Comment on "Covariant Tolman-Oppenheimer-Volkoff equations. II. The anisotropic case"

A. A. Isayev*

Kharkov Institute of Physics and Technology, Academicheskaya Street 1, Kharkov, 61108, Ukraine

(Received 2 August 2018; published 30 October 2018)

Recently, the covariant formulation of the Tolman-Oppenheimer-Volkoff (TOV) equations for studying the equilibrium structure of a spherically symmetric compact star in the presence of the pressure anisotropy in the interior of a star was presented in Ref. [1]. It was suggested there that the anisotropic solution of these equations can be obtained by finding, first, the solution of the common TOV equations for the isotropic pressure, and then by solving the differential equation for the anisotropic pressure whose particular form was established on the basis of the covariant TOV equations. It turns out that the anisotropic pressure determined according to this scheme has a nonremovable singularity $\Pi \sim \frac{1}{r^2}$ in the center of a star, and, hence, the corresponding anisotropic solution cannot represent a physically relevant model of an anisotropic compact star. A new scheme for constructing the anisotropic solution, based on the covariant TOV equations is suggested, which leads to the regularly behaved physical quantities in the interior of a star. A new algorithm is applied to build model anisotropic strange quark stars with the MIT bag model equation of state.

DOI: 10.1103/PhysRevD.98.088503

The presence of the pressure anisotropy in the interior of a compact star should be considered more like a common feature than an exception. The sources of the pressure anisotropy can be very different such as the existence of a solid core, the relativistic nature of the nuclear interaction at high densities, the presence of strong magnetic fields inside a star, etc. The equilibrium configuration of a spherically symmetric anisotropic compact star can be studied on the basis of the TOV equations, generalized considering the pressure anisotropy [2]. Recently, these equations were presented in the covariant form with the help of $1+1+2$ covariant formalism in Ref. [1]. There are several strategies to find solutions to these equations. One strategy is to set the anisotropy parameter in the specific preassigned form, which, together with the equation of state (EOS), can be used to find the energy density μ, the transverse $p_\perp$ and radial p_r pressures, and the unknown metric functions A, B (in notations of Ref. [1]). An example of such an approach can be found, e.g., in Ref. [2]. The other strategy is to somehow reduce the problem to the isotropic case, and then to build the anisotropic solution by properly modifying the obtained isotropic one. Such an approach was followed, e.g., recently in Refs. [3,4]. Yet another strategy was suggested in Ref. [1] and is based on the separation of the isotropic and anisotropic degrees of freedom in the covariant TOV equations. After the separation, these equations take the form [cf. Eqs. (35) in Ref. [1]]

$$P_{,\rho} + P^2 - P\left(\mathbb{M} - 3\mathcal{K} + \frac{7}{4}\right)$$
$$-\left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M} = -\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P}\left(-2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right),$$
$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right), \qquad (1)$$

where $P = B(\rho)p$ and $\mathbb{P} = B(\rho)\Pi$, $p \equiv \frac{p_r + 2p_\perp}{3}$ and $\Pi \equiv \frac{2(p_r - p_\perp)}{3}$ are the isotropic and anisotropic pressure, respectively. The isotropic pressure terms are gathered in the left-hand side (lhs) and the anisotropic pressure terms are collected in the right-hand side (rhs) of the first equation. Note the difference in sign in the last term in the lhs, and in the first term and in the term with P in the brackets in the rhs of the first equation, compared to the corresponding Eq. (35) in Ref. [1]. Equations (1) are nothing else than the rewritten Eqs. (23) of Ref. [1], and the difference in sign can be readily checked. Based on the structure of the covariant TOV equations (1), the following algorithm for constructing the anisotropic solution was proposed in Ref. [1]. First, to find the solution of the common TOV equations, supplemented with the EOS $\mathbb{M} = \mathbb{M}(P)$, for the isotropic pressure P at vanishing $\mathbb{P}$. In this way, there will be determined the coordinate dependence of the functions P, $\mathcal{K}$, and $\mathbb{M}$. Then the obtained solution of the TOV equations in the isotropic case should be substituted into the equation for the anisotropic pressure $\mathbb{P}$:

$$\mathbb{P}_{,\rho} + \mathbb{P}^2 - \mathbb{P}\left(-2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right) = 0. \qquad (2)$$

*isayev@kipt.kharkov.ua

While the equation for $\mathcal{K}$ in this scheme is of the Bernoulli type and admits formal integration, the equation for the isotropic pressure P, in view of the arbitrariness of the EOS $\mathbb{M} = \mathbb{M}(P)$, can be integrated only numerically. For this reason, finding the anisotropic pressure $\mathbb{P}$, in the general case, also requires numerical integration.

In order to check this algorithm for constructing the anisotropic solution, we will rewrite the covariant TOV equations in this scheme in the usual form with the radial coordinate r as an independent variable. The TOV equations in the isotropic case read (using the notations of Ref. [1] and the system of units with $c = 1$)

$$p_{,r} + G\frac{(\mu + p)(m(r) + 4\pi p r^3)}{r(r - 2Gm(r))} = 0, \tag{3}$$

$$m_{,r} = 4\pi\mu r^2. \tag{4}$$

Note that the differential equation for the metric function $B(r)$ was rewritten in terms of the local mass function $m(r)$ which are related by $B(r) = (1 - \frac{2Gm(r)}{r})^{-1}$. Equations (3) and (4), supplemented with the EOS $\mu = \mu(p)$, should be solved together with the initial conditions $p(0) = p_0$, $m(0) = 0$, p_0 being the central isotropic pressure. Equation (2), after straightforward transformations, takes the form

$$\Pi_{,r} + \frac{\Pi}{r}\left(G\frac{m(r) + 4\pi(\mu + 2p + \Pi)r^3}{r - 2Gm(r)} + 3\right) = 0. \tag{5}$$

Note that solution of Eqs. (3)–(5) should lead to the singularity-free physical quantities in the interior of a star. As follows from Eq. (5), the gradient $\Pi_{,r}$ will be finite at $r = 0$, if $\Pi \sim r^\alpha$, $\alpha \geq 1$ at $r \to 0$. Given this asymptotic behavior and taking into account that $m(r) \sim r^3$ at $r \to 0$, in the leading order approximation on small r, Eq. (5) reads $\Pi_{,r} + \frac{3\Pi}{r} = 0$, which can be fulfilled only if $\alpha = -3$, i.e., $\Pi \sim \frac{1}{r^3}$ at $r \to 0$, which contradicts to the constraint $\alpha \geq 1$. If the anisotropic pressure Π has a singularity at the origin, then the term with Π in the numerator of the fraction in the brackets in the lhs of Eq. (5) is of relevance as well, and, after retaining it, in the leading order approximation one gets

$$\Pi_{,r} + \frac{3\Pi}{r} + 4\pi G\Pi^2 r = 0. \tag{6}$$

Substituting in the last equation $\Pi = Cr^\alpha$, one can see that it can be satisfied at any small r, only if $\alpha = -2$, and, simultaneously, $C = \frac{2}{\pi G}$. Therefore, the anisotropic pressure Π, determined according to Eq. (5), has a nonremovable singularity $\Pi \sim \frac{1}{r^2}$ at $r = 0$, and, hence, the corresponding anisotropic solution cannot represent a physically relevant model of an anisotropic compact star. Note that this conclusion was reached on the basis of a general asymptotic analysis of Eq. (5) only under the assumption that the energy density μ and the isotropic pressure p are regular functions at the center of a star. This analysis does not rely on any specific form of the functions μ and p.

In order to get the regularly behaved anisotropic pressure at the center of a star, let us present the equation relating the isotropic P and anisotropic $\mathbb{P}$ pressures in Eq. (1) in a different form by carrying the term $(\frac{1}{4} - \mathcal{K})\mathbb{M}$ to the rhs:

$$P_{,\rho} + P^2 - P\left(\mathbb{M} - 3\mathcal{K} + \frac{7}{4}\right) = -\mathbb{P}_{,\rho} - \mathbb{P}^2 + \mathbb{P}\left(-2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right) + \left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M}. \tag{7}$$

A new algorithm for constructing the anisotropic solution of the covariant TOV equations consists of the following. First, it is necessary to find the isotropic pressure from the differential equations

$$P_{,\rho} + P^2 - P\left(\mathbb{M} - 3\mathcal{K} + \frac{7}{4}\right) = 0,$$
$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right), \tag{8}$$

supplemented by the EOS $\mathbb{M} = \mathbb{M}(P)$. Obtained in this way functions P, $\mathcal{K}$, and $\mathbb{M}$ should be substituted in the differential equation for the anisotropic pressure

$$\mathbb{P}_{,\rho} + \mathbb{P}^2 - \mathbb{P}\left(-2P + \mathbb{M} - 3\mathcal{K} + \frac{1}{4}\right) - \left(\frac{1}{4} - \mathcal{K}\right)\mathbb{M} = 0. \tag{9}$$

Note that in this new setup Eqs. (8) for finding the isotropic pressure are different from the common TOV equations in the isotropic case. Precisely, the equation for the isotropic pressure with the radial coordinate as an independent variable reads [cf. Eq. (3) for the common TOV equations]

$$p_{,r} + G\frac{(\mu + p)(m(r) + 4\pi p r^3)}{r(r - 2Gm(r))} - \frac{Gm(r)\mu}{r(r - 2Gm(r))} = 0, \tag{10}$$

while the second equation in (8) goes over to Eq. (4). It is also interesting to notice that Eq. (9) has no trivial solution with $\mathbb{P} \equiv 0$, and, hence the pressure anisotropy represents an essential feature of such a class of compact stars.

After straightforward transformations, Eq. (9) for the anisotropic pressure with the radial coordinate as an independent variable takes the form

$$\Pi_{,r} + \frac{\Pi}{r}\left(G\frac{m(r) + 4\pi(\mu + 2p + \Pi)r^3}{r - 2Gm(r)} + 3\right) + \frac{Gm(r)\mu}{r(r - 2Gm(r))} = 0. \tag{11}$$

Now, assuming that $\Pi = C' r^\alpha$, $\alpha \geq 1$ at $r \to 0$, in the leading order approximation one gets

$$\Pi_{,r} + \frac{3\Pi}{r} + \frac{4\pi G\mu_0^2 r}{3} = 0, \tag{12}$$

where $\mu_0 \equiv \mu(0)$ is the energy density at the center of a star. Equation (12) can be satisfied at small r, only if $\alpha = 2$ and $C' = -\frac{4\pi G\mu_0^2}{15}$, i.e., $\Pi \sim r^2$ at $r \to 0$. Therefore, the anisotropic pressure, determined according to Eq. (11) in a new scheme, is the regularly behaved function at the center of a star.

In order to test this new algorithm, let us consider anisotropic strange quark stars within the MIT bag model with the massless quarks and the EOS $\mu = 3p + 4B$, B being the bag constant. The radius of a spherically symmetric anisotropic star is determined from the condition $p_r(R) = p(R) + \Pi(R) = 0$, where isotropic $p(r)$ and anisotropic $\Pi(r)$ pressures, together with the mass function $m(r)$, are obtained by solving the differential equations (4), (10), (11) with the initial conditions $p(0) = p_0, m(0) = 0$, $\Pi(0) = 0$. The total mass of a compact star is found as $M = m(R)$. Table I presents the results of the numerical determination of the total mass of an anisotropic strange quark star in dependence on the central isotropic pressure p_0 at $B = 57\ \mathrm{MeV/fm^3}$. It is seen that even small central pressures $p_0 < 1\ \mathrm{MeV/fm^3}$ produce heavy strange quark stars with $M > 6\ M_\odot$ ($M_\odot$ being the solar mass). This is because the isotropic pressure, determined according to Eq. (10), decreases considerably slower with the radial coordinate r compared to that determined from the common TOV equation (3). Another peculiarity is that the total mass M decreases with the central pressure p_0 for the whole range of the central pressures under consideration, contrary to the stability constraint $\frac{dM}{dp_0} > 0$. This means that the obtained model massive anisotropic strange quark stars are unstable with respect to radial oscillations.

TABLE I. The mass M of an anisotropic strange quark star (in solar mass units), determined according to Eqs. (4), (10), and (11), for different values of the central isotropic pressure p_0 within the MIT bag model at $B = 57\ \mathrm{MeV/fm^3}$.

p_0, $\frac{\mathrm{MeV}}{\mathrm{fm^3}}$	$M/M_\odot$
0.001	6.7338
0.01	6.7337
0.1	6.7321
1	6.7163
10	6.5722
20	6.4362
30	6.3199
40	6.2190
50	6.1305
60	6.0523
70	5.9827
80	5.9203
90	5.8640
100	5.8131

To summarize, the proposed algorithm in Ref. [1] for constructing the anisotropic solution of the TOV equations, based on solving the common TOV equations for the isotropic pressure and Eq. (2) for the anisotropic pressure (in the covariant formulation), leads to the singularly behaved anisotropic pressure at the center of a star, and, hence, the corresponding anisotropic solution cannot represent a physically relevant model of an anisotropic compact star. In this Comment, suggest a new scheme for constructing the anisotropic solution, based on the covariant TOV equations, which leads to the regularly behaved anisotropic pressure in the interior of a star. This algorithm has been tested with respect to anisotropic strange quark stars within the MIT bag model. It turns out that the new algorithm gives rise to massive anisotropic strange quark stars with $M \gtrsim 5\ M_\odot$, which are unstable with respect to radial oscillations.

In the end, it would also be correct to note that Ref. [1] contains a number of other suggestions on constructing solutions of the TOV equations in the anisotropic case, mainly, in the analytical form, whose discussion is, however, beyond the scope of this Comment.

[1] S. Carloni and D. Vernieri, Covariant Tolman-Oppenheimer-Volkoff equations. II. The anisotropic case, Phys. Rev. D **97**, 124057 (2018).

[2] R. L. Bowers and E. P. T. Liang, Anisotropic spheres in general relativity, Astrophys. J. **188**, 657 (1974).

[3] F. Shojai, M. Kohandel, and A. Stepanian, On the relativistic anisotropic configurations, Eur. Phys. J. C **76**, 347 (2016).

[4] A. A. Isayev, General relativistic polytropes in anisotropic stars, Phys. Rev. D **96**, 083007 (2017).