Are there narrow flavor-exotic tetraquarks in large-N_c QCD?

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A salient feature shared by all tetraquark candidates observed in experiment is the absence of flavor-exotic states of the type $\bar{a}b\bar{c}d$, with four different quark flavors. This phenomenon may be understood from the properties of large-N_c QCD. On the one hand, previous work shows that consistency conditions for flavor-exotic Green functions, potentially containing these tetraquark poles, require the existence of two tetraquarks T_A and T_B: Each of them should decay dominantly via a single two-meson channel, $T_A \to M_{\bar{a}b}M_{\bar{c}d}$ and $T_B \to M_{\bar{a}d}M_{\bar{c}b}$, with suppressed rates $T_A \to M_{\bar{a}d}M_{\bar{c}b}$ and $T_B \to M_{\bar{a}b}M_{\bar{c}d}$. On the other hand, we have at hand only one diquark-antidiquark flavor structure $(\bar{a}\,\bar{c})(bd)$ that might produce a compact tetraquark bound state. Taking into account that the diquark-antidiquark structure is the only viable candidate for a compact tetraquark state, one concludes that it is impossible to obtain two different narrow tetraquarks decaying dominantly into distinct two-meson channels. This contradiction suggests that large-N_c QCD does not support the existence of narrow flavor-exotic tetraquarks. This argument does not rule out the possible existence of broad molecular-type flavor-exotic states or of molecular-type bound states lying very close to the two-meson thresholds.

I. MOTIVATION

In recent years, many narrow near-threshold hadron resonances which have a favorable interpretation as tetraquark and pentaquark hadrons (i.e., hadrons with minimal parton configurations consisting of four and five quarks, respectively) have been observed experimentally [1,2]. An intriguing feature shared by all exotic candidates is the absence of states with flavor-exotic structure, i.e., with four different quark flavors, which cannot be realized in a single ordinary hadron. The only flavor-exotic tetraquark candidate $X(5568)$ of D0 [3] was not confirmed by LHCb [4], CMS [5], CDF [6], and ATLAS [7]. Lattice calculations also seem to rule out the existence of tetraquarks with such a structure [8], at least with the quark content $(cs\bar{u}\bar{d})$. In this paper, we attempt to understand this phenomenon from the large-N_c

perspective and give arguments why in large-N_c QCD no narrow compact flavor-exotic states may exist.

QCD with a large number of colors N_c [i.e., SU(N_c) gauge theory for large N_c, with quarks in the fundamental representation] with a simultaneously decreasing coupling $\alpha_s \sim 1/N_c$ [9,10] has proven to be a useful theoretical tool to explain the essential properties of hadron interactions and, in particular, the properties of possibly existing tetraquark and pentaquark hadrons [11–19].

Recently, we have formulated rigorous criteria to be satisfied by the four-point Green functions of bilinear quark currents appropriate for tetraquark poles [17,18]: Any diagram which contributes to the potential tetraquark pole in the Mandelstam variable s, at $s = M_T^2$, where M_T is the tetraquark mass, should satisfy the following two almost self-evident criteria: (i) The diagram should have a non-trivial (i.e., nonpolynomial) dependence on s. (ii) It should support four-quark intermediate states and corresponding cuts starting at $s = (m_1 + m_2 + m_3 + m_4)^2$, where m_i are the masses of the quarks forming the tetraquark bound state. The presence or absence of this cut is established by solving the Landau equations for the corresponding diagram [20]. Hereafter, we refer to diagrams which satisfy these criteria as *tetraquark-phile* diagrams.

Here, we take a closer look at the four-point Green functions of quark bilinear currents (omitting spin and Lorentz indices, which do not play a fundamental role) of exotic flavor content. We show that the tetraquark-phile diagrams have a cylinder topology: For the direct (D) Green functions $\langle T\{j_{\bar{a}b}j_{\bar{c}d}j_{\bar{a}b}^{\dagger}j_{\bar{c}d}^{\dagger}\}\rangle$ and $\langle T\{j_{\bar{a}d}j_{\bar{c}b}j_{\bar{a}d}^{\dagger}j_{\bar{c}b}^{\dagger}\}\rangle$, the N_c-leading contributions are given by diagrams with multiple nonintersecting gluon exchanges lying on the tube; all these diagrams behave like $O(N_c^0)$. Cylinder diagrams with k gluonic handles, which avoid intersection of gluon lines on the tube, behave like $O(N_c^{-2k})$. For the recombination (R) Green functions $\langle T\{j_{\bar{a}b}j_{\bar{c}d}j_{\bar{a}d}^{\dagger}j_{\bar{c}b}^{\dagger}\}\rangle$, the N_c-leading tetraquark-phile diagrams are those containing one gluonic handle and they behave like $O(N_c^{-1})$; adding k gluonic handles leads to suppressed diagrams of order $O(N_c^{-1-2k})$. The above classification of behaviors does not take into account the eventual presence of quark loops. The insertion of a quark loop inside gluon lines lowers the scaling power in N_c by one unit.

Because of the different N_c scalings (even powers for D and odd powers for R) of the tetraquark-phile Green functions, one finds that narrow tetraquarks with a fixed mass at large N_c may emerge only in pairs, T_A and T_B, each of them decaying via one preferred two-meson channel [17,18]. Next, since the only viable flavor structure of narrow compact tetraquark bound states, resulting from confinement, is the diquark structure [21–29] and in the flavor-exotic case one has only one flavor diquark-antidiquark combination, $(\bar{a}\bar{c})(bd)$ (product of antisymmetric representations in color space), one encounters a contradiction with the requirement of the existence of two different tetraquarks. One thus has to conclude that, without the presence of a dynamical fine-tuning mechanism, narrow flavor-exotic compact states do not exist in large-N_c QCD.

II. NARROW FLAVOR-EXOTIC TETRAQUARK STATES AT LARGE N_c

We consider the case of possibly existing narrow flavor-exotic tetraquark states and study the properties of tetraquark-phile diagrams for direct Green functions $\langle T\{j_{\bar{a}b}j_{\bar{c}d}j_{\bar{a}b}^{\dagger}j_{\bar{c}d}^{\dagger}\}\rangle$ and $\langle T\{j_{\bar{a}d}j_{\bar{c}b}j_{\bar{a}d}^{\dagger}j_{\bar{c}b}^{\dagger}\}\rangle$ and recombination Green functions $\langle T\{j_{\bar{a}b}j_{\bar{c}d}j_{\bar{a}d}^{\dagger}j_{\bar{c}b}^{\dagger}\}\rangle$.

A. Direct four-point Green functions

According to the formulated criteria for selecting tetraquark-phile diagrams (which give necessary, but not sufficient conditions for the existence of tetraquark poles), the lowest-order diagrams in α_s are cylinder diagrams of Fig. 1 with two-gluon exchanges between the quark loops. They have cylinder topology, contain two color loops and are of order $O(\alpha_s^2 N_c^2) = O(N_c^0)$ at large N_c.

We now add one more gluon. As an example, we consider adding a gluon in the diagram of Fig. 1(a): The
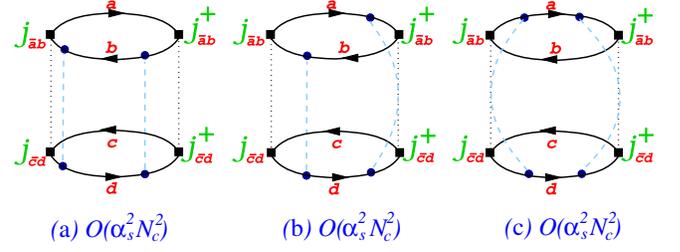
FIG. 1. Tetraquark-phile diagrams for the direct amplitude with two gluon exchanges. All diagrams have cylinder topology and two color loops and are of order $\alpha_s^2 N_c^2$. Light-blue dashed lines denote planar gluons (lying on the cylinder tube); gray dotted lines indicate the contours of the tube.
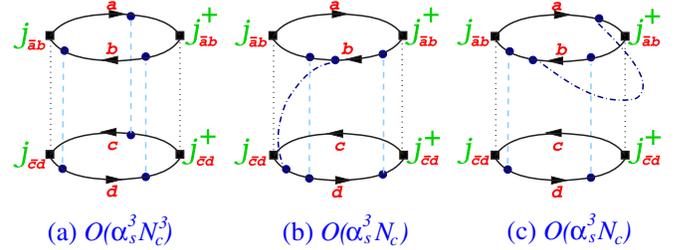
FIG. 2. Diagrams for a direct amplitude, appropriate for flavor-exotic tetraquarks, of order $O(\alpha_s^3)$, with three gluon exchanges. Light-blue dashed lines denote planar gluons (lying on the cylinder tube); dark-blue dash-dotted lines denote nonplanar gluons (lying outside the tube); gray dotted lines indicate the contours of the cylinder. Diagram (a) is cylindric, has three color loops, and is of order $\alpha_s^3 N_c^3$; diagrams (b) and (c) represent cylinders with one gluonic handle, have only one color loop, and are of order $\alpha_s^3 N_c$.

diagram of Fig. 2(a) is a cylinder diagram and has the same order in N_c as the diagrams of Fig. 1. The diagrams of Figs. 2(b) and 2(c) have the topology of a cylinder with one handle: Adding one handle reduces the number of color loops by one and adds one power of α_s; the two diagrams get a reduction factor $1/N_c^2$.

It is easy to establish the connection between the cylinder diagrams of Fig. 2 and the planar diagrams of Fig. 3: We break the quark lines a and d and put the two separated points of each line at $-\infty$ and $+\infty$, respectively. Then the classification planar and nonplanar gluon exchanges becomes obvious. When calculating the color factors, one has to take into account that the left and the right ends of the line a (d) are in fact joined together and form a color loop.

B. Recombination four-point Green function

The N_c-leading tetraquark-phile diagram for the recombination Green function is shown in Fig. 4: It is a cylinder with one handle, has one color loop, and is of order $\alpha_s^2 N_c$. Interestingly, there are no tetraquark-phile diagrams without handles in the recombination channel.
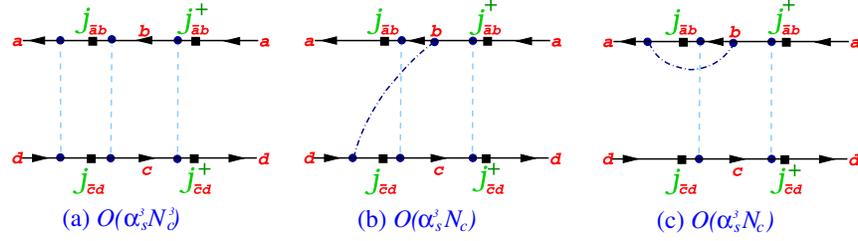
FIG. 3. Redrawing the cylinder diagrams of Fig. 2 as planar diagrams (with handles).
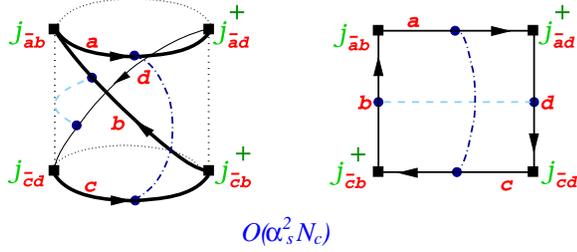
FIG. 4. N_c-leading $O(\alpha_s^2)$ diagram for the recombination amplitude appropriate for flavor-exotic tetraquarks.

FIG. 5. Effective tree-level meson-meson interactions in the direct channels: (a) Four-meson contact term. (b) Glueball exchange.

III. COMPARISON OF DIRECT AND RECOMBINATION GREEN FUNCTIONS

Applying our criteria for tetraquark-phile diagrams, one finds that the N_c-leading direct and recombination diagrams have different large-N_c behaviors. If tetraquark poles emerge at all, they should emerge in the N_c-leading tetraquark-phile diagrams (any different setup is difficult to justify from the perspective of bound-state equations for tetraquarks). Then, one needs two narrow tetraquark states T_A and T_B, each decaying via one preferred meson-meson channel, in order to satisfy the consistency conditions between D and R Green functions [17,18]:

$$A(T_A \to M_{\bar{a}b} M_{\bar{c}d}) = O(N_c^{-1}),$$
$$A(T_A \to M_{\bar{a}d} M_{\bar{c}b}) = O(N_c^{-2}),$$
$$A(T_B \to M_{\bar{a}b} M_{\bar{c}d}) = O(N_c^{-2}),$$
$$A(T_B \to M_{\bar{a}d} M_{\bar{c}b}) = O(N_c^{-1}). \qquad (3.1)$$

If the bound states exist, their widths $\Gamma(T_{A,B})$ will be determined by the dominant channels, which yields $\Gamma(T_{A,B}) = O(N_c^{-2})$, thus confirming the narrow-width property of the tetraquark candidate states.

The N_c-matching conditions also allow us to deduce the properties of the effective tree-level meson-meson interactions. The direct-channel N_c-leading connected diagrams are Okubo-Zweig-Iizuka-suppressed [10,11] and the corresponding effective meson-meson interactions come out to be of order $1/N_c^2$, resulting either from contact terms [Fig. 5(a)] or from glueball exchanges [Fig. 5(b)] [18].

On the other hand, the recombination-channel interactions are of the generic order $1/N_c$, resulting either from

contact terms [Fig. 6(a)] or from meson exchanges [Fig. 6(b)] [18]. (They are provided by the N_c-leading diagrams of the recombination channel.)

Taking into account the above properties and Eq. (3.1), one deduces the dominant structure of each of the two tetraquarks: T_A has the structure $(\bar{a}d)(\bar{c}b)$, while T_B has the structure $(\bar{a}b)(\bar{c}d)$, both of them being the product of two color-singlet clusters; their dominant decay channel proceeds through the recombination (quark-exchange) process rather than through the dissociation one.

However, in the diquark-antidiquark mechanism of the tetraquark formation [21–25], we have at our disposal only one flavor-exotic combination $(\bar{a}\bar{c})(bd)$, in the form of the product of color-antisymmetric representations, which leads to a contradiction with the above requirement of the existence of two tetraquarks. One thus is led to conclude that in large-N_c QCD flavor-exotic compact narrow tetraquarks might not exist.
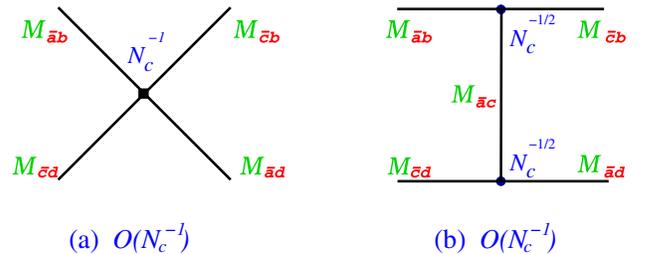
FIG. 6. Effective tree-level meson-meson interactions in the recombination channel: (a) Four-meson contact term. (b) Meson exchange (to be completed with a similar diagram in the u channel).

This conclusion rests on the observed different behaviors of the tetraquark-phile contributions to direct and recombination Green functions. Can one formulate different consistent criteria for selecting tetraquark-phile diagrams? It is conceivable that for some dynamical reasons, tetraquarks do not necessarily contribute to the generic leading diagrams that have been taken into account. The authors of [19] consider such a possibility by imposing more stringent selection rules. These are based on two main assumptions: (i) Tetraquark-phile diagrams have a nonplanar topology with one gluonic handle. (ii) Only one class of diagrams, either D or R, contributes to the tetraquark formation. For phenomenological reasons, it is channel D that is chosen as admissible for tetraquark emergence. Then a single tetraquark may accommodate the consistency conditions, with a coupling of order N_c^{-2} to the two sets of available meson pairs.

Without intending to discard the possibility of a selection mechanism as described in [19], which demands, however, a more detailed investigation on dynamical grounds, we would like to draw attention to one argument that does not seem well founded.

The main justification in [19] of imposing on the channel D tetraquark-phile diagrams to have a handle is based on the assertion that D-type planar diagrams do not describe mutual interactions of meson or $(\bar q q)$ pairs. However, this is contradicted by the existence of two-meson intermediate states contributing through planar diagrams to meson-meson scattering. First, the N_c-leading diagrams of the R channel, which do not have four-quark singularities, still contribute to the effective meson-meson interaction through the diagrams of Fig. 6, the global coupling being of order N_c^{-1}. Second, the unitarity condition requires that the diagrams of Fig. 6 generate meson loop diagrams of the type of Fig. 7, which are genuine parts of the meson-meson scattering amplitude in channel D. They are of order N_c^{-2}, i.e., of the same order as the leading planar diagrams of channel D. This could not happen if their underlying QCD diagrams were not of the planar type.

A typical such diagram is presented in Fig. 8. The intermediate states obtained from a vertical cut are precisely those corresponding to the quark-exchange process.
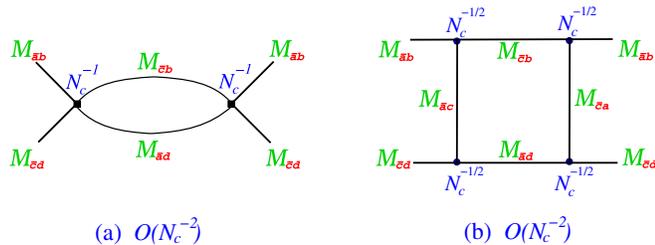
FIG. 7.   Contributions of two-meson intermediate states to the meson-meson scattering amplitude in a direct channel. The intermediate states are those produced by the recombination process at lower order.

$$O(N_c^{-2})$$
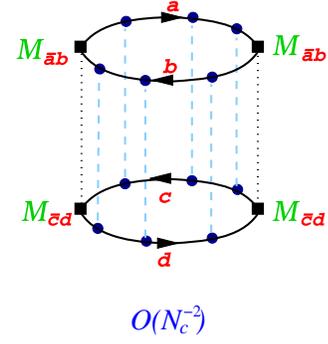
FIG. 8.   A typical cylindric QCD diagram [cf. Fig. 2(a)] contributing to the two-meson intermediate states in the meson-meson scattering amplitude in a direct channel. Gluon exchanges around the corners of the loops, going from one line of a loop to the other line, are parts of the external meson states and are not drawn. In contrast to the diagrams of Figs. 1 and 2, where the Green functions for interpolating currents are shown, this figure presents the meson-meson amplitudes and therefore contains an extra factor $1/N_c^2$, corresponding to $1/f_M^4$, where $f_M = O(\sqrt{N_c})$ is the meson decay constant.

Here, color rearrangement plays a physical role by converting the pair of initial mesons into the other pair.

This shows that the leading-order planar diagrams of channel D are not physically empty and describe a part of the meson-meson scattering process. The question whether a compact tetraquark pole may emerge from such a process or not still remains a relevant issue. Meson-meson interactions are expected to be short range, and if a tetraquark pole exists in the corresponding scattering amplitude, as a bound state or a resonance, it should be loosely bound and would presumably correspond to a molecular-type state. This possibility is examined in Sec. IV.

The possible existence of a hidden dynamical mechanism which favors the emergence of compact tetraquarks in fully exotic channels is an open question and deserves further study. We also emphasize that the conclusions obtained in the main part of this section apply only to the fully exotic case (four different quark flavors). For systems with a smaller number of quark flavors, additional QCD diagrams, not present in the fully exotic case, may invalidate some of the results obtained above and may allow for the existence of one tetraquark [17,18].

IV. MOLECULAR STATES AT LARGE N_c

Tetraquarks may also have a molecular structure, resulting from meson-meson interactions and existing either in the form of bound states or of resonances [30–36]. Meson-meson interactions are generally formulated in the form of effective Lagrangians or of empirical potentials. In the large-N_c limit, these interactions are expected to scale as $1/N_c$ [10,11].

In the case of mesons made of light quarks (u, d, s) and, in particular, involving the lightest pseudoscalar mesons,

chiral perturbation theory (ChPT) [37,38] provides the general effective Lagrangian suited for the description of the corresponding interactions at low energies. An extension of the energy domain of validity of ChPT is done with the aid of the unitarization condition of the scattering amplitudes, together with the use of dispersion relations [39,40].

In sectors involving heavy quarks, the masses of the latter introduce new scales in the system, which must be taken into account. The heavy mesons, in addition to their contact-type interactions, also interact through light-meson exchanges of Yukawa type [34,35,41,42] [cf. Fig. 6(b), where the heavy quarks correspond to those denoted b and d]. The latter interactions provide additional opportunities for the emergence of bound states or of resonances. A more systematic study can be done with the use of heavy quark effective field theory and the associated spin and flavor symmetries [36,43–48].

The possible dynamical emergence of bound states or of resonances from meson-meson interactions is studied by summing in the evaluation of the scattering amplitude chains of diagrams of similar structure; Figure 9, where we have explicitly factored out the N_c dependence of the effective coupling constant, schematically displays the summation of bubble diagrams. These diagrams are generally divergent and, accordingly, the effective coupling constants undergo renormalization [46,49,50]. According to the signs of the renormalized coupling constants, which might be determined from other experimental data, a bound state or a resonance pole may emerge.

The important qualitative feature of the resultant dynamical pole is that its mass squared, in the case of a resonance, or binding energy, in the case of a bound state, are essentially proportional, up to small corrections, to the inverse of the effective coupling constant [46,49,50]. Therefore, assuming that the renormalized effective coupling constant remains, as the bare one, inversely proportional to N_c, the resonance mass will be pushed towards infinity with a broad width, while the bound state will disappear from the spectrum.

In the case of light quarks, a detailed study of the problem has been presented in [39,40]. The general result is that, in the scalar-isoscalar s channel of $\pi\pi$ scattering, a dynamical resonance, corresponding to the observed $f_0(500)$ resonance, emerges, having a dominant structure of two quarks and two antiquarks, in distinction from the
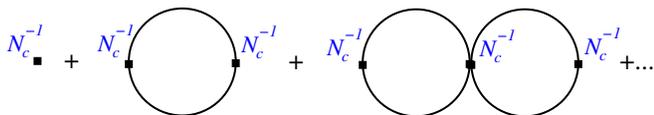
FIG. 9.   Chain of bubble diagrams participating in the realization of elastic unitarity and in the creation of a bound state or of a resonance pole; the N_c dependence of the effective meson-meson interaction has been factored out.

ordinary mesons [51]. This result has also been confirmed by a direct solution of the four-quark Bethe-Salpeter equation [52]. At large N_c, the mass and the width of the resonance behave as $\sqrt{N_c}$, as expected from the general qualitative features outlined above. Generalization of the calculations with other combinations of the light quarks is expected to provide similar qualitative conclusions.

In the case of sectors involving heavy quarks, the above conclusions would remain true in the formal limit of N_c going to infinity, but for finite values of N_c the observable effects might be less striking, since the mass gap between the two-meson threshold and the resonance position would be relatively reduced as compared to the light-quark-sector case.

Can molecular-type tetraquarks contribute to the large-N_c analysis of Green functions? The answer is negative, since the emergence of a molecular-type pole necessitates the summation of a chain of diagrams with different orders in $1/N_c$ (Fig. 9). This is in contrast to the case of compact tetraquarks, where the summation of planar diagrams is done at the same order in N_c together with the creation of the pole; this makes possible the matching of the pole contributions in the Green function, on the one hand, and in the Feynman diagrams, on the other. For molecular-type tetraquarks, the latter contribute to Feynman diagrams at leading order in N_c only through their contact terms and one-meson exchange terms (Fig. 6), and therefore a matching-type analysis is not possible. In that case, one has to deduce the properties of the tetraquark from the explicit summation itself.

A particular attention should be paid to the case of molecular-type bound states, also called deuteronlike, which may emerge very close to the two-meson threshold, with an unnaturally small binding energy [36,53]. They are also characterized by a large (negative) value of the S-wave scattering length, much greater than the natural scale provided by the physical parameters of the system, and exhibiting universality properties [54]. According to the general properties of the emergence of dynamical poles in the scattering amplitude as outlined above, these bound states might appear only in the strong-coupling limit of the effective theory, while the large-N_c limit drives the theory to its weak-coupling limit. Therefore, in the formal limit of large N_c, one also predicts the disappearance of these states. In general, the details of the creation mechanism of these states not being well known, it is admitted that underlying fine-tuning processes might be at work for their existence [54]. This would mean that they are very sensitive to variations of their physical parameters and, in particular, of N_c. Results obtained at large N_c might not correctly describe their physical properties at finite values of N_c.

Coming now back to the case of flavor-exotic tetraquarks, we recall that the direct-channel effective meson-meson interactions are actually of order $1/N_c^2$ (cf. Fig. 5); i.e., they are much weaker than in the generic case ($\sim 1/N_c$).

The recombination-channel interactions remain of order $1/N_c$ (cf. Fig. 6); however, since they represent off-diagonal-type contributions in a coupled-channel formalism, their effective contributions to the resonance or the bound-state pole formation will still be as in the direct-channel case. Therefore, the possibly existing resonance-pole positions will be pushed even more strongly to infinity than in the generic cases, while bound-state poles will be absent from the spectrum.

In conclusion, narrow-width molecular-type tetraquarks, with masses that remain fixed at large N_c, are not generally expected to occur in flavor-exotic sectors. Assuming that the continuation to finite values of N_c remains a smooth operation in the theory, this statement would still be valid in the physical world, except possibly in the particular case of a bound state lying very close to the two-meson threshold.

V. CONCLUSIONS

We have considered, in the large-N_c limit of QCD, the possibility of the existence of narrow four-quark states of an exotic flavor content, involving four quarks of different flavors (that requires two quarks and two antiquarks as a minimal parton configuration). The two cases of compact and molecular tetraquarks have been examined.

Compact tetraquarks are the genuine candidates for the quest for narrow-width states at large N_c. In the sectors of flavor-exotic states, the consistency constraints, coming from the direct and recombination (or quark-exchange) type channels, require the existence of two different tetraquarks, each having a structure made of two color-singlet clusters or mesons and decaying in a preferred two-meson channel, fixed by the dominance of the recombination-type effective interaction. On the other hand, the formation mechanism of tetraquarks through a primary formation of diquarks and antidiquarks predicts the existence of one tetraquark, decaying with equal weights, up to small corrections, into the two different two-meson channels. This contradiction suggests that compact tetraquarks do not exist in flavor-exotic sectors, unless some hidden dynamical mechanism favors their emergence [19,26,29].

Molecular tetraquarks, because of the weakening of the effective meson-meson interactions at large N_c, might only exist as resonances with masses and widths that increase like $\sqrt{N_c}$. In the case of the presence of heavy mesons, the mass gap between the resonance position and the two-meson thresholds might be substantially reduced at finite values of N_c. In the flavor-exotic case, the effective interactions are much weaker than in the generic cases, and, because of this feature, the masses of the possibly existing resonances are repelled to higher values. Therefore, at large N_c, no molecular-type tetraquarks, with fixed masses and narrow widths, are expected to emerge. An exceptional case might occur, at finite N_c, with the emergence of a single bound state lying very close to the two-meson threshold.

Up to now, experimental data, as well as lattice calculations, do not provide evidence for the existence of flavor-exotic tetraquarks in sectors involving one heavy quark, c or b. Flavor-exotic sectors involving two heavy quarks, c and b, seem to be yet unexplored. Therefore, experimental data and lattice calculations for these sectors would be of great help for the understanding of the underlying dynamics of QCD.

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[1] A. Ali, J. S. Lange, and S. Stone, Prog. Part. Nucl. Phys. **97,** 123 (2017).

[2] S. L. Olsen, T. Skwarnicki, and D. Zieminska, Rev. Mod. Phys. **90,** 015003 (2018).

[3] V. M. Abazov *et al.* (D0 Collaboration), Phys. Rev. Lett. **117,** 022003 (2016); Phys. Rev. D **97,** 092004 (2018).

[4] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **117,** 152003 (2016); **118,** 109904(A) (2017).

[5] A. M. Sirunyan *et al.* (CMS Collaboration), Phys. Rev. Lett. **120,** 202005 (2018).

[6] T. Aaltonen *et al.* (CDF Collaboration), Phys. Rev. Lett. **120,** 202006 (2018).

[7] M. Aaboud *et al.* (ATLAS Collaboration), Phys. Rev. Lett. **120,** 202007 (2018).

[8] Y. Ikeda *et al.* (HAL QCD collaboration), Phys. Lett. B **729,** 85 (2014).

[9] G. 't Hooft, Nucl. Phys. **B72,** 461 (1974).

[10] E. Witten, Nucl. Phys. **B160,** 57 (1979).

[11] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985), Chap. 8.

[12] S. Weinberg, Phys. Rev. Lett. **110,** 261601 (2013).

[13] M. Knecht and S. Peris, Phys. Rev. D **88,** 036016 (2013).

[14] T. D. Cohen and R. F. Lebed, Phys. Rev. D **90,** 016001 (2014).

[15] L. Maiani, A. D. Polosa, and V. Riquer, J. High Energy Phys. 06 (2016) 160.

[16] A. Esposito, A. Pilloni, and A. D. Polosa, Phys. Rep. **668,** 1 (2017).

[17] W. Lucha, D. Melikhov, and H. Sazdjian, Phys. Rev. D **96**, 014022 (2017).

[18] W. Lucha, D. Melikhov, and H. Sazdjian, Eur. Phys. J. C **77**, 866 (2017).

[19] L. Maiani, A. D. Polosa, and V. Riquer, Phys. Rev. D **98**, 054023 (2018).

[20] L. D. Landau, Nucl. Phys. **13**, 181 (1959).

[21] T. Schäfer, E. V. Shuryak, and J. J. M. Verbaarschot, Nucl. Phys. **B412**, 143 (1994).

[22] R. Jaffe and F. Wilczek, Phys. Rev. Lett. **91**, 232003 (2003).

[23] S. Nussinov, arXiv:hep-ph/0307357.

[24] E. V. Shuryak and I. Zahed, Phys. Lett. B **589**, 21 (2004).

[25] L. Maiani, F. Piccinini, A. D. Polosa, and V. Riquer, Phys. Rev. D **71**, 014028 (2005).

[26] S. J. Brodsky, D. S. Hwang, and R. F. Lebed, Phys. Rev. Lett. **113**, 112001 (2014).

[27] M. Karliner and J. L. Rosner, Phys. Rev. Lett. **119**, 202001 (2017).

[28] E. J. Eichten and C. Quigg, Phys. Rev. Lett. **119**, 202002 (2017).

[29] L. Maiani, A. D. Polosa, and V. Riquer, Phys. Lett. B **778**, 247 (2018).

[30] M. B. Voloshin and L. B. Okun, Pis'ma Zh. Eksp. Teor. Fiz. **23**, 369 (1976) [JETP Lett. **23**, 333 (1976)].

[31] M. Bander, G. L. Shaw, P. Thomas, and S. Meshkov, Phys. Rev. Lett. **36**, 695 (1976).

[32] A. De Rujula, H. Georgi, and S. L. Glashow, Phys. Rev. Lett. **38**, 317 (1977).

[33] N. A. Törnqvist, Z. Phys. C **61**, 525 (1994).

[34] C. Amsler and N. A. Törnqvist, Phys. Rep. **389**, 61 (2004).

[35] E. S. Swanson, Phys. Rep. **429**, 243 (2006).

[36] F.-K. Guo, C. Hanhart, U.-G. Meissner, Q. Wang, Q. Zhao, and B.-S. Zou, Rev. Mod. Phys. **90**, 015004 (2018).

[37] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).

[38] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

[39] J. R. Pelaez and G. Rios, Phys. Rev. Lett. **97**, 242002 (2006).

[40] J. R. Pelaez, Phys. Rep. **658**, 1 (2016).

[41] M. Pavon Valderrama, Phys. Rev. D **85**, 114037 (2012).

[42] M. Karliner and J. L. Rosner, Nucl. Phys. **A954**, 365 (2016).

[43] H. Georgi, Phys. Lett. B **240**, 447 (1990).

[44] M. B. Wise, Phys. Rev. D **45**, R2188 (1992).

[45] M. Neubert, Phys. Rep. **245**, 259 (1994).

[46] M. E. Luke and A. V. Manohar, Phys. Rev. D **55**, 4129 (1997).

[47] M. T. AlFiky, F. Gabbiani, and A. A. Petrov, Phys. Lett. B **640**, 238 (2006).

[48] V. Baru, A. Epelbaum, J. Gegelia, C. Hanhart, U.-G. Meissner, and A. V. Nefediev, arXiv:1810.06921.

[49] S. Weinberg, Nucl. Phys. **B363**, 3 (1991).

[50] R. Jackiw, in *M.A.B. Bég Memorial Volume*, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991), p. 25.

[51] R. L. Jaffe, Nucl. Phys. **A804**, 25 (2008).

[52] G. Eichmann, C. S. Fischer, and W. Heupel, Phys. Lett. B **753**, 282 (2016).

[53] S. Weinberg, Phys. Rev. **137**, B672 (1965).

[54] E. Braaten and H.-W. Hammer, Phys. Rep. **428**, 259 (2006).