Universe as an oscillator

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We apply the idea of using a matter-time gauge in quantum gravity to quantum cosmology. For a Friedmann-Lemaître-Robertson-Walker (FLRW) universe with dust and a cosmological constant Λ, we show that the dynamics maps exactly to the simple harmonic oscillator in the dust-time gauge. For $\Lambda > 0$ the oscillator frequency is imaginary, for $\Lambda < 0$ it is real, and for $\Lambda = 0$ the universe is a free particle. This result provides (i) a simple and general demonstration of nonperturbative singularity avoidance in FLRW quantum cosmology for all Λ, (ii) an exact Lorentzian Hartle-Hawking wave function, and (iii) the present age of the universe as the characteristic decay time of the propagator.

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I. INTRODUCTION

The application of quantum theory to gravity is pursued using a number of different approaches (see, e.g., Ref. [1] for a recent survey). These can be broadly divided into two categories: those that are "background dependent" and those that are not [2]. The term refers to what structures in the classical theory are to be held fixed in the passage to quantum theory. The canonical quantization approach formulated by DeWitt [3] is considered to be the defining case of a background-independent approach to quantum gravity; this was also the paper where the very first quantization of the Friedmann-Lemaître-Robertson-Walker (FLRW) model was described.

The canonical quantization program is naturally divided into two distinct approaches. These are referred to as (i) Dirac quantization, where the Hamiltonian constraint is imposed as an operator condition on wave function(al)s, and (ii) reduced phase space quantization, where time and spatial coordinate gauges are fixed in the classical theory before proceeding to quantization. It is the former that leads to the Wheeler-DeWitt equation. Solutions in either case are referred to as "wave functions of the Universe."

In its more recent incarnations, the Dirac quantization condition is approached via a path integral as in the Hartle-Hawking method [4], or by imposing the condition directly as in loop quantum gravity [5,6]. In reduced phase space quantization, a phase space variable is first selected as a clock. Its conjugate variable provides the physical nonvanishing Hamiltonian. Quantization then proceeds as in conventional quantum theory with a time-dependent Schrodinger equation (or path integral). This division has led to much debate about the role of time in quantum gravity at both the philosophical and physical levels, and questions about the equivalence of the two methods [7–9].

Because of the difficulty in solving the Wheeler-DeWitt equation in the former case and the time-dependent Schrodinger equation in the latter case, nearly all concrete calculations are restricted to either homogeneous cosmological models or inhomogeneous perturbations of these models. Examples of early work on such models include Refs. [10,11]. The more recent works are in the framework of loop quantum cosmology (LQC) [12] and on Lorentzian versions of the Hartle-Hawking prescription [13,14].

In this paper we revisit the flat, homogeneous, and isotropic cosmology with dust and a cosmological constant Λ. This remains the typical model to consider since current observations suggest that our Universe is modeled well by a FLRW cosmology with zero spatial curvature and a very small positive cosmological constant, $\Lambda \sim 3 \times 10^{-122} l_P^{-2}$ [15]. We study the model using the reduced phase method in the dust-time gauge [16–20]; a recent study via Dirac-Wheeler-DeWitt quantization appeared in Ref. [21]. In the context of matter-time gauges, there have also been several studies that used scalar field time in quantum gravity and cosmology; a representative selection is Refs. [22–25].

In the Arnowitt-Deser-Misner (ADM) canonical formalism, we show that dust-time gauge leads to a surprising result: the corresponding physical Hamiltonian, after a canonical transformation, becomes exactly that of a simple harmonic oscillator; the oscillator's frequency is determined by $\sqrt{\Lambda}$. The corresponding quantum theory is therefore immediate.

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For $\Lambda < 0$ the potential is that of the usual oscillator, whereas for $\Lambda > 0$ it is the inverted oscillator. The former case describes universes either as stationary states, or as wave packets that expand and contract *ad infinitum*. The latter case has only scattering solutions that give universes with a single bounce. Depending on the choice of canonical parametrization, the oscillator is either on the half or the full line. All cases provide singularity avoidance, for all choices of self-adjoint extensions of the Hamiltonian. Our work also exhibits one of the situations where Dirac and reduced phase space quantizations give similar results for a particular choice of operator ordering in the Wheeler-DeWitt equation.

We begin by reviewing the general formalism for the dust-time gauge, followed by its application to cosmology in the following sections.

II. DUST-TIME GAUGE

The model we consider is general relativity coupled to a pressureless dust field T. The action

$$S=\int d^4x\sqrt{-g}R-\int d^4x\frac{1}{2}M\sqrt{-g}(g^{ab}\partial_a T\partial_b T+1) \quad (1)$$

where g_{ab} is the 4-metric, R is the 4-Ricci scalar, and M is the dust energy density, leads to the canonical ADM action

$$S=\int d^3xdt(\pi^{ab}\dot{q}_{ab}+p_T\dot{T}-N\mathcal{H}-N^a\mathcal{C}_a), \quad (2)$$

where

$$\begin{aligned}\mathcal{H}&\equiv\mathcal{H}_G+\mathcal{H}_D\\&=\frac{1}{\sqrt{q}}\left(\pi^{ab}\pi_{ab}-\frac{1}{2}\pi^2\right)+\sqrt{q}(\Lambda-{}^{(3)}R)\\&\quad+\mathrm{sgn}(M)p_T\sqrt{1+q^{ab}\partial_a T\partial_b T},\end{aligned} \quad (3a)$$

$$\mathcal{C}_a\equiv-D_b\pi^b{}_a+p_T\partial_a T, \quad (3b)$$

q_{ab} is the 3-metric, π^{ab} is its conjugate momentum, p_T is the dust conjugate momentum, N is the lapse, N^a is the shift, and the metric is of the ADM form

$$ds^2=-N^2dt^2+(dx^a+N^adt)(dx^b+N^bdt)q_{ab}. \quad (4)$$

We define the canonical dust-time gauge by

$$T=\epsilon t \quad (5)$$

with $\epsilon=\pm1$. The requirement that the gauge be preserved in time gives

$$\dot{T}=\epsilon=\left\{T,\int d^3xN\mathcal{H}\right\}\Big|_{T=t}=\mathrm{sgn}(M)N. \quad (6)$$

The physical Hamiltonian $\mathcal{H}_\mathrm{p}$ is obtained by substituting the gauge into the dust symplectic term in the canonical action, which identifies $\mathcal{H}_\mathrm{p}\equiv-\epsilon p_T$. Solving the Hamiltonian constraint

$$\mathcal{H}_G+\mathrm{sgn}(M)p_T=0 \quad (7)$$

then identifies the physical Hamiltonian

$$\mathcal{H}_\mathrm{p}=-\epsilon p_T=\epsilon\mathrm{sgn}(M)\mathcal{H}_G=N\mathcal{H}_G, \quad (8)$$

using Eq. (6) for the last equality. It is also useful to note, using $p_T=\sqrt{q}\dot{T}M/N$ and Eq. (6), the relation

$$p_T=\epsilon\sqrt{q}\frac{M}{N}=\epsilon\sqrt{q}\frac{\mathrm{sgn}(M)}{N}|M|=\sqrt{q}|M|, \quad (9)$$

which shows that $p_T>0$ for $M\neq0$, and

$$\mathcal{H}_\mathrm{p}=-\epsilon\sqrt{q}|M|=N\mathcal{H}_G. \quad (10)$$

Thus the requirement that the dust Hamiltonian satisfy $\mathcal{H}_D=\mathrm{sgn}(M)p_T\geq0$ implies $\mathrm{sgn}(M)=+1$, since $p_T=\sqrt{q}|M|\geq0$. This means that the dust field satisfies the weak energy condition. With this choice Eq. (6) gives $N=\epsilon$. In the following we make the choice $N=\epsilon=-1$ which gives the manifestly positive physical Hamiltonian density

$$\mathcal{H}_\mathrm{p}=\sqrt{q}|M|=-\mathcal{H}_G\geq0. \quad (11)$$

A. Application to cosmology

Let us now consider the reduction of the dust-time gauge theory to homogeneous and isotropic cosmology. This is obtained by setting

$$\begin{aligned}q_{ab}&=a^2(t)e_{ab},\\\pi^{ab}&=\frac{p_a(t)}{6a(t)}e^{ab},\end{aligned} \quad (12)$$

where $e_{ab}=\mathrm{diag}(1,1,1)$ is a fiducial flat metric. The reduced phase space coordinates are (a,p_a), and we take $a\in(0,\infty)$ and $p_a\in\mathbb{R}$ as the definition of this parametrization [since we must have $\det(q_{ab})=a^3>0$].

The physical Hamiltonian (11) for the flat case then becomes

$$\mathcal{H}_\mathrm{p}=\frac{p_a^2}{24a}-\Lambda a^3. \quad (13)$$

To briefly recap, this FLRW model started with a four-dimensional constrained phase space—that of the dust field and scale factor. After fixing the time gauge and solving the Hamiltonian constraint, the reduced phase space becomes two dimensional, with canonical coordinates (a, p_a). This is unlike the vacuum de Sitter model (see, e.g., Ref. [26]), which actually has no physical degrees of freedom (d.o.f.); the physical meaning of "wave functions of the Universe" without additional d.o.f. is therefore unclear.

Let us now note the canonical transformation

$$p = \frac{p_a}{\sqrt{12a}}, \qquad x = \frac{4}{\sqrt{3}} a^{3/2}, \tag{14}$$

and the rescaling $\Lambda \to 4\Lambda/\sqrt{3}$ transforms the Hamiltonian to

$$\mathcal{H}_\mathrm{p} = \frac{1}{2}(p^2 - \Lambda x^2). \tag{15}$$

Thus, there are three cases of interest: $\Lambda = 0$ is a free particle, $\Lambda < 0$ is the oscillator, and $\Lambda > 0$ is the inverted oscillator.

III. QUANTIZATION AND WAVE FUNCTIONS OF THE UNIVERSE

This section consists of two parts where we describe quantization in the dust-time gauge for two choices of the configuration space. These lead to quantum theories on either the half-line or the full line. In the former case there is a one-parameter family of self-adjoint extensions of the physical Hamiltonian.

A. Quantization on the half-line

The classical theory is on the half-line, $x \in (0, \infty)$, so the obvious choice for the Hilbert space is $L^2(\mathbb{R}^+, dx)$. In this space it is known that Hamiltonians of the form $p^2 + V(x)$ have self-adjoint extensions. Specifically, it is readily checked that the physical Hamiltonian (15) is symmetric in the usual representation $\hat{p} \to -i\partial_x$, i.e., that $(\psi, \widehat{\mathcal{H}_\mathrm{p}}\phi) = (\widehat{\mathcal{H}_\mathrm{p}}\psi, \phi)$, provided $\lim_{x\to\infty}\phi = 0$ and

$$\lim_{x\to 0}[\psi^*\phi' - \phi\psi^{*\prime}] = 0. \tag{16}$$

This gives the boundary condition $\phi'(0) = \alpha\phi(0)$, $\alpha \in \mathbb{R}$. Thus, there is a one-parameter (α) family of self-adjoint extensions of $\widehat{\mathcal{H}_\mathrm{p}}$ on the half-line, so the Hilbert space is the subspace specified by

$$\mathbb{H}_\alpha = \{\phi \in \mathcal{L}^2(\mathbb{R}^+, dx) \Big| \lim_{x\to 0}(\ln\phi)' = \alpha \in \mathbb{R}\}. \tag{17}$$

We are interested in solving the time-dependent Schrodinger equation,

$$i\frac{\partial}{\partial t}\phi(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x,t) - \frac{1}{2}\Lambda x^2\phi(x,t), \tag{18}$$

with the boundary condition mentioned above. (In this equation all variables are dimensionless or, equivalently, written in Planck units.)

$\Lambda = 0$: There are two types of elementary solutions. The first are the ingoing and outgoing waves of fixed energy (in the dust-time gauge), and satisfying the above boundary condition,

$$\phi_{\alpha k}(x,t) = e^{-ik^2t/2}\left[e^{ikx} - \left(\frac{\alpha - ik}{\alpha + ik}\right)e^{-ikx}\right]. \tag{19}$$

Normalizable wave functions are constructed in the usual manner as

$$\psi_\alpha(x,t) = \int_{-\infty}^{\infty} dk f(k)\phi_{\alpha k}(x,t). \tag{20}$$

All such solutions describe universes with singularity avoidance and a bounce at the origin with a phase shift given by α.

The second type of solution is a bound state,

$$\phi(x,t) = e^{i\kappa^2 t/2}e^{-\kappa x}, \qquad \kappa > 0. \tag{21}$$

This corresponds to $\alpha = -\kappa$, a choice permitted by the boundary conditions. The universe this describes is ruled out by experiment, since $\langle a^{3/2}\rangle \sim \langle x\rangle = (2\kappa)^{-1}$ which has the interpretation of an emergent flat spacetime from the expectation value of the metric.

$\Lambda < 0$: This is the oscillator on the half-line with the boundary condition $\psi'(0) - \alpha\psi(0) = 0$. With $\Lambda = -1/l^2$ and $\zeta = t/l$, the propagator on $\mathbb{R}$ is a basic result,

$$K(x,\zeta;x',0) = \sqrt{\frac{1}{2\pi i l\sin\zeta}}\exp\left\{\frac{i[(x^2 + x'^2)\cos\zeta - 2xx']}{2l\sin\zeta}\right\}. \tag{22}$$

For the half-line problem at hand, given initial data $\psi(x,0) = f(x)$ for $x > 0$, the solution with the required boundary condition at $x = 0$ may be obtained by extending the given initial data $f(x)$ on $\mathbb{R}^+$ to the region $x < 0$, such that

$$f'(x) - \alpha f(x) = -(f'(-x) - \alpha f(-x)), \qquad x < 0, \tag{23}$$

i.e., imposing antisymmetry on the boundary condition function. Solving this equation gives the required extension

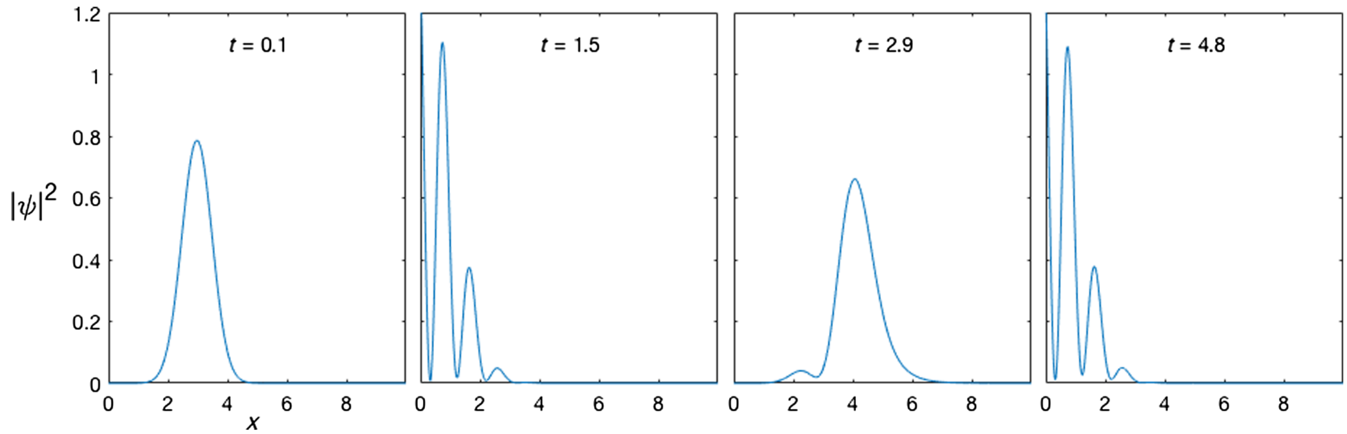

FIG. 1. Snapshots of $|\psi(x,t)|^2$ with the initial data $f(x) = \frac{e^{-(x-3)^2}}{\sqrt[4]{\pi/2}}$, and parameters $\Lambda = -1$ and $\alpha = 1.0$. The universe moves toward the origin ($t = 0.1$–1.5), expands asymmetrically ($t = 2.9$), and contracts again ($t = 4.8$). The profiles at $t = 1.5$ and $t = 4.8$ are nearly identical.

$$f_L(x) \equiv e^{\alpha x}\int_x^0 du e^{-\alpha u}[f'(-u) - \alpha f(-u)] + e^{\alpha x} f(0), \qquad x < 0, \tag{24}$$

where the integration constant is chosen such that $f_L(0) = f(0)$.

Convoluting this extended data with the full-line propagator (22) then gives the solution

$$\psi(x,\zeta) = \int_{-\infty}^0 dx' K(x,\zeta;x',0) f_L(x') + \int_0^\infty dx' K(x,\zeta;x',0) f(x'), \qquad x > 0. \tag{25}$$

It is straightforward to construct explicit examples of such solutions; all describe universes that expand out to a maximum size, recollapse, and bounce again. This is of course expected since wave packets are confined in the half-oscillator potential. Figure 1 shows the dynamics of a representative Gaussian wave function with $\Lambda = -1$ and $\alpha = 1$. The asymmetric bounce is evident, and the second and fourth frames demonstrate the multiple bounce feature.

$\Lambda > 0$: The Hamiltonian is not bounded from below. However, the unitary evolution operator is still well defined since the Hamiltonian has self-adjoint extensions. The propagator on $\mathbb{R}$ is obtained by the replacement $l \to il$ to give

$$\bar{K}(x,\zeta;x',0) = \sqrt{\frac{1}{2\pi i l \sinh\zeta}} \exp\left\{\frac{i[(x^2+x'^2)\cosh\zeta - 2xx']}{2l\sinh\zeta}\right\}. \tag{26}$$

Solutions of the time-dependent Schrodinger equation with the boundary condition $\phi'(0) - \alpha\phi(0) = 0$ are found in the same way as above by extending the initial data function to $x < 0$. It is evident that the propagator is damped for large times ζ due to the prefactor. However, for the very small Λ that is experimentally observed, the decay time would be very large. (It is useful to note that the issue of convergence of the Euclidean functional integral for the inverted oscillator was studied in Ref. [27], where it was shown that the integral for the propagator converges if the propagation time is bounded by a factor of the oscillator frequency.) Figure 2 shows the propagation of the same initial Gaussian wave packet as that in Fig. 1, but now for positive Λ. The wave packet moves outward and spreads rapidly.

B. Quantization on $\mathbb{R}$

In the above we started with the standard canonical parametrization for the FLRW cosmology which led to the oscillator on the half-line. There is an alternative parametrization that directly gives the oscillator on the real line after a rescaling of variables. This is

$$q_{ab} = A^{4/3}(t) e_{ab}, \qquad \pi^{ab} = \frac{1}{4A^{1/3}(t)} P_A(t) e^{ab}, \tag{27}$$

where the phase space (A, P_A) is now $\mathbb{R}^2$.

In this parametrization there is an exact Lorentzian "Hartle-Hawking" (HH) wave function, which is the "amplitude for a three-geometry given by a path integral over all compact positive-definite four-geometries which have the three-geometry as a boundary" [4]:

$$\psi[q] = \int D[g] D[\phi] \exp(-S[g,\phi]), \tag{28}$$

where S is the Euclidean action for matter and gravity, and the gravity measure is designed to reflect the definition above.

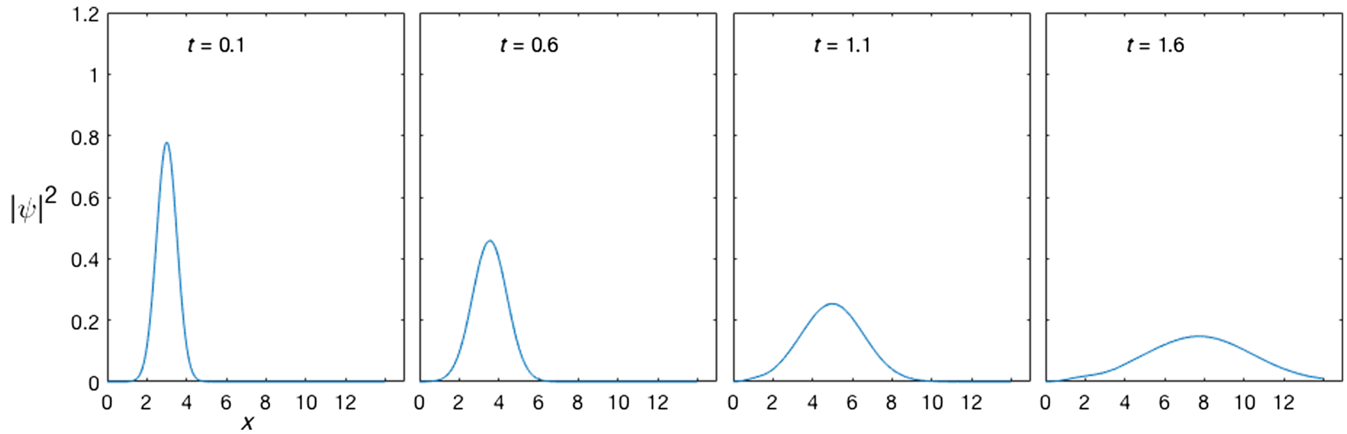

FIG. 2. Snapshots of $|\psi(x,t)|^2$ with the initial data $f(x) = \frac{e^{-(x-3)^2}}{\sqrt[4]{\pi/2}}$, and parameters $\Lambda = 1$ and $\alpha = 1.0$. The initial wave packet travels outwards and spreads.

In our case, we deploy the boundary condition obtained by setting $x' = 0$ in Eq. (26); this is the closest to the HH condition in Lorenzian theory:

$$\Psi_{\rm HH} \equiv \bar{K}(A,\zeta;0,0) = \sqrt{\frac{1}{2\pi i l \sinh\zeta}}\exp\left(-\frac{iA^2}{2l\tanh\zeta}\right), \tag{29}$$

where $A^4 = \det(q_{ab}) \equiv q$, and since we are now on the full line, $A \in \mathbb{R}$. This expression is just the oscillator propagator on the real line for $\Lambda = 1/l^2$ with $A_0 = \zeta_0 = 0$. For large times $\zeta = t/l$ this is

$$\bar{K}(q,\zeta;0,0) \to \frac{1}{\sqrt{\pi i l}}\exp\left(-\frac{i\sqrt{q}+t}{2l}\right). \tag{30}$$

This is oscillatory in 3-volume, and decays exponentially in time t.

IV. DISCUSSION

The basic result in this paper is that in general relativity coupled to pressureless dust in the dust-time gauge, the FLRW model with a cosmological constant has a physical Hamiltonian that is exactly that of a harmonic oscillator with frequency determined by $\sqrt{\Lambda}$. The Hamiltonian has a one-parameter (α) set of self-adjoint extensions, and explicit solutions of the time-dependent Schrodinger equation are readily constructed. All cases give singularity avoidance, which here means that wave functions describing the universe bounce at small spatial volume for any value of α, regardless of whether the configuration space is the half-line or the full line.

It is interesting to compare these results with those obtained in LQC [12] using the connection-triad variables. There the $\Lambda = 0$ case was studied with scalar field time, where the form of the Hamiltonian is such that wave function dynamics requires numerical study. It was subsequently studied in the dust time [18]. In both of these cases the Hamiltonian is essentially self-adjoint. In our case the bounce occurs for all self-adjoint extensions, and can be asymmetric in the sense that there is a phase shift at the bounce determined by α. Only the $\alpha = 0$ case gives a symmetric bounce.

For comparison with Dirac quantization, the corresponding quantum theory also resembles the oscillator, but only for the Laplace-Beltrami operator ordering in the kinetic term in the Wheeler-DeWitt operator [21]; that paper only considered $\Lambda = 1$, and did not address the most general self-adjoint extension with Robin boundary conditions. (This work was pointed out to us after the present work was posted to the arXiv.) Nevertheless, it is one of the few cases where it seems possible to rigorously establish equivalence between Dirac and reduced phase space quantizations. It would be interesting to study this issue for full quantum gravity with dust time [17].

Our consideration and results are entirely in the Lorentzian theory, and as such may be compared with similar models that invoke the Hartle-Hawking prescription in Lorentzian time, in particular the recent debate concerning integration contours for the propagator [13,14]. The latter work reported a suppression factor $\exp(-\Lambda l_p^2)$ in the propagator for the no-boundary wave function of the Universe in the semiclassical approximation. We found a similar result, but our state is *exact* (i.e., not just a semiclassical approximation), and also has explicit (dust) time dependence: Eq. (30) has the factor $\exp(-t/2l)$, which for fixed $t = t_0$ exhibits an exponential decay. Therefore, from the currently observed value of Λ, $l \sim 10^{60} l_p$, the characteristic decay time is $\sim 10^{60}$ Planck times, which is close to the age of the Universe.

The model with spatial curvature $k \neq 0$ and additional matter fields such as the minimally coupled scalar field is not exactly solvable. The physical Hamiltonian for this case in the dust-time gauge [after the canonical transformation (14)] is

$$\mathcal{H}_{\mathrm{p}}^{k} = \frac{1}{2}(p^2 - \Lambda x^2) + kx^{2/3} + \frac{p_\phi^2}{2x^2} + x^2 V(\phi). \quad (31)$$

Models such as this demonstrate that it is useful to consider matter-time gauges in the cosmological setting. Gravitational perturbations can be added to the physical Hamiltonian in a similar way, while retaining the oscillator form of the homogeneous part of the kinetic term. This may provide a useful starting point for studying singularity avoidance in dust-time gauge in the inhomogeneous setting.

We note that it is an important consideration to extend the model we have studied in two ways: to include anisotropy and, beyond that, inhomogeneity. The former is a larger minisuperspace model with a few more phase space d.o.f. A classical analysis in dust-time gauge appeared in Ref. [28]. The inclusion of general inhomogeneities is of course more difficult in that it involves studying field-theoretic models such as the Gowdy cosmologies [29]. The importance of such extensions is of current interest due to the issue of whether the no-boundary wave function is stable to perturbations: there are arguments for [30] and against such stability [31]. These works do not use the dust-time gauge and physical Hamiltonian that we study here, so an extension of our approach beyond FLRW to include a gravitational perturbation of fixed wave number along the lines studied in these papers would be potentially useful.

Last, the $\Lambda < 0$ case may be of interest in the context of the AdS/CFT conjecture and holography. Specifically, the idea of using matter (or other) time gauge in the bulk might provide a useful mechanism to probe bulk dynamics and the holographic signatures of resolved singularities in such settings [32], something which appears so far to be largely unexplored.

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[1] M. J. Duff and C. J. Isham, *Quantum Structure of Space and Time* (Cambridge University Press, Cambridge, England, 2012).
[2] C. J. Isham, Lect. Notes Phys. **434**, 1 (1994).
[3] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[4] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[5] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).
[6] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2007).
[7] J. B. Hartle and K. V. Kuchar, in *Quantum Theory of Gravity*, edited by S. M. Christensen (CRC Press, Boca Raton, FL, 1984), p. 315.
[8] K. Schleich, Classical Quantum Gravity **7**, 1529 (1990).
[9] K. Giesel and A. Oelmann, Acta Phys. Pol. B Proc. Suppl. **10**, 339 (2017).
[10] C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).
[11] W. F. Blyth and C. J. Isham, Phys. Rev. D **11**, 768 (1975).
[12] I. Agullo and P. Singh, in *Loop Quantum Gravity: The First 30 Years*, edited by A. Ashtekar and J. Pullin (World Scientific, Singapore, 2017), p. 183.
[13] J. Feldbrugge, J.-L. Lehners, and N. Turok, Phys. Rev. D **95**, 103508 (2017).
[14] J. D. Dorronsoro, J. J. Halliwell, J. B. Hartle, T. Hertog, and O. Janssen, Phys. Rev. D **96**, 043505 (2017).
[15] P. A. R. Ade *et al.* (Planck Collaboration), Astron. Astrophys. **594**, A13 (2016).
[16] J. D. Brown and K. V. Kuchar, Phys. Rev. D **51**, 5600 (1995).
[17] V. Husain and T. Pawlowski, Phys. Rev. Lett. **108**, 141301 (2012).
[18] V. Husain and T. Pawlowski, Classical Quantum Gravity **28**, 225014 (2011).
[19] K. Giesel and T. Thiemann, Classical Quantum Gravity **32**, 135015 (2015).
[20] M. Ali, V. Husain, S. Rahmati, and J. Ziprick, Classical Quantum Gravity **33**, 105012 (2016).
[21] H. Maeda, Classical Quantum Gravity **32**, 235023 (2015).
[22] C. Rovelli and L. Smolin, Phys. Rev. Lett. **72**, 446 (1994).
[23] J. Feinberg and Y. Peleg, Phys. Rev. D **52**, 1988 (1995).
[24] A. Nakonieczna and J. Lewandowski, Phys. Rev. D **92**, 064031 (2015).
[25] M. Assanioussi, J. Lewandowski, and I. Mäkinen, Phys. Rev. D **96**, 024043 (2017).
[26] J. J. Halliwell and J. Louko, Phys. Rev. D **39**, 2206 (1989).
[27] M. Carreau, E. Farhi, S. Gutmann, and P. F. Mende, Ann. Phys. (N.Y.) **204**, 186 (1990).
[28] M. Ali and V. Husain, Phys. Rev. D **96**, 044032 (2017).
[29] B. E. Navascués, M. Martín-Benito, and G. A. M. Marugán, Phys. Rev. D **92**, 024007 (2015).
[30] J. D. Dorronsoro, J. J. Halliwell, J. B. Hartle, T. Hertog, O. Janssen, and Y. Vreys, Phys. Rev. Lett. **121**, 081302 (2018).
[31] A. Di Tucci and J.-L. Lehners, arXiv:1806.07134.
[32] N. Bodendorfer, F. M. Mele, and J. Münch, arXiv:1804.01387.