Self-duality, helicity conservation, and normal ordering in nonlinear QED

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We give a proof of the equivalence of the electric-magnetic duality on one side and helicity conservation of the tree-level amplitudes on the other side within general models of nonlinear electrodynamics. Using modified Feynman rules derived from a generalized normal ordered Lagrangian, we discuss the interrelation of the above two properties of the theory also at higher loops. As an illustration we present two explicit examples; namely we find the generalized normal ordered Lagrangian for the Born-Infeld theory and derive a semiclosed expression for the Lagrangian of the Bossard-Nicolai model (in terms of the weak field expansion with explicitly known coefficients) from its normal ordered form.

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I. INTRODUCTION

The electric-magnetic duality is a remarkable on shell symmetry of the equation of motion of the Maxwell theory. It also holds for some of its nonlinear generalization the most famous of which is the one constructed by Born and Infeld in the 1930s [1]. The general aspects of this type of duality and of its extensions were studied in detail by Gaillard and Zumino in a seminal paper [2], where also the famous Noether-Gaillard-Zumino (NGZ) identity expressing the necessary and sufficient condition for the Lagrangian of the duality invariant theory was obtained. An iterative solution of the NGZ condition in terms of one arbitrary function was found in [3] proving at the same time that Maxwell and Born-Infeld (BI) theories are not only self-dual cases but also that there is an infinite class of such theories. Also various further generalizations were suggested. The supersymmetric extensions of self-dual theories were constructed by Kuzenko and Theisen in [4,5]. In [5] also self-dual higher derivative theories were discussed for the first time. The analysis of the case of noncommutative Maxwell type theories was performed by Banerjee in [6,7].

The general solution of the NGZ condition was found by Gaillard and Zumino in [8]. Their implicit construction of the solution is again parametrized by means of one arbitrary function of one external variable. This variable is implicitly determined as a solution of certain (in a general case transcendental) equation. Another alternative solution of the NGZ condition of this type was constructed by Hatsuda, Kamimura, and Sekiya in [9], where also nontrivial explicit examples of the self-dual Lagrangians were given in a closed form. A different approach to construction of the Lagrangian of self-dual theories, which is based on covariant perturbative deformations of $U(1)$ duality invariant theories, was considered by Kuzenko and Theisen in [5] and further elaborated by Bossard and Nicolai in [10] and by Carrasco, Kallosh, and Roiban in [11]. In the latter reference, the solutions of the corresponding nonlinear twisted self-duality constraints for the Maxwell case, Born-Infeld, and the Bossard-Nicolai (BN) model were discussed in detail. An interesting insight into the construction of self-dual theories was provided by Ivanov and Zupnik [12,13] using the bispinor auxiliary fields. The authors also proved that this approach appears to be fully equivalent to the one based on the nonlinear twisted self-duality constraint. Both the latter two approaches parametrize the general solution of the NGZ condition in terms of one arbitrary functional which has manifest $U(1)$ rotational symmetry. However, the physical meaning of this functional is not completely clear. Note also that, in spite of the progress in understanding the self-duality, only a few Lagrangians leading to self-dual theories beyond the BI one are known in a closed form. For instance, for the Lagrangian of the BN model only the first eight terms of the weak field expansion have been calculated explicitly [11] and its closed form is not known yet.

The BI theory is a prominent member of the class of self-dual theories and since its birth it has been subject of countless studies. The renewed interest in this theory was inspired by strings and D-branes: as was shown by Fradkin and Tseytlin in [14], the BI Lagrangian can be interpreted as a low energy effective theory describing the vector field

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coupled to the string ending on a D-brane (see also [15,16]). It also naturally appears as a bosonic sector of the effective theory which corresponds to the spontaneous breaking of the $\mathcal{N} = 2$ to $\mathcal{N} = 1$ SUSY (see [17,18,16]). Also the tree-level amplitudes in BI theory are special. First, they conserve helicity; i.e. the amplitudes with nonequal number of helicity plus and helicity minus photons (when all particles are assumed to be outgoing) vanish identically. This was proved in [19], where this property was interpreted as a consequence of the self-duality of the theory, and independently in [20] using Feynman diagrams. This results allow one to conclude, that the helicity violating one-loop amplitudes in the BI theory have vanishing imaginary parts and should be rational functions which hopefully vanish too. However, the general proof of the possible helicity conservation of the higher loop amplitudes in BI theory still seems to be an open problem. The second interesting property of tree-level amplitudes in BI theory has been established quite recently in [21]. Namely, the tree-level amplitudes have a unique soft behavior (they vanish) in the multichiral soft limit when all the particles with the same helicity[1] become simultaneously soft. This soft behavior constrains the amplitude strongly enough to fix uniquely the BI theory (up to the choice of the units[2]). Note also that the BI theory belongs to the class of theories for which the Cachazo–He–Yuan representation of the tree-level amplitudes exists [22].

It is a natural question which of the above properties of the BI theory are connected intimately with the self-duality only and can be thus proved for any self-dual theory. In this study we concentrate mainly on the connection of the helicity conservation and self-duality at the tree and loop level in the general nonlinear QED. As a byproduct we find a physical interpretation of the $U(1)$ rotational invariant generating functional which appears in the auxiliary field method of Ivanov and Zupnik (or equivalently in the method of the nonlinear twisted self-duality constraint of Carrasco, Kallosh, and Roiban) in terms of a certain generalization of normal ordering, which simplifies the Feynman rules for perturbative calculation of the S-matrix. As an explicit example we find a normal ordered version of the BI Lagrangian in a closed form (and rederive at the same time the hypergeometric form of the BI Lagrangian found originally in [23,24]) and calculate a semiclosed form of the BN Lagrangian (in terms of infinite series corresponding to the weak field expansion with explicit coefficients). We also briefly discuss a general form of the self-dual Lagrangian in terms of its normal ordered form.

The article is organized as follows. In Sec. II we shortly remind the reader of the basics of the nonlinear QED and duality transformation and also fix our notation. In Sec. III we briefly discuss various representations of the general solutions of the NGZ identity (including a new one) and give some examples of the self-dual Lagrangians beyond the Maxwell and BI case. In Sec. IV we discuss the quantization of nonlinear QED with stress on various versions of the Feynman rules within different representations of the Lagrangian. Section V is devoted to the proof of the tree-level helicity conservation in a general self-dual QED. In Sec. VI we introduce the normal ordering and modified Feynman rules and discuss the helicity conservation at the loop level. We also find the normal ordered form of the BI Lagrangian and the semiclosed form of the BN Lagrangian and give a general prescription for the transformation of the normal ordered Lagrangian into the usual form. In Sec. VII we summarize the results.

[1]Here we again assume all the particles to be outgoing.

[2]For example, up to one dimensionful parameter which corresponds at the classical level to the maximal intensity of the electric field.

II. NONLINEAR ELECTRODYNAMICS AND DUALITY

In what follows we will consider models of the nonlinear electrodynamic in four dimensions, the Lagrangian of which is a functions of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ only. The most general such Lagrangian can be written in the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\rm int}(F_{\mu\nu}), \tag{2.1}$$

where $\mathcal{L}_{\rm int}(F_{\mu\nu}) = O(F^4)$. From the phenomenological point of view, such models can appear as the leading order in the derivative expansion of the nonlocal effective action obtained by means of integrating out the massive charged degrees of freedom. Let us mention in this context the famous Euler-Heisenberg Lagrangian [25] (see also [26] for a comprehensive review) which describes effective interactions of the low-energy photons at energy scale $p \ll m_e$ where m_e is the electron mass,

$$\mathcal{L}^{\rm EH}_{\rm int}(F_{\mu\nu}) = \frac{\alpha^2}{90 m_e^4}\left[(F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(\tilde{F}^{\mu\nu}F_{\mu\nu})^2\right] + O\left(\left(\frac{\alpha}{m_e}\right)^3\right). \tag{2.2}$$

Here

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{2.3}$$

is the dual field strengths and we have written explicitly only the leading term in the fine structure constant α. Another example is the Born-Infeld modification of the Maxwell electrodynamics [1] designed originally in order to solve the problem of the infinite electromagnetic self-energy of the point charge

$$\mathcal{L}^{\mathrm{BI}} = -\Lambda^4\sqrt{-\det\left(\eta_{\mu\nu} + \frac{F_{\mu\nu}}{\Lambda^2}\right)} + \Lambda^4$$
$$= -\Lambda^4\sqrt{1 + \frac{2}{\Lambda^4}\mathcal{F} - \frac{1}{\Lambda^8}\mathcal{G}^2} + \Lambda^4. \qquad (2.4)$$

Here the two independent invariants $\mathcal{F}$ and $\mathcal{G}$ (in four dimensions any other invariant is a function of these two) are defined as

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}. \qquad (2.5)$$

The dimensionful scale Λ sets a limit on the maximal possible intensity of the electric field, $E_{\mathrm{max}} = \Lambda^2$. The Lagrangian $\mathcal{L}^{\mathrm{BI}}$ also appears as an effective action describing fluctuations of the massless degrees of freedom of an open string ending on a D-brane corresponding to the string excitations longitudinal to the brane. In this context $\Lambda^{-2} = 2\pi\alpha' = T^{-1}$ where T is the string tension and α' is the Regge slope.

For further consideration it will be useful to reformulate the Lagrangian in terms of the symmetric spinor fields ϕ_{AB} and $\bar{\phi}_{\dot{A}\dot{B}}$ defined as

$$F_{\mu\nu}\bar{\sigma}^{\mu}_{A\dot{A}}\bar{\sigma}^{\nu}_{B\dot{B}} = \phi_{AB}\varepsilon_{\dot{A}\dot{B}} + \bar{\phi}_{\dot{A}\dot{B}}\varepsilon_{AB} \qquad (2.6)$$

where as usual $\sigma^{\mu} = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^{\mu} = (1, -\boldsymbol{\sigma})$. Let us note that

$$\tilde{F}_{\mu\nu}\bar{\sigma}^{\mu}_{A\dot{A}}\bar{\sigma}^{\nu}_{B\dot{B}} = \mathrm{i}\phi_{AB}\varepsilon_{\dot{A}\dot{B}} - \mathrm{i}\bar{\phi}_{\dot{A}\dot{B}}\varepsilon_{AB}. \qquad (2.7)$$

As a consequence of the Cayley-Hamilton theorem for two-by-two traceless matrices $\boldsymbol{\Phi} \equiv \phi^{A}{}_{B} = \varepsilon^{AC}\phi_{CB}$ and $\bar{\boldsymbol{\Phi}} \equiv \bar{\phi}^{\dot{A}}{}_{\dot{B}} = \varepsilon^{\dot{A}\dot{C}}\bar{\phi}_{\dot{C}\dot{B}}$ we get

$$\boldsymbol{\Phi}^2 = -\det\boldsymbol{\Phi} = -\frac{1}{2}\phi^{AD}\phi_{AD} \equiv -\frac{1}{2}\phi^2 \qquad (2.8)$$

and thus

$$\mathrm{Tr}\boldsymbol{\Phi}^{2n} = \mathrm{Tr}\left(-\frac{1}{2}\phi^2\right)^n = \frac{(-1)^n}{2^{n-1}}(\phi^2)^n, \qquad \mathrm{Tr}\boldsymbol{\Phi}^{2n+1} = 0 \qquad (2.9)$$

and similarly for $\bar{\boldsymbol{\Phi}}$. Therefore the most general invariant built from $F_{\mu\nu}$ only can be expressed as a function of two independent invariants ϕ^2 and $\bar{\phi}^2$. For instance for the above two invariants $\mathcal{F}$ and $\mathcal{G}$ we get

$$\mathcal{F} = \frac{1}{8}(\phi^2 + \bar{\phi}^2), \qquad \mathcal{G} = \frac{\mathrm{i}}{8}(\phi^2 - \bar{\phi}^2), \qquad (2.10)$$

and the most general Lagrangian (2.1) can be expressed as a function of two variables $\mathcal{L}_{\mathrm{int}}(\phi^2, \bar{\phi}^2)$ in the form

$$\mathcal{L} = -\frac{1}{8}(\phi^2 + \bar{\phi}^2) + \mathcal{L}_{\mathrm{int}}(\phi^2, \bar{\phi}^2), \qquad (2.11)$$

where

$$\mathcal{L}_{\mathrm{int}} = \sum_{n+m>1} c_{nm}(\phi^2)^n(\bar{\phi}^2)^m \qquad (2.12)$$

and where Hermiticity requires $c^*_{nm} = c_{mn}$.

The classical equations of motion without sources expressed in terms of $F_{\mu\nu}$ consist of the Bianchi identity and the Euler-Lagrange equation

$$\partial_{\mu}\tilde{F}^{\mu\nu} = 0, \qquad \partial_{\mu}G^{\mu\nu} = 0 \qquad (2.13)$$

where

$$G_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}} \qquad (2.14)$$

is the Lorentz covariant constitutive equation. The above equations (2.13) are invariant with respect to the famous duality transformation written in the infinitesimal form as

$$\delta F_{\mu\nu} = \tilde{G}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}, \qquad (2.15)$$

$$\delta G_{\mu\nu} = \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \qquad (2.16)$$

provided the Lagrangian satisfies the Noether-Gaillard-Zumino relation [2]. The NGZ relation expresses consistency of the transformation (2.15) and (2.16) with definition of $G_{\mu\nu}$ (2.14). Here we write it in the form which appeared for the first time in [3] (see also [8,27]):

$$\varepsilon^{\mu\nu\alpha\beta}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\frac{\partial\mathcal{L}}{\partial F^{\alpha\beta}} = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + C, \qquad (2.17)$$

where C is an arbitrary constant. Provided we require the weak field expansion of the constitutive relation of the form

$$G_{\mu\nu} = F_{\mu\nu} + O(F^2),$$

i.e. the theory can be approximated in this limit by Maxwell electromagnetism, we get for the constant $C = 0$. In what follows we restrict ourselves to this case and refer to the theories satisfying (2.17) with $C = 0$ as self-dual theories.[3]

[3]Let us also note that, as proved in [4], the theories satisfying (2.17) for $C = 0$ have a unique $\mathcal{N} = 1$ supersymmetric extension.

Note, however, that the duality transformation (2.15), (2.16) is not an off-shell symmetry of the Lagrangian, but an on-shell symmetry of the equations of motion.

In terms of the spinors ϕ_{AB} and $\bar{\phi}_{\dot{A}\dot{B}}$ and analogous spinors Γ_{AB} and $\bar{\Gamma}_{\dot{A}\dot{B}}$ where

$$\bar{\sigma}^{\mu}_{A\dot{A}}\bar{\sigma}^{\nu}_{B\dot{B}}G_{\mu\nu} = \Gamma_{AB}\varepsilon_{\dot{A}\dot{B}} + \bar{\Gamma}_{\dot{A}\dot{B}}\varepsilon_{AB} \tag{2.18}$$

$$\Gamma_{AB} = -8\frac{\partial\mathcal{L}}{\partial\phi^2}\phi_{AB}, \tag{2.19}$$

$$\bar{\Gamma}_{\dot{A}\dot{B}} = -8\frac{\partial\mathcal{L}}{\partial\bar{\phi}^2}\bar{\phi}_{\dot{A}\dot{B}} \tag{2.20}$$

we can rewrite the duality transformation as

$$\delta\phi_{AB} = -\mathrm{i}\Gamma_{AB}, \qquad \delta\bar{\phi}_{\dot{A}\dot{B}} = \mathrm{i}\bar{\Gamma}_{\dot{A}\dot{B}}, \tag{2.21}$$

$$\delta\Gamma_{AB} = \mathrm{i}\phi_{AB}, \qquad \delta\bar{\Gamma}_{\dot{A}\dot{B}} = -\mathrm{i}\bar{\phi}_{\dot{A}\dot{B}}. \tag{2.22}$$

The NGZ relation (2.17) in these variables reads (note that the Lagrangian is a function of the invariants ϕ^2 and $\bar{\phi}^2$)

$$\phi^2\left(\frac{\partial\mathcal{L}}{\partial\phi^2}\right)^2 - \bar{\phi}^2\left(\frac{\partial\mathcal{L}}{\partial\bar{\phi}^2}\right)^2 - \frac{1}{64}(\phi^2-\bar{\phi}^2) = 0; \tag{2.23}$$

see also [5], where it first appeared in this form. It is straightforward to verify that the Born-Infeld Lagrangian, which in the variables ϕ_{AB} and $\bar{\phi}_{\dot{A}\dot{B}}$ reads

$$\mathcal{L}_{\mathrm{BI}} = -\Lambda^4\sqrt{1+\frac{1}{4\Lambda^4}(\phi^2+\bar{\phi}^2)+\frac{1}{64\Lambda^8}(\phi^2-\bar{\phi}^2)^2} + \Lambda^4, \tag{2.24}$$

satisfy the NGZ relation (2.23) and the theory is therefore self-dual.

The NGZ relation (2.23) can be further simplified using formally the variables

$$X_{\pm} = \frac{1}{2}\left(\sqrt{\phi^2} \pm \sqrt{\bar{\phi}^2}\right) \tag{2.25}$$

to the form

$$\frac{\partial\mathcal{L}}{\partial X_+}\frac{\partial\mathcal{L}}{\partial X_-} = \frac{1}{4}X_+X_- \tag{2.26}$$

or

$$\frac{\partial\mathcal{L}}{\partial X_+^2}\frac{\partial\mathcal{L}}{\partial X_-^2} = \frac{1}{16}. \tag{2.27}$$

This is the most suitable form for further consideration. In the next section we will discuss the solution of this equation in more detail and give some explicit examples of self-dual Lagrangians beyond the BI theory.

III. GENERAL SOLUTIONS OF THE NGZ IDENTITY

The NGZ identity written in the form (2.27) is a partial differential equation of the first order and as such it can be solved using standard methods. Of course not all of its solutions are physically acceptable. We typically require analyticity of the resulting Lagrangian in the variables ϕ^2 and $\bar{\phi}^2$ at the origin and we also expect that the weak field limit should reproduce the Maxwell electrodynamics. In this section we give a general prescription and also formulate the necessary condition for the above analyticity requirement.

According to the general methods for a solution of the first order partial differential equations by means of characteristics, the general solution $\mathcal{L}(X_+, X_-)$ of the equation (2.27) can be expressed implicitly in terms of four functions $p_{\pm}(u)$ and $x_{\pm}(u)$ which play the role of the one parametric set of the initial values of the characteristics, namely

$$\pm 4\mathcal{L}(X_+,X_-) = p_-(u)[X_-^2 - x_-(u)] + p_+(u)[X_+^2 - x_+(u)] + \int \mathrm{d}u[p_+(u)x_+'(u) + p_-(u)x_-'(u)]. \tag{3.1}$$

Here the prime denotes a derivative with respect to the parameter u. These functions are subject of the constraints

$$p_+(u)p_-(u) = 1 \tag{3.2}$$

$$p_+(u)[X_+^2 - x_+(u)] = p_-(u)[X_-^2 - x_-(u)]. \tag{3.3}$$

The first constraint reduces the number of independent functions to three while the second one allows one to determine the parameter u in terms of the variables $X_{\pm}$. For instance, the Maxwell theory can be reproduced by the choice

$$x_{\pm}^M(u) = 0, \qquad p_{\pm}^M(u) = -1. \tag{3.4}$$

Because the functions $p_{\pm}(u)$ and $x_{\pm}(u)$ appear in the above expressions in very special combinations, the above formula can be further simplified in such a way that there is only one arbitrary function left. Using integration by parts we get

$$\pm 4\mathcal{L}(X_+,X_-) = p_-(u)X_-^2 + p_+(u)X_+^2 - \int \mathrm{d}u[p_+'(u)x_+(u) + p_-'(u)x_-(u)]. \tag{3.5}$$

Writing the explicit solution of the first constraint (3.2) in the form

$$p_-(u) = p(u), \qquad p_+(u) = \frac{1}{p(u)} \tag{3.6}$$

we have

$$\pm 4\mathcal{L}(X_+, X_-) = p(u)X_-^2 + \frac{1}{p(u)}X_+^2 + \int \mathrm{d}u \frac{p'(u)}{p(u)} F(u), \tag{3.7}$$

where u is the solution of the second constraint (3.3), which can be written in the form

$$\frac{1}{p(u)}X_+^2 - p(u)X_-^2 = F(u) \tag{3.8}$$

and the function $F(u)$ is defined as

$$F(u) \equiv \frac{1}{p(u)}x_+(u) - p(u)x_-(u). \tag{3.9}$$

Introducing a new variable $z = p(u)$ and denoting $f(z) = F(u(z))$ we can rewrite the second constraint (3.3) as

$$\frac{1}{z}X_+^2 - zX_-^2 = f(z) \tag{3.10}$$

and finally we get the Lagrangian represented implicitly in terms of one arbitrary function $f(z)$

$$\pm 4\mathcal{L}(X_+, X_-) = zX_-^2 + \frac{1}{z}X_+^2 + \int \frac{\mathrm{d}z}{z} f(z), \tag{3.11}$$

where z is the solution of (3.10).

In what follows we will mainly restrict ourselves to the case when $\mathcal{L}(X_+, X_-)$ is analytic in ϕ^2 and $\bar{\phi}^2$. Because the resulting Lagrangian (3.11) depends on $X_\pm$ only through[4] $X_\pm^2$, assuming analyticity in $X_\pm^2$ the necessary condition for such an analyticity can be expressed as a symmetry condition

$$\mathcal{L}(X_+, X_-) = \mathcal{L}(X_-, X_+). \tag{3.12}$$

This can be achieved by the choice of function $f(z)$ which satisfies

$$f(z) = -f\left(\frac{1}{z}\right). \tag{3.13}$$

Indeed, in such a case the solution of (3.10) with X_+ and X_- interchanged is just $1/z$ where z is the original solution of (3.10) and the sum of the first two terms on the right-hand side of (3.11) is therefore invariant. For the second term we get immediately, provided we fix the lower limit of the integration appropriately,[5]

$$\int_1^{1/z} \frac{\mathrm{d}u}{u} f(u) = -\int_1^z \frac{\mathrm{d}w}{w^2} w f\left(\frac{1}{w}\right) = \int_1^z \frac{\mathrm{d}w}{w} f(w). \tag{3.14}$$

Let us now give some simple examples. Taking

$$f^{\mathrm{BI}}(z) = 2\Lambda^4\left(z - \frac{1}{z}\right) \tag{3.15}$$

and arranging the integration constant we reproduce the BI Lagrangian

$$\mathcal{L}^{\mathrm{BI}} = -\Lambda^4\sqrt{1 + \frac{1}{2\Lambda^4}(X_+^2 + X_-^2) + \frac{1}{4\Lambda^8}X_+^2X_-^2} + \Lambda^4. \tag{3.16}$$

The apparently simplest one parametric deformation of the BI Lagrangian can be obtained in this representation using

$$f^{\mathrm{MBI}}(z, a) = f^{\mathrm{BI}}(z) - 4a\Lambda^4.$$

The resulting Lagrangian reads

$$\mathcal{L}^{\mathrm{MBI}} = -\Lambda^4\left[r(X_+^2, X_-^2) - a\ln\left(\frac{a + r(X_+^2, X_-^2)}{1 + \frac{1}{2\Lambda^4}X_-^2}\right) - c\right] \tag{3.17}$$

where

$$r(X_+^2, X_-^2) = \sqrt{1 + a^2 + \frac{1}{2\Lambda^4}(X_+^2 + X_-^2) + \frac{1}{4\Lambda^8}X_+^2X_-^2}, \tag{3.18}$$

$$c = \sqrt{1 + a^2} - a\ln\left(a + \sqrt{1 + a^2}\right). \tag{3.19}$$

Note, however, that because $f^{\mathrm{MBI}}(z, a)$ does not satisfy (3.13) the Lagrangian $\mathcal{L}^{\mathrm{MBI}}$ is not analytic for $\phi^2 = \bar{\phi}^2 = 0$. Indeed, the weak field expansion reads now

$$\begin{aligned}\mathcal{L}^{\mathrm{MBI}} = &-\frac{\sqrt{1 + a^2}}{8}(\phi^2 + \bar{\phi}^2) + \frac{a}{4}\sqrt{\phi^2\bar{\phi}^2} \\ &- \frac{a}{32\Lambda^4}\sqrt{\phi^2\bar{\phi}^2}(\phi^2 + \bar{\phi}^2) + \frac{a^2}{128\Lambda^4\sqrt{1 + a^2}} \\ &\times [\phi^4 + \bar{\phi}^4 + 2(3 - 2a^{-2})\phi^2\bar{\phi}^2] + \ldots.\end{aligned} \tag{3.20}$$

[4]Note also that $X_\pm^2 = 2(\mathcal{F} \pm \sqrt{\mathcal{F}^2 + \mathcal{G}^2})$.

[5]Here we tacitly assume that either $f(1) = 0$ or $f(z)$ has at most integrable singularity for $z = 1$.

Note also the noncanonical normalization of the kinetic term.

Let us now relate the above construction of the self-dual Lagrangian to those known from the literature. The representation (3.11), (3.10) can be compared with the general solution found in [8] defining a new variable $w = -f(z)/z$. Expressing z in terms of w we get

$$\pm 4\mathcal{L}(X_+, X_-) = z(w)X_-^2 + \frac{1}{z(w)}X_+^2 - wz(w) + \int \mathrm{d}w z(w) \tag{3.21}$$

and w is a solution of

$$X_-^2 = \frac{1}{z(w)^2}X_+^2 + w. \tag{3.22}$$

Finally, using this equation in (3.21) we get

$$\pm 4\mathcal{L}(X_+, X_-) = \frac{2}{z(w)}X_+^2 + \int \mathrm{d}w z(w) \tag{3.23}$$

which is nothing else but the Gaillard-Zumino representation which expresses the solution in terms of arbitrary function $z(w)$. The latter representation has the advantage that it allows one to find the function $z(w)$ once the Lagrangian $\mathcal{L}(X_+, X_-)$ is known: for $X_+ = 0$ we get $w = X_-^2$ and thus

$$z(w) = \pm \frac{\mathrm{d}}{\mathrm{d}w} 4\mathcal{L}(0, \sqrt{w}). \tag{3.24}$$

For instance, for the BI theory we get immediately

$$z^{\mathrm{BI}}(w) = \frac{1}{\sqrt{1 + \frac{1}{2\Lambda^4}w}}. \tag{3.25}$$

Let us now define in (3.10) and (3.11) the following variable

$$u = \frac{z^2 - 1}{z^2 + 1} \tag{3.26}$$

and define in terms of this variable the following new function

$$G(u) = \frac{1}{2}\frac{zf(z)}{z^2 + 1}. \tag{3.27}$$

It is then an easy exercise to rewrite (3.10) into the form

$$\frac{1}{4}(X_+^2 - X_-^2) - \frac{1}{4}(X_+^2 + X_-^2)u = G(u) \tag{3.28}$$

and (3.11)

$$\pm\mathcal{L}(X_+, X_-) = \frac{1}{4}X_+^2\left(\sqrt{1-u^2} - \frac{u(1-u)}{\sqrt{1-u^2}}\right) + \frac{1}{4}X_-^2\left(\sqrt{1-u^2} + \frac{u(1+u)}{\sqrt{1-u^2}}\right) + \int \mathrm{d}u \frac{G(u)}{(1-u^2)^{3/2}}. \tag{3.29}$$

Using now (3.28) we get

$$\pm\mathcal{L}(X_+, X_-) = \frac{1}{4}(X_+^2 + X_-^2)\sqrt{1-u^2} - \frac{uG(u)}{\sqrt{1-u^2}} + \int \mathrm{d}u \frac{G(u)}{(1-u^2)^{3/2}} \tag{3.30}$$

and finally

$$\pm\mathcal{L}(X_+, X_-) = \frac{1}{4}(X_+^2 + X_-^2)\sqrt{1-u^2} - \int \mathrm{d}u \frac{uG'(u)}{\sqrt{1-u^2}}. \tag{3.31}$$

The latter formula together with the algebraic equation (3.28) corresponds to Hatsuda-Kamimura-Sekiya representation of the self-dual Lagrangian developed in [9] in terms of arbitrary function $G(u)$. Note that under the transformation $z \to 1/z$ the variable u transforms as $u \to -u$. Thus we get for the function $G(u)$

$$G(-u) = \frac{1}{2}\frac{zf(\frac{1}{z})}{z^2 + 1} \tag{3.32}$$

and the necessary condition for analyticity of the Lagrangian reads now

$$G(-u) = -G(u). \tag{3.33}$$

The BI Lagrangian is then reconstructed using

$$G^{\mathrm{BI}}(u) = \Lambda^4 u$$

and in [9] four more explicit examples were given, Note however, that only one of them [namely the example 4 with $G(u) = u(1 + au^2/3)/b$] satisfied the condition (3.33) and lead to the analytic Lagrangian.

IV. QUANTIZATION OF THE NONLINEAR QED

The usual formulation of the perturbation theory for the nonlinear electrodynamics at the quantum level requires a gauge fixing. This procedure sets the form of the propagator which then corresponds to the internal lines of the

Feynman graphs. In the Feynman gauge, which is manifestly Lorentz covariant, we get a simple propagator of the form

$$\langle 0|TA^{\mu}(x)A^{\nu}(y)|0\rangle = -\mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\frac{\eta^{\mu\nu}}{p^2+\mathrm{i}0} \tag{4.1}$$

The Feynman rules for the vertices are read off from the interaction Lagrangian $\mathcal{L}_{\mathrm{int}}(F_{\mu\nu})$ treated as a functional of the field $A_{\mu}(x)$ and the polarization vectors $\varepsilon_h^{\mu}(p)$ and their complex conjugates are attached to the incoming and outgoing external on-shell lines respectively. However, for practical purposes of amplitude calculation, the direct manipulation with the field $A^{\mu}(x)$ is rather clumsy because for the simple form of the propagator we have to pay with relatively complicated (usually infinitely many) interaction vertices which depend on the derivatives of $A^{\mu}(x)$. For the general interaction Lagrangian of the form $\mathcal{L}_{\mathrm{int}}(F_{\mu\nu})$ it is therefore much more convenient to work directly with the field $F^{\mu\nu}(x)$. The covariant propagator of the field $F^{\mu\nu}(x)$ can be derived from (4.1) by means of taking appropriate derivatives with the result

$$\langle 0|TF^{\mu\nu}(x)F^{\alpha\beta}(y)|0\rangle = \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\frac{P^{\mu\nu\alpha\beta}(p)}{p^2+\mathrm{i}0}, \tag{4.2}$$

where $P^{\mu\nu\alpha\beta}(p)$ is given by the expression

$$P^{\mu\nu\alpha\beta}(p) = -p^{\mu}p^{\alpha}\eta^{\nu\beta} + p^{\mu}p^{\beta}\eta^{\nu\alpha} + p^{\nu}p^{\alpha}\eta^{\mu\beta} - p^{\nu}p^{\beta}\eta^{\mu\alpha}. \tag{4.3}$$

Note, however, that the propagator of the field $F^{\mu\nu}(x)$ cannot be derived from any local kinetic term for the field $F_{\mu\nu}(x)$. Indeed, the tensor $P^{\mu\nu\alpha\beta}(p)$ can be rewritten as

$$P^{\mu\nu\alpha\beta}(p) = p^2\left[\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}) - \Pi^T_{\mu\nu\alpha\beta}\right],$$

where

$$\Pi^T_{\mu\nu\alpha\beta} = \frac{1}{2}(P^T_{\mu\alpha}P^T_{\nu\beta} - P^T_{\mu\beta}P^T_{\nu\alpha}), \qquad P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}. \tag{4.4}$$

Therefore $\Pi^T_{\mu\nu\alpha\beta}$ is the transverse projector in the space of the antisymmetric tensors and $P^{\mu\nu\alpha\beta}(p)$ is proportional to the longitudinal projector which has no inversion.

Working directly with the fields $F_{\mu\nu}(x)$ the interaction vertices are considerably simpler—they correspond to nonderivative couplings of the field $F^{\mu\nu}(x)$, now for the price of a slightly more complicated propagator. Also the external legs are now equipped with more complicated polarization tensors

$$\varepsilon_h^{\mu\nu}(p) = -\mathrm{i}p^{\mu}\varepsilon_h^{\nu}(p) + \mathrm{i}p^{\nu}\varepsilon_h^{\mu}(p). \tag{4.5}$$

Nevertheless the resulting Feynman rules for the S-matrix are completely equivalent to those based on the propagator (4.1) and we get manifest gauge invariance term by term for each Feynman diagram separately.

An even more efficient treatment, which shares the latter property, is to decompose the propagator of the field $F^{\mu\nu}(x)$ into the spinor basis $\phi_{AB}(x)$ and $\bar{\phi}_{\dot{C}\dot{D}}(x)$ [cf. (2.6)]. The free operators $\phi_{AB}(x)$ and $\bar{\phi}_{\dot{C}\dot{D}}(x)$ are directly connected with helicity: $\phi_{AB}(x)$ annihilates helicity plus and creates helicity minus states while $\bar{\phi}_{\dot{C}\dot{D}}(x)$ annihilates helicity minus and creates helicity plus states.[6] Another advantage of this decomposition is that the most general interaction Lagrangian $\mathcal{L}_{\mathrm{int}}(F_{\mu\nu})$ can be rewritten in the form (2.12) and therefore can be treated as a function of ϕ^2 and $\bar{\phi}^2$. The contraction of the spinor indices is thus considerably simpler than the contraction of the original Lorentz indices and the structure of the interaction vertices is then much more transparent within this formalism.

The decomposition of the propagator (4.2) in the spinor basis reads (see also [20])

$$\langle 0|T\phi_{AB}(x)\bar{\phi}_{\dot{C}\dot{D}}(y)|0\rangle = \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\frac{[p_{A\dot{C}}p_{B\dot{D}} + p_{A\dot{D}}p_{B\dot{C}}]}{p^2+\mathrm{i}0} \tag{4.6}$$

$$\langle 0|T\phi_{AB}(x)\phi_{CD}(y)|0\rangle = -\mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}[\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}] \tag{4.7}$$

$$\langle 0|T\phi_{\dot{A}\dot{B}}(x)\phi_{\dot{C}\dot{D}}(y)|0\rangle = -\mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}[\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}]. \tag{4.8}$$

Here as usual

$$p_{A\dot{B}} = \bar{\sigma}^{\mu}_{A\dot{B}}p_{\mu} \tag{4.9}$$

and $p_{A\dot{B}} = p_A p_{\dot{B}}$ on shell.

Note that only the "mixed" propagator $\langle T\phi\bar{\phi}\rangle$ possesses one particle pole with residue

$$[p_{A\dot{C}}p_{B\dot{D}} + p_{A\dot{D}}p_{B\dot{C}}]_{\text{on-shell}} = 2p_A p_B p_{\dot{A}} p_{\dot{B}}. \tag{4.10}$$

[6] In fact, the fields ϕ_{AB} and $\bar{\phi}_{\dot{C}\dot{D}}$ correspond to the self-dual combinations $F_{\pm\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm \mathrm{i}\tilde{F}_{\mu\nu})$. The relation of the latter fields to the helicity states and coherent states has been studied in detail in [28].

The corresponding Feynman rules associate therefore the combination $\sqrt{2}p_A p_B$ to each outgoing helicity minus and ingoing helicity plus external leg and $\sqrt{2}p_{\dot{A}} p_{\dot{B}}$ to each outgoing helicity plus and ingoing helicity minus external leg. The remaining two propagators (4.7) and (4.8) are pure contact terms proportional to the delta function of its space-time arguments.[7] Their presence is a direct consequence of the simple form of the original covariant propagators (4.1) and (4.2) we have started with.

V. HELICITY CONSERVATION IN NONLINEAR QED AT TREE LEVEL

Let us now assume the S-matrix of the general nonlinear QED. For our purposes it is convenient to treat it as a functional $\mathcal{S}[\phi, \bar{\phi}]$ of the classical off-shell fields $\phi_{AB}(x)$ and $\bar{\phi}_{\dot{C}\dot{D}}(x)$. The functional $\mathcal{S}[\phi, \bar{\phi}]$ is sometimes called the normal symbol of the S-matrix: once we know $\mathcal{S}[\phi, \bar{\phi}]$, the operator S-matrix $\hat{S}$ can be obtained by means of replacing the functional arguments with free fields operators $\phi_{AB}(x)_I$ and $\bar{\phi}_{\dot{C}\dot{D}}(x)_I$ in the interaction picture and then applying the usual normal ordering

$$\hat{S} = :\mathcal{S}[\phi_I, \bar{\phi}_I]:. \tag{5.1}$$

The analogous normal symbol $\mathcal{T}[\phi, \bar{\phi}]$ of the connected S-matrix is related to $\mathcal{S}[\phi, \bar{\phi}]$ via the relation

$$\mathcal{S}[\phi, \bar{\phi}] = \exp(\mathrm{i}\mathcal{T}[\phi, \bar{\phi}]). \tag{5.2}$$

The scattering amplitudes can be obtained directly form $\mathcal{T}[\phi, \bar{\phi}]$ applying appropriate differential operators (for pedagogical treatment of this formalism see e.g. [29]).

In what follows we will concentrate on theories the scattering amplitudes of which conserve helicity. This means that, when we treat all the external particles as outgoing, the amplitudes vanish provided the total number of helicity plus particles does not match the total number of helicity minus particles:

$$A(1^+, 2^+, \ldots, n^+, (n+1)^-, (n+2)^-, \ldots, (n+m)^-) = 0 \quad \text{for } n \neq m. \tag{5.3}$$

Because the fields $\phi_{AB}(x)$ and $\bar{\phi}_{\dot{C}\dot{D}}(x)$ are associated with helicity minus and helicity plus outgoing particles respectively, the requirement of helicity conservation necessitates the functional $\mathcal{T}[\phi, \bar{\phi}]$ to be invariant with respect to the global $U(1)$ transformation

$$\phi'_{AB} = \mathrm{e}^{\mathrm{i}\alpha}\phi_{AB}, \qquad \bar{\phi}'_{\dot{A}\dot{B}} = \mathrm{e}^{-\mathrm{i}\alpha}\bar{\phi}_{\dot{A}\dot{B}}. \tag{5.4}$$

Infinitesimally this means

$$\int \mathrm{d}^4x \left[\phi_{AB}(x)\frac{\delta\mathcal{T}[\phi,\bar{\phi}]}{\delta\phi_{AB}(x)} - \bar{\phi}_{\dot{A}\dot{B}}(x)\frac{\delta\mathcal{T}[\phi,\bar{\phi}]}{\delta\bar{\phi}_{\dot{A}\dot{B}}(x)}\right] = 0. \tag{5.5}$$

Let us now prove that duality invariance is a necessary and sufficient condition for helicity conservation of tree-level amplitudes, i.e. that the leading order term of the functional $\mathcal{T}[\phi, \bar{\phi}]$ in the quasiclassical expansion

$$\mathcal{T}[\phi, \bar{\phi}] = \mathcal{T}^{\mathrm{tree}}[\phi, \bar{\phi}] + O(\hbar) \tag{5.6}$$

satisfies (5.5) if and only if the theory is self-dual.

Let us first note that the perturbative construction of $\mathcal{S}[\phi, \bar{\phi}]$ is encoded in the representation

$$\mathcal{S}[\phi, \bar{\phi}] = \exp O\left[\frac{\delta}{\delta\phi}, \frac{\delta}{\delta\bar{\phi}}\right] \exp(\mathrm{i}S_{\mathrm{int}}[\phi, \bar{\phi}]) \tag{5.7}$$

where the differential operator in the functional derivatives

$$O\left[\frac{\delta}{\delta\phi}, \frac{\delta}{\delta\bar{\phi}}\right] = -\mathrm{i}\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\phi} - \mathrm{i}\frac{\delta}{\delta\bar{\phi}}\cdot\frac{\delta}{\delta\bar{\phi}} + \mathrm{i}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}}, \tag{5.8}$$

with

$$\Delta_{AB\dot{C}\dot{D}}(x, y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\frac{[p_{A\dot{C}}p_{B\dot{D}} + p_{A\dot{D}}p_{B\dot{C}}]}{p^2 + \mathrm{i}0} \tag{5.9}$$

implements the Wick theorem with propagators (4.6), (4.7) and (4.8). Here and in what follows we use condensed notation, e.g.

$$\Delta\cdot\frac{\delta}{\delta\bar{\phi}} \equiv \int \mathrm{d}^4y\Delta_{AB\dot{C}\dot{D}}(x, y)\frac{\delta}{\delta\bar{\phi}_{\dot{C}\dot{D}}(y)}, \tag{5.10}$$

$$\frac{\delta}{\delta\phi}\cdot\Delta = \int \mathrm{d}^4x\mathrm{d}^4y\frac{\delta}{\delta\phi_{AB}(y)}\Delta_{AB\dot{C}\dot{D}}(y, x), \tag{5.11}$$

$$\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}} \equiv \int \mathrm{d}^4x\mathrm{d}^4y\frac{\delta}{\delta\phi_{AB}(x)}\Delta_{AB\dot{C}\dot{D}}(x, y)\frac{\delta}{\delta\bar{\phi}_{\dot{C}\dot{D}}(y)}, \tag{5.12}$$

$$\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\phi} \equiv \int \mathrm{d}^4x\frac{\delta}{\delta\phi_{AB}(x)}\varepsilon_{AC}\varepsilon_{BD}\frac{\delta}{\delta\phi_{CD}(x)}, \tag{5.13}$$

[7] In fact these helicity violating propagators are unnecessary and can be discarded by means of the procedure of normal ordering which we describe in the next section.

etc.[8] We also tacitly assume some implicit UV regularization,[9] typically the dimensional regularization.[10] The details of this regularization are not essential when we restrict ourselves to the tree level. We get then using (5.2)

$$\frac{\delta\mathcal{T}[\phi,\bar{\phi}]}{\delta\phi_{AB}} = \mathrm{e}^{-\mathrm{i}\mathcal{T}[\phi,\bar{\phi}]}\mathrm{e}^{O[\frac{\delta}{\delta\phi},\frac{\delta}{\delta\bar{\phi}}]}\frac{\delta S_{\mathrm{int}}[\phi,\bar{\phi}]}{\delta\phi_{AB}}\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\phi,\bar{\phi}]}$$
$$= \mathrm{e}^{-\mathrm{i}\mathcal{T}[\phi,\bar{\phi}]}\mathrm{e}^{[O_\phi+O_\chi+O_{\chi\phi}]}\frac{\delta S_{\mathrm{int}}[\phi,\bar{\phi}]}{\delta\phi_{AB}}\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\chi,\bar{\chi}]}|_{\chi=\phi,\bar{\chi}=\bar{\phi}} \tag{5.14}$$

where we have doubled the number of fields in order to separate the action of the operator $\mathrm{e}^{O[\frac{\delta}{\delta\phi},\frac{\delta}{\delta\bar{\phi}}]}$ on both factors and introduced commuting operators

$$O_\phi = O\left[\frac{\delta}{\delta\phi},\frac{\delta}{\delta\bar{\phi}}\right],\qquad O_\chi = O\left[\frac{\delta}{\delta\chi},\frac{\delta}{\delta\bar{\chi}}\right] \tag{5.15}$$

$$O_{\chi\phi} = -2\mathrm{i}\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\chi} - 2\mathrm{i}\frac{\delta}{\delta\bar{\phi}}\cdot\frac{\delta}{\delta\bar{\chi}} + \mathrm{i}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\chi}} + \mathrm{i}\frac{\delta}{\delta\chi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}}. \tag{5.16}$$

Acting now with the diagonal operators e^{O_ϕ} and e^{O_χ} on $\delta S_{\mathrm{int}}[\phi,\bar{\phi}]/\delta\phi_{AB}$ and $\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\chi,\bar{\chi}]}$ respectively we get

$$\frac{\delta\mathcal{T}[\phi,\bar{\phi}]}{\delta\phi_{AB}} = \mathrm{e}^{-\mathrm{i}\mathcal{T}[\phi,\bar{\phi}]}\mathrm{e}^{O_{\chi\phi}}\frac{\delta S_{\mathrm{int}}[\phi,\bar{\phi}]}{\delta\phi_{AB}}\mathrm{e}^{\mathrm{i}\mathcal{T}[\chi,\bar{\chi}]}|_{\chi=\phi,\bar{\chi}=\bar{\phi}}. \tag{5.17}$$

Note that within dimensional regularization

$$O_\phi\frac{\delta S_{int}[\phi,\bar{\phi}]}{\delta\phi_{AB}} = 0 \tag{5.18}$$

because $\delta S_{\mathrm{int}}[\phi,\bar{\phi}]/\delta\phi_{AB}$ is local and therefore O_ϕ generates massless tadpoles. Equivalently we can set the left-hand side of (5.18) effectively to zero when we are interested in tree-level graphs only. The action of the operator $\mathrm{e}^{O_{\chi\phi}}$ on $\delta S_{\mathrm{int}}[\phi,\bar{\phi}]/\delta\phi_{AB}$ shifts its functional arguments according to

$$\phi\to\phi-2\mathrm{i}\frac{\delta}{\delta\chi}+\mathrm{i}\Delta\cdot\frac{\delta}{\delta\bar{\chi}},\qquad \bar{\phi}\to\bar{\phi}-2\mathrm{i}\frac{\delta}{\delta\bar{\chi}}+\mathrm{i}\frac{\delta}{\delta\chi}\cdot\Delta \tag{5.19}$$

and commuting the functional $\mathrm{e}^{\mathrm{i}\mathcal{T}[\chi,\bar{\chi}]}$ (treated as an operator) with functional derivatives $\delta/\delta\chi$ and $\delta/\delta\bar{\chi}$ shifts these derivatives

$$\frac{\delta}{\delta\chi}\mathrm{e}^{\mathrm{i}\mathcal{T}[\chi,\bar{\chi}]} = \mathrm{e}^{\mathrm{i}\mathcal{T}[\chi,\bar{\chi}]}\left(\frac{\delta}{\delta\chi}+\mathrm{i}\frac{\delta\mathcal{T}[\chi,\bar{\chi}]}{\delta\chi}\right). \tag{5.20}$$

As a result of this operation, we get $\delta\mathcal{T}/\delta\phi_{AB}$ in the form of an action of the differential operation on trivial functional $F[\chi,\bar{\chi}]=1$

$$\frac{\delta\mathcal{T}}{\delta\phi_{AB}} = \frac{\delta S_{\mathrm{int}}}{\delta\phi_{AB}}[\mathcal{J},\bar{\mathcal{J}}]1|_{\chi=\phi,\bar{\chi}=\bar{\phi}} \tag{5.21}$$

and similarly

$$\frac{\delta\mathcal{T}}{\delta\bar{\phi}_{\dot{A}\dot{B}}} = \frac{\delta S_{\mathrm{int}}}{\delta\bar{\phi}_{\dot{A}\dot{B}}}[\mathcal{J},\bar{\mathcal{J}}]1|_{\chi=\phi,\bar{\chi}=\bar{\phi}} \tag{5.22}$$

where $\mathcal{J}$ and $\bar{\mathcal{J}}$ are differential operators in functional derivatives given by

$$\mathcal{J} = \phi + 2\frac{\delta\mathcal{T}[\chi,\bar{\chi}]}{\delta\chi} - \Delta\cdot\frac{\delta\mathcal{T}[\chi,\bar{\chi}]}{\delta\bar{\chi}} + \hbar\frac{\delta}{\delta\chi} + \mathrm{i}\hbar\Delta\cdot\frac{\delta}{\delta\bar{\chi}} \tag{5.23}$$

$$\bar{\mathcal{J}} = \bar{\phi} + 2\frac{\delta\mathcal{T}[\chi,\bar{\chi}]}{\delta\bar{\chi}} - \frac{\delta\mathcal{T}[\chi,\bar{\chi}]}{\delta\chi}\cdot\Delta + \hbar\frac{\delta}{\delta\bar{\chi}} + \mathrm{i}\hbar\frac{\delta}{\delta\chi}\cdot\Delta. \tag{5.24}$$

In these formulas we restored the dependence on the Planck constant. At tree level therefore, writing

$$S_{\mathrm{int}}[\phi,\bar{\phi}] = \int \mathrm{d}^4x\mathcal{L}_{\mathrm{int}}(\phi(x)^2,\bar{\phi}(x)^2), \tag{5.25}$$

and taking the leading terms in the expansion in $\hbar$ on both sides of (5.21), (5.22) we get

$$\frac{\delta\mathcal{T}^{\mathrm{tree}}}{\delta\phi_{AB}} = \frac{\delta S_{\mathrm{int}}}{\delta J_{AB}}[J,\bar{J}] = 2J^{AB}\frac{\partial\mathcal{L}_{\mathrm{int}}(J^2,\bar{J}^2)}{\partial J^2}, \tag{5.26}$$

$$\frac{\delta\mathcal{T}^{\mathrm{tree}}}{\delta\bar{\phi}_{\dot{A}\dot{B}}} = \frac{\delta S_{\mathrm{int}}}{\delta\bar{J}_{\dot{A}\dot{B}}}[J,\bar{J}] = 2\bar{J}^{\dot{A}\dot{B}}\frac{\partial\mathcal{L}_{\mathrm{int}}(J^2,\bar{J}^2)}{\partial\bar{J}^2}. \tag{5.27}$$

Here we denoted (using our condensed notation)

$$J = \mathcal{J}|_{\hbar\to0,\chi=\phi,\bar{\chi}=\bar{\phi}} = \phi + 2\frac{\delta\mathcal{T}^{\mathrm{tree}}}{\delta\phi} - \Delta\cdot\frac{\delta\mathcal{T}^{\mathrm{tree}}}{\delta\bar{\phi}} \tag{5.28}$$

[8]The helicity consrevation condition (5.5) reads within this notation

$$\phi\cdot\frac{\delta\mathcal{T}^{\mathrm{tree}}}{\delta\phi} - \bar{\phi}\cdot\frac{\delta\mathcal{T}^{\mathrm{tree}}}{\delta\bar{\phi}} = 0.$$

[9]We also assume that S_{int} contains all the necessary counterterms. Note, however, that these are of the order $O(\hbar)$ and higher and can be effectively set to zero in what follows when we restrict ourselves to the tree level.

[10]In this case, in order to preserve the four-dimensional spinor algebra, we assume a dimensional reduction scheme.

$$\bar{J} = \bar{\mathcal{J}}|_{\hbar\to 0,\chi=\phi,\bar{\chi}=\bar{\phi}} = \bar{\phi} + 2\frac{\delta\mathcal{T}^{\rm tree}}{\delta\bar{\phi}} - \frac{\delta\mathcal{T}^{\rm tree}}{\delta\phi}\cdot\Delta. \quad (5.29)$$

Note that the right-hand sides of (5.26) and (5.27) are local when expressed in terms of the variables J and $\bar{J}$. From (5.26) and (5.27) it follows

$$\phi\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\phi} - \bar{\phi}\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\bar{\phi}} = \int {\rm d}^4x\left[2\phi_{AB}J^{AB}\frac{\partial\mathcal{L}_{\rm int}}{\partial J^2} - 2\bar{\phi}_{\dot{A}\dot{B}}\bar{J}^{\dot{A}\dot{B}}\frac{\partial\mathcal{L}_{\rm int}}{\partial\bar{J}^2}\right]. \quad (5.30)$$

Using (5.28) and (5.29) and with help of (5.26) and (5.27) we can express ϕ_{AB} as a functional of J_{AB} and $\bar{J}_{\dot{A}\dot{B}}$

$$\phi = J - 2\frac{\delta\mathcal{T}^{\rm tree}}{\delta\phi} + \Delta\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\bar{\phi}} = J - 4J\frac{\partial\mathcal{L}_{\rm int}}{\partial J^2} + 2\Delta\cdot\bar{J}\frac{\partial\mathcal{L}_{\rm int}}{\partial\bar{J}^2} \quad (5.31)$$

and similarly

$$\bar{\phi} = \bar{J} - 4J\frac{\partial\mathcal{L}_{\rm int}}{\partial\bar{J}^2} + 2\frac{\partial\mathcal{L}_{\rm int}}{\partial J^2}J\cdot\Delta \quad (5.32)$$

and with the help of (5.30)

$$\phi\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\phi} - \bar{\phi}\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\bar{\phi}} = \int {\rm d}^4x\left[2J^2\frac{\partial\mathcal{L}_{\rm int}}{\partial J^2} - 2\bar{J}^2\frac{\partial\mathcal{L}_{\rm int}}{\partial\bar{J}^2} - 8J^2\left(\frac{\partial\mathcal{L}_{\rm int}}{\partial J^2}\right)^2 + 8\bar{J}^2\left(\frac{\partial\mathcal{L}_{\rm int}}{\partial\bar{J}^2}\right)^2\right]. \quad (5.33)$$

Note that the right-hand side is local again when expressed in terms of J and $\bar{J}$: the nonlocal terms containing the mixed propagator $\Delta_{\dot{A}\dot{B}AB}$ completely canceled each other. Inserting now [cf. (2.11)]

$$\frac{\partial\mathcal{L}_{\rm int}}{\partial J^2} = \frac{1}{8} + \frac{\partial\mathcal{L}}{\partial J^2}, \qquad \frac{\partial\mathcal{L}_{\rm int}}{\partial\bar{J}^2} = \frac{1}{8} + \frac{\partial\mathcal{L}}{\partial\bar{J}^2} \quad (5.34)$$

into (5.33) we get finally

$$\phi\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\phi} - \bar{\phi}\cdot\frac{\delta\mathcal{T}^{\rm tree}}{\delta\bar{\phi}} = \int {\rm d}^4x\left[-8J^2\left(\frac{\partial\mathcal{L}}{\partial J^2}\right)^2 + 8\bar{J}^2\left(\frac{\partial\mathcal{L}}{\partial\bar{J}^2}\right)^2 + \frac{1}{8}J^2 - \frac{1}{8}\bar{J}^2\right]. \quad (5.35)$$

But the on the right-hand side of this equation we recognize the NGZ constraint (2.23) vanishing of which is the necessary and sufficient condition for self-dual theories. Therefore at tree level the helicity is conserved if and only if the theory is self-dual.

VI. NORMAL ORDERING AND MODIFIED FEYNMAN RULES

Writing in the formula (5.7) for the normal symbol of the S-matrix[11]

$$\exp O\left[\frac{\delta}{\delta\phi},\frac{\delta}{\delta\bar{\phi}}\right] = \exp\left({\rm i}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}}\right) \times\exp\left(-{\rm i}\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\phi} - {\rm i}\frac{\delta}{\delta\bar{\phi}}\cdot\frac{\delta}{\delta\bar{\phi}}\right), \quad (6.1)$$

we can rearrange the calculation of $\mathcal{S}[\phi,\bar{\phi}]$ as

$$\mathcal{S}[\phi,\bar{\phi}] = \exp\left({\rm i}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}}\right)\exp\left({\rm i}S^{\rm NO}_{\rm int}[\phi,\bar{\phi}]\right). \quad (6.2)$$

Here the normal ordered[12] interaction action $S^{\rm NO}_{\rm int}[\phi,\bar{\phi}]$ is defined as

$$\exp\left({\rm i}S^{\rm NO}_{\rm int}[\phi,\bar{\phi}]\right) = \exp\left(-{\rm i}\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\phi} - {\rm i}\frac{\delta}{\delta\bar{\phi}}\cdot\frac{\delta}{\delta\bar{\phi}}\right) \times\exp\left({\rm i}S_{\rm int}[\phi,\bar{\phi}]\right). \quad (6.3)$$

Because the operator

$$O_{\rm local} \equiv -{\rm i}\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\phi} - {\rm i}\frac{\delta}{\delta\bar{\phi}}\cdot\frac{\delta}{\delta\bar{\phi}} \quad (6.4)$$

is local and does not generate space-time derivatives of ϕ, $\bar{\phi}$, the functional $S^{\rm NO}_{\rm int}[\phi,\bar{\phi}]$ is also local, i.e.

$$S^{\rm NO}_{\rm int}[\phi,\bar{\phi}] = \int {\rm d}^4x\mathcal{L}^{\rm NO}_{\rm int}[\phi,\bar{\phi}], \quad (6.5)$$

and $\mathcal{L}_{\rm int}[\phi,\bar{\phi}]$ is a function of the invariants ϕ^2 and $\bar{\phi}^2$ only. Provided the normal ordered interaction action $S^{\rm NO}_{\rm int}[\phi,\bar{\phi}]$ is known, we can calculate the S-matrix equivalently using formula (6.2), i.e. using only the mixed propagator $\langle T\phi\bar{\phi}\rangle$ for the internal lines. This approach is much more effective and also more physical because only the mixed propagator has the one-particle pole. The contributions of the contact propagator terms are naturally accumulated in the normal ordered interaction vertices derived from $\mathcal{L}^{\rm NO}_{\rm int}[\phi,\bar{\phi}]$.

Note that Eq. (6.3) has the same structure as formula (5.7). Therefore, $S^{\rm NO}_{\rm int}[\phi,\bar{\phi}]$ is represented with

[11]Note that the individual terms in (5.8) are all commuting operators.

[12]The term "normal ordered" here should not be confused with the usual operator normal ordering with respect to the creation and annihilation operators.

connected graphs generated by the Wick contractions encoded in the operator $\exp O_{\mathrm{local}}$. Moreover, because the operator O_{local} generates local contractions, the loop graphs are proportional to $\delta^{(4)}(0)$, which vanish in dimensional regularization. So that within dimensional regularization, which we will implicitly assume in what follows, only the tree graphs contribute. We can therefore start with (6.3) and repeat all the manipulations which lead us from (5.7) to Eqs. (5.26) and (5.27). The results is

$$\frac{\delta S^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\delta\phi_{AB}} = \frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\partial\phi_{AB}} = 2J^{AB}\frac{\partial\mathcal{L}_{\mathrm{int}}(J^2,\bar{J}^2)}{\partial J^2}, \quad (6.6)$$

$$\frac{\delta S^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\delta\bar{\phi}_{\dot{A}\dot{B}}} = \frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\partial\bar{\phi}_{\dot{A}\dot{B}}} = 2\bar{J}^{\dot{A}\dot{B}}\frac{\partial\mathcal{L}_{\mathrm{int}}(J^2,\bar{J}^2)}{\partial\bar{J}^2}, \quad (6.7)$$

in this case with

$$J = \phi + 2\frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\partial\phi}, \quad \bar{J} = \bar{\phi} + 2\frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\partial\bar{\phi}}. \quad (6.8)$$

Expressing now ϕ and $\bar{\phi}$ in terms of J and $\bar{J}$ we get therefore an analog of (5.33) and finally an analog of (5.35), again with the NGZ constraint on the right-hand side:

$$\phi_{AB}\frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\partial\phi_{AB}} - \bar{\phi}_{\dot{A}\dot{B}}\frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]}{\partial\bar{\phi}_{\dot{A}\dot{B}}} = -8J^2\left(\frac{\partial\mathcal{L}}{\partial J^2}\right)^2 + 8\bar{J}^2\left(\frac{\partial\mathcal{L}}{\partial\bar{J}^2}\right)^2 + \frac{1}{8}J^2 - \frac{1}{8}\bar{J}^2.$$

We can thus conclude that the theory is self-dual if and only if the corresponding normal ordered interaction Lagrangian $\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]$ is $U(1)$ invariant, i.e. provided its dependence on ϕ and $\bar{\phi}$ is through the combination $\phi^2\bar{\phi}^2$ only: $\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}] = L(\phi^2\bar{\phi}^2)$.

A. Helicity conservation at higher loops

The latter statement allows us to enlarge the validity of the conclusion of the previous section concerning tree-level helicity conservation to all higher loop graphs with vertices derived from self-dual Lagrangian of the type (2.11) satisfying the NGZ condition. Indeed, note that in the formula (6.2) the operator $\exp(\mathrm{i}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}})$ preserves helicity. Therefore provided the theory is self-dual, the corresponding normal ordered interaction Lagrangian $\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]$ is $U(1)$ invariant, and therefore so must be the S-matrix $\mathcal{S}[\phi,\bar{\phi}]$. This implies the helicity conservation. Of course, here we assume that the implicit regularization does not violate the $U(1)$ symmetry and therefore only $U(1)$ symmetric counterterms are needed.

Remarkably, this can be formally understood also on the Lagrangian level. Note that the mixed propagator $\langle T\phi\bar{\phi}\rangle = \mathrm{i}\Delta$ satisfies

$$\int \mathrm{d}^4z\Delta_{AB\dot{C}\dot{D}}(x-z)\epsilon^{\dot{C}\dot{G}}\epsilon^{\dot{D}\dot{H}}\Delta_{EF\dot{G}\dot{H}}(z-y) = 2\delta^{(4)}(x-y)[\epsilon_{AE}\epsilon_{BF}+\epsilon_{AF}\epsilon_{BE}] \quad (6.9)$$

$$\int \mathrm{d}^4z\Delta_{AB\dot{C}\dot{D}}(x-z)\epsilon^{AE}\epsilon^{BF}\Delta_{EF\dot{G}\dot{H}}(z-y). = 2\delta^{(4)}(x-y)[\epsilon_{\dot{C}\dot{G}}\epsilon_{\dot{D}\dot{H}}+\epsilon_{\dot{C}\dot{H}}\epsilon_{\dot{C}\dot{G}}]. \quad (6.10)$$

Therefore, introducing formal functional Gaussian integration

$$\exp\left(\mathrm{i}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\bar{\phi}}\right) = \int D\varphi D\bar{\varphi}\exp\left(\frac{\mathrm{i}}{4}\varphi\cdot\Delta\cdot\bar{\varphi}+\varphi\cdot\frac{\delta}{\delta\phi}+\bar{\varphi}\cdot\frac{\delta}{\delta\bar{\phi}}\right), \quad (6.11)$$

we can represent the right-hand side of the formula (6.2) as

$$\begin{aligned}\mathcal{S}[\phi,\bar{\phi}] &= \int D\varphi D\bar{\varphi}\exp\left(\frac{\mathrm{i}}{4}\varphi\cdot\Delta\cdot\bar{\varphi}+\varphi\cdot\frac{\delta}{\delta\phi}+\bar{\varphi}\cdot\frac{\delta}{\delta\bar{\phi}}\right)\exp\left(\mathrm{i}S^{\mathrm{NO}}_{\mathrm{int}}[\phi,\bar{\phi}]\right)\\ &= \int D\varphi D\bar{\varphi}\exp\left(\frac{\mathrm{i}}{4}\varphi\cdot\Delta\cdot\bar{\varphi}\right)\exp\left(\mathrm{i}S^{\mathrm{NO}}_{\mathrm{int}}[\phi+\varphi,\bar{\phi}+\bar{\varphi}]\right)\\ &= \int D\varphi D\bar{\varphi}\exp\left(\frac{\mathrm{i}}{4}(\varphi-\phi)\cdot\Delta\cdot(\bar{\varphi}-\bar{\phi})\right)\exp\left(\mathrm{i}S^{\mathrm{NO}}_{\mathrm{int}}[\varphi,\bar{\varphi}]\right).\end{aligned} \quad (6.12)$$

The latter formula formally corresponds to the functional integral representation of the S-matrix in theory with "classical normal ordered action" $S^{\mathrm{NO}}[\varphi,\bar{\varphi}]$ of the form

$$S^{\mathrm{NO}}[\varphi,\bar{\varphi}] = \frac{1}{4}\varphi\cdot\Delta\cdot\bar{\varphi}+S^{\mathrm{NO}}_{\mathrm{int}}[\varphi,\bar{\varphi}], \quad (6.13)$$

which is formulated solely in terms of gauge invariant fields φ and $\bar{\varphi}$ without necessity to relate it to the potential A_μ. This is in contrast to the original action $S[\varphi, \bar{\varphi}]$

$$S[\varphi, \bar{\varphi}] = -\frac{1}{8}\varphi^2 - \frac{1}{8}\bar{\varphi}^2 + S_{\mathrm{int}}[\varphi, \bar{\varphi}], \tag{6.14}$$

for which the path integral quantization needs φ and $\bar{\varphi}$ to be expressed in terms of A_μ and a gauge fixing term has to be added.

For self-dual theories the action $S^{\mathrm{NO}}[\varphi, \bar{\varphi}]$ shows manifest $U(1)$ symmetry and implies therefore also manifestly the helicity conservation. Note, however, that the kinetic term of this action is nonlocal; nevertheless, it generates formally the right propagator $\langle T\phi\bar{\phi}\rangle = \mathrm{i}\Delta$. This is the price we pay for working directly with the variables φ and $\bar{\varphi}$.

B. Calculation of the normal ordered Lagrangian

According to the definition (6.3), the normal ordered interaction Lagrangian $\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi, \bar{\phi}]$ can be obtained as a sum of connected graphs with vertices from $S_{\mathrm{int}}[\phi, \bar{\phi}]$ and internal lines corresponding only to the contact propagators $\langle T\phi\phi\rangle$ and $\langle T\bar{\phi}\bar{\phi}\rangle$ [see (4.7) and (4.8)]. Because of the locality, the loops are proportional to $\delta^{(4)}(0)$ which vanishes in the dimensional regularization and thus only the tree-level graphs are relevant. These can be summed up as follows. Let us rewrite (6.3) in the form

$$\begin{aligned}\exp&(\mathrm{i}S^{\mathrm{NO}}_{\mathrm{int}}[\phi, \bar{\phi}])\\ &= \int D\varphi D\bar{\varphi} \mathrm{e}^{(-\frac{\mathrm{i}}{4}\varphi\cdot\varphi - \frac{\mathrm{i}}{4}\bar{\varphi}\cdot\bar{\varphi} + \varphi\cdot\frac{\delta}{\delta\phi} + \bar{\varphi}\cdot\frac{\delta}{\delta\bar{\phi}})} \mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\phi, \bar{\phi}]}\\ &= \int D\varphi D\bar{\varphi} \mathrm{e}^{(-\frac{\mathrm{i}}{4}\varphi\cdot\varphi - \frac{\mathrm{i}}{4}\bar{\varphi}\cdot\bar{\varphi})} \mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\phi+\varphi, \bar{\phi}+\bar{\varphi}]}\\ &= \int D\varphi D\bar{\varphi} \mathrm{e}^{(-\frac{\mathrm{i}}{4}(\varphi-\phi)\cdot(\varphi-\phi) - \frac{\mathrm{i}}{4}(\bar{\varphi}-\bar{\phi})\cdot(\bar{\varphi}-\bar{\phi}) + \mathrm{i}S_{\mathrm{int}}[\varphi, \bar{\varphi}])}.\end{aligned} \tag{6.15}$$

The result of the tree-level calculation of the functional integral then corresponds to

$$\begin{aligned}S^{\mathrm{NO}}_{\mathrm{int}}[\phi, \bar{\phi}] = S_{\mathrm{int}}[\varphi, \bar{\varphi}] &- \frac{1}{4}(\varphi - \phi)\cdot(\varphi - \phi)\\ &- \frac{1}{4}(\bar{\varphi} - \bar{\phi})\cdot(\bar{\varphi} - \bar{\phi}),\end{aligned} \tag{6.16}$$

where φ, $\bar{\varphi}$ satisfy the classical equation of motion

$$\begin{aligned}-\frac{1}{2}(\varphi - \phi)_{AB} + \frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\varphi^{AB}} &= 0,\\ -\frac{1}{2}(\bar{\varphi} - \bar{\phi})_{\dot{A}\dot{B}} + \frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\bar{\varphi}^{\dot{A}\dot{B}}} &= 0.\end{aligned} \tag{6.17}$$

Therefore

$$\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\phi, \bar{\phi}] = \mathcal{L}_{\mathrm{int}}[\varphi, \bar{\varphi}] - \frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\varphi^{AB}}\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\varphi_{AB}} - \frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\bar{\varphi}^{\dot{A}\dot{B}}}\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\bar{\varphi}_{\dot{A}\dot{B}}} \tag{6.18}$$

and φ, $\bar{\varphi}$ are solutions of (6.17). Note that both $\mathcal{L}_{\mathrm{int}}[\varphi, \bar{\varphi}]$ and $\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}[\varphi, \bar{\varphi}]$ are functions of the invariants φ^2 and $\bar{\varphi}^2$, therefore,

$$\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\varphi^{AB}} = 2\varphi_{AB}\frac{\partial\mathcal{L}_{\mathrm{int}}}{\partial\varphi^2}. \tag{6.19}$$

Finally we rewrite (6.18) as

$$\begin{aligned}\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}(\phi^2, \bar{\phi}^2) = \mathcal{L}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2) &- 4\varphi^2\left(\frac{\partial\mathcal{L}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2)}{\partial\varphi^2}\right)^2\\ &- 4\bar{\varphi}^2\left(\frac{\partial\mathcal{L}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2)}{\partial\bar{\varphi}^2}\right)^2,\end{aligned} \tag{6.20}$$

and the equations (6.17) can be written in the form

$$\begin{aligned}\phi^2 &= \varphi^2\left(1 - 4\frac{\partial\mathcal{L}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2)}{\partial\varphi^2}\right)^2,\\ \bar{\phi}^2 &= \bar{\varphi}^2\left(1 - 4\frac{\partial\mathcal{L}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2)}{\partial\bar{\varphi}^2}\right)^2.\end{aligned} \tag{6.21}$$

The summation of the tree graphs with the contact propagators $\langle T\phi\phi\rangle$ and $\langle T\bar{\phi}\bar{\phi}\rangle$ is therefore equivalent to the solution of the algebraic equations (6.21) with respect to φ^2 and $\bar{\varphi}^2$ and inserting then the solution into (6.20).

Let us note that the relation (6.3) connecting the original Lagrangian with the normal ordered one is invertible, namely

$$\begin{aligned}\exp(\mathrm{i}S_{\mathrm{int}}[\phi, \bar{\phi}]) = \exp&\left(\mathrm{i}\frac{\delta}{\delta\phi}\cdot\frac{\delta}{\delta\phi} + \mathrm{i}\frac{\delta}{\delta\bar{\phi}}\cdot\frac{\delta}{\delta\bar{\phi}}\right)\\ &\times \exp(\mathrm{i}S^{\mathrm{NO}}_{\mathrm{int}}[\phi, \bar{\phi}]).\end{aligned} \tag{6.22}$$

Repeating the above formal manipulation we can write the result of the inversion as

$$\begin{aligned}\mathcal{L}_{\mathrm{int}}(\phi^2, \bar{\phi}^2) = \mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2) &+ 4\varphi^2\left(\frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2)}{\partial\varphi^2}\right)^2\\ &+ 4\bar{\varphi}^2\left(\frac{\partial\mathcal{L}^{\mathrm{NO}}_{\mathrm{int}}(\varphi^2, \bar{\varphi}^2)}{\partial\bar{\varphi}^2}\right)^2,\end{aligned} \tag{6.23}$$

where now φ^2 and $\bar{\varphi}^2$ are solutions of algebraic equations

$$\phi^2 = \varphi^2\left(1 + 4\frac{\partial \mathcal{L}^{\rm NO}_{\rm int}(\varphi^2,\bar\varphi^2)}{\partial\varphi^2}\right)^2,$$
$$\bar\phi^2 = \bar\varphi^2\left(1 + 4\frac{\partial \mathcal{L}^{\rm NO}_{\rm int}(\varphi^2,\bar\varphi^2)}{\partial\bar\varphi^2}\right)^2. \quad (6.24)$$

Let us note that the starting point for derivation of Eqs. (6.23) and (6.24) is the analog of (6.16) and (6.17), namely

$$S_{\rm int}[\phi,\bar\phi] = S^{\rm NO}_{\rm int}[\varphi,\bar\varphi] + \frac{1}{4}(\varphi-\phi)\cdot(\varphi-\phi) + \frac{1}{4}(\bar\varphi-\bar\phi)\cdot(\bar\varphi-\bar\phi), \quad (6.25)$$

where φ, $\bar\varphi$ satisfy the classical equation of motion

$$\frac{1}{2}(\varphi-\phi)_{AB} + \frac{\partial\mathcal{L}^{\rm NO}_{\rm int}}{\partial\varphi^{AB}} = 0, \qquad \frac{1}{2}(\bar\varphi-\bar\phi)_{\dot A\dot B} + \frac{\partial\mathcal{L}^{\rm NO}_{\rm int}}{\partial\bar\varphi^{\dot A\dot B}} = 0. \quad (6.26)$$

This can be directly compared with the auxiliary field construction of the self-dual actions of Ivanov and Zupnik [12,13]. Up to a different normalization of the fields, the normal ordered action can be identified with the $U(1)$ invariant off-shell action from their construction where φ, $\bar\varphi$ play the role of the auxiliary fields.

In the next two subsections we will illustrate the application of the correspondence $\mathcal{L}_{\rm int}\leftrightarrow\mathcal{L}^{\rm NO}_{\rm int}$ in two special cases for which we can obtain the solution of both problems in a closed form.

C. Normal ordered form of the Born-Infeld Lagrangian

As the first illustration, let us find the normal ordered form of the BI Lagrangian. In the case of BI theory it is convenient to use the following change of variables (first introduced in [30]):

$$\varphi_{AB} = \frac{1}{\sqrt{2}}\psi_{AB}\frac{1+\bar\eta}{1-\eta\bar\eta}, \quad \bar\varphi_{\dot A\dot B} = \frac{1}{\sqrt{2}}\bar\psi_{\dot A\dot B}\frac{1+\eta}{1-\eta\bar\eta} \quad (6.27)$$

where

$$\eta = \frac{\psi^2}{16\Lambda^4}, \qquad \bar\eta = \frac{\bar\psi^2}{16\Lambda^4}, \quad (6.28)$$

and therefore

$$\varphi^2 = 8\Lambda^4\eta\left(\frac{1+\bar\eta}{1-\eta\bar\eta}\right)^2, \quad \bar\varphi^2 = 8\Lambda^4\bar\eta\left(\frac{1+\eta}{1-\eta\bar\eta}\right)^2. \quad (6.29)$$

In [31] it was found that, when expressed in terms of these new fields, the BI Lagrangian simplifies to a rational function of η and $\bar\eta$

$$\mathcal{L}_{\rm BI} = -\Lambda^4\frac{\eta+\bar\eta+2\eta\bar\eta}{1-\eta\bar\eta}, \quad (6.30)$$

and the interaction part looks like

$$\mathcal{L}_{\rm BI,int} \equiv \mathcal{L}_{\rm BI} + \frac{1}{8}(\varphi^2+\bar\varphi^2) = -\Lambda^4\eta\bar\eta\frac{(1+\eta)(1+\bar\eta)}{(1-\eta\bar\eta)^2}. \quad (6.31)$$

Inserting now the new parametrization into (6.20) we get

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar\phi^2) = 2\Lambda^4\eta\bar\eta\frac{1+\eta\bar\eta}{(1-\eta\bar\eta)^2} \quad (6.32)$$

and the algebraic equations (6.21) are transformed to

$$\phi^2 = 8\Lambda^4\frac{\eta}{(1-\eta\bar\eta)^2}, \qquad \bar\phi^2 = 8\Lambda^4\frac{\bar\eta}{(1-\eta\bar\eta)^2}. \quad (6.33)$$

From the latter equation we get

$$\phi^2\bar\phi^2 = 64\Lambda^8\frac{\eta\bar\eta}{(1-\eta\bar\eta)^4}, \quad (6.34)$$

or

$$4w(1-z)^4 - z = 0, \quad (6.35)$$

where

$$z = \eta\bar\eta, \qquad w = \frac{\phi^2\bar\phi^2}{(16\Lambda^4)^2}. \quad (6.36)$$

Therefore, the normal ordered BI Lagrangian reads

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar\phi^2) = 8\Lambda^4\frac{\phi^2\bar\phi^2}{(16\Lambda^4)^2}(1+\eta\bar\eta)(1-\eta\bar\eta)^2 = 8\Lambda^4 w(1+z)(1-z)^2, \quad (6.37)$$

where z is a solution of (6.35). This quartic equation has four solutions; however, only one of them is analytic for $w=0$. The proper solution can be inserted into right-hand side of (6.37) using the general formula (see also [24] where this approach has been used in similar context)

$$f(z_0) = \frac{1}{2\pi{\rm i}}\int_C {\rm d}z\frac{f(z)}{F(z)}F'(z) \quad (6.38)$$

where $f(z)$ is an analytic function at z_0 while z_0 is a simple zero of $F(z)$, i.e.

$$F(z_0) = 0, \qquad F'(z_0) \neq 0$$

and there is no other zero of $F(z)$ inside the closed curve C. Choosing $f = \mathcal{L}^{\rm NO}_{\rm BI,int}(z)$ and $F(z) = z - 4w(1-z)^4$ we get

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2) = 8\Lambda^4 w\frac{1}{2\pi \mathrm{i}}\oint_{|z|=\varepsilon}\mathrm{d}z\frac{(1+z)(1-z)^2}{z-4w(1-z)^4}\times(1+16w(1-z)^3). \tag{6.39}$$

The contour in the last formula picks up the solution of (6.35) which vanishes for ϕ, $\bar{\phi}\to 0$ provided ε is small enough. The integrand can be expanded in powers of w and we get

$$\begin{aligned}\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2) &= 8\Lambda^4 w\frac{1}{2\pi \mathrm{i}}\oint_{|z|=\varepsilon}\frac{\mathrm{d}z}{z}(1+z)(1-z)^2(1+16w(1-z)^3)\\ &\quad\times\sum_{n=0}^{\infty}4^n z^{-n}(1-z)^{4n}w^n\end{aligned} \tag{6.40}$$

and after some straightforward algebra

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2) = 8\Lambda^4\sum_{n=0}^{\infty}4^n w^{n+1}\frac{1}{2\pi \mathrm{i}}\oint_{|z|=\varepsilon}\frac{\mathrm{d}z}{z}(1+4z+3z^2) \tag{6.41}$$

$$\times\sum_{k=0}^{4n+1}\binom{4n+1}{k}(-1)^k z^{-n+k}. \tag{6.42}$$

Calculating the residue at $z=0$ we get in the end

$$\begin{aligned}&\frac{1}{2\pi \mathrm{i}}\oint_{|z|=\varepsilon}\frac{\mathrm{d}z}{z}(1+4z+3z^2)\sum_{k=0}^{4n+1}\binom{4n+1}{k}(-1)^k z^{-n+k}\\ &= (-1)^n\left[\binom{4n+1}{n}-4\binom{4n+1}{n-1}+3\binom{4n+1}{n-2}\right]\\ &= (-1)^n 2\frac{(4n+1)!}{(3n+2)!(n+1)!}\end{aligned}$$

and thus

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2) = 16\Lambda^4 w\sum_{n=0}^{\infty}\frac{(4n+1)!}{(3n+2)!(n+1)!}(-4w)^n \tag{6.43}$$

where

$$w = \frac{\phi^2\bar{\phi}^2}{(16\Lambda^4)^2}. \tag{6.44}$$

The power series (6.43) can be summed up and $\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2)$ is given in a closed form as

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2) = -\frac{3}{2}\Lambda^4\left\{{}_3F_2\left[\left(-\frac{1}{2},-\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{3},\frac{2}{3}\right),-\frac{2^2}{3^3}\frac{\phi^2\bar{\phi}^2}{\Lambda^8}\right]-1\right\}. \tag{6.45}$$

Remarkably the same function appears in the expression for the hypergeometric form of the BI Lagrangian found in [23,24]. Of course this is not an accidental coincidence, as we have discussed above.

Explicitly we get for the weak field expansion

$$\mathcal{L}^{\rm NO}_{\rm BI,int}(\phi^2,\bar{\phi}^2) = \frac{\phi^2\bar{\phi}^2}{32\Lambda^4}-\frac{[\phi^2\bar{\phi}^2]^2}{2048\Lambda^{12}}+3\frac{[\phi^2\bar{\phi}^2]^3}{131072\Lambda^{20}}+\ldots. \tag{6.46}$$

The simple form of (6.46) should be compared with the expansion of the original Lagrangian (2.24)

$$\begin{aligned}\mathcal{L}_{\rm BI,int}(\phi^2,\bar{\phi}^2) &= \frac{\phi^2\bar{\phi}^2}{32\Lambda^4}-\frac{\phi^2\bar{\phi}^2[\phi^2+\bar{\phi}^2]}{256\Lambda^8}\\ &\quad+\frac{\phi^2\bar{\phi}^2[(\phi^2)^2+3\phi^2\bar{\phi}^2+(\bar{\phi}^2)^2]}{2048\Lambda^{12}}\\ &\quad-\frac{\phi^2\bar{\phi}^2(\phi^2+\bar{\phi}^2)[(\phi^2)^2+5\phi^2\bar{\phi}^2+(\bar{\phi}^2)^2]}{16384\Lambda^{16}}\\ &\quad+\frac{\phi^2\bar{\phi}^2}{131072\Lambda^{20}}[(\phi^2)^4+10(\phi^2)^3\bar{\phi}^2\\ &\quad+20(\phi^2\bar{\phi}^2)^2+10\phi^2(\bar{\phi}^2)^3+(\bar{\phi}^2)^4]+\ldots\end{aligned}$$

for which the helicity conservation is a result of subtle cancellations of direct and induced contact terms (i.e. those stemming from gluing together the original vertices with local parts of the propagators).

D. The simplest helicity conserving theory and the Bossard-Nicolai model

Let us now illustrate the inverse problem: suppose that the normal ordered Lagrangian is known and try to identify the original one. Our example will be the apparently simplest helicity conserving theory which corresponds to normal ordered Lagrangian with only one quartic vertex [such a theory was assumed in [32] as the simplest interaction (SI) model]

$$\mathcal{L}^{\rm NO}_{\rm int}(\phi^2,\bar{\phi}^2) = \frac{\lambda}{4}\phi^2\bar{\phi}^2. \tag{6.47}$$

In this case we get the formula for the original Lagrangian (6.23) in the form

$$\mathcal{L}_{\rm int}(\phi^2,\bar{\phi}^2)=\frac{\lambda}{4}\varphi^2\bar{\varphi}^2+4\varphi^2\left(\frac{\lambda}{4}\bar{\varphi}^2\right)^2+4\bar{\varphi}^2\left(\frac{\lambda}{4}\varphi^2\right)^2$$
$$=\frac{\lambda}{4}\varphi^2\bar{\varphi}^2[(1+\lambda\varphi^2)(1+\lambda\bar{\varphi}^2)-\lambda^2\varphi^2\bar{\varphi}^2] \quad (6.48)$$

while the algebraic equations determining $\varphi^2,\bar{\varphi}^2$ as a functions of ϕ^2 and $\bar{\phi}^2$ (6.24) simplifies to

$$\varphi^2(1+\lambda\bar{\varphi}^2)^2-\phi^2=0,\quad \bar{\varphi}^2(1+\lambda\varphi^2)^2-\bar{\phi}^2=0. \quad (6.49)$$

Let us denote for short $z=\varphi^2$ and $\bar{z}=\bar{\varphi}^2$. The generalization of (6.38) to the case of two variables reads in our case

$$\mathcal{L}_{\rm int}(\phi^2,\bar{\phi}^2)=\frac{1}{(2\pi{\rm i})^2}\oint_{|z|,|\bar{z}|=\varepsilon}\frac{{\rm d}z{\rm d}\bar{z}}{h(z,\bar{z})\bar{h}(z,\bar{z})}\times\det\frac{\partial(h,\bar{h})}{\partial(z,\bar{z})}f(z,\bar{z}), \quad (6.50)$$

where we again choose the double contour in order to pick up the right solution. In the above formula [see (6.48)]

$$f(z,\bar{z})=\frac{\lambda}{4}z\bar{z}[(1+\lambda\bar{z})(1+\lambda z)-\lambda^2 z\bar{z}], \quad (6.51)$$

and [see (6.49)]

$$h(z,\bar{z})=z(1+\lambda\bar{z})^2-\phi^2,\qquad \bar{h}(z,\bar{z})=\bar{z}(1+\lambda z)^2-\bar{\phi}^2. \quad (6.52)$$

For the Jacobian we have then

$$\det\frac{\partial(h,\bar{h})}{\partial(z,\bar{z})}=(1+\lambda\bar{z})(1+\lambda z)[(1+\lambda\bar{z})(1+\lambda z)-4\lambda^2 z\bar{z}]. \quad (6.53)$$

Expanding $\mathcal{L}_{\rm int}(\phi^2,\bar{\phi}^2)$ given by (6.50) in powers of ϕ^2 and $\bar{\phi}^2$ we get

$$\mathcal{L}_{\rm int}(\phi^2,\bar{\phi}^2)=\frac{1}{(2\pi{\rm i})^2}\frac{\lambda}{4}\sum_{n,m}(\phi^2)^n(\bar{\phi}^2)^m\oint_{|z|,|\bar{z}|=\varepsilon}\frac{{\rm d}z{\rm d}\bar{z}}{z^n(1+\lambda\bar{z})^{2n+1}\bar{z}^m(1+\lambda z)^{2m+1}}\times[(1+\lambda\bar{z})(1+\lambda z)-4\lambda^2 z\bar{z}][(1+\lambda\bar{z})(1+\lambda z)-\lambda^2 z\bar{z}] \quad (6.54)$$

and after some algebra the double integral can be rewritten a form of the linear combination of factorized single variable integrals

$$\mathcal{L}_{\rm int}(\phi^2,\bar{\phi}^2)=\frac{1}{(2\pi{\rm i})^2}\frac{\lambda}{4}\sum_{n,m}(\phi^2)^n(\bar{\phi}^2)^m\oint_{|z|,|\bar{z}|=\varepsilon}{\rm d}z{\rm d}\bar{z}[f_{n,2m-1}(z)f_{m,2n-1}(\bar{z})-5\lambda^2 f_{n-1,2m}(z)f_{m-1,2n}(\bar{z})+4\lambda^4 f_{n-1,2m}(z)f_{m-1,2n}(\bar{z})] \quad (6.55)$$

where

$$f_{k,l}(x)=\frac{1}{x^k(1+\lambda x)^l}. \quad (6.56)$$

The resulting single variable integrals can be evaluated using residue theorem

$$\frac{1}{2\pi{\rm i}}\oint_{|z|=\varepsilon}{\rm d}zf_{k,l}(z)=\frac{1}{(k-1)!}(-\lambda)^{k-1}l(l+1)\ldots(l+k-2)=(-\lambda)^{k-1}\binom{l+k-2}{k-1}. \quad (6.57)$$

As a result

$$\mathcal{L}_{\rm int}(\phi^2,\bar{\phi}^2)=\sum_{n,m\geq 1}c_{nm}(\phi^2)^n(\bar{\phi}^2)^m \quad (6.58)$$

where the coefficients are explicitly given as

$$c_{nm}=(-1)^{n+m}\frac{\lambda^{n+m-1}}{4}\left[\binom{n+2m-3}{n-1}\binom{m+2n-3}{m-1}-5\binom{n+2m-3}{n-2}\binom{m+2n-3}{m-2}+4\binom{n+2m-3}{n-3}\binom{m+2n-3}{m-3}\right]. \quad (6.59)$$

After a simple rearrangement of the binomial coefficients we get finally the original Lagrangian corresponding to the normal ordered one (6.47) in the form

$$\mathcal{L}_{\text{int}} = -\frac{1}{4}\sum_{n,m\geq 1}\frac{(-\lambda)^{n+m-1}}{nm}\binom{n+2m-2}{n-1}\binom{m+2n-2}{m-1}(\phi^2)^n(\bar{\phi}^2)^m. \tag{6.60}$$

For identification of this theory let us express back the variables ϕ^2 and $\bar{\phi}^2$ in terms of the invariants $\mathcal{F}$ and $\mathcal{G}$ [see (2.10)]

$$\phi^2 = 4(\mathcal{F} - \mathrm{i}\mathcal{G}), \qquad \bar{\phi}^2 = 4(\mathcal{F} + \mathrm{i}\mathcal{G}). \tag{6.61}$$

Fixing now $\lambda = g^2/8$ we get from (6.60)

$$\begin{aligned}\mathcal{L}_{\text{int}} = {}& \frac{1}{2}g^2(\mathcal{F}^2+\mathcal{G}^2) - \frac{1}{2}g^4\mathcal{F}(\mathcal{F}^2+\mathcal{G}^2) + \frac{1}{4}g^6(\mathcal{F}^2+\mathcal{G}^2)(3\mathcal{F}^2+\mathcal{G}^2) - \frac{1}{8}g^8\mathcal{F}(\mathcal{F}^2+\mathcal{G}^2)(11\mathcal{F}^2+7\mathcal{G}^2)\\ &+\frac{1}{32}g^{10}(\mathcal{F}^2+\mathcal{G}^2)(91\mathcal{F}^4+86\mathcal{F}^2\mathcal{G}^2+11\mathcal{G}^4) - \frac{1}{8}g^{12}\mathcal{F}(\mathcal{F}^2+\mathcal{G}^2)(51\mathcal{F}^4+64\mathcal{F}^2\mathcal{G}^2+17\mathcal{G}^4)\\ &+\frac{1}{64}g^{14}(\mathcal{F}^2+\mathcal{G}^2)(969\mathcal{F}^6+1517\mathcal{F}^4\mathcal{G}^2+623\mathcal{F}^2\mathcal{G}^4+43\mathcal{G}^6)+\ldots\end{aligned} \tag{6.62}$$

which can be identified with the first seven terms of the expansion of the interaction Lagrangian of the Bossard-Nicolai model; these terms were calculated explicitly in [11] using a different method. We can therefore conclude that the Lagrangian (6.60) corresponds to the BN model.

E. From normal ordering to original Lagrangian: The general case of self-dual theory

According to the previous subsections, in the general case, the self-dual theory is obtained form the manifestly $U(1)$ invariant interaction Lagrangian as

$$\begin{aligned}\mathcal{L}_{\text{int}}(\phi^2,\bar{\phi}^2) = {}& \mathcal{L}_{\text{int}}^{\text{NO}}(\varphi^2,\bar{\varphi}^2) + 4\varphi^2\left(\frac{\partial\mathcal{L}_{\text{int}}^{\text{NO}}(\varphi^2,\bar{\varphi}^2)}{\partial\varphi^2}\right)^2\\ &+4\bar{\varphi}^2\left(\frac{\partial\mathcal{L}_{\text{int}}^{\text{NO}}(\varphi^2,\bar{\varphi}^2)}{\partial\bar{\varphi}^2}\right)^2,\end{aligned} \tag{6.63}$$

where φ^2, $\bar{\varphi}^2$ are solutions of

$$\begin{aligned}\phi^2 &= \varphi^2\left(1+4\frac{\partial\mathcal{L}_{\text{int}}^{\text{NO}}(\varphi^2,\bar{\varphi}^2)}{\partial\varphi^2}\right)^2,\\ \bar{\phi}^2 &= \bar{\varphi}^2\left(1+4\frac{\partial\mathcal{L}_{\text{int}}^{\text{NO}}(\varphi^2,\bar{\varphi}^2)}{\partial\bar{\varphi}^2}\right)^2.\end{aligned} \tag{6.64}$$

Note that duality invariance and Lorentz invariance requires that $\mathcal{L}_{\text{int}}(\varphi^2,\bar{\varphi}^2)$ is a function of the invariant combination $\varphi^2\bar{\varphi}^2$

$$\mathcal{L}_{\text{int}}^{\text{NO}}(\varphi^2,\bar{\varphi}^2) \equiv L(\varphi^2\bar{\varphi}^2) \tag{6.65}$$

and thus we can write

$$\begin{aligned}\mathcal{L}_{\text{int}} &= L + 4\varphi^2(\bar{\varphi}^2L')^2 + 4\bar{\varphi}^2\varphi^2(\varphi^2L')^2\\ &= L + 4\varphi^2\bar{\varphi}^2(\bar{\varphi}^2+\varphi^2)L'^2\end{aligned} \tag{6.66}$$

where the prime means a derivative of L with respect to $\varphi^2\bar{\varphi}^2$. The algebraic equations defining φ^2 and $\bar{\varphi}^2$ in terms of ϕ^2 and $\bar{\phi}^2$ are then

$$\phi^2 = \varphi^2(1+4\bar{\varphi}^2L')^2, \quad \bar{\phi}^2 = \bar{\varphi}^2(1+4\varphi^2L')^2, \tag{6.67}$$

or taking the square root

$$\sqrt{\phi^2} = \sqrt{\varphi^2}(1+4\bar{\varphi}^2L'), \qquad \sqrt{\bar{\phi}^2} = \sqrt{\bar{\varphi}^2}(1+4\varphi^2L'). \tag{6.68}$$

Let us introduce new variables

$$x_\pm = \frac{1}{2}\left(\sqrt{\varphi^2}\pm\sqrt{\bar{\varphi}^2}\right), \qquad X_\pm = \frac{1}{2}\left(\sqrt{\phi^2}\pm\sqrt{\bar{\phi}^2}\right), \tag{6.69}$$

in terms of which we get

$$\sqrt{\varphi^2}\sqrt{\bar{\varphi}^2} = x_+^2 - x_-^2, \quad \varphi^2+\bar{\varphi}^2 = 2(x_+^2+x_-^2), \tag{6.70}$$

$$X_\pm = x_\pm(1\pm 4\sqrt{\varphi^2}\sqrt{\bar{\varphi}^2}L'(\varphi^2\bar{\varphi}^2)). \tag{6.71}$$

Let us further abbreviate $z = \sqrt{\varphi^2}\sqrt{\bar{\varphi}^2}$. The interaction Lagrangian is then expressed in a compact form

$$\mathcal{L}_{\text{int}} = L(z^2) + 8z^2L'(z^2)^2(x_+^2+x_-^2) \tag{6.72}$$

where $x_\pm$ are solutions of

$$X_\pm = x_\pm(1\pm 4zL'(z^2)). \tag{6.73}$$

Note that we do not need to know $x_\pm$ individually but only in the combinations $x_+^2 + x_-^2$ and z. The latter equations imply for these

$$x_+^2 + x_-^2 = \frac{X_+^2}{(1+4zL'(z^2))^2} + \frac{X_-^2}{(1-4zL'(z^2))^2}, \tag{6.74}$$

$$z = \frac{X_+^2}{(1+4zL'(z^2))^2} - \frac{X_-^2}{(1-4zL'(z^2))^2}. \tag{6.75}$$

For the interaction Lagrangian we get therefore

$$\mathcal{L}_{\rm int} = L(z^2) + 8z^2L'(z^2)^2 \times \left[\frac{X_+^2}{(1+4zL'(z^2))^2} + \frac{X_-^2}{(1-4zL'(z^2))^2}\right], \tag{6.76}$$

where z is solution of single equation (6.75). Using this equation we can finally simplify $\mathcal{L}_{\rm int}$ to the form

$$\mathcal{L}_{\rm int} = L(z^2) + 2zL'(z^2) \times \left(\frac{X_+^2}{1+4zL'(z^2)} - \frac{X_-^2}{1-4zL'(z^2)} - z\right). \tag{6.77}$$

Remarkably, we can immediately make sure that the complete Lagrangian $\mathcal{L} = -\frac{1}{4}X_+^2 - \frac{1}{4}X_-^2 + \mathcal{L}_{\rm int}$ represents the solution of the NGZ condition. Indeed, it is an easy exercise to show that this Lagrangian can be reconstructed according to the general representation (3.7), (3.8) with the identification

$$p(u) = \frac{1+4uL'(u^2)}{1-4uL'(u^2)}, \qquad F(u) = u[1-16u^2L'(u^2)^2], \tag{6.78}$$

from which all the other representations discussed in Sec. III can be in principle derived. For instance, the BN model, for which we have $L(z) = \frac{\lambda}{4}z$, can be constructed according to (3.10) and (3.11) using the function[13]

$$f^{BN}(z) = 4\frac{z}{\lambda}\frac{z-1}{(z+1)^3}. \tag{6.79}$$

Let us now return to the general case. In order to insert the right solution of (6.75) into (6.77) we use the same trick as in the previous subsections and write

$$\mathcal{L}_{\rm int} = \frac{1}{2\pi \mathrm{i}}\oint_{|z|=\varepsilon} \frac{\mathrm{d}z}{h(z)} h'(z)L_{\rm int}(z), \tag{6.80}$$

[13]See Sec. III for passing form $p(u)$ and $F(u)$ to $f(z)$.

where now

$$h(z) = z - \left[\frac{X_+^2}{(1+4zL'(z^2))^2} - \frac{X_-^2}{(1-4zL'(z^2))^2}\right] \tag{6.81}$$

and $L_{\rm int}(z)$ is the right-hand side of (6.77). Expanding the integrand in powers of $X_\pm$ we get

$$\mathcal{L}_{\rm int} = \frac{1}{2\pi \mathrm{i}}\oint_{|z|=\varepsilon} \frac{\mathrm{d}z}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}\left[\frac{X_+^2}{(1+4zL'(z^2))^2} - \frac{X_-^2}{(1-4zL'(z^2))^2}\right]^n h'(z)L_{\rm int}(z) \tag{6.82}$$

and finally using the residue theorem

$$\mathcal{L}_{\rm int} = \sum_{n=0}^{\infty}\lim_{z\to 0}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\frac{X_+^2}{(1+4zL'(z^2))^2} - \frac{X_-^2}{(1-4zL'(z^2))^2}\right]^n h'(z)L_{\rm int}(z). \tag{6.83}$$

The latter formula allows us to calculate $\mathcal{L}_{\rm int}$ to any desired order in $X_\pm^2 = 2(\mathcal{F} \pm \sqrt{\mathcal{F}^2+\mathcal{G}^2})$ or ϕ^2 and $\bar\phi^2$. Writing

$$L(z) = \Lambda^4 \sum_{n=1}^{\infty}\frac{1}{n!}\frac{c_{4n}}{\Lambda^{8n}}z^n, \tag{6.84}$$

where Λ is dimensionful scale, we get explicitly

$$\begin{aligned}\mathcal{L}_{\rm int} &= \frac{c_4}{\Lambda^4}\phi^2\bar\phi^2 - 4\frac{c_4^2}{\Lambda^8}\phi^2\bar\phi^2(\phi^2+\bar\phi^2) \\ &\quad + \frac{1}{2\Lambda^{12}}\phi^2\bar\phi^2\{32c_4^3[(\phi^2)^2 + 4\phi^2\bar\phi^2 + (\bar\phi^2)^2] \\ &\quad + c_8\phi^2\bar\phi^2\} - 8\frac{c_4}{\Lambda^{16}}\phi^2\bar\phi^2(\phi^2+\bar\phi^2) \\ &\quad \times \{8c_4^3[(\phi^2)^2 + 9\phi^2\bar\phi^2 + (\bar\phi^2)^2] + c_8\phi^2\bar\phi^2\} \\ &\quad + \ldots.\end{aligned} \tag{6.85}$$

As expected, the couplings at individual terms are related. Notice e.g. the relation between the four-point and six-point interaction. This relation implies that any two (analytic) self-dual theories which have the same four-point interaction (once the coupling constants of the four-point terms are adjusted appropriately) have also the same six-point vertex. This explains e.g. the equivalence of the BI and BN models up to $O(F_{\mu\nu}^8)$, which might seem to be an accidental coincidence. It also prevents any one-loop effective Euler-Heisenberg Lagrangian to be self-dual beyond the four-point interaction term due to the mismatch of the powers of the fine structure constants at the four-point and six-point vertexes.[14]

F. Normal ordered form of implicitly defined self-dual Lagrangians

In the previous subsection we mentioned that, once the normal ordered interaction Lagrangian $\mathcal{L}_{\rm int}^{\rm NO}(\phi^2,\bar{\phi}^2) = L(\phi^2\bar{\phi}^2)$ for the self-dual theory is known, we can at least in principle construct the original Lagrangian using the formula for the general solution of the NGZ condition (3.7), (3.8) with the identification

$$p(u) = \frac{1+4uL'(u^2)}{1-4uL'(u^2)}, \qquad F(u) = u[1-16u^2L'(u^2)^2]. \tag{6.86}$$

Quite remarkably, this relation works also in the reversed direction. Suppose e.g. that the solution of the NGZ identity is given by Eqs. (3.10) and (3.11) with the known function $f(z)$ and let us derive the normal ordered Lagrangian directly from this function. Let us rewrite the first equation of (6.86) as

$$4uL'(u^2) = \frac{p-1}{p+1}, \tag{6.87}$$

and suppose it can be solved in order to express u as a function of p. Using now the identification $F(u) = f(p(u))$ [see (3.8) and (3.9)], the second relation of (6.86) can be rewritten in terms of the variable p

$$f(p) = u(p)\left[1-\left(\frac{p-1}{p+1}\right)^2\right]. \tag{6.88}$$

The above solution $u(p)$ has to be therefore given by

$$u(p) = f(p)\frac{(p+1)^2}{4p}. \tag{6.89}$$

Inserting this back into (6.87) and multiplying by $u'(p)$ given explicitly by (6.89) we get

$$2u'(p)u(p)L'(u(p)^2) = \frac{1}{2}u'(p)\frac{p-1}{p+1} \tag{6.90}$$

[14]A similar observation was made already in [33], where the matching of the BI and various Euler Heisenberg Lagrangians was discussed. Note also that in the case of supersymmetric QED, it was shown in [34] that the corresponding Euler-Heisenberg Lagrangian conserves helicity for the four-point amplitude and, moreover, not only the Euler-Heisenberg Lagrangian itself but also the complete one-loop effective action vanishes for the self-dual configurations $F_{\mu\nu} = \pm \mathrm{i}\tilde{F}_{\mu\nu}$ as a consequence of supersymmetry.

where the right-hand side is now known. Finally, up to an inessential constant, we get

$$L(\phi^2\bar{\phi}^2) = \frac{1}{2}\int \mathrm{d}p\,\frac{p-1}{p+1}\frac{\mathrm{d}}{\mathrm{d}p}\left[f(p)\frac{(p+1)^2}{4p}\right]\Bigg|_{p=p(\phi^2\bar{\phi}^2)} \tag{6.91}$$

where $p(\phi^2\bar{\phi}^2)$ is a solution of (6.89) written in the form

$$\sqrt{\phi^2\bar{\phi}^2} = f(p)\frac{(p+1)^2}{4p}, \tag{6.92}$$

with respect to p. Of course, to get the normal ordered interaction Lagrangian in a closed form we have to be able to solve the latter equation explicitly.

Let us give a simple example of the application of this general prescription. Take a solution of the NGZ condition in the form (3.10) and (3.11) with

$$f(z) = 4\Lambda^4\sqrt{z}\frac{1-z}{(1+z)^2}, \tag{6.93}$$

which satisfies the analyticity condition (3.13). Note, however, that the closed form of this solution is not accessible since Eq. (3.10) is the eight order polynomial equation for z. Inserting this function into (6.91) we get

$$L(\phi^2\bar{\phi}^2) = -\frac{\Lambda^4}{2}\left(\sqrt{p}+\frac{1}{\sqrt{p}}\right)+C, \tag{6.94}$$

where C is an integration constant and p is a solution of

$$\sqrt{\phi^2\bar{\phi}^2} = \Lambda^4\left(\frac{1}{\sqrt{p}}-\sqrt{p}\right) \tag{6.95}$$

or explicitly

$$p(\phi^2\bar{\phi}^2) = 1+\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}\pm\sqrt{\frac{\phi^2\bar{\phi}^2}{\Lambda^8}+\left(\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}\right)^2} = \left[1+\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}\mp\sqrt{\frac{\phi^2\bar{\phi}^2}{\Lambda^8}+\left(\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}\right)^2}\right]^{-1}. \tag{6.96}$$

Finally we get for the normal ordered interaction Lagrangian $\mathcal{L}_{\rm int}^{\rm NO}(\phi^2,\bar{\phi}^2) = L(\phi^2\bar{\phi}^2)$

$$\mathcal{L}_{\mathrm{int}}^{\mathrm{NO}}(\phi^2,\bar{\phi}^2)=\Lambda^4-\frac{\Lambda^4}{2}\sqrt{1+\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}+\sqrt{\frac{\phi^2\bar{\phi}^2}{\Lambda^8}+\left(\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}\right)^2}}-\frac{\Lambda^4}{2}\sqrt{1+\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}-\sqrt{\frac{\phi^2\bar{\phi}^2}{\Lambda^8}+\left(\frac{\phi^2\bar{\phi}^2}{2\Lambda^8}\right)^2}} \tag{6.97}$$

where we adjusted the integration constant to get $L(0)=0$. Therefore, although the original Lagrangian is not known in a closed form, we have enough information on the model e.g. for calculation of the scattering amplitudes using the known normal ordered Lagrangian and the modified Feynman rules.

VII. SUMMARY AND CONCLUSION

In this paper, we presented a general proof of the equivalence of two apparently disconnected aspects of the models of nonlinear quantum electrodynamics, namely the classical duality invariance of the field equations, which is expressed on the Lagrangian level by the NGZ condition, and the helicity conservation of the tree-level amplitudes. We have shown that the tree-level S-matrix is invariant with respect to the $U(1)$ rotational symmetry, which expresses the helicity conservation, if and only if the Lagrangian of the theory satisfies the NGZ conditions. On the level of the traditional Feynman rules, the helicity conservation is a result of subtle cancellations between contributions of different Feynman graphs and as such is far from being manifest. Using a reorganization of the perturbative calculation by means of generalized normal ordering of the Lagrangian and introducing a corresponding modification of the Feynman rules, we have shown that for the self-dual models the helicity conservation can be made manifest on the level of individual Feynman graphs. The general arguments follow two steps: first we have proved that the normal ordered Lagrangian is invariant with respect to the $U(1)$ rotational symmetry if and only if the NGZ identity for the original Lagrangian is satisfied and then we have shown that the modified Feynman rules manifestly respect this symmetry. This allows us to enlarge the above statement on helicity conservation also to higher loops.

The transformation leading from the original Lagrangian to the normal ordered one and vice versa can be reformulated as a calculation of the tree-level functional integral over auxiliary fields, i.e. as a substitution of solutions of the classical equation of motions, which become algebraic (generally transcendental), into a generating Lagrangian. This enables us to identify the normal ordered Lagrangian with the off-shell $U(1)$ invariant interaction part of the auxiliary field Lagrangian developed by Ivanov and Zupnik [12,13] (and with its equivalent within the approach of Carrasco, Kallosh, and Roiban in [11]). This gives the latter constructions of the self-dual Lagrangians a clear physical interpretation.

We have also discussed several aspects of the generalized normal ordering. Namely we gave a general formula for the coefficients of the weak field expansion of the original Lagrangian of the self-dual theory provided the normal ordered Lagrangian is known and we also find the general prescription for the normal ordered Lagrangian derived form the implicit representation of the general solution of the NGZ condition.

As an illustration of the above concepts we have calculated two explicit examples. Namely, as the first one we have found the normal ordered form of the BI Lagrangian and recovered in this way the hypergeometric form of this theory presented in [23,24]. As the second example we gave two new representations of the BN model. The first one corresponds to the implicit construction of the general solution of the GNZ condition for which we found the generating function $f^{BN}(z)$. As the second one we calculated explicitly the Lagrangian of the BN model in a form of weak field expansion with explicitly known coefficients.

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