Reply to "Comment on 'Can accretion disk properties observationally distinguish black holes from naked singularities?'"

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(Received 3 August 2018; published 16 October 2018)

In the Comment on "Can accretion disk properties observationally distinguish black holes from naked singularities?", by Bertrand Chauvineau, the author did show that the metric used in Z. Kovács and T. Harko, Phys. Rev. D **82**, 124047 (2010), and initially introduced in K. D. Krori and D. R. Bhattacharjee, J. Math. Phys. **23**, 637 (1982) and K. K. Nandi, P. M. Alsing, J. C. Evans, and T. B. Nayak, Phys. Rev. D **63**, 084027 (2001), does not satisfy the Einstein gravitational field equations with a minimally coupled scalar field. In our reply we would like to point out that this result is actually not new, but it was already published in the literature. Moreover, a rotating solution that generalizes the Kerr metric for a nonminimally coupled scalar field does exist. We briefly discuss the nature of the singularities for the generalized metric, and point out that it can be used as a testing ground to differentiate black holes from naked singularities. Moreover, we mention the existence of some other typing or technical errors existing in the literature.

DOI: 10.1103/PhysRevD.98.088502

In the preceding Comment [1], the author did show that the metric introduced in Refs. [2,3] and used in Ref. [4] to perform a comparative study of the accretion disk properties of rotating naked singularities and Kerr-type black holes does not satisfy the Einstein field equations with a nonminimally coupled scalar field as a matter source. The findings of the Comment are *undoubtedly correct, and we fully agree with them*. However, we would like to first point out that this result is not new, and it has been already known for some time, being published first in Ref. [5]. When discussing the metric of Krori and Bhattacharjee [2] the authors of [5] explicitly mention that "However although this type of metric has been used in a number of later articles....one can check that the original metric derived by Krori and Bhattacharjee does not satisfy the field equations..." [5]. Unfortunately, when writing our paper [4] *we were not aware that the results by Krori and Bhattacharjee* [2] *and Nandi et al.* [3] *are erroneous*, and thus we have adopted their proposed rotating geometries as examples of metrics that could help in distinguishing observationally between black hole and naked singularity properties. Of course we also take full responsibility for not checking carefully these previously published results in the literature. We would also like to emphasize that *the use of a metric that is not an exact solution of the Einstein field equations could have serious implications on the validity of the results of* [4], from both theoretical and observational point of view.

On the other hand, a rotating solution of the gravitational field equations in the framework of the Brans-Dicke theory,

$$R_{\mu\nu} = \frac{\omega}{\phi^2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{\phi}\nabla_\mu\nabla_\nu\phi, \tag{1}$$

and

$$\Box\phi = 0, \tag{2}$$

respectively, with the action given by [6]

$$S = \int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi\right), \tag{3}$$

where ϕ is a scalar field that makes Newton's gravitational constant dynamical, was also presented in [5] (a similar solution was obtained earlier in [7]). The solution is of the form

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$$ds^2 = (\bar{\Delta}\sin^2\theta)^{-2/(2\omega+3)}\left[-f dt^2 - \frac{4mar}{\rho}\sin^2\theta dt d\phi + \left(r^2 + a^2 + \frac{2ma^2 r}{\rho}\sin^2\theta\right)\sin^2\theta d\phi^2\right] + (\bar{\Delta}\sin^2\theta)^{2/(2\omega+3)}\rho\left(\frac{dr^2}{\Delta} + d\theta^2\right), \tag{4}$$

where we have defined

$$f(r,\theta) = 1 - \frac{2mr}{\rho}, \qquad \rho(r,\theta) = r^2 + a^2\cos^2\theta,$$
$$\Delta(r) = r^2 + a^2 - 2mr, \quad \bar{\Delta} = \frac{\Delta}{m^2}. \tag{5}$$

Note that we should assume $\omega \neq -3/2$. In the above metric m and a are two arbitrary constants, related to the mass and the angular momentum of the black hole, respectively. The scalar field can be obtained as

$$\phi = (\bar{\Delta}^2\sin^4\theta)^{1/(2\omega+3)}, \tag{6}$$

and it satisfies Eq. (2). By using a conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega = 1/\sqrt{\phi}$ and a redefinition of the scalar field given by $\tilde{\phi} = \sqrt{2\omega+3}\ln\phi$, the metric (4) becomes [5]

$$ds^2 = -f dt^2 - \frac{4mar}{\rho}\sin^2\theta dt d\phi + \left(r^2 + a^2 + \frac{2ma^2 r}{\rho}\sin^2\theta\right)\sin^2\theta d\phi^2 + (\bar{\Delta}\sin^2\theta)^{4/(2\omega+3)}\rho\left(\frac{dr^2}{\Delta} + d\theta^2\right). \tag{7}$$

This metric satisfies the field equations [5]

$$R_{\mu\nu} = \frac{1}{2}\tilde{\phi}_\mu\tilde{\phi}_\nu, \qquad \Box\tilde{\phi} = 0, \tag{8}$$

with the scalar field given by

$$\tilde{\phi} = \frac{2}{\sqrt{2\omega+3}}\ln(\bar{\Delta}\sin^2\theta). \tag{9}$$

The singularities of the space-time described by the rotating metric (4) occur at $\Delta = 0$, and $f = 0$ and $\rho = 0$, respectively, which give

$$r_\pm = m\left(1 \pm \sqrt{1 - a_\star^2\cos^2\theta}\right),$$
$$r_{s,n} = m\left(1 \pm \sqrt{1 - a_\star^2}\right), \tag{10}$$

where $a_\star = a/m$. Note that the $r_\pm$ are the surfaces of infinite redshift and $r_{s,n}$ are the null surfaces, and we always have $r_+ \geq r_s$.

The Kretchmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be computed as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{512}{\rho^6(2\omega+3)^4}(\Delta^2\sin^4\theta)^{-\frac{2\omega+5}{2\omega+3}}g(r,\theta), \tag{11}$$

where $g(r,\theta)$ is a polynomial in r and $\cos\theta$. One can see that $\rho = 0$ is a curvature singularity, which corresponds to $r = 0$ and $\theta = \pi/2$, and it resembles a ringlike singularity. In the range $\omega < -3/2$ or $\omega > -1/2$, the $r_{s,n}$ are the Killing horizons [5].

In the range $-5/2 < \omega < -3/2$, $R(r_{s,n}) = 0$ and we have no curvature singularity in this case. In the opposite case where $\omega > -3/2$ or $\omega < -5/2$, we have $R(r_{s,n}) \to \infty$ and we have two curvature singularities.

From the above relations, we deduce that the curvature singularities $r_{s,n}$ are covered by the horizon for $\omega < -3/2$ and $\omega > -1/2$. In the case $-\frac{3}{2} < \omega \leq -\frac{1}{2}$ there is no horizon and we have three naked singularities $r = 0$, $r = r_{s,n}$.

The field equations (1) admit a static black hole solution of the form

$$ds^2 = -F^{2/\lambda}dt^2 + \left(1 + \frac{B}{r}\right)^4 F^{2(\lambda - C - 1)/\lambda}[dr^2 + r^2 d\Omega^2], \tag{12}$$

where $F = (1 - B/r)/(1 + B/r)$, $\phi = \phi_0 F^{C/\lambda}$, with C, b and λ constants related to each other as

$$\lambda = \sqrt{(C+1)^2 - C\left(1 - \frac{\omega C}{2}\right)}. \tag{13}$$

It should be noted that the original paper [6] had a sign typo on the scalar field (the scalar field was written in the form $\phi = \phi_0 F^{-C/\lambda}$), which was corrected by Brans himself in [8]. It is interesting that [8] has also a typo in the (00) component of the metric tensor.

A metric similar to the Krori and Bhattacharjee metric [2] was considered in [9], but it does not satisfy the Einstein gravitational field equations in the presence of a massless scalar field.

To conclude, rotating Kerr-like solutions of the gravitational field equations for a minimally coupled scalar field do exist. These solutions reduce to the standard Kerr metric of general relativity in the limit $\omega \to \infty$, and they can describe both black hole and naked singularity geometries. Therefore, as suggested in [4], these metrics are the ideal candidates for the investigation of the Penrose conjecture, according to which a cosmic censor who forbids the occurrence of naked singularities does exist in nature. They can also offer a possibility of theoretically and observationally differentiating rotating naked singularities from Kerr-type black holes through the comparative study of their accretion thin disk electromagnetic emission properties.

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