Comment on "Can accretion disk properties observationally distinguish black holes from naked singularities?"

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One points out that a metric used as a General Relativity filled by a massless scalar φ solution in several studies, and that would have been a generalization of the Janis-Newman-Winicour spherical solution to the rotating case, does not solve the corresponding Einstein equation $R_{ab} = \partial_a\varphi\partial_b\varphi$. As a consequence, the (conformally related) Krori-Bhattacharjee spacetime is not a vacuum Brans-Dicke solution.

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Let us consider a spacetime filled by a massless scalar φ. The general relativity (GR) equation reads

$$R_{ab} = \partial_a\varphi\partial_b\varphi. \tag{1}$$

Regardless of (1), or of any other field equation, the following metric

$$ds^2 = -V^\eta(dt - wd\phi)^2 - 2w(dt - wd\phi)d\phi + V^{1-\eta}\Sigma\left(\frac{dr^2}{\Delta} + d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2}$$

i.e., writing in extenso the metric tensor components, with the usual index notation $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$,

$$\begin{aligned} g_{00} &= -V^\eta, \\ g_{03} &= -w(1 - V^\eta), \\ g_{33} &= 2w^2 - w^2V^\eta + V^{1-\eta}\Sigma\sin^2\theta, \\ g_{11} &= V^{1-\eta}\frac{\Sigma}{\Delta}, \\ g_{22} &= V^{1-\eta}\Sigma, \end{aligned} \tag{3}$$

where $\eta \in [0, 1]$ and

$$\begin{aligned} w &= a\sin^2\theta, \\ \Sigma &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2 - 2\mu r + a^2, \\ V(r, \theta) &= 1 - \frac{2\mu r}{\Sigma}, \end{aligned} \tag{4}$$

describes a naked singularity spacetime for $\eta < 1$ (μ being an integration constant, related to the field's Arnowitt-Deser-Misner mass m by $m = \eta\mu$). The $\eta = 1$ case corresponds to the Kerr metric, with angular momentum a, in Boyer-Lindquist coordinates. The $a = 0$ case corresponds to the Janis-Newman-Winicour (JNW) metric [1] but written using Campanelli-Lousto (CL) coordinates [2].[1] The metric (2) can then be interpreted as a rotating generalization of the JNW metric.

For the purpose of studying astrophysical issues related to the (hypothetical) existence of naked singularities, the metric (2), considered as a solution of (1) with the scalar

$$\varphi(r, \theta) = k\sqrt{1 - \eta^2}\ln V(r, \theta) \tag{5}$$

in previous papers (see later), where k is a numerical factor, was used in Ref. [3] (dealing with accretion disk dynamics), in Ref. [4] (dealing with lensing effects), and also recently in Ref. [5] (dealing with frame dragging and tidal effects). However, any scalar metric that solves (1) should also satisfy the Klein-Gordon equation

$$\Box\varphi = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\varphi) = 0 \tag{6}$$

as a direct consequence (stress tensor conservation) of (1). The point is that a lengthy, but straightforward, calculation yields

$$\begin{aligned} \Box(\ln V) &= \frac{1}{\sqrt{-g}}\left[\partial_1\left(\frac{\sqrt{-g}}{Vg_{11}}\partial_1 V\right) + \partial_2\left(\frac{\sqrt{-g}}{Vg_{22}}\partial_2 V\right)\right] \\ &= -4\mu^2a^2V^\eta\frac{P}{Q}, \end{aligned} \tag{7}$$

where

[1]The link between the isotropic form and the CL form of the JNW metric can be obtained from $4r\bar{r} = (2\bar{r} + \mu)^2$, where $\bar{r}$ is the isotropic radial coordinate and r the (CL like) radial coordinate used in this Comment [2].

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$$P = (1 + 3\cos^2\theta)r^4 - 8\mu r^3\cos^2\theta + 2a^2(1 + \cos^2\theta)r^2\cos^2\theta + a^4\sin^2\theta\cos^4\theta,$$
$$Q = (r^2 - 2\mu r + a^2\cos^2\theta)^3(r^2 + a^2\cos^2\theta)^2. \quad (8)$$

Thence, Eq. (6) is not solved by (5) with the metric (2). Therefore, the metric (2) cannot solve (1) with the scalar (5).

Let us also remark that, even specifying to the $\theta = \pi/2$ symmetry plane, one has $\Box(\ln V) \neq 0$. Incidentally, the calculation to perform then is substantially shorter directly specifying to this plane, taking into account that θ enters the metric components (3) just through $\sin^2\theta$ terms. (See the main steps of the detailed calculation in the Appendix.)

To clarify where the scalar metric (2)–(5), used in Refs. [3–5], is coming from, it is worth reminding that massless scalar filled GR solutions are related to vacuum Brans-Dicke (BD) solutions. Indeed, considering a scalar Φ and a metric $\bar{g}_{ab}$, it is well known that, in a four-dimensional spacetime, the conformal transformation

$$g_{ab} = \Phi\bar{g}_{ab}$$

leads, for any constant ω, to the identity

$$\int \left[\Phi\bar{R} - \frac{\omega}{\Phi}(\bar{\partial}\Phi)^2\right]\sqrt{-\bar{g}}d^4x = \int \left(R - \frac{1}{2}(\partial\varphi)^2\right)\sqrt{-g}d^4x,$$

where

$$\varphi = \sqrt{2\omega + 3}\ln\Phi.$$

This means that the vacuum BD action of the BD gravitational field $(\Phi, \bar{g}_{ab})$ identifies with the GR action, with gravitational field g_{ab}, but filled by the (matter source) massless scalar φ. From this correspondence, it is shown in Ref. [6] that (2)–(5) are conformally related to a scalar metric, claimed in Ref. [7] to be a vacuum BD solution. However, it is worth pointing out that the scalar-metric "solution" proposed in Ref. [7] was obtained by the authors from the Brans class I spherical solution [8,9], by using a method that allowed Newman and Janis to recover Kerr's metric from Schwarzschild's (by performing some complex coordinate transformation) [10]. The point is that the authors of Ref. [7] never proved that the same method, but applied to a BD vacuum solution, should return another vacuum BD solution.[2] The fact that the metric (2) does not solve (1) with the scalar (5) shows that the scalar metric proposed by Ref. [7] is actually not a vacuum BD solution, despite the [7] authors' claim.[3]

While (7) proves that the metric (2) does not solve (1) with the scalar (5), one could ask if there is nevertheless another scalar function $\psi(r,\theta)$, such that $R_{ab} = \partial_a\psi\partial_b\psi$ for the metric components (3). If it were the case, one should have $R_{00} = R_{03} = R_{33} = 0$ everywhere in the spacetime. However, a direct calculation [the explicit expressions of the connection components of (3) are required here], specified for convenience to the $\theta = \pi/2$ plane, shows that

$$R_{00} = -2\frac{\eta(1-\eta)\mu^2a^2}{r^6}V^{2\eta-3}.$$

Hence, the metric (2) cannot solve (1), whatever the considered stationary axisymmetric scalar $\varphi(r,\theta)$.

All the expressions obtained here are consistent with the fact that one should have (i) $R_{00} = 0$ for $\eta = 1$ [Kerr solution, for which $R_{ab} = 0$, and in this case let us note that the r.h.s. of (5) vanishes] and (ii) $R_{00} = 0$ and $\Box\varphi = 0$ for $a = 0$ (JNW solution).

APPENDIX: EXPLICIT CALCULATION OF $\Box(\ln V)$ IN THE $\theta = \pi/2$ PLANE

We detail here the main steps of the calculation of $\Box(\ln V)$ (that does not require explicit connection component calculations) in the symmetry plane. Substantial simplifications occur then, thanks to the fact that θ enters the relevant functions through $\sin^2\theta$ terms. Indeed, any first derivative with respect to θ then yields $\sin\theta\cos\theta$ as a factor, that vanishes in the plane. This is not true of course for second derivatives with respect to θ, i.e., for $\partial_2\partial_2$ terms.

From the first line of (7), and using (3),

$$\Box(\ln V) = \frac{1}{\Delta g_{11}\sin\theta}\left[\sin\theta\partial_1\left(\frac{\Delta}{V}\partial_1 V\right) + \sin\theta\left(\frac{1}{V}\partial_2\partial_2 V + \partial_2\left(\frac{1}{V}\right)\partial_2 V\right) + \frac{\cos\theta}{V}\partial_2 V\right].$$

Hence, in the symmetry plane

$$\Box(\ln V) = \frac{1}{\Delta g_{11}}\left[\partial_1\left(\frac{\Delta}{V}\partial_1 V\right) + \frac{1}{V}\partial_2\partial_2 V\right].$$

For the $\partial_2\partial_2 V$ term, one obtains

[2] Let us emphasize that the authors of Ref. [10] explicitly wrote in their paper: "there is no simple, clear reason for the series of operations performed on the tetrad to yield a new (different from Schwarzschild) solution of the Einstein's equations." They just quoted that this happens to be the case in the very specific example they considered (GR and its Schwarzschild solution).

[3] Let us mention that the Brans class I solution reported in Refs. [8,9] differ by a sign in the exponent entering the scalar (the correct form being the Ref. [9] one). The form used in Ref. [7] to derive their "solution" is the Ref. [8] one, as it can be shown using the Campanelli-Lousto [2] radial coordinate. The sign is corrected in Ref. [6] for making the solution proposed by Ref. [7] coherent with the Brans class I solution, but as reported in Ref. [9], in the nonrotating case.

$$\partial_2\partial_2 V = 2\mu r \partial_2\left(\frac{1}{\Sigma^2}\partial_2\Sigma\right)$$
$$= \frac{2\mu}{r^3}\partial_2\partial_2\Sigma$$
$$= -\frac{4\mu a^2}{r^3}\partial_2(\sin\theta\cos\theta)$$
$$= \frac{4\mu a^2}{r^3}.$$

For the $\partial_1(\frac{\Delta}{V}\partial_1 V)$ term, one can directly replace θ by $\pi/2$ in the involved quantities, since no derivative with respect to θ is involved. A direct calculation yields, since $V = 1 - \frac{2\mu}{r}$, then

$$\partial_1\left(\frac{\Delta}{V}\partial_1 V\right) = 4\mu a^2\frac{\mu - r}{r^4 V^2}.$$

Hence, reinserting, one obtains

$$\Box(\ln V) = -4\frac{\mu^2 a^2}{r^6}V^{\eta-3},$$

which is the expression obtained from (7) and (8) in the $\theta = \pi/2$ case.

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