at some finite ϵ. We nevertheless obtained the critical exponents of the IR fixed points to the order ϵ^4. The $\mathcal{O}(1)$ mixed-stable fixed point becomes weakly coupled in the right corners of both conformal windows, where it can be understood as another example of a Banks-Zaks type of fixed point.

It should be emphasized that the disappearance of the IR fixed points in the theory (1) before the physical (integer) dimensions such as $d = 5$ necessarily comes at a price: since at small ϵ a real IR fixed point exists both for $N = 2$ and $N = 3$, it can be moved into the complex plane only through a collision with another fixed point with mixed stability. As a matter of principle, such an additional fixed point may either appear at a finite ϵ, or exist already at $d = 6$. We find generally the latter to be true in two- and higher-loop beta functions for the tensor representation. At $N = 2$ there is no $\mathcal{O}(\epsilon)$ fixed point other than the decoupled Priest-Lubensky fixed point, whereas for $N = 3$ there is another nontrivial fixed point with both $g_1 \sim \epsilon$ and $g_2 \sim \epsilon$ available for a collision, but the IR-stable fixed point has a larger emergent symmetry. In both cases one of the two $\mathcal{O}(\epsilon)$ fixed points which could in principle collide lie at the RG-invariant higher-symmetry lines: the mixed-stable Priest-Lubensky fixed point at the decoupled line $g_1 = 0$ for $N = 2$, and the IR-stable fixed point at the $U(3)$-symmetric $g_1 + g_2 = 0$ line for $N = 3$. This feature prevents their annihilation by any fixed point which is not on the same special line, and the only event in which these points can participate is an exchange of stability [30]. Therefore both for $N = 2$ and $N = 3$ the only option for the ultimate annihilation of the IR-stable fixed point is by some $\mathcal{O}(1)$ fixed point of mixed stability, which is precisely what we find. The existence of the mixed-stable fixed point may of course be simply an artifact of the perturbative expressions for the beta functions. If that is the case, however, we hope that the above discussion makes it clear that the theory (1) will have IR fixed points at $d = 5$, at least within the usual scheme of ϵ expansion. If not, the infrared triviality in $d = 5$ is a consequence of the existence of a mixed-stable nontrivial fixed point in $d = 6$, which is almost equally interesting. The reader should note that the situation in the scalar representation [4] is radically different: there with a decrease of N the IR fixed point always collides with another $\mathcal{O}(\epsilon)$ fixed point, so they either both exist in say $d = 5$ or they both do not. This is similar to what one finds in scalar electrodynamics, for example [28,31,32].

It seems also worth noting that the cases of $N = 2$ and of $N = 3$ are different in another important respect: the mixed-stable fixed point that annihilates the IR-stable fixed point at $N = 2$ is connected continuously to the perturbative Banks-Zaks-like fixed point near $N = 2.653$. Provided the Banks-Zaks fixed point survives the deformation of N from $N = 2.653$ to $N = 2$ it may indeed annihilate the Wilson-Fisher fixed point at some finite critical ϵ. This critical value, however, does not necessarily have to be such that $d_c > 5$. We find that the critical value of ϵ at $N = 2$, however, to be much smaller than unity, independently of the approximation we take. At $N = 3$, on the other hand, the $\mathcal{O}(1)$ mixed-stable fixed point responsible for the annihilation of the IR-stable fixed point is not connected to the Banks-Zaks fixed point near $N = 4$, due to the stability exchange between the Wilson-Fisher and the Priest-Lubensky fixed points around $N = 3.684$. Nevertheless, the fixed point that annihilates the IR-stable fixed point at $N < 3.684$ eventually also becomes weakly coupled, but at a higher value of $N = 4.0057$. As far as we can tell this additional characteristic value of N turns out to be close to $N = 4$ purely accidentally.

If we would ignore the observed loss of full IR stability of the $U(3)$-symmetric fixed point with increase of epsilon, it alone would in fact be more robust, and would become complex due to a collision with an another $U(3)$-symmetric fixed point at a somewhat higher value of $\epsilon = 0.21$, in two loops for example. The fixed point that collides with it, however, does not appear to be of Banks-Zaks variety, and its nonperturbative existence may be questioned. Altogether, it is tempting to speculate that the Wilson-Fisher fixed point at $N = 3$ has a higher chance of surviving in $d = 5$ than its $N = 2$ cousin.

In view of the above uncertainties it would obviously be interesting to apply nonperturbative methods to the model (1), such as recently developed conformal bootstrap [33]. Such a study directly in integer dimension has been already performed for the scalar representation [34]. Extending functional renormalization group to the tensorial representation (1) is also desirable. Such a calculation was also done previously on the scalar representation [35], as well as on the tensorial representation, but only for the special case $N = 2$ [12]. In the latter case the functional renormalization group agreed with the two-loop calculation on the collision of the $\mathcal{O}(\epsilon)$ IR and the $\mathcal{O}(1)$ UV fixed point at a small critical value of ϵ.

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APPENDIX: $O(N)$ RENORMALIZATION GROUP FUNCTIONS

In this appendix we record the full expressions for the various renormalization group functions for the $O(N)$ theory. These are more involved compared with those of the Hubbard-Stratonovich decomposition of [4]. When compared to the specific expression for $N = 3$ in the main text, the couplings g_i in the appendix should be understood as ig_i in the theory (1).

The anomalous dimensions of the two fields are

$$
\begin{aligned}
\gamma_\phi(g_i) = {} & \left[-\frac{1}{6}g_1^2 - \frac{1}{6}Ng_1^2 + \frac{1}{3N}g_1^2\right] \\
& + \left[-\frac{4}{3N}g_1^3 g_2 - \frac{8}{9}g_1^3 g_2 - \frac{22}{27N^2}g_1^2 g_2^2 - \frac{11}{72}Ng_1^2 g_2^2 - \frac{5}{36N}g_1^4 - \frac{11}{216}N^2 g_1^2 g_2^2 - \frac{11}{216}N^2 g_1^4 - \frac{1}{24}Ng_1^4 - \frac{1}{108}g_1^4 \right. \\
& \left. + \frac{1}{9}N^2 g_1^3 g_2 + \frac{13}{54N^2}g_1^4 + \frac{1}{3}Ng_1^3 g_2 + \frac{11}{27}g_1^2 g_2^2 + \frac{11}{18N}g_1^2 g_2^2 + \frac{16}{9N^2}g_1^3 g_2\right] \\
& + \left[-\frac{5333}{108N^2}g_1^4 g_2^2 - \frac{64}{3N^3}\zeta_3 g_1^4 g_2^2 - \frac{32207}{1944N}g_1^4 g_2^2 - \frac{920}{81N^3}g_1^2 g_2^4 - \frac{544}{81N^3}g_1^3 g_2^3 - \frac{20}{3}\zeta_3 g_1^4 g_2^2 - \frac{880}{243N^2}g_1^6 \right. \\
& - \frac{989}{324}g_1^3 g_2^3 - \frac{445}{162}g_1^2 g_2^4 - \frac{239}{162N^2}g_1^5 g_2 - \frac{4}{3N^3}\zeta_3 g_1^6 - \frac{3355}{2592}Ng_1^6 - \frac{755}{648}N^2 g_1^4 g_2^2 - \frac{727}{648N}g_1^5 g_2 - \frac{581}{648}g_1^5 g_2 \\
& - \frac{785}{1296}Ng_1^3 g_2^3 - \frac{1}{2}\zeta_3 g_1^6 - \frac{545}{1296}Ng_1^2 g_2^4 - \frac{1}{3}\zeta_3 N g_1^4 g_2^2 - \frac{2591}{7776}N^2 g_1^6 - \frac{685}{2592}Ng_1^5 g_2 - \frac{931}{7776}N^3 g_1^4 g_2^2 \\
& - \frac{193}{7776}N^3 g_1^6 - \frac{1}{216}N^3 g_1^2 g_2^4 + \frac{89}{1296}N^3 g_1^3 g_2^3 + \frac{91}{1296}N^2 g_1^2 g_2^4 + \frac{217}{2592}N^3 g_1^5 g_2 + \frac{1}{6}\zeta_3 N^2 g_1^6 + \frac{205}{648}N^2 g_1^5 g_2 \\
& + \frac{53}{162}N^2 g_1^3 g_2^3 + \frac{1}{3}\zeta_3 N^2 g_1^4 g_2^2 + \frac{2}{3}\zeta_3 N g_1^6 + \frac{233}{288}g_1^6 + \frac{12289}{7776}Ng_1^4 g_2^2 + \frac{2765}{1296N}g_1^6 + \frac{1133}{486N^3}g_1^6 \\
& + \frac{973}{324N}g_1^3 g_2^3 + \frac{272}{81N^3}g_1^5 g_2 + \frac{1489}{324N}g_1^2 g_2^4 + \frac{20}{3N}\zeta_3 g_1^4 g_2^2 + \frac{565}{81N^2}g_1^3 g_2^3 + \frac{799}{81N^2}g_1^2 g_2^4 + \frac{22039}{1296}g_1^4 g_2^2 \\
& \left. + \frac{64}{3N^2}\zeta_3 g_1^4 g_2^2 + \frac{11821}{243N^3}g_1^4 g_2^2 - \frac{1}{N}\zeta_3 g_1^6 + \frac{2}{N^2}\zeta_3 g_1^6\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-\frac{17920}{9N^4}\zeta_5 g_1^4 g_2^4-\frac{73520}{81N^3}\zeta_3 g_1^4 g_2^4-\frac{2560}{3N}\zeta_5 g_1^4 g_2^4-\frac{576706}{729N^3}g_1^3 g_2^5-\frac{5164846}{6561N^3}g_1^4 g_2^4-\frac{4807832}{6561N^4}g_1^2 g_2^6\\
&-\frac{18008927}{26244N^3}g_1^6 g_2^2-\frac{5152}{9N^3}\zeta_4 g_1^4 g_2^4-\frac{5120}{9N^4}\zeta_5 g_1^6 g_2^2-\frac{14080}{27N^4}\zeta_3 g_1^3 g_2^5-\frac{1280}{3N^4}\zeta_4 g_1^3 g_2^5-\frac{1280}{3N^3}\zeta_4 g_1^5 g_2^3\\
&-\frac{856511}{2187N^3}g_1^5 g_2^3-\frac{25912}{81N^2}\zeta_3 g_1^4 g_2^4-\frac{7808}{27N^4}\zeta_3 g_1^5 g_2^3-\frac{3216407}{13122N}g_1^2 g_2^6-\frac{334133}{1458N^2}g_1^3 g_2^5-\frac{645791}{2916N^2}g_1^5 g_2^3\\
&-\frac{17504}{81N^3}\zeta_3 g_1^2 g_2^6-\frac{640}{3N}\zeta_5 g_1^6 g_2^2-\frac{17252}{81N^3}\zeta_3 g_1^6 g_2^2-\frac{5506}{27N}\zeta_3 g_1^5 g_2^3-\frac{4852}{27N}\zeta_3 g_1^3 g_2^5-\frac{1600}{9N^2}\zeta_5 g_1^6 g_2^2\\
&-\frac{726199}{6561N^2}g_1^4 g_2^4-\frac{224}{3N^3}\zeta_4 g_1^2 g_2^6-\frac{20239}{324}\zeta_3 N g_1^4 g_2^4-\frac{5000}{81N^2}\zeta_3 g_1^2 g_2^6-\frac{399727}{6561}g_1^6 g_2^2-\frac{544}{9N^2}\zeta_3 g_1^7 g_2\\
&-\frac{3792575}{69984}N g_1^5 g_2^3-\frac{160}{3N^2}\zeta_5 g_1^8-\frac{765971}{17496}g_1^7 g_2-\frac{2215}{54}\zeta_3 g_1^6 g_2^2-\frac{2082131}{52488N^3}g_1^8-\frac{89}{3}\zeta_4 g_1^5 g_2^3\\
&-\frac{5779879}{209952}g_1^4 g_2^4-\frac{1463}{54}\zeta_3 g_1^5 g_2^3-\frac{234439}{8748N^3}g_1^7 g_2-\frac{6445}{243N^4}g_1^7 g_2-\frac{64}{3N^2}\zeta_4 g_1^2 g_2^6-\frac{160}{9N^4}\zeta_5 g_1^8\\
&-\frac{47}{3N}\zeta_3 g_1^7 g_2-\frac{91}{6}\zeta_4 N g_1^4 g_2^4-\frac{1196}{81N^3}\zeta_3 g_1^8-\frac{4741}{324}\zeta_3 N g_1^6 g_2^2-\frac{146525}{11664}N g_1^3 g_2^5-\frac{110}{9N^3}\zeta_4 g_1^6 g_2^2\\
&-\frac{1584155}{139968}N g_1^7 g_2-\frac{98}{9}\zeta_4 g_1^6 g_2^2-\frac{32}{3N^2}\zeta_4 g_1^5 g_2^3-\frac{509}{54}\zeta_3 g_1^8-\frac{1277473}{139968N}g_1^8-\frac{80}{9}\zeta_5 N^2 g_1^6 g_2^2\\
&-\frac{25}{3}\zeta_5 N g_1^8-\frac{3274453}{419904}g_1^8-\frac{139}{18}\zeta_4 g_1^8-\frac{3112547}{419904}N g_1^4 g_2^4-\frac{131}{18N}\zeta_4 g_1^6 g_2^2-\frac{20}{3}\zeta_4 N g_1^5 g_2^3\\
&-\frac{339929}{52488}g_1^2 g_2^6-\frac{50}{9}\zeta_5 N^2 g_1^8-\frac{875}{162}\zeta_3 N g_1^2 g_2^6-\frac{452057}{93312}N^2 g_1^5 g_2^3-\frac{323}{72}\zeta_3 N^2 g_1^7 g_2-\frac{40}{9}\zeta_5 N^3 g_1^4 g_2^4\\
&-\frac{2861}{648}\zeta_3 N^2 g_1^4 g_2^4-\frac{91}{27}\zeta_3 g_1^3 g_2^5-\frac{82}{27N^4}\zeta_3 g_1^8-\frac{2288509}{839808}N^3 g_1^6 g_2^2-\frac{7}{3N}\zeta_4 g_1^7 g_2-\frac{5}{3}\zeta_4 g_1^4 g_2^4\\
&-\frac{29}{27}\zeta_3 N^3 g_1^5 g_2^3-\frac{29}{36}\zeta_3 N^2 g_1^3 g_2^5-\frac{64301}{104976}N^2 g_1^2 g_2^6-\frac{5}{9}\zeta_5 N^3 g_1^8-\frac{5}{9}\zeta_5 N^4 g_1^4 g_2^4-\frac{1}{4}\zeta_4 N^3 g_1^3 g_2^5\\
&-\frac{179209}{839808}N^3 g_1^2 g_2^6-\frac{151673}{839808}N^3 g_1^8-\frac{17}{108}\zeta_3 N^3 g_1^3 g_2^5-\frac{62419}{419904}N^4 g_1^6 g_2^2-\frac{1}{12}\zeta_4 N^2 g_1^3 g_2^5-\frac{1}{12}\zeta_4 N^3 g_1^7 g_2\\
&-\frac{335}{5832}N^4 g_1^3 g_2^5-\frac{13855}{839808}N^4 g_1^8-\frac{1}{162}\zeta_3 N^4 g_1^2 g_2^6-\frac{1}{216}\zeta_3 N^4 g_1^7 g_2+\frac{1}{216}\zeta_3 N^4 g_1^5 g_2^3+\frac{9437}{839808}N^4 g_1^2 g_2^6\\
&+\frac{1}{72}\zeta_4 N^2 g_1^2 g_2^6+\frac{5}{324}\zeta_3 N^4 g_1^6 g_2^2+\frac{1}{36}\zeta_4 N^3 g_1^8+\frac{1}{27}\zeta_3 N^4 g_1^3 g_2^5+\frac{1}{24}\zeta_4 N^3 g_1^2 g_2^6+\frac{7153}{93312}N^4 g_1^7 g_2\\
&+\frac{65}{648}\zeta_3 N^3 g_1^2 g_2^6+\frac{4691}{46656}N^3 g_1^3 g_2^5+\frac{31975}{279936}N^4 g_1^5 g_2^3+\frac{11}{72}\zeta_4 N^3 g_1^6 g_2^2+\frac{107}{648}\zeta_3 N^2 g_1^2 g_2^6+\frac{17519}{104976}N^4 g_1^4 g_2^4\\
&+\frac{1}{4}\zeta_4 N^3 g_1^5 g_2^3+\frac{5}{18}\zeta_4 N g_1^8+\frac{11}{36}\zeta_3 N^4 g_1^4 g_2^4+\frac{13}{36}\zeta_4 N^3 g_1^4 g_2^4+\frac{305}{648}\zeta_3 N^3 g_1^8+\frac{401323}{839808}N^3 g_1^4 g_2^4\\
&+\frac{25}{36}\zeta_4 N^2 g_1^4 g_2^4+\frac{25}{36}\zeta_4 N^2 g_1^8+\frac{239}{324}\zeta_3 N g_1^8+\frac{5}{6N}\zeta_4 g_1^8+\frac{5}{6}\zeta_4 g_1^2 g_2^6+\frac{809}{648}\zeta_3 N^3 g_1^6 g_2^2\\
&+\frac{115}{72}\zeta_4 N^2 g_1^6 g_2^2+\frac{761255}{419904}N^2 g_1^8+\frac{85925}{46656}N^2 g_1^3 g_2^5+\frac{130603}{69984}N^3 g_1^7 g_2+\frac{313}{162}\zeta_3 g_1^2 g_2^6+\frac{22549}{11664}g_1^3 g_2^5\\
&+\frac{71}{36}\zeta_4 N g_1^6 g_2^2+\frac{1694359}{839808}N^2 g_1^4 g_2^4+\frac{157619}{69984}N^3 g_1^5 g_2^3+\frac{527}{216}\zeta_3 N^2 g_1^5 g_2^3+\frac{187}{72}\zeta_3 N^2 g_1^8+\frac{248801}{93312}N^2 g_1^6 g_2^2\\
&+\frac{11}{4}\zeta_4 N^2 g_1^5 g_2^3+\frac{25}{9}\zeta_5 N^2 g_1^4 g_2^4+\frac{1913}{648}\zeta_3 N^3 g_1^4 g_2^4+\frac{28}{9N^4}\zeta_4 g_1^8+\frac{41}{12}\zeta_4 N g_1^7 g_2+\frac{1397}{324N}\zeta_3 g_1^8\\
&+\frac{2987}{648}\zeta_3 N^2 g_1^6 g_2^2+\frac{209}{36}\zeta_3 N g_1^7 g_2+\frac{1725895}{279936}N^2 g_1^7 g_2+\frac{411389}{46656}N g_1^8+\frac{256}{27N^4}\zeta_3 g_1^7 g_2+\frac{104}{9N^4}\zeta_4 g_1^6 g_2^2\\
&+\frac{320}{27}\zeta_3 N g_1^3 g_2^5+\frac{136}{9N^2}\zeta_4 g_1^6 g_2^2+\frac{3178777}{209952}N g_1^2 g_2^6+\frac{151}{9N^2}\zeta_4 g_1^8+\frac{172}{9N^2}\zeta_3 g_1^8+\frac{1136993}{52488N^4}g_1^8
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{386885}{17496N}g_1^7g_2+\frac{70}{3}\zeta_5g_1^8+\frac{1285873}{52488N^2}g_1^8+\frac{11459573}{419904}Ng_1^6g_2^2+\frac{1757}{54}\zeta_3g_1^7g_2+\frac{304}{9N^3}\zeta_3g_1^7g_2\\
&+\frac{1825}{54}\zeta_3Ng_1^5g_2^3+\frac{6721}{162}\zeta_3g_1^4g_2^4+\frac{3618361}{69984}g_1^5g_2^3+\frac{5032}{81N^2}\zeta_3g_1^6g_2^2+\frac{560}{9N^3}\zeta_5g_1^8+\frac{640}{9N^4}\zeta_4g_1^2g_2^6\\
&+\frac{5996}{81N}\zeta_3g_1^2g_2^6+\frac{1330523}{17496N^2}g_1^6g_2^2+\frac{1368163}{17496N^2}g_1^7g_2+\frac{7436}{81N}\zeta_3g_1^6g_2^2+\frac{280}{3}\zeta_5Ng_1^4g_2^4+\frac{2781593}{26244N}g_1^6g_2^2\\
&+\frac{8816}{81N^4}\zeta_3g_1^6g_2^2+\frac{1040}{9}\zeta_5g_1^6g_2^2+\frac{388}{3N}\zeta_4g_1^5g_2^3+\frac{424}{3N^2}\zeta_3g_1^5g_2^3+\frac{4168}{27N^2}\zeta_3g_1^3g_2^5+\frac{16768}{81N^4}\zeta_3g_1^2g_2^6\\
&+\frac{1431505}{6561N^2}g_1^2g_2^6+\frac{22969285}{104976N}g_1^4g_2^4+\frac{179635}{729N}g_1^3g_2^5+\frac{4875121}{17496N}g_1^5g_2^3+\frac{742328}{2187N^4}g_1^5g_2^3+\frac{1024}{3N^4}\zeta_4g_1^5g_2^3\\
&+\frac{9280}{27N^3}\zeta_3g_1^5g_2^3+\frac{34793}{81N}\zeta_3g_1^4g_2^4+\frac{4736}{9N^4}\zeta_4g_1^4g_2^4+\frac{3529919}{6561N^4}g_1^6g_2^2+\frac{14560}{27N^3}\zeta_3g_1^3g_2^5+\frac{5120}{9N^2}\zeta_5g_1^4g_2^4\\
&+\frac{4667252}{6561N^4}g_1^4g_2^4+\frac{4933030}{6561N^3}g_1^2g_2^6+\frac{570512}{729N^4}g_1^3g_2^5+\frac{22144}{27N^4}\zeta_3g_1^4g_2^4+\frac{2560}{3N^3}\zeta_5g_1^6g_2^2+\frac{19840}{9N^3}\zeta_5g_1^4g_2^4\\
&-\frac{156}{N}\zeta_4g_1^3g_2^5-\frac{136}{N^2}\zeta_4g_1^4g_2^4-\frac{40}{N^3}\zeta_4g_1^7g_2-20\zeta_5g_1^4g_2^4-\frac{14}{N^3}\zeta_4g_1^8-10\zeta_4g_1^7g_2-5\zeta_4g_1^3g_2^5-2\zeta_4Ng_1^2g_2^6\\
&-\zeta_3N^3g_1^7g_2+\zeta_4N^2g_1^7g_2+12\zeta_4Ng_1^3g_2^5+\frac{16}{N^2}\zeta_4g_1^7g_2+\frac{26}{N}\zeta_4g_1^2g_2^6+\frac{32}{N^4}\zeta_4g_1^7g_2+\frac{128}{N^2}\zeta_4g_1^3g_2^5\\
&+\frac{198}{N}\zeta_4g_1^4g_2^4+\frac{448}{N^3}\zeta_4g_1^3g_2^5\Bigg]+\mathcal{O}(g_i^{10})
\end{aligned}
$$

$$
\begin{aligned}
\gamma_z(g_i)=&\left[-\frac{1}{3}g_2^2-\frac{1}{6}g_1^2-\frac{1}{6}Ng_2^2+\frac{4}{3N}g_2^2\right]\\
&+\Bigg[-\frac{152}{27N}g_2^4-\frac{16}{9N}g_1^3g_2-\frac{13}{9}g_2^4-\frac{11}{54}g_1^2g_2^2-\frac{11}{108}Ng_1^2g_2^2-\frac{11}{108}Ng_1^4-\frac{1}{54N}g_1^4+\frac{1}{108}g_1^4+\frac{1}{108}N^2g_2^4\\
&\quad+\frac{2}{9}Ng_1^3g_2+\frac{7}{27}Ng_2^4+\frac{4}{9}g_1^3g_2+\frac{22}{27N}g_1^2g_2^2+\frac{400}{27N^2}g_2^4\Bigg]\\
&+\Bigg[-\frac{40784}{81N^2}g_2^6-\frac{1280}{3N^3}\zeta_3g_2^6-\frac{5168}{81N}g_2^6-\frac{8815}{162N^2}g_1^4g_2^2-\frac{68}{3}\zeta_3g_2^6-\frac{1456}{81N^2}g_1^3g_2^3-\frac{40}{3N}\zeta_3g_1^4g_2^2-\frac{2789}{486N}g_1^2g_2^4\\
&\quad-\frac{2929}{1296}Ng_1^4g_2^2-\frac{148}{81}g_1^5g_2-\frac{5}{3}\zeta_3g_1^4g_2^2-\frac{119}{81}g_1^2g_2^4-\frac{9025}{7776}g_1^6-\frac{535}{486N^2}g_1^6-\frac{541}{648}N^2g_2^6-\frac{2}{3N}\zeta_3g_1^6\\
&\quad-\frac{35}{96}Ng_1^6-\frac{1}{3}\zeta_3Ng_2^6-\frac{709}{2592}N^2g_1^4g_2^2-\frac{409}{7776}N^2g_1^6-\frac{13}{324}Ng_1^3g_2^3+\frac{25}{7776}N^3g_2^6+\frac{67}{7776}N^2g_1^2g_2^4\\
&\quad+\frac{11}{162}Ng_2^6+\frac{13}{162}N^2g_1^3g_2^3+\frac{1}{6}\zeta_3Ng_1^6+\frac{20}{81}N^2g_1^5g_2+\frac{1019}{3888}Ng_1^2g_2^4+\frac{1}{2}\zeta_3g_1^6+\frac{1}{2}\zeta_3N^2g_2^6+\frac{44}{81N^2}g_1^5g_2\\
&\quad+\frac{65}{108}Ng_1^5g_2+\frac{2}{3N^2}\zeta_3g_1^6+\frac{5}{6}\zeta_3Ng_1^4g_2^2+\frac{65}{54}g_1^3g_2^3+\frac{257}{144N}g_1^6+\frac{7211}{1296}g_1^4g_2^2+\frac{494}{81N}g_1^3g_2^3+\frac{3674}{243N^2}g_1^2g_2^4\\
&\quad+\frac{80}{3N^2}\zeta_3g_1^4g_2^2+\frac{17695}{648N}g_1^4g_2^2+\frac{42745}{972}g_2^6+\frac{704}{3N^2}\zeta_3g_2^6+\frac{227408}{243N^3}g_2^6-\frac{1}{N}g_1^5g_2+\frac{32}{N}\zeta_3g_2^6\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-\frac{655360}{9N^4}\zeta_5 g_2^8-\frac{10404128}{243N^3}g_2^8-\frac{54272}{9N^3}\zeta_3 g_2^8-\frac{52480}{9N}\zeta_5 g_2^8-\frac{22746338}{6561N^3}g_1^4g_2^4-\frac{21760}{9N^2}\zeta_5 g_1^4g_2^4\\
&-\frac{65800}{81N^2}g_2^8-\frac{6976}{9N^3}\zeta_4 g_1^4g_2^4-\frac{1498816}{2187N^3}g_1^3g_2^5-\frac{50752}{81N^3}\zeta_3 g_1^4g_2^4-\frac{49792}{81N^3}\zeta_3 g_1^2g_2^6-\frac{5120}{9N^2}\zeta_5 g_1^6g_2^2\\
&-\frac{1204666}{2187N^2}g_1^2g_2^6-\frac{1280}{3N^3}\zeta_4 g_1^5g_2^3-\frac{2779244}{6561N^3}g_1^6g_2^2-\frac{3392}{9N^2}\zeta_3 g_2^8-\frac{3200}{9N}\zeta_5 g_1^4g_2^4-\frac{323263}{972}g_2^8\\
&-\frac{3057539}{13122}g_1^4g_2^4-\frac{1408}{9N^2}\zeta_4 g_1^2g_2^6-\frac{3944}{27N^2}\zeta_3 g_1^5g_2^3-\frac{3441205}{26244N}g_1^6g_2^2-\frac{10496}{81N^3}\zeta_3 g_1^6g_2^2-\frac{40585}{324}Ng_2^8\\
&-\frac{992}{9N^2}\zeta_3 g_1^3g_2^5-\frac{6653}{81}\zeta_3 g_1^4g_2^4-\frac{1885}{27}\zeta_3 Ng_2^8-\frac{200}{3N}\zeta_3 g_1^5g_2^3-\frac{142994}{2187N}g_1^2g_2^6-\frac{904931}{17496N}g_1^7g_2\\
&-\frac{4133}{81N}\zeta_3 g_1^6g_2^2-\frac{421}{9}\zeta_4 g_1^4g_2^4-\frac{416}{9N^2}\zeta_3 g_1^7g_2-\frac{347}{9}\zeta_3 g_1^6g_2^2-\frac{331889}{8748}g_1^5g_2^3-\frac{2408}{81}\zeta_3 g_1^2g_2^6\\
&-\frac{256}{9N^3}\zeta_4 g_1^6g_2^2-\frac{80}{3N^2}\zeta_5 g_1^8-\frac{430079}{17496}g_1^7g_2-\frac{67}{3}\zeta_4 Ng_2^8-\frac{64}{3N^2}\zeta_4 g_2^8-\frac{64}{3N}\zeta_4 g_1^2g_2^6-\frac{56}{3}\zeta_4 g_1^5g_2^3\\
&-\frac{39760}{2187N^3}g_1^5g_2^3-\frac{160}{9}\zeta_5 Ng_1^6g_2^2-\frac{140}{9}\zeta_5 N^2g_1^4g_2^4-\frac{316}{27}\zeta_3 g_1^3g_2^5-\frac{937}{81}\zeta_3 Ng_1^4g_2^4-\frac{595}{54N}g_1^3g_2^5\\
&-\frac{242705}{26244N^3}g_1^8-\frac{1932971}{209952}Ng_1^4g_2^4-\frac{213961}{23328N}g_1^8-\frac{74}{9}\zeta_3 Ng_1^3g_2^5-\frac{181}{27}\zeta_3 Ng_1^7g_2-\frac{55}{9}\zeta_5 N^3g_2^8\\
&-\frac{109}{18N}\zeta_4 g_1^8-\frac{160}{27}\zeta_3 g_2^8-\frac{905}{162N}\zeta_3 g_1^8-\frac{255955}{46656}N^2g_1^6g_2^2-\frac{321547}{69984}Ng_1^5g_2^3-\frac{40}{9}\zeta_5 g_1^8-\frac{20}{9}\zeta_5 N^3g_1^4g_2^4\\
&-\frac{67}{36}\zeta_3 N^2g_1^7g_2-\frac{14}{9N^3}\zeta_4 g_1^8-\frac{34}{27}\zeta_3 N^2g_2^8-\frac{8887}{7776}N^2g_1^3g_2^5-\frac{16553}{17496}Ng_1^2g_2^6-\frac{31}{36}\zeta_3 g_1^8\\
&-\frac{54811}{69984}N^2g_1^2g_2^6-\frac{5}{9}\zeta_5 N^2g_1^8-\frac{5}{9}\zeta_5 N^4g_2^8-\frac{73123}{139968}N^3g_1^5g_2^3-\frac{27593}{69984}N^3g_1^3g_2^5-\frac{154823}{419904}N^2g_1^8\\
&-\frac{1}{3}\zeta_4 g_1^8-\frac{1}{3}\zeta_4 N^2g_1^2g_2^6-\frac{1}{3}\zeta_4 N^2g_1^7g_2-\frac{14}{81}\zeta_3 N^3g_1^6g_2^2-\frac{13643}{104976}N^3g_1^6g_2^2-\frac{13}{324}\zeta_3 N^3g_1^2g_2^6\\
&-\frac{1831}{52488}N^3g_1^8+\frac{1}{324}\zeta_3 N^3g_1^8+\frac{1}{36}\zeta_4 N^2g_1^8+\frac{5105}{139968}N^3g_1^2g_2^6+\frac{1}{12}\zeta_3 N^2g_1^5g_2^3+\frac{253}{1728}N^4g_2^8\\
&+\frac{26539}{139968}N^3g_1^7g_2+\frac{2}{9}\zeta_4 Ng_1^2g_2^6+\frac{5}{18}\zeta_4 N^2g_1^6g_2^2+\frac{1}{3}\zeta_4 Ng_1^4g_2^4+\frac{19}{54}\zeta_3 N^4g_2^8+\frac{43}{108}\zeta_3 N^3g_1^3g_2^5\\
&+\frac{11}{27}\zeta_3 Ng_1^2g_2^6+\frac{67819}{139968}N^2g_1^5g_2^3+\frac{1}{2}\zeta_4 N^2g_1^5g_2^3+\frac{1}{2}\zeta_4 N^3g_2^8+\frac{19}{36}\zeta_3 N^2g_1^8+\frac{5}{9}\zeta_3 N^2g_1^2g_2^6\\
&+\frac{91}{162}\zeta_3 N^3g_1^4g_2^4+\frac{46}{81N^3}\zeta_3 g_1^8+\frac{71}{108}\zeta_3 N^3g_1^5g_2^3+\frac{2}{3}\zeta_4 Ng_1^8+\frac{19}{27}\zeta_3 Ng_1^5g_2^3+\frac{5}{6}\zeta_4 Ng_1^7g_2\\
&+\frac{29017}{23328}Ng_1^8+\frac{23}{18}\zeta_3 N^2g_1^3g_2^5+\frac{49}{36}\zeta_4 N^2g_1^4g_2^4+\frac{605131}{419904}N^3g_1^4g_2^4+\frac{7183}{3888}N^3g_2^8+\frac{7}{3}\zeta_4 g_1^6g_2^2\\
&+\frac{391}{162}\zeta_3 Ng_1^8+\frac{421}{162}\zeta_3 N^2g_1^6g_2^2+\frac{353}{108}\zeta_3 N^3g_2^8+\frac{485339}{139968}N^2g_1^7g_2+\frac{65}{18}\zeta_4 Ng_1^6g_2^2+\frac{11}{3}\zeta_4 g_1^7g_2\\
&+\frac{829}{162}\zeta_3 N^2g_1^4g_2^4+\frac{24437}{4374}g_1^3g_2^5+\frac{56}{9N^2}\zeta_4 g_1^8+\frac{179753}{26244}Ng_1^6g_2^2+\frac{15}{2}\zeta_4 N^2g_2^8+\frac{3284909}{419904}g_1^8\\
&+\frac{80}{9N^3}\zeta_5 g_1^8+\frac{311099}{34992}Ng_1^7g_2+\frac{942539}{104976}N^2g_1^4g_2^4+\frac{6097}{648}N^2g_2^8+\frac{21016}{2187N^3}g_1^7g_2+\frac{3361}{324}\zeta_3 Ng_1^6g_2^2\\
&+\frac{281}{27N^2}\zeta_3 g_1^8+\frac{127853}{11664}Ng_1^3g_2^5+\frac{331873}{26244N^2}g_1^8+\frac{397}{27}\zeta_3 g_1^7g_2+\frac{136}{9}\zeta_4 g_1^2g_2^6+\frac{160}{9N}\zeta_5 g_1^8\\
&+\frac{160}{9}\zeta_5 g_1^6g_2^2+\frac{160}{9N}\zeta_3 g_1^3g_2^5+\frac{176}{9N^2}\zeta_4 g_1^6g_2^2+\frac{45601}{2187N^2}g_1^5g_2^3+\frac{200}{9}\zeta_5 Ng_1^4g_2^4+\frac{956}{27N}\zeta_3 g_1^7g_2\\
&+\frac{376}{9N}\zeta_3 g_1^2g_2^6+\frac{99248}{2187}g_1^2g_2^6+\frac{184067}{3888}g_1^6g_2^2+\frac{484}{9N}\zeta_4 g_1^4g_2^4+\frac{281221}{4374N^2}g_1^7g_2+\frac{812323}{8748N}g_1^5g_2^3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{12854}{81N}\zeta_3 g_1^4 g_2^4+\frac{1600}{9N}\zeta_5 g_1^6 g_2^2+\frac{9622837}{52488N} g_1^4 g_2^4+\frac{5152}{27N^3}\zeta_3 g_1^5 g_2^3+\frac{16364}{81N^2}\zeta_3 g_1^6 g_2^2+\frac{680}{3}\zeta_5 N g_2^8\\
&+\frac{2560}{9N^3}\zeta_4 g_1^2 g_2^6+\frac{208178}{729N^2} g_1^3 g_2^5+\frac{8320}{27N^3}\zeta_3 g_1^3 g_2^5+\frac{25688}{81N^2}\zeta_3 g_1^4 g_2^4+\frac{703817}{2187N^2} g_1^6 g_2^2+\frac{8960}{27N^2}\zeta_3 g_1^2 g_2^6\\
&+\frac{3200}{9}\zeta_5 g_2^8+\frac{1328}{3N^2}\zeta_4 g_1^4 g_2^4+\frac{5120}{9N^3}\zeta_5 g_1^6 g_2^2+\frac{25456}{27N}\zeta_3 g_2^8+\frac{756712}{729N^3} g_1^2 g_2^6+\frac{3856}{3N}\zeta_4 g_2^8\\
&+\frac{12800}{9N^2}\zeta_5 g_2^8+\frac{14320676}{6561N^2} g_1^4 g_2^4+\frac{35840}{9N^3}\zeta_5 g_1^4 g_2^4+\frac{39574}{9N} g_2^8+\frac{214016}{27N^4}\zeta_3 g_2^8+\frac{65536}{3N^4}\zeta_4 g_2^8\\
&+\frac{5034688}{81N^4} g_2^8-\frac{14592}{N^3}\zeta_4 g_2^8-168\zeta_4 g_2^8-\frac{48}{N}\zeta_4 g_1^5 g_2^3-\frac{24}{N}\zeta_4 g_1^6 g_2^2-\frac{16}{N^3}\zeta_4 g_1^7 g_2-\frac{14}{N}\zeta_4 g_1^7 g_2\\
&-10\zeta_5 N^2 g_2^8-5\zeta_5 N g_1^8+7\zeta_4 N g_1^5 g_2^3+\frac{20}{N^2}\zeta_4 g_1^7 g_2+39\zeta_3 g_1^5 g_2^3+320\zeta_5 g_1^4 g_2^4+\frac{320}{N^2}\zeta_4 g_1^5 g_2^3+\frac{51200}{N^3}\zeta_5 g_2^8\Bigg]+\mathcal{O}(g_i^{10}). \qquad \text{(A1)}
\end{aligned}
$$

The large number of terms in comparison with the same anomalous dimensions of [4] derive from the group theory defined by (4). The two four loop beta functions are

$$
\begin{aligned}
\beta_1(g_i)=&-\frac{1}{2}\epsilon g_1+\left[-\frac{2}{3N}g_1^3-\frac{1}{6}g_1 g_2^2-\frac{1}{6}N g_1^3-\frac{1}{12}N g_1 g_2^2+\frac{1}{4}g_1^3+\frac{1}{2}N g_1^2 g_2+\frac{2}{3N}g_1 g_2^2-\frac{4}{N}g_1^2 g_2+g_1^2 g_2\right]\\
&+\Bigg[-\frac{1276}{27N^2}g_1^3 g_2^2-\frac{104}{9N^2}g_1^2 g_2^3-\frac{20}{3N^2}g_1^4 g_2-\frac{76}{27N}g_1 g_2^4-\frac{67}{27N^2}g_1^5-\frac{481}{216}g_1^5-\frac{455}{216}N g_1^3 g_2^2-\frac{19}{18}g_1^4 g_2\\
&\quad-\frac{13}{18}g_1 g_2^4-\frac{113}{216}N g_1^5-\frac{65}{216}N^2 g_1^3 g_2^2-\frac{11}{216}N^2 g_1^5+\frac{1}{216}N^2 g_1 g_2^4+\frac{1}{36}N g_1^2 g_2^3+\frac{5}{72}N^2 g_1^2 g_2^3\\
&\quad+\frac{7}{54}N g_1 g_2^4+\frac{5}{24}N^2 g_1^4 g_2+\frac{11}{36}N g_1^4 g_2+\frac{2}{3}g_1^2 g_2^3+\frac{23}{9N}g_1^4 g_2+\frac{365}{108N}g_1^5+\frac{34}{9N}g_1^2 g_2^3+\frac{191}{36}g_1^3 g_2^2+\frac{200}{27N^2}g_1 g_2^4+\frac{644}{27N}g_1^3 g_2^2\Bigg]\\
&+\Bigg[-\frac{121040}{81N^3}g_1^2 g_2^5-\frac{4048}{9N^3}g_1^4 g_2^3-\frac{90602}{243N^3}g_1^5 g_2^2-\frac{7040}{27N^3}g_1^3 g_2^4-\frac{20392}{81N^2}g_1 g_2^6-\frac{640}{3N^3}\zeta_3 g_1 g_2^6\\
&\quad-\frac{417439}{3888N}g_1^5 g_2^2-\frac{8411}{108}g_1^4 g_2^3-\frac{22333}{324}g_1^2 g_2^5-\frac{12415}{216N}g_1^6 g_2-\frac{44387}{972N}g_1^3 g_2^4-\frac{2584}{81N}g_1 g_2^6-\frac{88}{3N^2}\zeta_3 g_1^5 g_2^2\\
&\quad-\frac{76}{3N}\zeta_3 g_1^7-\frac{46559}{2592N}g_1^7-\frac{461}{27}g_1^6 g_2-\frac{7855}{486N^3}g_1^7-\frac{34}{3}\zeta_3 g_1 g_2^6-\frac{3193}{324}N g_1^4 g_2^3-\frac{125}{27N^3}g_1^6 g_2\\
&\quad-\frac{17}{4}\zeta_3 g_1^7-\frac{7}{2}\zeta_3 g_1^5 g_2^2-\frac{5959}{1728}N^2 g_1^5 g_2^2-\frac{3}{2}\zeta_3 N^2 g_1^2 g_2^5-\frac{4}{3N^3}\zeta_3 g_1^7-\frac{13145}{15552}N^2 g_1^3 g_2^4-\frac{1}{2}\zeta_3 N^2 g_1^4 g_2^3\\
&\quad-\frac{6755}{15552}N^2 g_1^7-\frac{541}{1296}N^2 g_1 g_2^6-\frac{1073}{3888}N^3 g_1^5 g_2^2-\frac{1}{6}\zeta_3 N g_1 g_2^6-\frac{73}{2592}N^3 g_1^2 g_2^5-\frac{193}{7776}N^3 g_1^7-\frac{5}{432}N^3 g_1^3 g_2^4\\
&\quad+\frac{25}{15552}N^3 g_1 g_2^6+\frac{11}{324}N g_1 g_2^6+\frac{7}{54}N^3 g_1^6 g_2+\frac{1}{4}N^3 g_1^4 g_2^3+\frac{1}{4}\zeta_3 N^2 g_1 g_2^6+\frac{2}{3}\zeta_3 N^2 g_1^7+\frac{73}{108}N g_1^2 g_2^5\\
&\quad+\frac{95}{72}N^2 g_1^2 g_2^5+\frac{8915}{5184}N g_1^7+\frac{11}{6}\zeta_3 N^2 g_1^5 g_2^2+\frac{2197}{864}N^2 g_1^6 g_2+\frac{4889}{1296}N^2 g_1^4 g_2^3+\frac{19}{4}\zeta_3 N g_1^7+\frac{44195}{7776}N g_1^3 g_2^4\\
&\quad+\frac{97}{12}\zeta_3 N g_1^5 g_2^2+\frac{3701}{432}N g_1^6 g_2+\frac{66715}{7776}N g_1^5 g_2^2+\frac{148115}{15552}g_1^7+\frac{1759}{162}g_1^3 g_2^4+\frac{49703}{2592}g_1^5 g_2^2+\frac{5170}{243N^2}g_1^7\\
&\quad+\frac{42745}{1944}g_1 g_2^6+\frac{79}{3N^2}\zeta_3 g_1^7+\frac{128}{3N^3}\zeta_3 g_1^5 g_2^2+\frac{2599}{36N^2}g_1^6 g_2+\frac{887}{9N}g_1^2 g_2^5+\frac{352}{3N^2}\zeta_3 g_1 g_2^6+\frac{10175}{81N}g_1^4 g_2^3\\
&\quad+\frac{35647}{243N^2}g_1^3 g_2^4+\frac{23948}{81N^2}g_1^4 g_2^3+\frac{34541}{108N^2}g_1^5 g_2^2+\frac{113704}{243N^3}g_1 g_2^6+\frac{21604}{27N^2}g_1^2 g_2^5-\frac{1024}{N^3}\zeta_3 g_1^3 g_2^4-\frac{704}{N^2}\zeta_3 g_1^2 g_2^5\\
&\quad-\frac{512}{N^3}\zeta_3 g_1^4 g_2^3-\frac{128}{N}\zeta_3 g_1^4 g_2^3-\frac{96}{N}\zeta_3 g_1^2 g_2^5-64\zeta_3 g_1^3 g_2^4-\frac{48}{N^3}\zeta_3 g_1^6 g_2-\frac{36}{N}\zeta_3 g_1^5 g_2^2-\frac{20}{N^2}\zeta_3 g_1^6 g_2\\
&\quad-\frac{2}{N}\zeta_3 g_1^6 g_2-2\zeta_3 N g_1^6 g_2-2\zeta_3 N^2 g_1^6 g_2+\zeta_3 N g_1^2 g_2^5+2\zeta_3 N^2 g_1^3 g_2^4+4\zeta_3 g_1^4 g_2^3+15\zeta_3 N g_1^4 g_2^3+\frac{16}{N}\zeta_3 g_1 g_2^6\\
&\quad+24\zeta_3 g_1^6 g_2+68\zeta_3 g_1^2 g_2^5+\frac{80}{N}\zeta_3 g_1^3 g_2^4+\frac{384}{N^2}\zeta_3 g_1^4 g_2^3+\frac{576}{N^2}\zeta_3 g_1^3 g_2^4+\frac{1280}{N^3}\zeta_3 g_1^2 g_2^5\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-\frac{230834240}{2187N^4}g_1^2g_2^7-\frac{181760}{3N^4}\zeta_5g_1^4g_2^5-\frac{372395888}{6561N^4}g_1^3g_2^6-\frac{327680}{9N^4}\zeta_5g_1g_2^8-\frac{888832}{27N^4}\zeta_3g_1^2g_2^7-\frac{867328}{27N^4}\zeta_3g_1^4g_2^5\\
&\quad-\frac{2514176}{81N^4}\zeta_3g_1^3g_2^6-\frac{5202064}{243N^3}g_1g_2^8-\frac{551936}{27N^4}\zeta_3g_1^5g_2^4-\frac{31253248}{2187N^3}g_1^4g_2^5-\frac{123712}{9N^3}\zeta_4g_1^3g_2^6-\frac{316268}{27N^2}\zeta_3g_1^5g_2^4\\
&\quad-\frac{8397230}{729N^4}g_1^6g_2^3-\frac{28096}{3N^3}\zeta_4g_1^4g_2^5-\frac{60317968}{6561N^4}g_1^5g_2^4-\frac{79360}{9N^4}\zeta_5g_1^5g_2^4-\frac{35732567}{4374N^2}g_1^6g_2^3-\frac{33418985}{4374N}g_1^2g_2^7\\
&\quad-\frac{488524}{81N}\zeta_3g_1^3g_2^6-\frac{38426596}{6561N}g_1^3g_2^6-\frac{16384}{3N^4}\zeta_3g_1^6g_2^3-\frac{12800}{3N^2}\zeta_5g_1^2g_2^7-\frac{12392}{3N}\zeta_3g_1^2g_2^7-\frac{36080}{9N^2}\zeta_5g_1^7g_2^2\\
&\quad-\frac{97760}{27N^2}\zeta_3g_1^6g_2^3-\frac{272896}{81N^4}\zeta_3g_1^7g_2^2-\frac{27136}{9N^3}\zeta_3g_1g_2^8-\frac{26240}{9N}\zeta_5g_1g_2^8-\frac{6998}{3N}\zeta_3g_1^6g_2^3-\frac{13898251}{6561N^4}g_1^7g_2^2\\
&\quad-\frac{2058517}{972N^2}g_1^7g_2^2-\frac{178650049}{104976N}g_1^5g_2^4-\frac{44780}{27N^2}\zeta_3g_1^8g_2-\frac{95542}{81N^2}\zeta_3g_1^7g_2^2-\frac{3280}{3}\zeta_5Ng_1^3g_2^6-\frac{3200}{3}\zeta_5g_1^2g_2^7\\
&\quad-\frac{36891121}{34992N}g_1^6g_2^3-\frac{2485}{3}\zeta_5Ng_1^4g_2^5-\frac{7360}{9N}\zeta_5g_1^5g_2^4-\frac{2020}{3}\zeta_5g_1^6g_2^3-\frac{4181207}{6561N^2}g_1^3g_2^6-\frac{395207}{648N^2}g_1^8g_2\\
&\quad-\frac{1610}{3}\zeta_5Ng_1^6g_2^3-\frac{8609149}{17496}g_1^4g_2^5-\frac{4352}{9N^2}\zeta_4g_1^3g_2^6-\frac{1376}{3N^3}\zeta_4g_1^5g_2^4-\frac{11480}{27N^4}\zeta_3g_1^8g_2-\frac{32900}{81N^2}g_1g_2^8\\
&\quad-\frac{3640}{9N}\zeta_5g_1^7g_2^2-\frac{31811}{81}\zeta_3g_1^3g_2^6-\frac{20297063}{52488}g_1^5g_2^4-\frac{1120}{3N^4}\zeta_5g_1^8g_2-\frac{38891}{108}\zeta_3Ng_1^5g_2^4-\frac{217837}{648}\zeta_3Ng_1^7g_2^2\\
&\quad-\frac{931}{3}\zeta_4g_1^4g_2^5-\frac{8242}{27}\zeta_3g_1^4g_2^5-\frac{880}{3N^2}\zeta_5g_1^8g_2-\frac{14983}{54N^2}\zeta_3g_1^9-\frac{28066511}{104976N^2}g_1^9-\frac{7057}{27}\zeta_3g_1^7g_2^2\\
&\quad-\frac{2266}{9N}\zeta_4g_1^7g_2^2-\frac{712}{3N^3}\zeta_4g_1^6g_2^3-\frac{706}{3N^3}\zeta_4g_1^8g_2-\frac{650}{3}\zeta_5g_1^8g_2-\frac{1175}{6}\zeta_5N^2g_1^4g_2^5-\frac{1696}{9N^2}\zeta_3g_1g_2^8\\
&\quad-\frac{12077}{72}\zeta_3N^2g_1^5g_2^4-\frac{323263}{1944}g_1g_2^8-\frac{20547673}{139968}Ng_1^6g_2^3-\frac{403}{3N}\zeta_4g_1^6g_2^3-\frac{1207}{9}\zeta_4g_1^3g_2^6-\frac{2287}{18}\zeta_4g_1^5g_2^4\\
&\quad-\frac{1635631}{13122N^4}g_1^9-\frac{880}{9N^4}\zeta_5g_1^9-\frac{280}{3}\zeta_5N^2g_1^3g_2^6-\frac{169}{2}\zeta_4g_1^6g_2^3-\frac{245}{3}\zeta_4g_1^8g_2-\frac{5595557}{69984}Ng_1^8g_2\\
&\quad-\frac{602}{9}\zeta_4Ng_1^3g_2^6-\frac{40585}{648}Ng_1g_2^8-\frac{355}{6}\zeta_5Ng_1^8g_2-\frac{526}{9N}\zeta_4g_1^5g_2^4-\frac{502}{9N^3}\zeta_4g_1^7g_2^2-\frac{7184041}{139968}N^2g_1^6g_2^3\\
&\quad-\frac{583}{12}\zeta_3Ng_1^9-\frac{430}{9N^3}\zeta_4g_1^9-\frac{2914}{81}\zeta_3N^3g_1^3g_2^6-\frac{1885}{54}\zeta_3Ng_1g_2^8-\frac{100}{3}\zeta_5N^2g_1^6g_2^3-\frac{280}{9N}\zeta_5g_1^9\\
&\quad-\frac{3053}{108}\zeta_3N^2g_1^8g_2-\frac{619}{24}\zeta_3N^3g_1^4g_2^5-\frac{1599959}{69984}Ng_1^9-\frac{293}{18N}\zeta_4g_1^9-\frac{40601}{2592}N^2g_1^8g_2-\frac{11992789}{839808}N^3g_1^3g_2^6\\
&\quad-\frac{936427}{69984}N^2g_1^2g_2^7-\frac{53}{4}\zeta_4N^2g_1^2g_2^7-\frac{110}{9}\zeta_5N^2g_1^7g_2^2-\frac{35}{3N}\zeta_4g_1^8g_2-\frac{67}{6}\zeta_4Ng_1g_2^8-\frac{32}{3N^2}\zeta_4g_1g_2^8\\
&\quad-\frac{739}{72}\zeta_3N^3g_1^2g_2^7-\frac{6047}{648}\zeta_3N^3g_1^5g_2^4-\frac{157}{18}\zeta_4N^2g_1^7g_2^2-\frac{1397}{162}\zeta_3N^2g_1^7g_2^2-\frac{605}{72}\zeta_3N^2g_1^9-\frac{27293}{3456}N^3g_1^7g_2^2\\
&\quad-\frac{1559}{216}\zeta_3N^3g_1^8g_2-\frac{948031}{139968}N^3g_1^4g_2^5-\frac{124}{27}\zeta_3g_1^2g_2^7-\frac{40}{9}\zeta_3N^4g_1^4g_2^5-\frac{356}{81}\zeta_3N^4g_1^3g_2^6-\frac{36535}{8748}N^3g_1^2g_2^7\\
&\quad-\frac{3234373}{839808}N^3g_1^5g_2^4-\frac{27}{8}\zeta_4Ng_1^8g_2-\frac{10}{3}\zeta_5N^3g_1^6g_2^3-\frac{55}{18}\zeta_5N^3g_1g_2^8-\frac{80}{27}\zeta_3g_1g_2^8-\frac{1108445}{419904}N^2g_1^9\\
&\quad-\frac{89}{36}\zeta_4N^3g_1^7g_2^2-\frac{793}{486}N^4g_1^4g_2^5-\frac{1354387}{839808}N^4g_1^3g_2^6-\frac{11}{8}\zeta_4N^2g_1^9-\frac{9}{8}\zeta_4N^3g_1^2g_2^7-\frac{239}{216}\zeta_3N^3g_1^9\\
&\quad-\frac{77}{72}\zeta_3N^4g_1^2g_2^7-\frac{17}{27}\zeta_3N^2g_1g_2^8-\frac{5}{9}\zeta_5N^4g_1^5g_2^4-\frac{29}{54}\zeta_3N^3g_1^6g_2^3-\frac{58967}{139968}N^4g_1^2g_2^7-\frac{258479}{839808}N^4g_1^7g_2^2\\
&\quad-\frac{5}{18}\zeta_5N^4g_1g_2^8-\frac{2}{9}\zeta_4N^3g_1^9-\frac{7411}{34992}N^2g_1^7g_2^2-\frac{130235}{839808}N^4g_1^5g_2^4-\frac{7}{72}\zeta_4N^3g_1^5g_2^4-\frac{2}{27}\zeta_3N^4g_1^6g_2^3\\
&\quad-\frac{56635}{839808}N^3g_1^9-\frac{13855}{839808}N^4g_1^9-\frac{1}{216}\zeta_3N^4g_1^8g_2+\frac{19}{648}\zeta_3N^4g_1^7g_2^2+\frac{253}{3456}N^4g_1g_2^8+\frac{949}{8748}N^4g_1^8g_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{19}{108}\zeta_3N^4g_1g_2^8+\frac{1}{4}\zeta_4N^3g_1g_2^8+\frac{1}{4}\zeta_4N^3g_1^4g_2^5+\frac{22571}{69984}N^3g_1^8g_2+\frac{20969}{46656}N^4g_1^6g_2^3+\frac{11}{24}\zeta_3N^4g_1^5g_2^4\\
&+\frac{7}{8}\zeta_4N^3g_1^3g_2^6+\frac{7183}{7776}N^3g_1g_2^8+\frac{10}{9}\zeta_5N^3g_1^9+\frac{4}{3}\zeta_4N^3g_1^8g_2+\frac{19}{12}\zeta_4Ng_1^6g_2^3+\frac{19}{12}\zeta_4N^3g_1^6g_2^3\\
&+\frac{353}{216}\zeta_3N^3g_1g_2^8+\frac{5}{3}\zeta_5N^4g_1^2g_2^7+\frac{16}{9N^4}\zeta_4g_1^9+\frac{5}{2}\zeta_5N^2g_1^9+\frac{319}{108}\zeta_3g_1^9+\frac{215}{72}\zeta_4g_1^9\\
&+\frac{227}{72}\zeta_4Ng_1^9+\frac{15}{4}\zeta_4N^2g_1g_2^8+\frac{6097}{1296}N^2g_1g_2^8+\frac{61}{12}\zeta_4N^2g_1^4g_2^5+\frac{97}{18}\zeta_4N^2g_1^3g_2^6+\frac{601}{108}\zeta_3N^3g_1^7g_2^2\\
&+\frac{20}{3}\zeta_5N^4g_1^3g_2^6+\frac{20}{3}\zeta_5N^4g_1^4g_2^5+\frac{787}{108}\zeta_3N^2g_1^2g_2^7+\frac{61}{8}\zeta_4N^2g_1^6g_2^3+\frac{276205}{34992}N^3g_1^6g_2^3+\frac{191}{24}\zeta_4N^2g_1^8g_2\\
&+\frac{49}{6}\zeta_4N^2g_1^5g_2^4+\frac{25}{3}\zeta_5N^3g_1^8g_2+\frac{100}{9}\zeta_5N^3g_1^5g_2^4+\frac{3519241}{279936}g_1^9+\frac{51}{4}\zeta_4Ng_1^5g_2^4+\frac{85}{6}\zeta_5Ng_1^9\\
&+\frac{6860935}{419904}N^2g_1^3g_2^6+\frac{2789}{162}\zeta_3N^2g_1^3g_2^6+\frac{55}{3}\zeta_5N^3g_1^2g_2^7+\frac{67}{3}\zeta_4Ng_1^4g_2^5+\frac{370873}{15552}N^2g_1^4g_2^5+\frac{579433}{23328}Ng_1^4g_2^5\\
&+\frac{236}{9N^2}\zeta_4g_1^7g_2^2+\frac{61489}{2187N^4}g_1^8g_2+\frac{65}{2}\zeta_5N^3g_1^4g_2^5+\frac{5155153}{139968}N^2g_1^5g_2^4+\frac{1006}{27N^4}\zeta_3g_1^9+\frac{235}{6}\zeta_5N^2g_1^8g_2\\
&+\frac{2933}{72}\zeta_4Ng_1^7g_2^2+\frac{763}{18N^2}\zeta_4g_1^9+\frac{389}{9}\zeta_3N^2g_1^6g_2^3+\frac{9703}{216}\zeta_3Ng_1^8g_2+\frac{415}{9}\zeta_5g_1^9+\frac{160}{3}\zeta_5N^3g_1^3g_2^6\\
&+\frac{184}{3N^4}\zeta_4g_1^8g_2+\frac{4747979}{69984}g_1^8g_2+\frac{640}{9N^3}\zeta_5g_1^7g_2^2+\frac{2543225}{34992}Ng_1^7g_2^2+\frac{224}{3N^4}\zeta_4g_1^6g_2^3+\frac{697}{9}\zeta_4g_1^7g_2^2\\
&+\frac{10405}{108}\zeta_3N^2g_1^4g_2^5+\frac{940}{9}\zeta_5Ng_1^5g_2^4+\frac{340}{3}\zeta_5Ng_1g_2^8+\frac{24206977}{209952}g_1^7g_2^2+\frac{3575}{27N}\zeta_3g_1^8g_2+\frac{1210}{9}\zeta_5Ng_1^7g_2^2\\
&+\frac{1600}{9}\zeta_5g_1g_2^8+\frac{76597949}{419904}Ng_1^5g_2^4+\frac{21445}{108N}\zeta_3g_1^9+\frac{9411335}{46656N}g_1^9+\frac{616}{3N^2}\zeta_4g_1^5g_2^4+\frac{680}{3N^3}\zeta_5g_1^8g_2\\
&+\frac{8194109}{34992}Ng_1^2g_2^7+\frac{6745957}{26244N^3}g_1^9+\frac{1129447}{4374N}g_1^7g_2^2+\frac{7522}{27}\zeta_3Ng_1^2g_2^7+\frac{3993463}{13122}g_1^3g_2^6+\frac{928}{3N^2}\zeta_4g_1^8g_2\\
&+\frac{34181}{108}\zeta_3Ng_1^6g_2^3+\frac{8800}{27}\zeta_3g_1^8g_2+\frac{3176}{9N^4}\zeta_4g_1^7g_2^2+\frac{12787787}{34992N}g_1^8g_2+\frac{20401}{54}\zeta_3g_1^6g_2^3+\frac{10279367}{26244}Ng_1^3g_2^6\\
&+\frac{12728}{27N}\zeta_3g_1g_2^8+\frac{1564}{3N^2}\zeta_4g_1^6g_2^3+\frac{4711087}{8748}g_1^2g_2^7+\frac{15233}{27N^3}\zeta_3g_1^9+\frac{103867}{162}\zeta_3Ng_1^3g_2^6+\frac{1928}{3N}\zeta_4g_1g_2^8\\
&+\frac{18790}{27}\zeta_3Ng_1^4g_2^5+\frac{6400}{9N^2}\zeta_5g_1g_2^8+\frac{7900}{9}\zeta_5g_1^7g_2^2+\frac{8285827}{8748N^3}g_1^8g_2+\frac{4544923}{4374N^2}g_1^5g_2^4+\frac{28106}{27N^3}\zeta_3g_1^8g_2\\
&+\frac{567739}{486N}g_1^4g_2^5+\frac{42951041}{34992}g_1^6g_2^3+\frac{4160}{3}\zeta_5g_1^3g_2^6+\frac{3335356}{2187N^2}g_1^2g_2^7+\frac{41752}{27N^2}\zeta_3g_1^4g_2^5+\frac{14080}{9N^4}\zeta_5g_1^7g_2^2\\
&+\frac{42343}{27N}\zeta_3g_1^5g_2^4+\frac{15104}{9N^4}\zeta_4g_1^5g_2^4+\frac{286175}{162N}\zeta_3g_1^7g_2^2+\frac{49024}{27N^2}\zeta_3g_1^2g_2^7+\frac{337373}{162}\zeta_3g_1^5g_2^4+\frac{19787}{9N}g_1g_2^8\\
&+\frac{7900}{3}\zeta_5g_1^4g_2^5+\frac{10913}{4N^3}g_1^7g_2^2+\frac{8896}{3N^2}\zeta_4g_1^4g_2^5+\frac{9280}{3N^2}\zeta_5g_1^6g_2^3+\frac{10240}{3N^4}\zeta_5g_1^6g_2^3+\frac{2503916}{729N^2}g_1^4g_2^5\\
&+\frac{107008}{27N^4}\zeta_3g_1g_2^8+\frac{117808}{27N^3}\zeta_3g_1^7g_2^2+\frac{45440}{9N^3}\zeta_5g_1^5g_2^4+\frac{20480}{3N}\zeta_5g_1^3g_2^6+\frac{572368}{81N^2}\zeta_3g_1^3g_2^6+\frac{25600}{3N^3}\zeta_5g_1^3g_2^6\\
&+\frac{56348995}{6561N^3}g_1^5g_2^4+\frac{28928}{3N^4}\zeta_4g_1^4g_2^5+\frac{29632}{3N^3}\zeta_3g_1^6g_2^3+\frac{32768}{3N^4}\zeta_4g_1g_2^8+\frac{2996960}{243N^4}g_1^4g_2^5+\frac{52480}{3N}\zeta_5g_1^2g_2^7\\
&+\frac{176896}{9N^4}\zeta_4g_1^3g_2^6+\frac{87147953}{4374N^3}g_1^6g_2^3+\frac{1677440}{81N^3}\zeta_3g_1^3g_2^6+\frac{1812992}{81N^3}\zeta_3g_1^5g_2^4+\frac{688256}{27N^3}\zeta_3g_1^2g_2^7+\frac{735392}{27N^3}\zeta_3g_1^4g_2^5\\
&+\frac{2517344}{81N^4}g_1g_2^8+\frac{295095736}{6561N^3}g_1^3g_2^6+\frac{154240}{3N^3}\zeta_5g_1^4g_2^5+\frac{159607016}{2187N^3}g_1^2g_2^7+\frac{655360}{3N^4}\zeta_5g_1^2g_2^7-\frac{153600}{N^3}\zeta_5g_1^2g_2^7\\
&-\frac{45568}{N^4}\zeta_4g_1^2g_2^7-\frac{20480}{N^4}\zeta_5g_1^3g_2^6-\frac{19200}{N^3}\zeta_5g_1^6g_2^3-\frac{16000}{N^2}\zeta_5g_1^4g_2^5-\frac{10240}{N^2}\zeta_5g_1^3g_2^6-\frac{7296}{N^3}\zeta_4g_1g_2^8
\end{aligned}
$$

$$-\frac{5956}{N}\zeta_3 g_1^4 g_2^5-\frac{2928}{N}\zeta_4 g_1^2 g_2^7-680\zeta_5 N g_1^2 g_2^7-\frac{640}{N^2}\zeta_5 g_1^5 g_2^4-\frac{320}{N^2}\zeta_5 g_1^9-\frac{200}{N^3}\zeta_5 g_1^9-140\zeta_5 g_1^5 g_2^4-84\zeta_4 g_1 g_2^8$$

$$-5\zeta_5 N^2 g_1 g_2^8+30\zeta_5 N^2 g_1^2 g_2^7+67\zeta_4 N g_1^2 g_2^7+85\zeta_5 N^2 g_1^5 g_2^4+\frac{140}{N}\zeta_4 g_1^4 g_2^5+\frac{256}{N^2}\zeta_4 g_1^2 g_2^7+320\zeta_4 g_1^2 g_2^7$$

$$+\frac{360}{N}\zeta_5 g_1^8 g_2+\frac{1600}{N}\zeta_5 g_1^4 g_2^5+\frac{1700}{N}\zeta_4 g_1^3 g_2^6+\frac{6640}{N}\zeta_5 g_1^6 g_2^3+\frac{25600}{N^3}\zeta_5 g_1 g_2^8+\frac{30848}{N^3}\zeta_4 g_1^2 g_2^7\Bigg]+\mathcal{O}(g_i^{11})$$

$$\beta_2(g_i)=-\frac{1}{2}\epsilon g_2+\Bigg[-\frac{1}{4}g_1^2 g_2+\frac{1}{4}N g_2^3+\frac{1}{2}g_1^3+\frac{3}{2}g_2^3-\frac{10}{N}g_2^3\Bigg]$$

$$+\Bigg[-\frac{2680}{9N^2}g_2^5-\frac{52}{9N}g_1^2 g_2^3-\frac{85}{36}N g_2^5-\frac{161}{72}g_1^4 g_2-\frac{5}{6}g_1^3 g_2^2-\frac{19}{36}N g_1^4 g_2-\frac{1}{12}g_1^5-\frac{5}{72}N^2 g_2^5-\frac{1}{24}N g_1^3 g_2^2$$

$$+\frac{5}{36}N g_1^2 g_2^3+\frac{1}{6N}g_1^5+\frac{7}{24}N g_1^5+\frac{31}{36}g_1^2 g_2^3+\frac{19}{3N}g_1^3 g_2^2+\frac{41}{6}g_2^5+\frac{323}{36N}g_1^4 g_2+\frac{794}{9N}g_2^5\Bigg]$$

$$+\Bigg[-\frac{1159112}{81N^3}g_2^7-\frac{4171}{16N}g_1^4 g_2^3-\frac{158879}{648}g_2^7-\frac{17153}{81N^2}g_1^2 g_2^5-\frac{2438}{27N^2}g_1^3 g_2^4-\frac{1147}{36N}g_1^5 g_2^2-\frac{21251}{864}g_1^4 g_2^3-\frac{44}{3N}g_2^7$$

$$-\frac{29}{2}\zeta_3 g_1^4 g_2^3-\frac{619}{72N}g_1^7-\frac{2201}{576}N g_1^6 g_2-\frac{553}{216}N g_1^3 g_2^4-\frac{3985}{2592}N g_1^2 g_2^5-\frac{3}{2}\zeta_3 N g_1^5 g_2^2-\frac{3}{2}\zeta_3 N g_1^7-\frac{869}{864}N^2 g_1^5 g_2^2$$

$$-\frac{535}{864}N^2 g_1^3 g_2^4-\frac{35}{5184}N^2 g_1^2 g_2^5+\frac{95}{5184}N^2 g_1^6 g_2+\frac{211}{5184}N^3 g_2^7+\frac{43}{432}N^2 g_1^7+\frac{119}{288}N g_1^5 g_2^2+\frac{155}{288N}g_1^6 g_2$$

$$+\frac{3}{4}\zeta_3 N^2 g_2^7+\frac{2827}{1728}N^2 g_1^4 g_2^3+\frac{7}{4}\zeta_3 N g_1^6 g_2+\frac{22}{9}N g_1^7+\frac{359}{144}N^2 g_2^7+\frac{713}{216N^2}g_1^7+\frac{199}{54}g_1^2 g_2^5+\frac{895}{216}g_1^5 g_2^2$$

$$+\frac{4319}{864}g_1^7+\frac{26111}{5184}g_1^6 g_2+\frac{23}{4}\zeta_3 N g_1^4 g_2^3+\frac{27}{4}\zeta_3 g_1^6 g_2+\frac{4319}{432}N g_1^4 g_2^3+\frac{815}{54}N g_2^7+\frac{295}{18}g_1^3 g_2^4+\frac{7961}{324N^2}g_1^6 g_2$$

$$+\frac{63}{2}\zeta_3 N g_2^7+\frac{2285}{54N}g_1^3 g_2^4+\frac{1247}{27N^2}g_1^5 g_2^2+\frac{19915}{324N}g_1^2 g_2^5+\frac{72311}{108N^2}g_1^4 g_2^3+\frac{170320}{27N^2}g_2^7-\frac{6784}{N^3}\zeta_3 g_2^7-\frac{368}{N}\zeta_3 g_2^7$$

$$-\frac{320}{N^2}\zeta_3 g_1^3 g_2^4-\frac{80}{N}\zeta_3 g_1^4 g_2^3-\frac{48}{N}\zeta_3 g_1^5 g_2^2-3\zeta_3 N g_1^3 g_2^4-\frac{1}{N}\zeta_3 g_1^6 g_2+\frac{1}{N^2}\zeta_3 g_1^6 g_2+8\zeta_3 g_1^3 g_2^4+10\zeta_3 g_2^7$$

$$+30\zeta_3 g_1^5 g_2^2+\frac{96}{N^2}\zeta_3 g_1^5 g_2^2+\frac{96}{N}\zeta_3 g_1^3 g_2^4+\frac{280}{N^2}\zeta_3 g_1^4 g_2^3+\frac{2528}{N^2}\zeta_3 g_2^7\Bigg]$$

$$+\Bigg[-\frac{10848256}{9N^4}\zeta_3 g_2^9-\frac{240076576}{243N^4}g_2^9-\frac{512000}{3N^4}\zeta_5 g_2^9-\frac{373760}{3N^2}\zeta_5 g_2^9-\frac{390584}{9N}\zeta_3 g_2^9-\frac{1096736}{27N^2}\zeta_3 g_2^9$$

$$-\frac{5870125}{162N}g_2^9-\frac{21869194}{729N^2}g_2^9-\frac{65078792}{2187N^2}g_1^4 g_2^5-\frac{141728}{9N^2}\zeta_3 g_1^3 g_2^6-\frac{413012}{27N^2}\zeta_3 g_1^4 g_2^5-\frac{8846024}{729N^3}g_1^3 g_2^6$$

$$-\frac{323840}{27N^3}\zeta_3 g_1^2 g_2^7-\frac{2749040}{243N^3}g_1^2 g_2^7-\frac{28160}{3N^3}\zeta_5 g_1^4 g_2^5-\frac{108626}{27N^2}\zeta_3 g_1^6 g_2^3-\frac{101477}{27N}\zeta_3 g_1^4 g_2^5-\frac{2365903}{729N^2}g_1^5 g_2^4$$

$$-\frac{5600}{3N^2}\zeta_4 g_1^2 g_2^7-\frac{5418287}{2916N^2}g_1^7 g_2^2-\frac{5542}{3N^2}\zeta_3 g_1^7 g_2^2-\frac{54418877}{34992N}g_1^4 g_2^5-\frac{4546}{3N}\zeta_3 g_1^5 g_2^4-\frac{2780}{3N^3}\zeta_4 g_1^6 g_2^3$$

$$-\frac{2140}{3N}\zeta_3 g_1^2 g_2^7-\frac{901973}{1458N}g_1^5 g_2^4-\frac{1760}{3}\zeta_5 N g_1^4 g_2^5-\frac{1085}{2}\zeta_5 N g_1^5 g_2^4-\frac{11268667}{23328}g_1^6 g_2^3-\frac{2418533}{5832}g_1^3 g_2^6$$

$$-\frac{6424625}{17496N^2}g_1^8 g_2-\frac{5993}{18}\zeta_3 N g_1^6 g_2^3-\frac{17929}{54}\zeta_3 g_1^6 g_2^3-\frac{5773}{18N^2}\zeta_3 g_1^8 g_2-\frac{535}{2}\zeta_5 N^2 g_2^9-\frac{4649}{24}\zeta_3 g_1^8 g_2$$

$$-\frac{1879}{12}g_1^2 g_2^7-\frac{29677}{216}\zeta_3 N^2 g_1^4 g_2^5-\frac{400}{3N}\zeta_5 g_1^6 g_2^3-\frac{257}{2N}\zeta_4 g_1^6 g_2^3-\frac{47225}{486N}g_1^2 g_2^7-\frac{274}{3N}\zeta_4 g_1^4 g_2^5$$

$$-\frac{4837}{54}\zeta_3 N^3 g_2^9-\frac{21086035}{279936}N^2 g_1^4 g_2^5-\frac{701}{12}\zeta_3 N^2 g_1^5 g_2^4-\frac{271543}{5184}N g_1^7 g_2^2-\frac{877}{18N^2}\zeta_3 g_1^9-\frac{523}{12N}\zeta_4 g_1^8 g_2$$

$$-\frac{313033}{7776}g_1^9-\frac{119}{3}\zeta_3 N g_1^3 g_2^6-\frac{225047}{5832N^2}g_1^9-\frac{1419031}{46656}N^3 g_2^9-\frac{80}{3N^3}\zeta_5 g_1^8 g_2-\frac{157}{6}\zeta_3 N^2 g_1^3 g_2^6$$

$$
\begin{aligned}
&-\frac{71}{3}\zeta_4 N g_1^2 g_2^7-\frac{2231}{108}\zeta_3 N^3 g_1^4 g_2^5-\frac{39}{2N^2}\zeta_4 g_1^9-\frac{858811}{46656}N g_1^5 g_2^4-\frac{637}{36}\zeta_3 N^2 g_2^9-\frac{387053}{23328}N^2 g_1^5 g_2^4\\
&-\frac{175}{12}\zeta_3 N^3 g_1^5 g_2^4-\frac{34085}{2592}N g_1^8 g_2-\frac{2713}{216}\zeta_3 N^2 g_1^6 g_2^3-\frac{25}{2}\zeta_5 g_1^9-\frac{15361}{1296}N g_1^9-\frac{31}{3}\zeta_4 g_1^2 g_2^7-\frac{19}{2}\zeta_4 N g_1^5 g_2^4\\
&-\frac{2653907}{279936}N^2 g_1^8 g_2-\frac{1007}{108}\zeta_3 N^4 g_2^9-\frac{2334161}{279936}N^3 g_1^4 g_2^5-\frac{59}{9}\zeta_3 N g_1^9-\frac{43}{8}\zeta_4 N g_1^8 g_2-\frac{235667}{46656}N^3 g_1^5 g_2^4\\
&-\frac{37}{8}\zeta_4 N g_1^9-\frac{47}{12}\zeta_3 N^2 g_1^9-\frac{23}{6}\zeta_4 N^2 g_1^6 g_2^3-\frac{327893}{93312}N^4 g_2^9-\frac{227}{72}\zeta_3 N^3 g_1^3 g_2^6-\frac{11}{6}\zeta_4 N^2 g_1^8 g_2\\
&-\frac{19}{12}\zeta_4 N^2 g_1^4 g_2^5-\frac{5}{4}\zeta_3 N^3 g_1^7 g_2^2-\frac{131}{108}\zeta_3 N g_1^8 g_2-\frac{45437}{46656}N^3 g_1^3 g_2^6-\frac{5}{6}\zeta_5 N^2 g_1^8 g_2-\frac{1}{2}\zeta_4 N^2 g_1^2 g_2^7\\
&-\frac{11}{24}\zeta_3 N^2 g_1^7 g_2^2-\frac{62069}{279936}N^3 g_1^8 g_2-\frac{7}{216}\zeta_3 N^3 g_1^9-\frac{1}{54}\zeta_3 N^3 g_1^2 g_2^7+\frac{377}{15552}N^3 g_1^2 g_2^7+\frac{1109}{15552}N^3 g_1^7 g_2^2\\
&+\frac{3463}{46656}N^3 g_1^9+\frac{73}{216}\zeta_3 N^3 g_1^8 g_2+\frac{25153}{69984}N^3 g_1^6 g_2^3+\frac{3}{8}\zeta_4 N^2 g_1^9+\frac{3}{8}\zeta_4 N^3 g_2^9+\frac{3589}{7776}N g_1^3 g_2^6+\frac{3}{4}\zeta_4 N^2 g_1^3 g_2^6\\
&+\frac{36557}{46656}N^2 g_1^9+\frac{16775}{15552}N^2 g_1^2 g_2^7+\frac{7}{6}\zeta_3 N^2 g_1^2 g_2^7+\frac{5}{4}\zeta_4 N g_1^4 g_2^5+\frac{445}{216}\zeta_3 N^3 g_1^6 g_2^3+\frac{19}{8}\zeta_4 N^2 g_1^7 g_2^2\\
&+\frac{33}{8}\zeta_4 N g_1^7 g_2^2+\frac{33703}{7776}N^2 g_1^3 g_2^6+\frac{43}{8}\zeta_4 N^2 g_1^5 g_2^4+\frac{1271}{216}\zeta_3 g_1^9+\frac{51}{8}\zeta_3 N^2 g_1^8 g_2+\frac{180395}{23328}N^2 g_1^7 g_2^2\\
&+\frac{67}{8}\zeta_4 g_1^9+\frac{117097}{11664}N^2 g_1^6 g_2^3+\frac{295}{27N^3}\zeta_3 g_1^9+\frac{85}{6}\zeta_5 N^4 g_2^9+\frac{7835}{486}N g_1^2 g_2^7+\frac{292831}{17496N^3}g_1^8 g_2+\frac{75}{4N}\zeta_4 g_1^9\\
&+\frac{149905}{7776N}g_1^9+\frac{251}{12}\zeta_4 N g_1^6 g_2^3+\frac{55}{2}\zeta_5 N g_1^9+\frac{233255}{7776}N^2 g_2^9+\frac{95}{3}\zeta_5 N^3 g_1^4 g_2^5+\frac{101}{3N^3}\zeta_4 g_1^8 g_2+\frac{3847}{108N}\zeta_3 g_1^8 g_2\\
&+\frac{212543}{5832N^3}g_1^9+\frac{1037}{27}\zeta_3 g_1^2 g_2^7+\frac{1681}{36N}\zeta_3 g_1^9+\frac{115}{2}\zeta_5 N^2 g_1^5 g_2^4+\frac{115}{2}\zeta_4 g_1^6 g_2^3+\frac{1091}{18}\zeta_3 N g_1^2 g_2^7+\frac{365}{6}\zeta_4 g_1^4 g_2^5\\
&+\frac{217}{3N^2}\zeta_4 g_1^8 g_2+\frac{22670675}{279936}g_1^8 g_2+\frac{739}{9}\zeta_3 N g_1^7 g_2^2+\frac{752}{9N}\zeta_3 g_1^3 g_2^6+\frac{169}{2}\zeta_4 N g_2^9+\frac{515}{6}\zeta_5 g_1^8 g_2\\
&+\frac{3134083}{34992}N g_1^6 g_2^3+\frac{4642033}{46656}g_1^7 g_2^2+\frac{310}{3}\zeta_5 N g_1^6 g_2^3+\frac{385}{3}\zeta_5 N^3 g_2^9+\frac{11360171}{69984}N g_1^4 g_2^5+\frac{500}{3}\zeta_5 g_1^6 g_2^3\\
&+\frac{18427}{108}\zeta_3 N g_1^4 g_2^5+\frac{6841}{36}\zeta_3 g_1^7 g_2^2+\frac{605}{3}\zeta_5 N^2 g_1^4 g_2^5+\frac{3829699}{15552N}g_1^8 g_2+\frac{6683}{27N^3}\zeta_3 g_1^8 g_2+\frac{802}{3N^2}\zeta_4 g_1^6 g_2^3\\
&+\frac{6431}{18}\zeta_3 N g_1^5 g_2^4+\frac{541387}{1458N^2}g_1^6 g_2^3+\frac{1280}{3N^2}\zeta_5 g_1^6 g_2^3+\frac{433877}{972N}g_1^3 g_2^6+\frac{1340}{3N}\zeta_5 g_1^8 g_2+\frac{3929587}{8748N}g_1^6 g_2^3\\
&+\frac{2713009}{5832}g_1^5 g_2^4+\frac{4606}{9N}\zeta_3 g_1^7 g_2^2+\frac{1437265}{2592N}g_1^7 g_2^2+\frac{2009}{3}\zeta_3 g_1^3 g_2^6+\frac{1572629}{2187N^3}g_1^6 g_2^3+\frac{18769}{18}\zeta_3 g_1^5 g_2^4\\
&+\frac{3232}{3N^3}\zeta_4 g_1^4 g_2^5+\frac{10136}{9N^3}\zeta_3 g_1^7 g_2^2+\frac{13584037}{11664}N g_2^9+\frac{4480}{3N^3}\zeta_5 g_1^6 g_2^3+\frac{1898587}{972N^3}g_1^7 g_2^2+\frac{114421}{54}\zeta_3 N g_2^9\\
&+\frac{115471}{54}\zeta_3 g_1^4 g_2^5+\frac{78273889}{34992}g_1^4 g_2^5+\frac{133711}{54N}\zeta_3 g_1^6 g_2^3+\frac{15937043}{5832}g_2^9+\frac{87008}{27}\zeta_3 g_2^9+\frac{2960806}{729N^3}g_1^5 g_2^4\\
&+\frac{39520}{9N^2}\zeta_3 g_1^2 g_2^7+\frac{1183219}{243N^2}g_1^2 g_2^7+\frac{134120}{27N^3}\zeta_3 g_1^6 g_2^3+\frac{15104}{3N^3}\zeta_4 g_1^2 g_2^7+\frac{48208}{9N^3}\zeta_3 g_1^5 g_2^4+\frac{1457795}{243N^2}g_1^3 g_2^6\\
&+\frac{21440}{3N}\zeta_5 g_1^4 g_2^5+\frac{21520}{3}\zeta_5 g_2^9+\frac{25600}{3N^2}\zeta_5 g_1^4 g_2^5+\frac{874720}{27N^3}\zeta_3 g_1^4 g_2^5+\frac{107200}{3N^3}\zeta_3 g_1^3 g_2^6+\frac{122777033}{2187N^3}g_1^4 g_2^5\\
&+\frac{214240}{3N}\zeta_5 g_2^9+\frac{439170008}{729N^3}g_2^9+\frac{19692160}{27N^3}\zeta_3 g_2^9-\frac{81920}{N^3}\zeta_5 g_2^9-\frac{81920}{N^3}\zeta_5 g_1^3 g_2^6-\frac{70912}{N^3}\zeta_4 g_2^9\\
&-\frac{13600}{N^2}\zeta_5 g_1^5 g_2^4-\frac{10112}{N^3}\zeta_4 g_1^3 g_2^6-4580\zeta_5 N g_2^9-3160\zeta_5 g_1^4 g_2^5-\frac{2844}{N^2}\zeta_3 g_1^5 g_2^4-1760\zeta_5 g_1^3 g_2^6-\frac{1280}{N^3}\zeta_5 g_1^7 g_2^2\\
&-940\zeta_5 g_1^5 g_2^4-784\zeta_4 g_2^9-\frac{664}{N}\zeta_4 g_1^3 g_2^6-\frac{640}{N^3}\zeta_4 g_1^5 g_2^4-610\zeta_5 g_1^7 g_2^2-\frac{584}{N^2}\zeta_4 g_1^4 g_2^5-\frac{230}{N}\zeta_4 g_1^7 g_2^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{216}{N^3}\zeta_4 g_1^7 g_2^2-\frac{190}{N^2}\zeta_5 g_1^8 g_2-156\zeta_4 g_1^5 g_2^4-40\zeta_5 N g_1^7 g_2^2-\frac{40}{N}\zeta_4 g_2^9-\frac{20}{N^3}\zeta_5 g_1^9-10\zeta_5 N g_1^8 g_2-10\zeta_5 N^2 g_1^7 g_2^2\\
&-\frac{5}{N}\zeta_5 g_1^9+\frac{1}{N^3}\zeta_4 g_1^9+5\zeta_5 N^2 g_1^9+5\zeta_5 N^3 g_1^3 g_2^6+7\zeta_4 g_1^7 g_2^2+18\zeta_4 N^2 g_2^9+19\zeta_4 g_1^8 g_2+20\zeta_5 N^3 g_1^5 g_2^4\\
&+\frac{30}{N^2}\zeta_5 g_1^9+30\zeta_5 N^2 g_1^6 g_2^3+40\zeta_5 N g_1^3 g_2^6+45\zeta_5 N^2 g_1^3 g_2^6+54\zeta_4 N g_1^3 g_2^6+59\zeta_4 g_1^3 g_2^6+\frac{280}{N}\zeta_4 g_1^2 g_2^7\\
&+\frac{336}{N^2}\zeta_4 g_1^5 g_2^4+\frac{370}{N^2}\zeta_4 g_1^7 g_2^2+\frac{400}{N}\zeta_5 g_1^7 g_2^2+\frac{440}{N}\zeta_4 g_1^5 g_2^4+\frac{640}{N}\zeta_5 g_1^3 g_2^6+\frac{2400}{N^2}\zeta_5 g_1^7 g_2^2+\frac{3616}{N^2}\zeta_4 g_1^3 g_2^6\\
&+\frac{5440}{N}\zeta_5 g_1^5 g_2^4+\frac{11552}{N^2}\zeta_4 g_2^9+\frac{22720}{N^3}\zeta_5 g_1^5 g_2^4+\frac{37120}{N^2}\zeta_5 g_1^3 g_2^6+\frac{135680}{N^4}\zeta_4 g_2^9\Bigg]+\mathcal{O}(g_i^{11}).
\end{aligned}
\tag{A2}
$$

Finally the entries of the mass mixing matrix are

$$
\begin{aligned}
-\gamma_{11}(g_i)=&\left[-\frac{2}{3N}g_1^2+\frac{1}{3}g_1^2+\frac{1}{3}Ng_1^2\right]\\
&+\Bigg[-\frac{67}{27N^2}g_1^4-\frac{20}{9N^2}g_1^3 g_2-\frac{29}{36}Ng_1^4-\frac{5}{9N}g_1^2 g_2^2-\frac{5}{12}Ng_1^3 g_2-\frac{10}{27}g_1^2 g_2^2-\frac{5}{36}N^2 g_1^3 g_2+\frac{1}{54}N^2 g_1^4\\
&\quad+\frac{5}{108}N^2 g_1^2 g_2^2+\frac{5}{36}Ng_1^2 g_2^2+\frac{41}{108}g_1^4+\frac{20}{27N^2}g_1^2 g_2^2+\frac{10}{9}g_1^3 g_2+\frac{5}{3N}g_1^3 g_2+\frac{26}{9N}g_1^4\Bigg]\\
&+\Bigg[-\frac{38006}{243N^3}g_1^4 g_2^2-\frac{20503}{324}g_1^4 g_2^2-\frac{4847}{81N^2}g_1^3 g_2^3-\frac{128}{3N^2}\zeta_3 g_1^4 g_2^2-\frac{8741}{324N}g_1^3 g_2^3-\frac{7855}{486N^3}g_1^6-\frac{40}{3N}\zeta_3 g_1^4 g_2^2\\
&\quad-\frac{7741}{648N}g_1^5 g_2-\frac{9289}{1296}g_1^6-\frac{13}{2}\zeta_3 g_1^6-\frac{394}{81N^2}g_1^2 g_2^4-\frac{2021}{486}Ng_1^4 g_2^2-\frac{2329}{648N}g_1^6-\frac{241}{81N^2}g_1^5 g_2-\frac{377}{162N}g_1^2 g_2^4\\
&\quad-\frac{1865}{1296}N^2 g_1^3 g_2^3-\frac{1741}{1296}N^2 g_1^5 g_2-\frac{4}{3N^3}\zeta_3 g_1^6-\frac{2}{3}\zeta_3 N^2 g_1^4 g_2^2-\frac{1247}{2592}N^3 g_1^5 g_2-\frac{19}{81}N^3 g_1^3 g_2^3+\frac{13}{648}N^3 g_1^2 g_2^4\\
&\quad+\frac{11}{324}N^2 g_1^2 g_2^4+\frac{85}{648}Ng_1^2 g_2^4+\frac{643}{3888}N^3 g_1^6+\frac{4427}{7776}N^3 g_1^4 g_2^2+\frac{2}{3}\zeta_3 Ng_1^4 g_2^2+\frac{175}{162}g_1^2 g_2^4+\frac{7}{6}\zeta_3 N^2 g_1^6\\
&\quad+\frac{9961}{7776}N^2 g_1^6+\frac{97}{32}Ng_1^6+\frac{4765}{1296}Ng_1^3 g_2^3+\frac{11741}{2592}Ng_1^5 g_2+\frac{4405}{864}N^2 g_1^4 g_2^2+\frac{17}{3}\zeta_3 Ng_1^6+\frac{160}{27N^3}g_1^2 g_2^4\\
&\quad+\frac{494}{81N^3}g_1^5 g_2+\frac{991}{162}g_1^5 g_2+\frac{40}{3}\zeta_3 g_1^4 g_2^2+\frac{1678}{81}g_1^3 g_2^3+\frac{10909}{486N^2}g_1^6+\frac{128}{3N^3}\zeta_3 g_1^4 g_2^2+\frac{11330}{243N}g_1^4 g_2^2\\
&\quad+\frac{5192}{81N^3}g_1^3 g_2^3+\frac{27791}{162N^2}g_1^4 g_2^2-\frac{32}{N^3}\zeta_3 g_1^5 g_2-\frac{17}{N}\zeta_3 g_1^6-6\zeta_3 g_1^5 g_2-2\zeta_3 Ng_1^5 g_2+\frac{16}{N}\zeta_3 g_1^5 g_2+\frac{18}{N^2}\zeta_3 g_1^6+\frac{24}{N^2}\zeta_3 g_1^5 g_2\Bigg]\\
&+\Bigg[-\frac{152960}{9N^3}\zeta_5 g_1^4 g_2^4-\frac{186496}{27N^4}\zeta_3 g_1^4 g_2^4-\frac{34351672}{6561N^4}g_1^4 g_2^4-\frac{41920}{9N^2}\zeta_5 g_1^4 g_2^4-\frac{10240}{3N^4}\zeta_5 g_1^5 g_2^3-\frac{271456}{81N}\zeta_3 g_1^4 g_2^4\\
&\quad-\frac{32009185}{13122N}g_1^4 g_2^4-\frac{11955619}{6561N^4}g_1^6 g_2^2-\frac{123136}{81N^4}\zeta_3 g_1^6 g_2^2-\frac{4160}{3N^3}\zeta_5 g_1^6 g_2^2-\frac{28576}{27N^3}\zeta_3 g_1^5 g_2^3-\frac{9472}{9N^4}\zeta_4 g_1^4 g_2^4\\
&\quad-\frac{76852}{81N}\zeta_3 g_1^6 g_2^2-\frac{2720}{3}\zeta_5 Ng_1^4 g_2^4-\frac{3947033}{4374N}g_1^5 g_2^3-\frac{624742}{729N^3}g_1^3 g_2^5-\frac{56072}{81N^2}\zeta_3 g_1^6 g_2^2-\frac{3658796}{6561N^3}g_1^2 g_2^6\\
&\quad-\frac{1360}{3N^2}\zeta_5 g_1^7 g_2-\frac{2833546}{6561N}g_1^6 g_2^2-\frac{11008}{27N^4}\zeta_3 g_1^3 g_2^5-\frac{3192103}{8748N^2}g_1^7 g_2-\frac{9643}{27N^2}\zeta_3 g_1^8-\frac{3160}{9N^2}\zeta_5 g_1^6 g_2^2\\
&\quad-\frac{8272}{27N^4}\zeta_3 g_1^7 g_2-\frac{2534}{9N^3}\zeta_4 g_1^6 g_2^2-\frac{2534}{9N}\zeta_3 g_1^5 g_2^3-\frac{4967}{18}\zeta_3 g_1^5 g_2^3-\frac{7144}{27N^2}\zeta_3 g_1^7 g_2-\frac{21112}{81}\zeta_3 g_1^4 g_2^4\\
&\quad-\frac{563068}{2187N^4}g_1^5 g_2^3-\frac{742}{3N}\zeta_4 g_1^5 g_2^3-\frac{351341}{1458N^2}g_1^3 g_2^5-\frac{5452}{27N}\zeta_3 g_1^3 g_2^5-\frac{1676}{9N^2}\zeta_4 g_1^6 g_2^2-\frac{497753}{2916N^2}g_1^5 g_2^3\\
&\quad-\frac{1067132}{6561N^2}g_1^2 g_2^6-\frac{1280}{9N^4}\zeta_4 g_1^2 g_2^6-\frac{22805}{162N}\zeta_3 g_1^8-\frac{14011867}{104976N^2}g_1^8-\frac{10496}{81N^4}\zeta_3 g_1^2 g_2^6-\frac{1635631}{13122N^4}g_1^8
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{355}{3}\zeta_4 g_1^7 g_2-\frac{1000}{9N^3}\zeta_5 g_1^8-\frac{3819251}{34992N}g_1^7 g_2-\frac{880}{9N^4}\zeta_5 g_1^8-\frac{211}{3N}\zeta_3 g_1^7 g_2-\frac{200}{3}\zeta_5 N^2 g_1^5 g_2^3-\frac{485}{9}\zeta_5 N^2 g_1^4 g_2^4\\
&-\frac{136}{3N^2}\zeta_4 g_1^5 g_2^3-\frac{3568}{81N}\zeta_3 g_1^2 g_2^6-\frac{119405}{2916}N g_1^3 g_2^5-\frac{16517473}{419904}N g_1^6 g_2^2-\frac{24895}{648}\zeta_3 N^2 g_1^6 g_2^2-\frac{1240519}{46656}N g_1^8\\
&-\frac{2849}{108}\zeta_3 N^3 g_1^5 g_2^3-\frac{16027}{648}\zeta_3 N^3 g_1^4 g_2^4-\frac{3949}{216}\zeta_3 N^2 g_1^7 g_2-\frac{1265}{72}\zeta_4 N g_1^6 g_2^2-\frac{448}{27}\zeta_3 g_1^3 g_2^5\\
&-\frac{4054259}{279936}N^2 g_1^7 g_2-\frac{12041831}{839808}N^3 g_1^4 g_2^4-\frac{474407}{34992}N^3 g_1^5 g_2^3-\frac{139}{12}\zeta_4 N g_1^7 g_2-\frac{100}{9}\zeta_5 g_1^6 g_2^2-\frac{8902499}{839808}N^2 g_1^8\\
&-\frac{972859}{104976}N g_1^2 g_2^6-\frac{5453}{648}\zeta_3 N g_1^8-\frac{25}{3}\zeta_5 N^2 g_1^7 g_2-\frac{577589}{69984N}g_1^8-\frac{1207}{162}\zeta_3 N^3 g_1^6 g_2^2-\frac{125}{18}\zeta_4 N^2 g_1^6 g_2^2\\
&-\frac{115}{18}\zeta_5 N^2 g_1^8-\frac{35}{6}\zeta_5 N g_1^8-\frac{1181}{216}\zeta_3 N^4 g_1^5 g_2^3-\frac{55}{12}\zeta_4 N^2 g_1^5 g_2^3-\frac{198325}{46656}N^2 g_1^3 g_2^5-\frac{71}{18}\zeta_3 N^3 g_1^7 g_2\\
&-\frac{421}{108}\zeta_3 N^2 g_1^8-\frac{271}{72}\zeta_4 N^2 g_1^8-\frac{41}{12N}\zeta_4 g_1^8-\frac{245}{72}\zeta_3 N^4 g_1^4 g_2^4-\frac{113}{36}\zeta_4 N^2 g_1^4 g_2^4-\frac{322637}{104976}g_1^4 g_2^4\\
&-\frac{55}{18}\zeta_5 N^2 g_1^6 g_2^2-\frac{348061}{139968}N^3 g_1^7 g_2-\frac{1553747}{839808}N^4 g_1^4 g_2^4-\frac{5}{3}\zeta_4 g_1^2 g_2^6-\frac{369623}{279936}N^4 g_1^5 g_2^3-\frac{35}{36}\zeta_4 N^3 g_1^4 g_2^4\\
&-\frac{101}{108}\zeta_3 N^3 g_1^3 g_2^5-\frac{775529}{839808}N^3 g_1^8-\frac{29}{36}\zeta_4 N^3 g_1^8-\frac{554189}{839808}N^4 g_1^6 g_2^2-\frac{5}{9}\zeta_5 N^3 g_1^8-\frac{43}{81}\zeta_3 g_1^2 g_2^6\\
&-\frac{31}{72}\zeta_4 N^3 g_1^6 g_2^2-\frac{91}{216}\zeta_3 N^4 g_1^7 g_2-\frac{89}{324}\zeta_3 N^2 g_1^2 g_2^6-\frac{1}{4}\zeta_4 N^3 g_1^5 g_2^3-\frac{17}{108}\zeta_3 N^4 g_1^3 g_2^5-\frac{89}{648}\zeta_3 N^4 g_1^6 g_2^2\\
&-\frac{1}{12}\zeta_4 N^3 g_1^2 g_2^6-\frac{5183}{104976}N^4 g_1^8-\frac{11}{324}\zeta_3 N^3 g_1^2 g_2^6-\frac{1}{36}\zeta_4 N^2 g_1^2 g_2^6-\frac{7907}{419904}N^4 g_1^2 g_2^6+\frac{1}{81}\zeta_3 N^4 g_1^2 g_2^6\\
&+\frac{2}{27}\zeta_3 N^4 g_1^8+\frac{35515}{419904}N^3 g_1^2 g_2^6+\frac{13765}{93312}N^4 g_1^7 g_2+\frac{7649}{46656}N^4 g_1^3 g_2^5+\frac{1}{6}\zeta_4 N^2 g_1^3 g_2^5+\frac{1}{2}\zeta_4 N^3 g_1^3 g_2^5\\
&+\frac{211}{324}\zeta_3 N^3 g_1^8+\frac{84637}{104976}N^2 g_1^2 g_2^6+\frac{6991}{5832}N^3 g_1^3 g_2^5+\frac{5}{3}\zeta_5 N^4 g_1^6 g_2^2+\frac{16}{9N^4}\zeta_4 g_1^8+\frac{13}{6}\zeta_4 N^3 g_1^7 g_2\\
&+\frac{7}{3}\zeta_3 N g_1^7 g_2+\frac{227}{81}\zeta_3 N g_1^2 g_2^6+\frac{53}{18}\zeta_3 N^2 g_1^3 g_2^5+\frac{10}{3}\zeta_5 N^3 g_1^7 g_2+\frac{43951}{13122}g_1^2 g_2^6+\frac{1610017}{419904}N^3 g_1^6 g_2^2\\
&+\frac{361537}{93312}N^2 g_1^5 g_2^3+\frac{71}{18}\zeta_4 N g_1^8+\frac{311}{72}\zeta_4 g_1^8+\frac{55}{9}\zeta_5 N^4 g_1^4 g_2^4+\frac{20}{3}\zeta_5 N^4 g_1^5 g_2^3+\frac{649073}{93312}N^2 g_1^6 g_2^2\\
&+\frac{15}{2}\zeta_5 N^3 g_1^6 g_2^2+\frac{3393971}{419904}N^2 g_1^4 g_2^4+\frac{157}{12}\zeta_4 N^2 g_1^7 g_2+\frac{79}{6}\zeta_4 N g_1^5 g_2^3+\frac{674}{27}\zeta_3 N g_1^3 g_2^5+\frac{3669337}{139968}N g_1^7 g_2\\
&+\frac{485}{18N^2}\zeta_4 g_1^8+\frac{319627}{11664}g_1^3 g_2^5+\frac{97}{3}\zeta_4 g_1^4 g_2^4+\frac{100}{3}\zeta_5 N^3 g_1^5 g_2^3+\frac{5551}{162}\zeta_3 N^2 g_1^4 g_2^4+\frac{106}{3}\zeta_4 N g_1^4 g_2^4\\
&+\frac{1006}{27N^4}\zeta_3 g_1^8+\frac{112}{3N^4}\zeta_4 g_1^7 g_2+\frac{3088}{81N^2}\zeta_3 g_1^2 g_2^6+\frac{128}{3N^2}\zeta_4 g_1^2 g_2^6+\frac{410}{9}\zeta_5 N^3 g_1^4 g_2^4+\frac{21546947}{419904}g_1^8\\
&+\frac{421}{8}\zeta_3 N^2 g_1^5 g_2^3+\frac{1013}{18}\zeta_4 g_1^6 g_2^2+\frac{4506535}{69984}g_1^5 g_2^3+\frac{197}{3}\zeta_4 g_1^5 g_2^3+\frac{5017}{72}\zeta_3 g_1^8+\frac{15024445}{209952}g_1^6 g_2^2\\
&+\frac{224}{3N}\zeta_4 g_1^7 g_2+\frac{8039}{81}\zeta_3 N g_1^6 g_2^2+\frac{438491}{4374N^2}g_1^6 g_2^2+\frac{3544}{27N^2}\zeta_3 g_1^3 g_2^5+\frac{10816}{81N^3}\zeta_3 g_1^2 g_2^6+\frac{9612793}{69984}g_1^7 g_2\\
&+\frac{1328}{9N}\zeta_4 g_1^6 g_2^2+\frac{448}{3N^3}\zeta_4 g_1^2 g_2^6+\frac{36694}{243N^4}g_1^7 g_2+\frac{991}{6}\zeta_3 g_1^7 g_2+\frac{500}{3N}\zeta_5 g_1^8+\frac{500}{3}\zeta_5 g_1^7 g_2\\
&+\frac{1149035}{6561N}g_1^2 g_2^6+\frac{771079}{4374N^3}g_1^7 g_2+\frac{3732565}{17496}N g_1^5 g_2^3+\frac{4357}{18}\zeta_3 N g_1^5 g_2^3+\frac{13291333}{52488N^3}g_1^8+\frac{764}{3N^2}\zeta_4 g_1^7 g_2\\
&+\frac{60174143}{209952}N g_1^4 g_2^4+\frac{2600}{9N^4}\zeta_4 g_1^6 g_2^2+\frac{523217}{1458N}g_1^3 g_2^5+\frac{9844}{27N^2}\zeta_3 g_1^5 g_2^3+\frac{32608}{81N^3}\zeta_3 g_1^8+\frac{7631}{18}\zeta_3 g_1^6 g_2^2\\
&+\frac{1340}{3N}\zeta_5 g_1^6 g_2^2+\frac{74671}{162}\zeta_3 N g_1^4 g_2^4+\frac{12640}{27N^3}\zeta_3 g_1^3 g_2^5+\frac{4468}{9N^3}\zeta_3 g_1^7 g_2+\frac{1600}{3N^2}\zeta_5 g_1^5 g_2^3+\frac{3610000}{6561N^4}g_1^2 g_2^6
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2000}{3N^3}\zeta_5 g_1^7 g_2+\frac{550784}{729N^4}g_1^3g_2^5+\frac{2384}{3N^3}\zeta_4 g_1^5g_2^3+\frac{2560}{3N^4}\zeta_4 g_1^3g_2^5+\frac{8896}{9N^4}\zeta_3 g_1^5g_2^3+\frac{2326564}{2187N^3}g_1^5g_2^3\\
&+\frac{11168}{9N^3}\zeta_4 g_1^4g_2^4+\frac{8596958}{6561N^2}g_1^4g_2^4+\frac{12160}{9N^4}\zeta_5 g_1^6g_2^2+\frac{55414141}{26244N^3}g_1^6g_2^2+\frac{197552}{81N^2}\zeta_3 g_1^4g_2^4+\frac{217408}{81N^3}\zeta_3 g_1^6g_2^2\\
&+\frac{8960}{3N^3}\zeta_5 g_1^5g_2^3+\frac{39952214}{6561N^3}g_1^4g_2^4+\frac{21440}{3N}\zeta_5 g_1^4g_2^4+\frac{616672}{81N^3}\zeta_3 g_1^4g_2^4+\frac{135680}{9N^4}\zeta_5 g_1^4g_2^4-\frac{896}{N^3}\zeta_4 g_1^3g_2^5\\
&-\frac{576}{N^4}\zeta_4 g_1^5g_2^3-\frac{436}{N}\zeta_4 g_1^4g_2^4-\frac{320}{N^4}\zeta_5 g_1^7g_2-320\zeta_5 N g_1^5g_2^3-\frac{256}{N^2}\zeta_4 g_1^3g_2^5-\frac{252}{N^3}\zeta_4 g_1^7g_2-55\zeta_5 N g_1^6g_2^2\\
&-55\zeta_5 N g_1^7 g_2-\frac{52}{N}\zeta_4 g_1^2g_2^6-\frac{40}{N^2}\zeta_5 g_1^8-\frac{29}{N^3}\zeta_4 g_1^8-24\zeta_4 N g_1^3g_2^5+4\zeta_4 N g_1^2g_2^6+10\zeta_4 g_1^3g_2^5+95\zeta_5 g_1^8\\
&+\frac{184}{N^2}\zeta_4 g_1^4g_2^4+240\zeta_5 g_1^5g_2^3+\frac{312}{N}\zeta_4 g_1^3g_2^5+340\zeta_5 g_1^4g_2^4\Bigg]+\mathcal{O}(g_i^{10})
\end{aligned}
$$

$$
\begin{aligned}
-\gamma_{21}(g_i)={}&g_1^2+\left[-\frac{3}{2}Ng_1^3g_2-\frac{1}{2}Ng_1^2g_2^2-\frac{1}{6}g_1^4+\frac{1}{3N}g_1^4+\frac{1}{3}Ng_1^4-3g_1^3g_2-g_1^2g_2^2+\frac{4}{N}g_1^2g_2^2+\frac{12}{N}g_1^3g_2\right]\\
&+\Bigg[-\frac{3362}{27N}g_1^3g_2^3-\frac{704}{9N^2}g_1^2g_2^4-\frac{547}{9N}g_1^4g_2^2-\frac{355}{9}g_1^3g_2^3-\frac{139}{9N}g_1^6-\frac{104}{9}g_1^4g_2^2-\frac{749}{108N}g_1^5g_2-\frac{1283}{432}Ng_1^5g_2\\
&-\frac{601}{432}N^2g_1^5g_2+\frac{229}{432}N^2g_1^6+\frac{89}{144}N^2g_1^4g_2^2+\frac{37}{36}N^2g_1^2g_2^4+\frac{7}{6}g_1^2g_2^4+\frac{67}{36}Ng_1^2g_2^4+\frac{449}{216}N^2g_1^3g_2^3+\frac{689}{144}Ng_1^6\\
&+\frac{1123}{216}Ng_1^4g_2^2+\frac{353}{54}g_1^5g_2+\frac{713}{108N^2}g_1^6+\frac{2953}{432}g_1^6+\frac{1159}{108}Ng_1^3g_2^3+\frac{190}{9N}g_1^2g_2^4+\frac{911}{27N^2}g_1^5g_2+\frac{3140}{27N^2}g_1^4g_2^2\\
&+\frac{7768}{27N^2}g_1^3g_2^3-\frac{96}{N}\zeta_3g_1^3g_2^3-\frac{32}{N}\zeta_3g_1^4g_2^2-24\zeta_3g_1^3g_2^3-4\zeta_3g_1^4g_2^2-2\zeta_3Ng_1^6+2\zeta_3Ng_1^4g_2^2+4\zeta_3Ng_1^3g_2^3\\
&+\frac{64}{N^2}\zeta_3g_1^4g_2^2+\frac{256}{N^2}\zeta_3g_1^3g_2^3\Bigg]\\
&+\Bigg[-\frac{2305340}{243N^2}g_1^3g_2^5-\frac{219520}{27N^2}\zeta_3g_1^3g_2^5-\frac{649568}{81N^3}g_1^2g_2^6-\frac{15680}{3N^2}\zeta_5g_1^4g_2^4-\frac{14080}{3N^2}\zeta_5g_1^5g_2^3-\frac{12032}{3N^3}\zeta_4g_1^3g_2^5\\
&-\frac{31280}{9N^2}\zeta_3g_1^4g_2^4-\frac{10240}{3N^3}\zeta_5g_1^6g_2^2-\frac{1003757}{324N^2}g_1^6g_2^2-\frac{232699}{81N^2}g_1^4g_2^4-\frac{72392}{27N^2}\zeta_3g_1^6g_2^2-\frac{19328}{9N^2}\zeta_3g_1^2g_2^6\\
&-\frac{380203}{243N}g_1^3g_2^5-\frac{334348}{243N^3}g_1^5g_2^3-\frac{35168}{27N}\zeta_3g_1^3g_2^5-\frac{28196}{27N}\zeta_3g_1^4g_2^4-\frac{2560}{3N^3}\zeta_4g_1^4g_2^4-\frac{2080}{3}\zeta_5g_1^6g_2^2\\
&-\frac{5744}{9N^2}\zeta_3g_1^5g_2^3-\frac{1424}{3N^2}\zeta_3g_1^7g_2-\frac{111998}{243N^2}g_1^7g_2-\frac{1280}{3N}\zeta_5g_1^5g_2^3-\frac{1024}{3N^3}\zeta_4g_1^5g_2^3-\frac{3008}{9N}\zeta_3g_1^2g_2^6-\frac{800}{3}\zeta_5g_1^3g_2^5\\
&-\frac{4759}{18}g_1^2g_2^6-\frac{85627}{324N}g_1^5g_2^3-\frac{512}{3N^3}\zeta_4g_1^6g_2^2-\frac{440}{3N^2}\zeta_5g_1^7g_2-\frac{3844}{27}\zeta_3g_1^5g_2^3-\frac{251129}{1944}g_1^5g_2^3-\frac{368}{3}\zeta_4g_1^3g_2^5\\
&-\frac{290}{3}\zeta_5Ng_1^4g_2^4-\frac{857}{9}\zeta_3Ng_1^5g_2^3-\frac{237713}{2916N^2}g_1^8-\frac{208}{3}\zeta_4g_1^4g_2^4-\frac{620}{9N}\zeta_3g_1^7g_2-\frac{344017}{5832}Ng_1^6g_2^2\\
&-\frac{438583}{7776}Ng_1^7g_2-\frac{160}{3N^3}\zeta_5g_1^7g_2-\frac{154}{3}\zeta_4g_1^5g_2^3-\frac{369593}{7776}g_1^8-\frac{90533}{1944}N^2g_1^3g_2^5-\frac{128}{3}\zeta_4g_1^6g_2^2-\frac{1004}{27}\zeta_3Ng_1^6g_2^2\\
&-\frac{95}{3}\zeta_5N^2g_1^5g_2^3-\frac{272}{9N^3}g_1^7g_2-\frac{725}{27}\zeta_3N^2g_1^4g_2^4-\frac{1369}{54}\zeta_3N^2g_1^3g_2^5-\frac{65}{3}\zeta_5g_1^8-\frac{50}{3N}\zeta_5g_1^8\\
&-\frac{293}{18}\zeta_3N^2g_1^6g_2^2-\frac{41189}{2592}N^2g_1^7g_2-\frac{10673}{972}N^2g_1^4g_2^4-\frac{78043}{7776}Ng_1^8-\frac{9113}{972N}g_1^4g_2^4-\frac{967}{108}\zeta_3N^2g_1^8\\
&-\frac{22861}{2592}N^2g_1^6g_2^2-\frac{20}{3}\zeta_5N^3g_1^5g_2^3-\frac{50}{9}\zeta_3N^2g_1^2g_2^6-\frac{6859}{1296}N^2g_1^2g_2^6-\frac{13661}{2592}N^3g_1^3g_2^5-\frac{5833}{1296}Ng_1^4g_2^4\\
&-\frac{65}{18}\zeta_3N^3g_1^3g_2^5-\frac{47}{18}\zeta_3N^3g_1^4g_2^4-\frac{9803}{3888}N^3g_1^4g_2^4-\frac{775}{324}N^3g_1^2g_2^6-\frac{3}{2}\zeta_4N^2g_1^7g_2-\frac{8}{9}\zeta_3N^3g_1^7g_2\\
&-\frac{20129}{23328}N^3g_1^6g_2^2-\frac{7}{9}\zeta_3N^3g_1^6g_2^2-\frac{7}{12}\zeta_4N^2g_1^8-\frac{1205}{2592}N^3g_1^7g_2+\frac{71}{2916}N^3g_1^8+\frac{5}{27}\zeta_3N^3g_1^8+\frac{1}{3}\zeta_4N^2g_1^4g_2^4
\end{aligned}
$$

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\begin{aligned}
&+\frac{1}{2}\zeta_4 N^2 g_1^6 g_2^2+\frac{5}{6}\zeta_4 N^2 g_1^3 g_2^5+\frac{5}{3}\zeta_4 N^2 g_1^5 g_2^3+\frac{11}{6}\zeta_4 N g_1^6 g_2^2+\frac{29473}{11664}N^2 g_1^8+\frac{10}{3}\zeta_5 N^3 g_1^4 g_2^4+\frac{10}{3}\zeta_5 N^3 g_1^6 g_2^2\\
&+\frac{8803}{2592}N^3 g_1^5 g_2^3+\frac{71}{18}\zeta_3 N^3 g_1^5 g_2^3+\frac{20341}{3888}N g_1^5 g_2^3+\frac{20}{3}\zeta_5 N g_1^6 g_2^2+\frac{20}{3}\zeta_5 N^3 g_1^3 g_2^5+\frac{23}{3}\zeta_4 N g_1^3 g_2^5\\
&+\frac{25}{3}\zeta_5 N^2 g_1^4 g_2^4+\frac{88}{9}\zeta_3 N g_1^2 g_2^6+\frac{31273}{2592}N^2 g_1^5 g_2^3+\frac{161}{12}\zeta_4 g_1^8+\frac{769}{54}\zeta_3 N^2 g_1^5 g_2^3+\frac{1873}{108}\zeta_3 g_1^8\\
&+\frac{11675}{648}N g_1^2 g_2^6+\frac{3193}{162N}g_1^8+\frac{181}{9}\zeta_3 N^2 g_1^7 g_2+\frac{590}{27N^3}\zeta_3 g_1^8+\frac{100}{3}\zeta_5 N^2 g_1^3 g_2^5+\frac{67}{2N}\zeta_4 g_1^8+\frac{110}{3}\zeta_5 N g_1^5 g_2^3\\
&+\frac{110}{3}\zeta_5 N^2 g_1^6 g_2^2+\frac{130}{3}\zeta_5 N g_1^8+\frac{212543}{2916N^3}g_1^8+\frac{2033}{27}\zeta_3 N g_1^3 g_2^5+\frac{227}{3}\zeta_3 N g_1^4 g_2^4+\frac{80923}{972}N g_1^3 g_2^5+\frac{1573}{18N}\zeta_3 g_1^8\\
&+\frac{304}{3N}\zeta_4 g_1^4 g_2^4+\frac{152729}{1296}g_1^7 g_2+\frac{560}{3N}\zeta_4 g_1^5 g_2^3+\frac{592}{3N}\zeta_4 g_1^3 g_2^5+\frac{640}{3N}\zeta_5 g_1^3 g_2^5+\frac{102509}{432}g_1^6 g_2^2+\frac{712}{3}\zeta_3 g_1^2 g_2^6\\
&+\frac{78631}{324}g_1^4 g_2^4+\frac{9784}{27}\zeta_3 g_1^6 g_2^2+\frac{117445}{324N}g_1^7 g_2+\frac{1096}{3N^2}\zeta_4 g_1^6 g_2^2+\frac{1120}{3N}\zeta_5 g_1^4 g_2^4+\frac{3584}{9N^3}\zeta_3 g_1^7 g_2+\frac{3656}{9N}\zeta_3 g_1^6 g_2^2\\
&+\frac{37805}{81N}g_1^2 g_2^6+\frac{1460}{3N}\zeta_5 g_1^7 g_2+\frac{1600}{3}\zeta_5 g_1^5 g_2^3+\frac{1657057}{2916N}g_1^6 g_2^2+\frac{16064}{27N^3}\zeta_3 g_1^5 g_2^3+\frac{16616}{27}\zeta_3 g_1^4 g_2^4\\
&+\frac{17792}{27N}\zeta_3 g_1^5 g_2^3+\frac{23788}{27}\zeta_3 g_1^3 g_2^5+\frac{1161749}{972}g_1^3 g_2^5+\frac{337690}{243N^2}g_1^5 g_2^3+\frac{54112}{27N^3}\zeta_3 g_1^6 g_2^2+\frac{6080}{3N^2}\zeta_4 g_1^3 g_2^5\\
&+\frac{2410019}{729N^3}g_1^6 g_2^2+\frac{11264}{3N^3}\zeta_3 g_1^2 g_2^6+\frac{12160}{3N^2}\zeta_5 g_1^6 g_2^2+\frac{336220}{81N^2}g_1^2 g_2^6+\frac{1075616}{243N^3}g_1^4 g_2^4+\frac{162560}{27N^3}\zeta_3 g_1^4 g_2^4\\
&+\frac{20480}{3N^3}\zeta_5 g_1^3 g_2^5+\frac{24320}{3N^3}\zeta_5 g_1^4 g_2^4+\frac{385024}{27N^3}\zeta_3 g_1^3 g_2^5+\frac{3928400}{243N^3}g_1^3 g_2^5-\frac{2560}{N^2}\zeta_5 g_1^3 g_2^5-\frac{1536}{N^3}\zeta_4 g_1^2 g_2^6\\
&-\frac{160}{N}\zeta_5 g_1^6 g_2^2-120\zeta_5 N g_1^3 g_2^5-100\zeta_3 g_1^7 g_2-95\zeta_5 N g_1^7 g_2-\frac{89}{N^2}\zeta_3 g_1^8-72\zeta_4 g_1^2 g_2^6-\frac{66}{N}\zeta_4 g_1^6 g_2^2-\frac{40}{N^3}\zeta_5 g_1^8\\
&-\frac{35}{N^2}\zeta_4 g_1^8-\frac{34}{N}\zeta_4 g_1^7 g_2-26\zeta_3 N g_1^8-20\zeta_5 N^2 g_1^7 g_2-15\zeta_4 N g_1^5 g_2^3-10\zeta_4 N g_1^4 g_2^4-9\zeta_4 N g_1^8-2\zeta_4 N g_1^2 g_2^6\\
&-2\zeta_4 N g_1^7 g_2+\zeta_4 N^2 g_1^2 g_2^6+\frac{2}{N^3}\zeta_4 g_1^8+5\zeta_4 g_1^7 g_2+10\zeta_5 N^2 g_1^8+\frac{56}{N^2}\zeta_4 g_1^7 g_2+\frac{60}{N^2}\zeta_5 g_1^8+\frac{64}{N^3}\zeta_4 g_1^7 g_2\\
&+65\zeta_3 N g_1^7 g_2+70\zeta_5 g_1^7 g_2+\frac{96}{N^2}\zeta_4 g_1^5 g_2^3+\frac{112}{N}\zeta_4 g_1^2 g_2^6+240\zeta_5 g_1^4 g_2^4+\frac{576}{N^2}\zeta_4 g_1^4 g_2^4+\frac{832}{N^2}\zeta_4 g_1^2 g_2^6\\
&\left.+\frac{7680}{N^3}\zeta_5 g_1^5 g_2^3\right]+\mathcal{O}(g_i^{10})
\end{aligned}
$$

$$
\begin{aligned}
-\gamma_{12}(g_i)=&\left[\frac{1}{2}g_1^2+\frac{1}{2}N g_1^2-\frac{1}{N}g_1^2\right]\\
&+\left[-\frac{9}{4}N g_1^3 g_2-\frac{10}{9N^2}g_1^4-\frac{8}{9N^2}g_1^2 g_2^2-\frac{3}{4}N^2 g_1^3 g_2-\frac{1}{6}N g_1^2 g_2^2-\frac{1}{9}N g_1^4-\frac{1}{18}N^2 g_1^2 g_2^2-\frac{1}{36}N^2 g_1^4+\frac{4}{9}g_1^2 g_2^2\right.\\
&\left.\quad+\frac{19}{36}g_1^4+\frac{2}{3N}g_1^2 g_2^2+\frac{13}{18N}g_1^4-\frac{12}{N^2}g_1^3 g_2+6g_1^3 g_2+\frac{9}{N}g_1^3 g_2\right]\\
&+\left[-\frac{13220}{81N^3}g_1^2 g_2^4-\frac{10589}{81N^3}g_1^4 g_2^2-\frac{37925}{648}g_1^2 g_2^4-\frac{487}{9N^3}g_1^5 g_2-\frac{62551}{1296}g_1^4 g_2^2-\frac{179}{12}g_1^3 g_2^3-\frac{2075}{144}g_1^5 g_2-\frac{8533}{648N}g_1^6\right.\\
&\quad-\frac{14005}{1296}N g_1^2 g_2^4-\frac{22415}{2592}N g_1^4 g_2^2-\frac{14743}{2592}g_1^6-\frac{7}{2}\zeta_3 N^2 g_1^2 g_2^4-\frac{869}{324N^3}g_1^6-\frac{5}{3}N g_1^3 g_2^3-\frac{3}{2}\zeta_3 N^2 g_1^5 g_2\\
&\quad-\frac{35}{48}N^2 g_1^3 g_2^3-\frac{1}{2}\zeta_3 N^3 g_1^2 g_2^4-\frac{1}{2}\zeta_3 N^3 g_1^5 g_2-\frac{17}{48}N^3 g_1^3 g_2^3-\frac{41}{288}N^3 g_1^5 g_2-\frac{1}{288}N g_1^5 g_2+\frac{755}{2592}N^3 g_1^6\\
&\quad+\frac{23}{72}N^2 g_1^5 g_2+\frac{1}{2}\zeta_3 N^3 g_1^3 g_2^3+\frac{1}{2}\zeta_3 N^3 g_1^4 g_2^2+\frac{973}{1296}N^3 g_1^4 g_2^2+\frac{2297}{2592}N^3 g_1^2 g_2^4+\frac{5569}{2592}N^2 g_1^6+\frac{7}{2}\zeta_3 N^2 g_1^3 g_2^3\\
&\quad+\frac{7}{2}\zeta_3 N^2 g_1^4 g_2^2+\frac{10615}{2592}N^2 g_1^4 g_2^2+\frac{12493}{2592}N^2 g_1^2 g_2^4+\frac{14135}{2592}N g_1^6+\frac{4421}{324N^2}g_1^6+\frac{1139}{72N}g_1^5 g_2+\frac{103}{3N}g_1^3 g_2^3
\end{aligned}
$$

$$+\frac{1891}{36N^2}g_1^5g_2+\frac{35809}{648N}g_1^4g_2^2+\frac{22631}{324N}g_1^2g_2^4+\frac{214}{3N^2}g_1^3g_2^3+\frac{41321}{324N^2}g_1^4g_2^2+\frac{12716}{81N^2}g_1^2g_2^4-\frac{192}{N^3}\zeta_3g_1^3g_2^3$$
$$-\frac{176}{N^2}\zeta_3g_1^2g_2^4-\frac{128}{N^3}\zeta_3g_1^4g_2^2-\frac{88}{N^3}g_1^3g_2^3-\frac{80}{N}\zeta_3g_1^2g_2^4-58\zeta_3g_1^3g_2^3-46\zeta_3g_1^4g_2^2-10\zeta_3Ng_1^3g_2^3-\frac{8}{N}\zeta_3g_1^5g_2$$
$$-6\zeta_3Ng_1^4g_2^2-\frac{4}{N^3}\zeta_3g_1^6-\frac{4}{N^2}\zeta_3g_1^6-3\zeta_3Ng_1^6-\zeta_3N^2g_1^6+3\zeta_3g_1^6+4\zeta_3Ng_1^5g_2+6\zeta_3g_1^5g_2+\frac{9}{N}\zeta_3g_1^6$$
$$+10\zeta_3Ng_1^2g_2^4+\frac{48}{N}\zeta_3g_1^4g_2^2+58\zeta_3g_1^2g_2^4+\frac{80}{N}\zeta_3g_1^3g_2^3+\frac{128}{N^2}\zeta_3g_1^4g_2^2+\frac{176}{N^2}\zeta_3g_1^3g_2^3+\frac{192}{N^3}\zeta_3g_1^2g_2^4\Bigg]$$
$$+\Bigg[-\frac{3285376}{243N^4}g_1^3g_2^5-\frac{39680}{3N^4}\zeta_5g_1^4g_2^4-\frac{25564160}{2187N^4}g_1^2g_2^6-\frac{28160}{3N^2}\zeta_5g_1^2g_2^6-\frac{190528}{27N^4}\zeta_3g_1^4g_2^4-\frac{18944}{3N^4}\zeta_4g_1^2g_2^6$$
$$-\frac{16000}{3N}\zeta_5g_1^4g_2^4-\frac{11437255}{2187N}g_1^2g_2^6-\frac{15616}{3N^4}\zeta_3g_1^2g_2^6-\frac{807383}{162N}g_1^3g_2^5-\frac{14080}{3N^3}\zeta_5g_1^6g_2^2-\frac{39680}{9N^4}\zeta_3g_1^3g_2^5$$
$$-\frac{12128}{3N^3}\zeta_4g_1^3g_2^5-\frac{30994}{9N}\zeta_3g_1^4g_2^4-\frac{6777062}{2187N^3}g_1^4g_2^4-\frac{25664}{9N^4}\zeta_3g_1^5g_2^3-\frac{6193529}{2187N^4}g_1^6g_2^2-\frac{7364}{3N}\zeta_4g_1^2g_2^6$$
$$-\frac{65000}{27N}\zeta_3g_1^2g_2^6-\frac{6440}{3N}\zeta_5g_1^5g_2^3-\frac{56432}{27N}\zeta_3g_1^3g_2^5-\frac{6256}{3N^3}\zeta_4g_1^4g_2^4-\frac{5120}{3N^4}\zeta_5g_1^3g_2^5-\frac{14384}{9N^4}\zeta_3g_1^6g_2^2$$
$$-\frac{946987}{729N^2}g_1^6g_2^2-\frac{11306}{9N}\zeta_3g_1^5g_2^3-\frac{3640}{3}\zeta_5Ng_1^2g_2^6-\frac{20834891}{17496N}g_1^6g_2^2-\frac{3397087}{2916N}g_1^5g_2^3-\frac{744536}{729N^4}g_1^5g_2^3$$
$$-\frac{25108}{27N}\zeta_3g_1^6g_2^2-\frac{2560}{3}\zeta_5g_1^6g_2^2-\frac{2240}{3N^2}\zeta_5g_1^7g_2-\frac{20140}{27N^2}\zeta_3g_1^6g_2^2-\frac{1840}{3N^3}\zeta_4g_1^5g_2^3-\frac{5420}{9N^2}\zeta_3g_1^5g_2^3$$
$$-\frac{4160}{9N^4}\zeta_3g_1^7g_2-\frac{893500}{2187N^2}g_1^4g_2^4-\frac{1168}{3N^2}\zeta_4g_1^4g_2^4-\frac{482105}{1458}g_1^5g_2^3-\frac{1146671}{4374}g_1^2g_2^6-\frac{748}{3N^3}\zeta_4g_1^6g_2^2$$
$$-\frac{2634569}{11664N}g_1^7g_2-\frac{1302539}{5832N^2}g_1^7g_2-\frac{1999}{9}\zeta_3g_1^4g_2^4-\frac{5849}{27}\zeta_3g_1^7g_2-\frac{7058677}{34992}Ng_1^4g_2^4-\frac{1715}{9N}\zeta_3g_1^7g_2$$
$$-\frac{36533}{216}g_1^3g_2^5-\frac{4427}{27N^2}\zeta_3g_1^8-\frac{4286}{27}\zeta_3g_1^3g_2^5-\frac{437}{3}\zeta_4Ng_1^4g_2^4-\frac{3593}{27}\zeta_3g_1^2g_2^6-\frac{2302967}{17496N^2}g_1^8$$
$$-\frac{380}{3}\zeta_5Ng_1^3g_2^5-\frac{332}{3}\zeta_4Ng_1^3g_2^5-\frac{323}{3}\zeta_4g_1^2g_2^6-\frac{320}{3N^3}\zeta_5g_1^8-\frac{455}{6}\zeta_5g_1^8-\frac{53962}{729N^4}g_1^7g_2-\frac{125}{2}\zeta_5Ng_1^7g_2$$
$$-\frac{160}{3}\zeta_5N^2g_1^2g_2^6-\frac{11371}{216}\zeta_3N^2g_1^6g_2^2-\frac{105}{2}\zeta_5N^2g_1^7g_2-\frac{157}{3N}\zeta_4g_1^6g_2^2-\frac{148}{3N^2}\zeta_4g_1^7g_2-\frac{2246273}{46656}N^2g_1^6g_2^2$$
$$-\frac{275}{6}\zeta_4g_1^6g_2^2-\frac{2537}{72}\zeta_3N^2g_1^5g_2^3-\frac{1094113}{34992}g_1^4g_2^4-\frac{268757}{8748N^4}g_1^8-\frac{802}{27N^4}\zeta_3g_1^8-\frac{175}{6}\zeta_5N^2g_1^5g_2^3$$
$$-\frac{1271875}{46656}N^2g_1^7g_2-\frac{14989}{576}Ng_1^8-\frac{145}{6}\zeta_5N^2g_1^4g_2^4-\frac{193895}{8748}N^3g_1^2g_2^6-\frac{511}{27}\zeta_3N^2g_1^8-\frac{2537461}{139968}Ng_1^6g_2^2$$
$$-\frac{373}{24}\zeta_3N^3g_1^4g_2^4-\frac{169}{12}\zeta_3Ng_1^7g_2-\frac{27}{2}\zeta_4Ng_1^7g_2-\frac{40}{3N^4}\zeta_5g_1^8-\frac{25}{2}\zeta_5N^3g_1^5g_2^3-\frac{10555}{972}N^3g_1^3g_2^5$$
$$-\frac{468427}{46656}N^3g_1^7g_2-\frac{233473}{23328}N^3g_1^5g_2^3-\frac{349}{36}\zeta_3N^3g_1^2g_2^6-\frac{29}{3}\zeta_4N^2g_1^4g_2^4-\frac{660215}{69984}N^2g_1^8-\frac{55}{6}\zeta_5N^3g_1^4g_2^4$$
$$-\frac{329}{36}\zeta_3N^3g_1^3g_2^5-\frac{25}{4}\zeta_4N^3g_1^6g_2^2-\frac{49}{8}\zeta_4N^2g_1^5g_2^3-\frac{145}{24}\zeta_4N^2g_1^7g_2-\frac{11}{2}\zeta_4N^3g_1^2g_2^6-\frac{1069}{216}\zeta_3Ng_1^8$$
$$-\frac{119}{36}\zeta_3N^3g_1^8-\frac{43291}{17496}N^4g_1^2g_2^6-\frac{17}{8}\zeta_4Ng_1^8-\frac{97}{54N}\zeta_3g_1^8-\frac{5}{3}\zeta_5N^4g_1^5g_2^3-\frac{5}{3}\zeta_4N^2g_1^3g_2^5$$
$$-\frac{29093}{17496}N^3g_1^8-\frac{161}{108}\zeta_3N^4g_1^3g_2^5-\frac{10}{9}\zeta_3N^3g_1^6g_2^2-\frac{55}{54}\zeta_3N^4g_1^2g_2^6-\frac{217}{216}\zeta_3N^4g_1^4g_2^4-\frac{7}{8}\zeta_4N^4g_1^6g_2^2$$
$$-\frac{9257}{11664}N^4g_1^5g_2^3-\frac{5761}{7776}N^4g_1^3g_2^5-\frac{131}{216}\zeta_3N^4g_1^7g_2-\frac{20137}{46656}N^4g_1^7g_2-\frac{1223}{2916}N^4g_1^4g_2^4-\frac{5}{12}\zeta_4N^4g_1^2g_2^6$$
$$-\frac{1}{6}\zeta_4N^2g_1^8-\frac{7945}{69984}N^4g_1^8+\frac{1}{12}\zeta_4N^4g_1^3g_2^5+\frac{13721}{139968}N^2g_1^4g_2^4+\frac{1}{9}\zeta_3N^4g_1^8+\frac{1}{8}\zeta_4N^4g_1^5g_2^3+\frac{1}{6N}\zeta_4g_1^8$$

$$
\begin{aligned}
&+\frac{7}{24}\zeta_4N^4g_1^7g_2+\frac{1}{2}\zeta_4N^3g_1^8+\frac{37}{72}\zeta_3N^4g_1^5g_2^3+\frac{35}{54}\zeta_3N^3g_1^5g_2^3+\frac{3}{4}\zeta_4N^3g_1^7g_2+\frac{19}{24}\zeta_4N^4g_1^4g_2^4\\
&+\frac{129877}{139968}N^4g_1^6g_2^2+\frac{349}{216}\zeta_3N^4g_1^6g_2^2+\frac{5}{3}\zeta_5N^4g_1^3g_2^5+\frac{11}{6}\zeta_4N^3g_1^5g_2^3+\frac{9}{4}\zeta_4N^3g_1^3g_2^5+\frac{20297}{7776}N^2g_1^3g_2^5\\
&+\frac{10}{3}\zeta_5N^3g_1^8+\frac{10}{3}\zeta_5N^4g_1^2g_2^6+\frac{29}{8}\zeta_4N^2g_1^6g_2^2+\frac{713935}{139968}N^3g_1^4g_2^4+\frac{768091}{139968}N^3g_1^6g_2^2+\frac{157}{24}\zeta_4N^3g_1^4g_2^4\\
&+\frac{43}{6}\zeta_3N^3g_1^7g_2+\frac{29}{4}\zeta_4N^2g_1^2g_2^6+\frac{25}{3}\zeta_5N^3g_1^3g_2^5+\frac{311}{24}\zeta_4g_1^8+\frac{40}{3}\zeta_5N^3g_1^6g_2^2+\frac{1909}{108}\zeta_3N^2g_1^2g_2^6\\
&+\frac{2299}{108}\zeta_3N^2g_1^4g_2^4+\frac{65}{3}\zeta_5N^2g_1^8+\frac{68}{3N^3}\zeta_4g_1^8+\frac{70}{3}\zeta_4g_1^4g_2^4+\frac{567917}{23328N}g_1^8+\frac{724}{27}\zeta_3N^2g_1^3g_2^5+\frac{55}{2}\zeta_5Ng_1^8\\
&+\frac{185}{6}\zeta_4g_1^7g_2+\frac{579887}{17496}N^2g_1^2g_2^6+\frac{855281}{23328}N^2g_1^5g_2^3+\frac{110}{3}\zeta_5N^3g_1^2g_2^6+\frac{5363345}{139968}g_1^8+\frac{9689}{216}\zeta_3N^2g_1^7g_2\\
&+\frac{841}{18}\zeta_3Ng_1^5g_2^3+\frac{160}{3N^4}\zeta_5g_1^7g_2+\frac{160}{3N}\zeta_5g_1^3g_2^5+\frac{161}{3}\zeta_4Ng_1^6g_2^2+\frac{166}{3}\zeta_4g_1^3g_2^5+\frac{6737}{108}\zeta_3Ng_1^6g_2^2\\
&+\frac{151}{2}\zeta_4g_1^5g_2^3+\frac{3612697}{46656}Ng_1^7g_2+\frac{280}{3}\zeta_5g_1^3g_2^5+\frac{20611}{216}\zeta_3g_1^8+\frac{1144}{9N^3}\zeta_3g_1^8+\frac{2395549}{17496N^3}g_1^8\\
&+\frac{428}{3N^2}\zeta_4g_1^6g_2^2+\frac{460}{3}\zeta_5Ng_1^5g_2^3+\frac{464}{3N^4}\zeta_4g_1^6g_2^2+\frac{520}{3N^2}\zeta_5g_1^8+\frac{4115309}{23328}g_1^7g_2+\frac{643}{3}\zeta_4Ng_1^2g_2^6+\frac{670}{3}\zeta_5g_1^5g_2^3\\
&+\frac{6673}{27}\zeta_3Ng_1^3g_2^5+\frac{770}{3}\zeta_5g_1^4g_2^4+\frac{2994941}{11664}Ng_1^5g_2^3+\frac{7429}{27}\zeta_3Ng_1^2g_2^6+\frac{8186}{27N^2}\zeta_3g_1^7g_2+\frac{223895}{729N^3}g_1^7g_2\\
&+\frac{1180}{3}\zeta_5g_1^7g_2+\frac{259655}{648}Ng_1^3g_2^5+\frac{1250}{3}\zeta_5Ng_1^4g_2^4+\frac{1280}{3N^2}\zeta_5g_1^3g_2^5+\frac{3857}{9}\zeta_3Ng_1^4g_2^4+\frac{923}{2}\zeta_3g_1^5g_2^3\\
&+\frac{25439}{54}\zeta_3g_1^6g_2^2+\frac{4760}{9N^3}\zeta_3g_1^7g_2+\frac{5110967}{8748}Ng_1^2g_2^6+\frac{1760}{3}\zeta_5g_1^2g_2^6+\frac{5944171}{8748}g_1^6g_2^2+\frac{2998}{3N}\zeta_4g_1^4g_2^4\\
&+\frac{1519463}{1458N^2}g_1^5g_2^3+\frac{869092}{729N^3}g_1^5g_2^3+\frac{12632}{9N^2}\zeta_3g_1^3g_2^5+\frac{4252}{3N}\zeta_4g_1^3g_2^5+\frac{3280451}{2187N}g_1^4g_2^4+\frac{5120}{3N^4}\zeta_5g_1^6g_2^2\\
&+\frac{6080}{3N^2}\zeta_5g_1^6g_2^2+\frac{1629736}{729N^4}g_1^4g_2^4+\frac{64160}{27N^2}\zeta_3g_1^4g_2^4+\frac{8000}{3N^2}\zeta_5g_1^4g_2^4+\frac{75404}{27N^3}\zeta_3g_1^6g_2^2+\frac{7870936}{2187N^2}g_1^2g_2^6\\
&+\frac{987116}{243N^2}g_1^3g_2^5+\frac{114368}{27N^3}\zeta_3g_1^5g_2^3+\frac{41134879}{8748N^3}g_1^6g_2^2+\frac{134752}{27N^3}\zeta_3g_1^3g_2^5+\frac{157888}{27N^3}\zeta_3g_1^2g_2^6+\frac{20224}{3N^3}\zeta_4g_1^2g_2^6\\
&+\frac{20800}{3N^3}\zeta_5g_1^5g_2^3+\frac{71216}{9N^3}\zeta_3g_1^4g_2^4+\frac{37760}{3N}\zeta_5g_1^2g_2^6+\frac{28407472}{2187N^3}g_1^2g_2^6+\frac{3455248}{243N^3}g_1^3g_2^5+\frac{45760}{3N^3}\zeta_5g_1^4g_2^4\\
&-\frac{33280}{N^3}\zeta_5g_1^2g_2^6-\frac{6400}{N^4}\zeta_5g_1^5g_2^3-\frac{1160}{N^2}\zeta_4g_1^3g_2^5-\frac{372}{N^2}\zeta_4g_1^5g_2^3-260\zeta_5Ng_1^6g_2^2-50\zeta_4Ng_1^5g_2^3-\frac{40}{N^2}\zeta_4g_1^8\\
&-\frac{30}{N}\zeta_5g_1^8-30\zeta_5N^2g_1^3g_2^5-\frac{28}{N^3}\zeta_4g_1^7g_2-5\zeta_5N^3g_1^7g_2+\frac{6}{N^4}\zeta_4g_1^8+\frac{16}{N^4}\zeta_4g_1^7g_2+\frac{49}{N}\zeta_4g_1^7g_2+60\zeta_5N^2g_1^6g_2^2\\
&+\frac{120}{N^3}\zeta_5g_1^7g_2+\frac{300}{N}\zeta_5g_1^7g_2+\frac{324}{N}\zeta_4g_1^5g_2^3+\frac{640}{N^4}\zeta_4g_1^5g_2^3+\frac{1280}{N^3}\zeta_5g_1^3g_2^5+\frac{1280}{N^2}\zeta_5g_1^5g_2^3+\frac{1600}{N^4}\zeta_4g_1^4g_2^4\\
&+\left.\frac{1616}{N^2}\zeta_3g_1^2g_2^6+\frac{1920}{N^2}\zeta_4g_1^2g_2^6+\frac{2000}{N}\zeta_5g_1^6g_2^2+\frac{3840}{N^4}\zeta_4g_1^3g_2^5+\frac{30720}{N^4}\zeta_5g_1^2g_2^6\right]+\mathcal{O}(g_i^{10})
\end{aligned}
$$

$$
\begin{aligned}
-\gamma_{22}(g_i) = & \left[-\frac{20}{3N}g_2^2-\frac{1}{6}g_1^2+\frac{5}{6}Ng_2^2+\frac{5}{3}g_2^2\right] \\
& +\left[-\frac{4640}{27N^2}g_2^4-\frac{146}{27}Ng_2^4-\frac{104}{27N}g_1^2g_2^2-\frac{49}{54}N^2g_2^4-\frac{20}{27}g_1^4-\frac{5}{9}g_1^3g_2-\frac{19}{54}Ng_1^4-\frac{5}{18}Ng_1^3g_2+\frac{13}{27}Ng_1^2g_2^2\right. \\
& \left.\quad+\frac{26}{27}g_1^2g_2^2+\frac{40}{27N}g_1^4+\frac{20}{9N}g_1^3g_2+\frac{194}{9}g_2^4+\frac{1936}{27N}g_2^4\right] \\
& +\left[-\frac{1982128}{243N^3}g_2^6-\frac{10496}{3N^3}\zeta_3g_2^6-\frac{478715}{972}g_2^6-\frac{500}{3}\zeta_3g_2^6-\frac{95399}{648N}g_1^4g_2^2-\frac{29662}{243N^2}g_1^2g_2^4-\frac{52153}{1296}g_1^4g_2^2\right. \\
& \quad-\frac{100}{3N}\zeta_3g_1^4g_2^2-\frac{7817}{324}Ng_2^6-\frac{65}{3}\zeta_3g_1^4g_2^2-\frac{1096}{81N^2}g_1^3g_2^3-\frac{586}{81N^2}g_1^5g_2-\frac{13}{3}\zeta_3Ng_2^6-\frac{325}{81}Ng_1^3g_2^3 \\
& \quad-\frac{14821}{3888}Ng_1^2g_2^4-\frac{14335}{7776}g_1^6-\frac{1177}{864}Ng_1^6-\frac{821}{648}N^2g_1^3g_2^3-\frac{2}{3N}\zeta_3g_1^6-\frac{4937}{7776}N^2g_1^2g_2^4-\frac{7}{18N}g_1^5g_2 \\
& \quad-\frac{155}{648}N^2g_1^5g_2-\frac{11}{54}Ng_1^5g_2+\frac{4253}{7776}N^2g_1^6+\frac{283}{432N}g_1^6+\frac{2}{3N^2}\zeta_3g_1^6+\frac{7}{6}\zeta_3Ng_1^6+\frac{11437}{7776}N^3g_2^6+\frac{5}{2}\zeta_3g_1^6 \\
& \quad+\frac{5}{2}\zeta_3N^2g_2^6+\frac{8147}{2592}N^2g_1^4g_2^2+\frac{1091}{324}g_1^5g_2+\frac{3475}{972N^2}g_1^6+\frac{35}{6}\zeta_3Ng_1^4g_2^2+\frac{479}{54}g_1^3g_2^3+\frac{1232}{81N}g_1^3g_2^3+\frac{2471}{162}g_1^2g_2^4 \\
& \quad+\frac{1243}{81}N^2g_2^6+\frac{10393}{648}Ng_1^4g_2^2+\frac{24715}{486N}g_1^2g_2^4+\frac{224}{3N^2}\zeta_3g_1^4g_2^2+\frac{44843}{162N^2}g_1^4g_2^2+\frac{55006}{81N}g_2^6+\frac{5696}{3N^2}\zeta_3g_2^6 \\
& \quad+\frac{371848}{81N^2}g_2^6-\frac{192}{N^2}\zeta_3g_1^3g_2^3-\frac{24}{N}\zeta_3g_1^5g_2-6\zeta_3Ng_1^3g_2^3-\zeta_3N^2g_1^3g_2^3-\zeta_3N^2g_1^5g_2+2\zeta_3N^2g_1^4g_2^2+8\zeta_3g_1^5g_2 \\
& \left.\quad+24\zeta_3g_1^3g_2^3+\frac{32}{N^2}\zeta_3g_1^5g_2+\frac{80}{N}\zeta_3g_1^3g_2^3+\frac{256}{N}\zeta_3g_2^6\right] \\
& +\left[-\frac{17251328}{27N^4}\zeta_3g_2^8-\frac{410707712}{729N^4}g_2^8-\frac{1907728}{27N}\zeta_3g_2^8-\frac{39417340}{729N}g_2^8-\frac{378880}{9N^4}\zeta_5g_2^8-\frac{110080}{3N^3}\zeta_5g_2^8\right. \\
& \quad-\frac{282880}{9N^2}\zeta_5g_2^8-\frac{111825646}{6561N^2}g_1^4g_2^4-\frac{97856}{9N^2}\zeta_3g_1^3g_2^5-\frac{79360}{9N^3}\zeta_5g_1^4g_2^4-\frac{619312}{81N^2}\zeta_3g_1^4g_2^4-\frac{22720}{3N^2}\zeta_5g_1^5g_2^3 \\
& \quad-\frac{4685968}{729N^3}g_1^2g_2^6-\frac{511744}{81N^3}\zeta_3g_1^2g_2^6-\frac{13840}{3}\zeta_5Ng_2^8-\frac{289726}{81N}\zeta_3g_1^4g_2^4-\frac{6734800}{2187N^3}g_1^3g_2^5-\frac{73283287}{26244N}g_1^4g_2^4 \\
& \quad-\frac{8320}{3N^2}\zeta_4g_2^8-\frac{198820}{81N^2}\zeta_3g_1^6g_2^2-\frac{16576}{9N}\zeta_3g_1^3g_2^5-\frac{14080}{9N^3}\zeta_5g_1^6g_2^2-\frac{12640}{9N^2}\zeta_4g_1^2g_2^6-\frac{2474768}{2187N^2}g_1^5g_2^3 \\
& \quad-\frac{24244}{27N}\zeta_3g_1^5g_2^3-\frac{7232}{9N^2}\zeta_3g_1^5g_2^3-\frac{4772141}{8748N}g_1^5g_2^3-\frac{4780}{9}\zeta_5Ng_1^4g_2^4-\frac{4760}{9N}\zeta_5g_1^6g_2^2-\frac{933137}{2187}g_1^3g_2^5 \\
& \quad-\frac{1225}{3}\zeta_5N^2g_2^8-\frac{3352}{9N^3}\zeta_4g_1^6g_2^2-\frac{799126}{2187}g_1^2g_2^6-\frac{3863993}{11664}g_1^6g_2^2-\frac{2685313}{8748N^2}g_1^6g_2^2-\frac{23360}{81}\zeta_3g_1^2g_2^6 \\
& \quad-\frac{826}{3}\zeta_4Ng_2^8-\frac{820}{3}\zeta_5Ng_1^5g_2^3-\frac{800}{3}\zeta_5g_1^7g_2-\frac{2084}{9N^2}\zeta_3g_1^7g_2-\frac{1856749}{8748N^2}g_1^7g_2-\frac{568}{3N}\zeta_4g_1^2g_2^6 \\
& \quad-\frac{27373}{162}\zeta_3Ng_1^6g_2^2-\frac{1406}{9N}\zeta_4g_1^4g_2^4-\frac{22733}{162}\zeta_3N^2g_1^4g_2^4-\frac{28227671}{209952}N^2g_1^4g_2^4-\frac{2099}{18}\zeta_3N^3g_2^8-\frac{248}{3N}\zeta_4g_1^7g_2 \\
& \quad-\frac{218413}{2916}N^3g_2^8-\frac{3851}{54}\zeta_3N^2g_1^5g_2^3-\frac{208}{3N^3}\zeta_4g_1^7g_2-\frac{1192085}{17496N}g_1^7g_2-\frac{764353}{11664}Ng_1^3g_2^5-\frac{560}{9}\zeta_5g_1^6g_2^2 \\
& \quad-\frac{170}{3N^2}\zeta_5g_1^8-\frac{145}{3}\zeta_4Ng_1^5g_2^3-\frac{306473}{6561N^2}g_1^8-\frac{1237}{27N^2}\zeta_3g_1^8-\frac{394}{9}\zeta_3N^2g_1^3g_2^5-\frac{4211087}{139968}N^2g_1^5g_2^3 \\
& \quad-\frac{1924}{81}\zeta_3N^3g_1^4g_2^4-\frac{8935697}{419904}N^3g_1^4g_2^4-\frac{185}{9N}\zeta_4g_1^8-\frac{39}{2}\zeta_4N^2g_2^8-\frac{55}{3}\zeta_3N^3g_1^5g_2^3-\frac{35}{2}\zeta_5Ng_1^8-\frac{121}{9}\zeta_3g_1^8 \\
& \quad-\frac{40}{3N}\zeta_4g_1^6g_2^2-\frac{40}{3}\zeta_4g_1^7g_2-\frac{112591}{8748}Ng_1^2g_2^6-\frac{1658371}{139968}N^2g_1^7g_2-\frac{103}{9}\zeta_3N^2g_1^7g_2-\frac{1229}{108}\zeta_3N^4g_2^8
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{89027}{11664}N^4g_2^8-\frac{133}{18}\zeta_4N^2g_1^6g_2^2-\frac{53}{9}\zeta_4N^2g_1^4g_2^4-\frac{154}{27}\zeta_3Ng_1^2g_2^6-\frac{152}{27}\zeta_3N^3g_1^3g_2^5-\frac{549997}{139968}N^3g_1^5g_2^3\\
&-\frac{1347413}{419904}N^2g_1^8-\frac{13}{6}\zeta_4N^3g_1^6g_2^2-\frac{49709}{23328}Ng_1^8-\frac{52}{27}\zeta_3N^3g_1^7g_2-\frac{11}{6}\zeta_4N^2g_1^2g_2^6-\frac{25}{18}\zeta_5N^2g_1^8\\
&-\frac{184205}{139968}N^3g_1^7g_2-\frac{13}{18}\zeta_4N^2g_1^8-\frac{146111}{419904}N^3g_1^6g_2^2-\frac{1}{3}\zeta_4N^3g_1^4g_2^4-\frac{29339}{104976}N^3g_1^8-\frac{1}{4}\zeta_4N^3g_1^3g_2^5\\
&+\frac{25}{162}\zeta_3N^3g_1^2g_2^6+\frac{1}{6}\zeta_4Ng_1^4g_2^4+\frac{25}{81}\zeta_3N^3g_1^8+\frac{4}{9N^3}\zeta_4g_1^8+\frac{2}{3}\zeta_4g_1^8+\frac{2}{3}\zeta_4N^3g_1^7g_2+\frac{68947}{69984}N^3g_1^2g_2^6\\
&+\frac{175}{162}\zeta_3Ng_1^8+\frac{5}{4}\zeta_4N^3g_2^8+\frac{34475}{26244}Ng_1^6g_2^2+\frac{17}{12}\zeta_4Ng_1^8+\frac{5}{3}\zeta_5N^2g_1^6g_2^2+\frac{25}{12}\zeta_4N^3g_1^5g_2^3+\frac{79}{36}\zeta_3N^2g_1^8\\
&+\frac{20}{9N^3}\zeta_5g_1^8+\frac{5}{2}\zeta_4N^2g_1^3g_2^5+\frac{254}{81}\zeta_3N^2g_1^6g_2^2+\frac{29}{9}\zeta_4Ng_1^2g_2^6+\frac{130469}{34992}N^3g_1^3g_2^5+\frac{133583}{34992}Ng_1^7g_2\\
&+\frac{47}{12}\zeta_4N^2g_1^7g_2+\frac{14}{3}\zeta_3N^2g_1^2g_2^6+\frac{29}{6}\zeta_3Ng_1^7g_2+\frac{1075171}{209952}g_1^8+\frac{601}{81}\zeta_3N^3g_1^6g_2^2+\frac{1345}{162N}\zeta_3g_1^8+\frac{26}{3}\zeta_4Ng_1^7g_2\\
&+\frac{82}{9N}\zeta_3g_1^7g_2+\frac{175}{18}\zeta_5g_1^8+\frac{32}{3N^3}\zeta_4g_1^5g_2^3+\frac{390847}{34992}N^2g_1^2g_2^6+\frac{137}{12}\zeta_4N^2g_1^5g_2^3+\frac{357235}{26244N^3}g_1^8+\frac{281}{18}\zeta_4Ng_1^6g_2^2\\
&+\frac{146}{9N^2}\zeta_4g_1^8+\frac{50}{3}\zeta_5N^2g_1^7g_2+\frac{50}{3}\zeta_5N^3g_1^5g_2^3+\frac{160}{9}\zeta_5N^4g_2^8+\frac{125}{6}\zeta_4g_1^6g_2^2+\frac{172577}{7776}N^2g_1^3g_2^5+\frac{1972}{81N^3}\zeta_3g_1^8\\
&+\frac{430423}{15552}N^2g_1^6g_2^2+\frac{43781}{1458}N^2g_2^8+\frac{104}{3N^2}\zeta_4g_1^5g_2^3+\frac{340}{9}\zeta_5N^3g_1^4g_2^4+\frac{242405}{5832N}g_1^8+\frac{500}{9}\zeta_5Ng_1^6g_2^2\\
&+\frac{1712}{27N^3}\zeta_3g_1^7g_2+\frac{4968941}{69984}Ng_1^5g_2^3+\frac{805}{9N}\zeta_5g_1^8+\frac{2617}{27}\zeta_3g_1^7g_2+\frac{1030}{9}\zeta_3Ng_1^3g_2^5+\frac{3193}{27}\zeta_3N^2g_2^8\\
&+\frac{4168655}{34992}g_1^7g_2+\frac{1108}{9}\zeta_4g_1^2g_2^6+\frac{1286}{9}\zeta_4g_1^4g_2^4+\frac{319897}{2187N^3}g_1^7g_2+\frac{4018}{27}\zeta_3g_1^6g_2^2+\frac{2755}{18}\zeta_5N^3g_2^8\\
&+\frac{1730}{9N^2}\zeta_4g_1^6g_2^2+\frac{1732}{9}\zeta_3Ng_1^5g_2^3+\frac{1750}{9}\zeta_5N^2g_1^4g_2^4+\frac{640}{3N^3}\zeta_5g_1^7g_2+\frac{3394969}{13122}Ng_1^4g_2^4+\frac{1040}{3N}\zeta_5g_1^7g_2\\
&+\frac{1076}{3N}\zeta_4g_1^5g_2^3+\frac{20209543}{52488N}g_1^6g_2^2+\frac{32348}{81}\zeta_3Ng_1^4g_2^4+\frac{3968}{9N}\zeta_3g_1^2g_2^6+\frac{8070263}{17496}g_1^5g_2^3+\frac{1094668}{2187N}g_1^2g_2^6\\
&+\frac{89015}{162N}g_1^3g_2^5+\frac{15608}{27}\zeta_3g_1^5g_2^3+\frac{67060}{81N}\zeta_3g_1^6g_2^2+\frac{1822484}{2187N^3}g_1^5g_2^3+\frac{5508667}{6561N^3}g_1^6g_2^2+\frac{23680}{27}\zeta_3g_2^8\\
&+\frac{25664}{27N^3}\zeta_3g_1^5g_2^3+\frac{35300}{27}\zeta_3g_1^3g_2^5+\frac{14560}{9N^2}\zeta_5g_1^6g_2^2+\frac{1337791}{729}g_2^8+\frac{170113}{81}\zeta_3g_1^4g_2^4+\frac{18944}{9N^3}\zeta_4g_1^4g_2^4\\
&+\frac{1601846}{729N^2}g_1^3g_2^5+\frac{205360}{81N^3}\zeta_3g_1^6g_2^2+\frac{23296}{9N^3}\zeta_4g_1^2g_2^6+\frac{136340503}{52488}g_1^4g_2^4+\frac{2030993}{729}Ng_2^8+\frac{91712}{27N^2}\zeta_3g_1^2g_2^6\\
&+\frac{7801508}{2187N^2}g_1^2g_2^6+\frac{34240}{9N}\zeta_5g_1^4g_2^4+\frac{37562}{9}\zeta_3Ng_2^8+\frac{42560}{9}\zeta_5g_2^8+\frac{18448}{3N}\zeta_4g_2^8+\frac{56960}{9N^2}\zeta_5g_1^4g_2^4\\
&+\frac{29440}{3N^3}\zeta_5g_1^5g_2^3+\frac{1005920}{81N^3}\zeta_3g_1^4g_2^4+\frac{12456544}{729N^2}g_2^8+\frac{501760}{27N^3}\zeta_3g_1^3g_2^5+\frac{170597692}{6561N^3}g_1^4g_2^4+\frac{732032}{27N^2}\zeta_3g_2^8\\
&+\frac{354560}{9N}\zeta_5g_2^8+\frac{179200}{3N^4}\zeta_4g_2^8+\frac{298418240}{729N^3}g_2^8+\frac{12938752}{27N^3}\zeta_3g_2^8-\frac{46080}{N^3}\zeta_5g_1^3g_2^5-\frac{44544}{N^3}\zeta_4g_2^8\\
&-\frac{4480}{N^3}\zeta_4g_1^3g_2^5-3200\zeta_5g_1^3g_2^5-2400\zeta_5g_1^4g_2^4-\frac{1208}{N^2}\zeta_4g_1^4g_2^4-240\zeta_5Ng_1^3g_2^5-200\zeta_4g_1^3g_2^5-155\zeta_4g_1^5g_2^3\\
&-20\zeta_5Ng_1^7g_2+10\zeta_5N^3g_1^3g_2^5+40\zeta_5N^2g_1^5g_2^3+72\zeta_4g_2^8+100\zeta_5N^2g_1^3g_2^5+\frac{100}{N^2}\zeta_4g_1^7g_2+\frac{160}{N^2}\zeta_5g_1^7g_2\\
&+280\zeta_5g_1^5g_2^3+\frac{304}{N}\zeta_4g_1^3g_2^5+\frac{1280}{N}\zeta_5g_1^5g_2^3+\frac{2400}{N^2}\zeta_4g_1^3g_2^5+\frac{4480}{N}\zeta_5g_1^3g_2^5+\frac{26880}{N^2}\zeta_5g_1^3g_2^5\Bigg]+\mathcal{O}(g_i^{10}). \qquad \text{(A3)}
\end{aligned}
$$

The full expressions for all the renormalization group functions are available in the Supplemental Material [24].

[1] M. Aizenman, Phys. Rev. Lett. **47**, 1 (1981).
[2] J. Fröhlich, Nucl. Phys. **B200**, 281 (1982).
[3] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, England, 2002).
[4] L. Fei, S. Giombi, and I. R. Klebanov, Phys. Rev. D **90**, 025018 (2014).
[5] E. Ma, Prog. Theor. Phys. **54**, 1828 (1975).
[6] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Phys. Rev. D **91**, 045011 (2015).
[7] J. A. Gracey, Phys. Rev. D **92**, 025012 (2015).
[8] I. F. Herbut and L. Janssen, Phys. Rev. D **93**, 085005 (2016).
[9] L. Janssen and I. F. Herbut, Phys. Rev. B **92**, 045117 (2015).
[10] H. Osborn and A. Stergiou, J. High Energy Phys. 05 (2018) 51.
[11] J. A. Gracey and R. M. Simms, Phys. Rev. D **96**, 025022 (2017).
[12] D. Roscher and I. F. Herbut, Phys. Rev. D **97**, 116019 (2018).
[13] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).
[14] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).
[15] J. A. M. Vermaseren, arXiv:math-ph/0010025.
[16] M. Tentyukov and J. A. M. Vermaseren, Comput. Phys. Commun. **181**, 1419 (2010).
[17] S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000).
[18] C. Studerus, Comput. Phys. Commun. **181**, 1293 (2010).
[19] A. von Manteuffel and C. Studerus, Phys. Rev. C **85**, 054001 (2012).
[20] O. V. Tarasov, Phys. Rev. D **54**, 6479 (1996).
[21] O. V. Tarasov, Nucl. Phys. **B502**, 455 (1997).
[22] P. A. Baikov and K. G. Chetyrkin, Nucl. Phys. **B837**, 186 (2010).
[23] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **303**, 334 (1993).
[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.98.096014 to this article for the electronic version of the four loop $O(N)$ renormalization group functions.
[25] A. J. McKane, D. J. Wallace, and R. K. P. Zia, Phys. Lett. **65B**, 171 (1976); O. de Alcantra Bonfim, J. Kirkham, and A. J. McKane, J. Phys. A **13**, L247 (1980); **13**, 3785 (1980); **14**, 2391 (1981).
[26] R. G. Priest and T. C. Lubensky, Phys. Rev. B **13**, 4159 (1976).
[27] I. F. Herbut and Z. Tešanović, Phys. Rev. Lett. **78**, 980 (1997).
[28] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, Cambridge, England, 2007).
[29] For other examples and further discussion of collision and annihilation of IR and UV fixed points, see H. Gies and J. Jaeckel, Eur. Phys. J. C **46**, 433 (2006); D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, Phys. Rev. D **80**, 125005 (2009); J. Braun, C. S. Fischer, and H. Gies, Phys. Rev. D **84**, 034045 (2011); I. F. Herbut, Phys. Rev. D **94**, 025036 (2016); V. Gorbenko, S. Rychkov, and B. Zan, J. High Energy Phys. 10 (2018) 108; arXiv:1808.04380.
[30] S. Gukov, Nucl. Phys. **B919**, 583 (2017).
[31] B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. **32**, 292 (1974).
[32] I. F. Herbut and Z. Tešanović, Phys. Rev. Lett. **76**, 4588 (1996).
[33] D. Poland, S. Rychkov, and A. Vichi, arXiv:1805.04405 [Rev. Mod. Phys. (to be published)].
[34] Y. Nakayama and T. Ohtsuki, Phys. Lett. B **734**, 193 (2014); S. M. Chester, S. S. Pufu, and R. Yacoby, Phys. Rev. D **91**, 086014 (2015).
[35] A. Eichorn, L. Janssen, and M. M. Scherer, Phys. Rev. D **93**, 125021 (2016).