Hadrons of $\mathcal{N}=2$ supersymmetric QCD in four dimensions from little string theory

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It was recently shown that non-Abelian vortex strings supported in a version of four-dimensional $\mathcal{N}=2$ supersymmetric QCD become critical superstrings. In addition to four translational moduli, non-Abelian strings under consideration have six orientational and size moduli. Together they form a ten-dimensional target space required for a superstring to be critical, namely, the product of the flat four-dimensional space and conifold—a noncompact Calabi-Yau threefold. In this paper we report on further studies of low-lying closed string states that emerge in four dimensions and identify them as hadrons of our four-dimensional $\mathcal{N}=2$ QCD. We use the approach based on little string theory, describing the critical string on the conifold as a noncritical $c=1$ string with the Liouville field and a compact scalar at the self-dual radius. In addition to the massless hypermultiplet found earlier we observe several massive vector multiplets and a massive spin-2 multiplet, all belonging to the long (non-BPS) representations of $\mathcal{N}=2$ supersymmetry in four dimensions. All the above states are interpreted as baryons formed by a closed string with confined monopoles attached. Our construction presents an example of a "reverse holography."

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I. INTRODUCTION

In this paper we continue studying the spectrum of four-dimensional "hadrons" formed by the closed critical string [1], which in turn can be obtained from a solitonic vortex string under an appropriate choice of the coupling constant [2]. One of our main tasks is to analyze the structure of the four-dimensional (4D) supermultiplets emerging from quantization of the closed string mentioned above. We start though from a brief review of the setup.

The problem of understanding confining gauge theories splits into two different equally fundamental tasks. The first one is to understand the physical nature of confinement and describe the formation of confining strings. There was a great progress in this direction in supersymmetric gauge theories due to the breakthrough papers by Seiberg and Witten [3,4], in which the monopole condensation was shown to occur in the monopole vacua of $\mathcal{N}=2$ supersymmetric QCD (SQCD). This leads to the formation of Abelian Abrikosov-Nielsen-Olesen (ANO) vortices [5], which confine color electric charges. Attempts to find a non-Abelian generalization of this mechanism led to the discovery of the so-called "instead-of-confinement" phase that occurs in the quark vacua of $\mathcal{N}=2$ SQCD; see [6] for a review. In this phase the (s)quarks condense while the monopoles are confined.

Once the nature of the confining string is understood the second task is to quantize this string in 4D theory outside the critical dimension to study the hadron spectrum. Most solitonic strings, such as the ANO strings, have a finite thickness manifesting itself in the presence of an infinite series of unknown higher-derivative corrections in the effective sigma model on the string world sheet. This makes the task of quantizing such a string virtually impossible.

Recent advances in this direction [2] demonstrated that the non-Abelian solitonic vortex in a particular version of 4D $\mathcal{N}=2$ SQCD becomes a critical superstring. This particular 4D SQCD has the U(2) gauge group, four quark flavors, and the Fayet-Iliopoulos (FI) [7] parameter ξ.

Non-Abelian vortices were first discovered in $\mathcal{N}=2$ SQCD with the U(N) gauge group and $N_f \geq N$ flavors of quark hypermultiplets [8–11]. In addition to four translational moduli characteristic of the ANO strings [5], the non-Abelian strings carry orientational moduli, as well as the size moduli if $N_f > N$ [8–11] (see [12–15] for reviews).

If $N_f > N$ their dynamics are described by the effective two-dimensional sigma model on the string world sheet with the target space

$$\mathcal{O}(-1)^{\oplus(N_f-N)}_{\mathbb{CP}^1}, \tag{1.1}$$

to which we refer to as the weighted CP model [WCP($N, N_f - N$)].

For $N_f = 2N$ the model becomes conformal. Moreover, for $N = 2$ the dimension of the orientational/size moduli space is 6 and they can be combined with four translational moduli to form a ten-dimensional space required for superstring to become critical.[1]

In this case the target space of the world-sheet two-dimensional (2D) theory on the non-Abelian vortex string is $\mathbb{R}^4 \times Y_6$, where Y_6 is a noncompact six-dimensional Calabi-Yau manifold, the so-called resolved conifold [16,17].

Since the non-Abelian vortex string on the conifold is critical it has a perfectly good UV behavior. This opens the possibility that it can become thin in a certain regime [2]. The string transverse size is given by $1/m$, where m is a typical mass scale of the four-dimensional fields forming the string. The string cannot be thin in a *weakly coupled* 4D theory because at weak coupling $m \sim g\sqrt{T}$ and is always small in the units of $\sqrt{T}$ where T is the tension. Here g is the gauge coupling constant of the 4D $\mathcal{N} = 2$ QCD and T is the string tension.

A conjecture was put forward in [2] that at strong coupling in the vicinity of a critical value of $g_c^2 \sim 1$ the non-Abelian string on the conifold becomes thin, and higher-derivative corrections in the action can be ignored. It is expected that the thin string produces linear Regge trajectories even for small spins [2]. The above conjecture implies[2] that $m(g^2) \to \infty$ at $g^2 \to g_c^2$.

A version of the string-gauge duality for 4D SQCD was proposed [2]: at weak coupling this theory is in the Higgs phase and can be described in terms of (s)quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as string states formed by the non-Abelian vortex string.

The vortices in the U(N) theories under consideration are topologically stable and cannot be broken. Therefore the finite-length strings are closed. Thus, we focus on the closed strings. The goal is to identify the closed string states with the hadrons of 4D $\mathcal{N} = 2$ SQCD.

The first step of this program, namely, identifying massless string states, was carried out in [18,19] using supergravity formalism. In particular, a single matter hypermultiplet associated with the deformation of the complex structure of the conifold was found as the only 4D *massless* mode of the string. Other states arising from the massless ten-dimensional graviton are not dynamical in four dimensions. In particular, the 4D graviton and unwanted vector multiplet associated with deformations of the Kähler form of the conifold are absent. This is due to noncompactness of the Calabi-Yau manifold we deal with and non-normalizability of the corresponding modes over six-dimensional space Y_6.

The next step was done in [1] where a number of massive states of the closed non-Abelian vortex string were found. This step required a change of strategy. The point is that the coupling constant $1/\beta$ of the world-sheet WCP(2,2) is not small. Moreover β tends to 0 once the 4D coupling g^2 approaches the critical value g_c^2 we are interested in. At $\beta \to 0$ the resolved conifold develops a conical singularity. The supergravity approximation does not work for massive states.[3]

To analyze the massive states the little string theories (LST) approach (see [20] for a review) was used in [1]. Namely, we used the equivalence between the critical string on the conifold and noncritical $c = 1$ string, which contains the Liouville field and a compact scalar at the self-dual radius [21,22]. The latter theory [in the mirror Wess-Zumino-Novikov-Witten (WZNW) formulation] can be analyzed by virtue of algebraic methods. This leads to identification of towers of massive states with spin 0 and spin 2 [1].

In this paper we focus on the 4D multiplet structure of the states found earlier in [1,19]. In addition to the massless BPS hypermultiplet associated with deformations of the complex structure of the conifold we identify several massive vector multiplets and a massive spin-2 multiplet, all belonging to long non-BPS representations of $\mathcal{N} = 2$ supersymmetry in four dimensions. We interpret all states we found as baryons formed by a closed string with confined monopoles attached. Note, that the relation between LST and certain partly confined 4D gauge theories was also studied in [23] using AdS/CFT approach. In particular, massive stringy states were discussed.

The paper is organized as follows. In Sec. II we review the description of the non-Abelian vortex as a critical superstring on a conifold and identify the massless string state. In Sec. III we review the LST approach in terms of noncritical $c = 1$ string and the spectrum of massive states. In Sec. IV we introduce 4D supercharges and construct the

[1]The non-Abelian vortex string is $1/2$ Bogomolny–Prasad–Sommerfelfd (BPS) saturated and, therefore, has $\mathcal{N} = (2,2)$ supersymmetry on its world sheet. Thus, we actually deal with a superstring in the case at hand.

[2]At $N_f = 2N$ the beta function of the 4D $\mathcal{N} = 2$ QCD is 0, so the gauge coupling g^2 does not run. Note, however, that conformal invariance in the 4D theory is broken by the FI parameter ξ, which does not run either.

[3]This is in contradistinction to the massless states. For the latter, we can perform computations at large β where the supergravity approximation is valid and then extrapolate to strong coupling. In the sigma-model language massless states correspond to chiral primary operators. They are protected by $\mathcal{N} = (2,2)$ world-sheet supersymmetry and their masses are not lifted by quantum corrections.

massless BPS hypermultiplet. In Sec. V we consider the lowest massive string excitations and show that they form a long vector supermultiplet. Section VI deals with the construction of the $\mathcal{N}=2$ spin-2 stringy supermultiplet. In Sec. VII we discuss linear Regge trajectories, while Sec. VIII summarizes our conclusions. In Appendix A we describe the Becchi–Rouet–Stora–Tyutin (BRST) operator and transitions between different pictures. In Appendix B we review long $\mathcal{N}=2$ supermultiplets in 4D.

II. NON-ABELIAN VORTEX STRING

A. Four-dimensional $\mathcal{N}=2$ SQCD

As was already mentioned non-Abelian vortex strings were first found in 4D $\mathcal{N}=2$ SQCD with the gauge group $\mathrm{U}(N)$ and $N_f \geq N$ flavors (i.e., the quark hypermultiplets) supplemented by the FI D term ξ [8–11]; see e.g., [14] for a detailed review of this theory. Here we just mention that at weak coupling, $g^2 \ll 1$, this theory is in the Higgs phase in which the scalar components of the quark multiplets (squarks) develop vacuum expectation values (VEVs). These VEVs break the $\mathrm{U}(N)$ gauge group Higgsing all gauge bosons. The Higgsed gauge bosons combine with the screened quarks to form long $\mathcal{N}=2$ multiplets with mass $m \sim g\sqrt{\xi}$.

The global flavor $\mathrm{SU}(N_f)$ is broken down to the so-called color-flavor locked group. The resulting global symmetry is

$$\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(N_f - N) \times \mathrm{U}(1)_B; \tag{2.1}$$

see [14] for more details.

The unbroken global $\mathrm{U}(1)_B$ factor above is identified with a baryonic symmetry. Note that what is usually identified as the baryonic U(1) charge is a part of our 4D theory gauge group. Our $\mathrm{U}(1)_B$ is an unbroken by squark VEVs combination of two U(1) symmetries: the first is a subgroup of the flavor $\mathrm{SU}(N_f)$ and the second is the global U(1) subgroup of $\mathrm{U}(N)$ gauge symmetry.

As was already noted, we consider $\mathcal{N}=2$ SQCD in the Higgs phase: N squarks condense. Therefore, non-Abelian vortex strings confine monopoles. In the $\mathcal{N}=2$ 4D theory these strings are 1/2 BPS saturated; hence, their tension is determined exactly by the FI parameter,

$$T = 2\pi\xi. \tag{2.2}$$

However, the monopoles cannot be attached to the string end points. In fact, in the $\mathrm{U}(N)$ theories confined monopoles are junctions of two distinct elementary non-Abelian strings [10,11] (see [14] for a review). As a result, in four-dimensional $\mathcal{N}=2$ SQCD we have monopole-antimonopole mesons in which the monopole and antimonopole are connected by two confining strings. In addition, in the $\mathrm{U}(N)$ gauge theory we can have baryons appearing as a closed necklace configurations of $N \times$ (integer) monopoles [14]. For the U(2) gauge group the lightest baryon presented by such a necklace configuration consists of two monopoles; see Fig. 1.

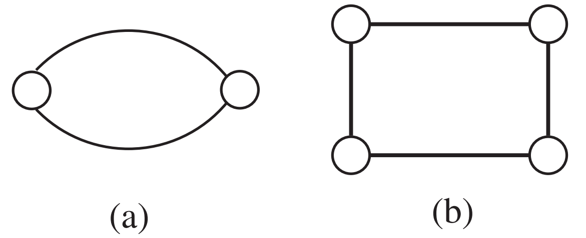

FIG. 1. Examples of the monopole “necklace” baryons: Open circles denote monopoles.

Both stringy monopole-antimonopole mesons and monopole baryons with spins $J \sim 1$ have mass determined by the string tension, $\sim\sqrt{\xi}$ and are heavier at weak coupling than perturbative states, which have mass $m \sim g\sqrt{\xi}$. However, according to our conjecture, at strong coupling near the critical point g_c^2 $m \to \infty$; see [2] and Sec. II C below. In this regime perturbative states decouple and we are left with hadrons formed by the closed string states.[4] All hadrons identified as closed string states in this paper turn out to be baryons and look like monopole necklaces; see Fig. 1.

B. World-sheet sigma model

The presence of color-flavor locked group $\mathrm{SU}(N)_{C+F}$ is the reason for the formation of the non-Abelian vortex strings [8–11] in our 4D SQCD. The most important feature of these vortices is the presence of the so-called orientational zero modes.

Let us briefly review the model emerging on the world sheet of the non-Abelian critical string [2,18,19]. If $N_f = N$ the dynamics of the orientational zero modes of the non-Abelian vortex, which become orientational moduli fields on the world sheet, is described by the two-dimensional $\mathcal{N}=(2,2)$ supersymmetric $\mathrm{CP}(N-1)$ model [14].

If one adds extra quark flavors, non-Abelian vortices become semilocal. They acquire size moduli [24]. In particular, for the non-Abelian semilocal vortex at hand, in addition to the orientational zero modes n^P ($P = 1, 2$), there are the so-called size moduli ρ^K ($K = 1, 2$) [8,11, 24–27]. The target space of the WCP(2,2) sigma model on the string world sheet is defined by the D-term condition

$$|n^P|^2 - |\rho^K|^2 = \beta, \tag{2.3}$$

and a U(1) phase is gauged away.

[4]There are also massless bifundamental quarks, charged with respect to both non-Abelian factors in (2.1). These are associated with the Higgs branch present in 4D QCD; see [14,19] for details.

The total number of real bosonic degrees of freedom (d.o.f.) in this model is 6, where we take into account the constraint (2.3) and the fact that one U(1) phase is gauged away. As was already mentioned, these six internal d.o.f. are combined with four translational moduli to form a ten-dimensional space needed for the superstring to be critical.

At weak coupling the world-sheet coupling constant β in (2.3) is related to the 4D SU(2) gauge coupling as follows: g^2

$$\beta \approx \frac{4\pi}{g^2}; \tag{2.4}$$

see [14]. Note that the first (and the only) coefficient is the same for the 4D SQCD and the world-sheet model β functions. Both vanish at $N_f = 2N$. This ensures that our world-sheet theory is conformal.

Since the non-Abelian vortex string is 1/2 BPS it preserves $\mathcal{N} = (2,2)$ in the world-sheet sigma model, which is necessary to have $\mathcal{N} = 2$ space-time supersymmetry [28,29]. Moreover, as was shown in [19], the string theory of the non-Abelian critical vortex is type IIA.

The global symmetry of the world-sheet sigma model is

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1), \tag{2.5}$$

i.e., exactly the same as the unbroken global group in the 4D theory; cf. (2.1), at $N = 2$ and $N_f = 4$. The fields n and ρ transform in the following representations:

$$n\colon\ (\mathbf{2}, 0, 0), \qquad \rho\colon\ (0, \mathbf{2}, 1). \tag{2.6}$$

C. Thin string regime

The coupling constant of 4D SQCD can be complexified,

$$\tau \equiv i\frac{4\pi}{g^2} + \frac{\theta_{4D}}{2\pi}, \tag{2.7}$$

where θ_{4D} is the four-dimensional θ angle. Note that the SU(N) version of the four-dimensional SQCD at hand possesses a strong-weak coupling duality, namely, $\tau \to -\frac{1}{\tau}$ [30,31]. This suggests that the self-dual point $g^2 = 4\pi$ would be a natural candidate for a critical value g_c^2, where our non-Abelian vortex string becomes thin.[5] However, as shown recently in [32] S-duality maps our U(N) theory to a theory in which a different U(1) subgroup of the flavor group is gauged. In particular, in our U(N) theory all quark flavors have equal charges with respect to the U(1) subgroup of the gauge group, while in the S-dual version only one flavor is charged with respect to the U(1) gauge group. As a result the S-dual version supports a different type of non-Abelian string [32]. This ensures that S-duality is broken in our 4D theory by the choice of U(1) subgroup that is gauged and we do not consider it here.

The two-dimensional coupling constant β can be naturally complexified too if we include the two-dimensional θ term,

$$\beta \to \beta + i\frac{\theta_{2D}}{2\pi}. \tag{2.8}$$

The exact relation between the complexified 4D and 2D couplings is as follows:

$$\exp(-2\pi\beta) = -h(\tau)[h(\tau) + 2], \tag{2.9}$$

where the function $h(\tau)$ is a special modular function of τ defined in terms of the θ-functions,

$$h(\tau) = \theta_1^4/(\theta_2^4 - \theta_1^4).$$

This function enters the Seiberg-Witten curve for our 4D theory [30,31]. The equation (2.9) generalizes the quasi-classical relation (2.4). It can be derived using 2D-4D correspondence: the match of BPS spectra of 4D theory at $\xi = 0$ and the world-sheet theory on the non-Abelian string [10,11,33,34]. Details of this derivation are presented elsewhere.[6]

According to the hypothesis formulated in [2], our critical non-Abelian string becomes infinitely thin at strong coupling in the critical point τ_c (or g_c^2). Moreover, we conjectured in [19] that τ_c corresponds to $\beta = 0$ in the world-sheet theory via relation (2.9). Thus we assume that $m \to \infty$ at $\beta = 0$, which corresponds to $g^2 = g_c^2$ in 4D SQCD.

At the point $\beta = 0$ the non-Abelian string becomes infinitely thin, higher-derivative terms can be neglected and the theory of the non-Abelian string reduces to the WCP(2,2) model. The point $\beta = 0$ is a natural choice because at this point we have a regime change in the 2D sigma model. This is the point where the resolved conifold defined by the D term constraint (2.3) develops a conical singularity [17].

D. Massless 4D baryon as deformation of the conifold complex structure

In this section we briefly review the only 4D massless state associated with the deformation of the conifold complex structure. It was found in [19]. As was already mentioned, all other modes arising from the massless ten-dimensional (10D) graviton have non-normalizable wave functions over the conifold. In particular, the 4D graviton is absent [19]. This result matches our expectations since

[5] We suggested this earlier in [18,19].

[6] Our result (2.9) is different from the one obtained in [32] using localization technique.

from the very beginning we started from $\mathcal{N}=2$ SQCD in the flat four-dimensional space without gravity.

The target space of the world-sheet WCP(2,2) model is defined by the D-term condition (2.3). We can construct the U(1) gauge-invariant "mesonic" variables

$$w^{PK}=n^P\rho^K. \tag{2.10}$$

These variables are subject to the constraint $\det w^{PK}=0$, or

$$\sum_{\alpha=1}^{4}w_\alpha^2=0, \tag{2.11}$$

where

$$w^{PK}\equiv\sigma_\alpha^{PK}w_\alpha,$$

and the σ matrices above are $(1,-i\tau^a)$, $a=1,\ 2,\ 3$. Equation (2.11) defines the conifold Y_6. It has the Kähler Ricci-flat metric and represents a noncompact Calabi-Yau manifold [16,17,35]. It is a cone that can be parametrized by the noncompact radial coordinate

$$\tilde r^2=\sum_{\alpha=1}^{4}|w_\alpha|^2 \tag{2.12}$$

and five angles; see [16]. Its section at fixed $\tilde r$ is $S_2\times S_3$.

At $\beta=0$ the conifold develops a conical singularity, so both S_2 and S_3 can shrink to 0. The conifold singularity can be smoothed out in two distinct ways: by deforming the Kähler form or by deforming the complex structure. The first option is called the resolved conifold and amounts to introducing a nonzero β in (2.3). This resolution preserves the Kähler structure and Ricci flatness of the metric. If we put $\rho^K=0$ in (2.3) we get the $CP(1)$ model with the S_2 target space (with the radius $\sqrt{\beta}$). The resolved conifold has no normalizable zero modes. In particular, the modulus β that becomes a scalar field in four dimensions has non-normalizable wave function over the Y_6 manifold [19].

As explained in [19,36], non-normalizable 4D modes can be interpreted as (frozen) coupling constants in the 4D theory. The β field is the most straightforward example of this, since the 2D coupling β is related to the 4D coupling; see Eq. (2.9).

If $\beta=0$ another option exists, namely a deformation of the complex structure [17]. It preserves the Kähler structure and Ricci flatness of the conifold and is usually referred to as the *deformed conifold*. It is defined by deformation of Eq. (2.11), namely,

$$\sum_{\alpha=1}^{4}w_\alpha^2=b, \tag{2.13}$$

where b is a complex number. Now the S_3 cannot shrink to 0, its minimal size is determined by b.

The modulus b becomes a 4D complex scalar field. The effective action for this field was calculated in [19] using the explicit metric on the deformed conifold [16,37,38],

$$S(b)=T\int d^4x|\partial_\mu b|^2\log\frac{T^2L^4}{|b|}, \tag{2.14}$$

where L is the size of $\mathbb{R}^4$ introduced as an infrared regularization of logarithmically divergent b field norm.[7]

We see that the norm of the b modulus turns out to be logarithmically divergent in the infrared. The modes with the logarithmically divergent norm are at the borderline between normalizable and non-normalizable modes. Usually such states are considered as "localized" on the string. We follow this rule. We can relate this logarithmic behavior to the marginal stability of the b state; see [19].

The field b, being massless, can develop a VEV. Thus, we have a new Higgs branch in 4D $\mathcal{N}=2$ SQCD that is developed only for the critical value of the coupling constant g_c^2.

The logarithmic metric in (2.14) in principle can receive both perturbative and nonperturbative quantum corrections in $1/\beta$, the sigma model coupling. However, in the $\mathcal{N}=2$ theory the nonrenormalization theorem of [31] forbids the dependence of the Higgs branch metric on the 4D coupling constant g^2. Since the 2D coupling β is related to g^2 we expect that the logarithmic metric in (2.14) will stay intact. This expectation is confirmed in [1].

In [19] the massless state b was interpreted as a baryon of 4D $\mathcal{N}=2$ SQCD. Let us explain this. From Eq. (2.13) we see that the complex parameter b (which is promoted to a 4D scalar field) is singlet with respect to both SU(2) factors in (2.5), i.e., the global world-sheet group.[8] What about its baryonic charge?

Since

$$w_\alpha=\frac{1}{2}\mathrm{Tr}[(\bar\sigma_\alpha)_{KP}n^P\rho^K] \tag{2.15}$$

we see that the b state transforms as

$$(1,1,2), \tag{2.16}$$

where we used (2.6) and (2.13). Three numbers above refer to the representations of (2.5). In particular it has the baryon charge $Q_B(b)=2$.

To conclude this section let us note that in our case of type IIA superstring the complex scalar associated with

[7] The infrared regularization on the conifold $\tilde r_{\max}$ translates into the size L of the 4D space because the variables ρ in (2.12) have an interpretation of the vortex string sizes, $\tilde r_{\max}\sim TL^2$.

[8] Which is isomorphic to the 4D global group (2.1) at $N=2$, $N_f=4$.

deformations of the complex structure of the Calabi-Yau space enters as a component of a massless 4D $\mathcal{N}=2$ hypermultiplet; see [39] for a review. Instead, for type IIB superstring it would be a component of a vector BPS multiplet. Nonvanishing baryonic charge of the b state confirms our conclusion that the string under consideration is a type IIA.

III. MASSIVE STATES FROM NONCRITICAL $c=1$ STRING

As was explained in Sec. I, the critical string theory on the conifold is hard to use for calculating the spectrum of massive string modes because the supergravity approximation does not work. In this section we review the results obtained in [1] based on the LST approach. Namely, in [1] we used the equivalent formulation of our theory as a noncritical $c=1$ string theory with the Liouville field and a compact scalar at the self-dual radius [21,22]. We intend to use the same formulation in this paper to analyze the 4D hypermultiplet structure of the massive states.

A. Noncritical $c=1$ string theory

Noncritical $c=1$ string theory is formulated on the target space

$$\mathbb{R}^4 \times \mathbb{R}_\phi \times S^1, \tag{3.1}$$

where $\mathbb{R}_\phi$ is a real line associated with the Liouville field ϕ and the theory has a linear in ϕ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{Q}{2}\phi}. \tag{3.2}$$

We determine Q in Eq. (3.7).

Generically the above equivalence is formulated in the so-called double scaling limit between the critical string on noncompact Calabi-Yau spaces with an isolated singularity on the one hand, and noncritical $c=1$ string with the additional Ginzburg-Landau $\mathcal{N}=2$ superconformal system [21], on the other hand. Following [21] we assume the double scaling limit when the string coupling constant of the conifold theory g_{con} and the deformation parameter of the conifold b simultaneously go to zero with the combination $b^{\frac{Q^2}{2}}/g_{con}$ fixed. In this limit non-trivial physics is localized near the singularity of the Calabi–Yau manifold. In the conifold case this extra Ginzburg–Landau factor in (3.1) is absent [40].

In [21,40,41] it was argued that noncritical string theories with the string coupling exponentially falling off at $\phi \to \infty$ are holographic. The string coupling goes to 0 in the bulk of the space-time and nontrivial dynamics (LST)[9] is localized on the "boundary." In our case the boundary is the four-dimensional space in which $\mathcal{N}=2$ SQCD is defined. (It is worth emphasizing that in our case the boundary 4D dynamics is the starting point while the extra six dimensions represent an auxiliary mathematical construct. Perhaps it can be referred to as a reverse holography.)

In other words, holography for our non-Abelian vortex string theory is most welcome and expected. We start with $\mathcal{N}=2$ SQCD in 4D space and study solitonic vortex strings. In our approach 10D space formed by 4D actual space and six internal moduli of the string is an artificial construction needed to formulate the string theory of a special non-Abelian vortex. Clearly we expect that all nontrivial actual physics should be localized exclusively on the 4D "boundary." In other words, we expect that LST in our case is nothing but 4D $\mathcal{N}=2$ SQCD at the critical value of the gauge coupling g_c^2 (in the hadronic description).

The linear dilaton in (3.2) implies that the bosonic stress tensor of $c=1$ matter coupled to 2D gravity is

$$T_{--} = -\frac{1}{2}[(\partial_-\phi)^2 + Q\partial_-^2\phi + (\partial_-Y)^2], \tag{3.3}$$

where $\partial_- = \partial_z$ The compact scalar Y represents $c=1$ matter and satisfies the following condition:

$$Y \sim Y + 2\pi Q. \tag{3.4}$$

Here we normalize the scalar fields in such a way that their propagators are

$$\langle\phi(z),\phi(0)\rangle = -\log z\bar{z}, \quad \langle Y(z), Y(0)\rangle = -\log z\bar{z}. \tag{3.5}$$

The central charge of the supersymmetrized $c=1$ theory above is

$$c^{\mathrm{SUSY}}_{\phi+Y} = 3 + 3Q^2. \tag{3.6}$$

The criticality condition for the string on (3.1) implies that this central charge should be equal to 9. This gives

$$Q = \sqrt{2}, \tag{3.7}$$

to be used in Eq. (3.2).

Deformation of the conifold (2.13) translates into adding the Liouville interaction to the world-sheet sigma model [21],

$$\delta L = b\int d^2\theta e^{-\frac{\phi+iY}{Q}}. \tag{3.8}$$

The conifold singularity at $b=0$ corresponds to the string coupling constant becoming infinitely large at $\phi \to -\infty$; see (3.2). At $b \neq 0$ the Liouville interaction regularizes the

[9] The main example of this behavior is nongravitational LST in the flat six-dimensional space formed by the world volume of parallel NS5 branes.

behavior of the string coupling preventing the string from propagating to the region of large negative ϕ.

In fact the $c=1$ noncritical string theory can also be described in terms of the two-dimensional black hole [42], which is the $\mathrm{SL}(2,R)/\mathrm{U}(1)$ coset WZNW theory [21,22,43,44] at level

$$k=\frac{2}{Q^2}. \tag{3.9}$$

In [45] it was shown that $\mathcal{N}=(2,2)$ $\mathrm{SL}(2,R)/\mathrm{U}(1)$ coset is a mirror description of the $c=1$ Liouville theory. The relation above implies in the case of the conifold ($Q=\sqrt{2}$) that

$$k=1, \tag{3.10}$$

where k is the total level of the Kač-Moody algebra in the supersymmetric version (the level of the bosonic part of the algebra is then $k_b=k+2=3$). The target space of this theory has the form of a semi-infinite cigar; the field ϕ associated with the motion along the cigar cannot take large negative values due to semi-infinite geometry. In this description the string coupling constant at the tip of the cigar is $g_s\sim 1/b$. In fact as was argued in [21] in the noncritical string theory by itself the parameter b does not have to be small. If we following [21] take b large the string coupling at the tip of the cigar will be small and the string perturbation theory becomes reliable, cf. [21,23]. In particular, we can use the tree-level approximation to obtain the spring spectrum. Note also that, as we already mentioned in the Introduction, the $\mathrm{SL}(2,R)/\mathrm{U}(1)$ WZNW model is exactly solvable.

In terms of 4D SQCD taking b large means moving along the Higgs branch far away from the origin.

B. Vertex operators

Vertex operators for the string theory on (3.1) are constructed in [21]; see also [40,43]. Primaries of the $c=1$ part for large positive ϕ (where the target space becomes a cylinder $\mathbb{R}_\phi\times S^1$) take the form

$$V^L_{j,m_L}\times V^R_{j,m_R}\approx \exp\left(\sqrt{2}j\phi+i\sqrt{2}(m_LY_L+m_RY_R)\right), \tag{3.11}$$

where we split ϕ and Y into left and right-moving parts, say $\phi=\phi_L+\phi_R$. For the self-dual radius (3.7) (or $k=1$) the parameter $2m$ in Eq. (3.11) is integer. For the left-moving sector $2m_L\equiv 2m$ is the total momentum plus the winding number along the compact dimension Y. For the right-moving sector we introduce $2m_R$, which is the winding number minus momentum. We see below that for our case type IIA string $m_R=-m$, while for type IIB string $m_R=m$.

The primary operator (3.11) is related to the wave function over "extra dimensions" as follows:

$$V_{j,m}=g_s\Psi_{j,m}(\phi,Y).$$

The string coupling (3.2) depends on ϕ. Thus,

$$\Psi_{j,m}(\phi,Y)\sim e^{\sqrt{2}(j+\frac{1}{2})\phi+i\sqrt{2}mY}. \tag{3.12}$$

We look for string states with normalizable wave functions over the extra dimensions, which we interpret as hadrons in 4D $\mathcal{N}=2$ SQCD. The condition for the string states to have normalizable wave functions reduces to[10]

$$j\le -\frac{1}{2}. \tag{3.13}$$

The scaling dimension of the primary operator (3.11) is

$$\Delta_{j,m}=m^2-j(j+1). \tag{3.14}$$

Unitarity implies that it should be positive,

$$\Delta_{j,m}>0. \tag{3.15}$$

Moreover, to ensure that conformal dimensions of left and right-moving parts of the vertex operator (3.11) are the same we impose that $m_R=\pm m_L$.

The spectrum of the allowed values of j and m in (3.11) was exactly determined by using the Kač-Moody algebra for the coset $\mathrm{SL}(2,R)/\mathrm{U}(1)$ in [43,46–49]; see [50] for a review. Both discrete and continuous representations were found. Parameters j and m determine the global quadratic Casimir operator and the projection of the spin on the third axis,

$$J^2|j,m\rangle=-j(j+1)|j,m\rangle,\qquad J^3|j,m\rangle=m|j,m\rangle, \tag{3.16}$$

where J^a ($a=1,2,3$) are the global $\mathrm{SL}(2,R)$ currents.

We focus on discrete representations with

$$j=-\frac{1}{2},-1,-\frac{3}{2},\ldots,\qquad m=\pm\{j,j-1,j-2,\ldots\}. \tag{3.17}$$

Discrete representations include the normalizable states localized near the tip of the cigar [see (3.13)], while the

[10]We include the case $j=-\frac{1}{2}$, which is at the borderline between normalizable and non-normalizable states. In [1] it is shown that $j=-\frac{1}{2}$ corresponds to the norm logarithmically divergent in the infrared in much the same way as the norm of the b state; see (2.14).

continuous representations contain non-normalizable states.

Discrete representations contain states with negative norm. To exclude these ghost states a restriction for spin j is imposed [46–50],

$$-\frac{k+2}{2} < j < 0. \tag{3.18}$$

Thus, for our value $k = 1$ we are left with only two allowed values of j,

$$j = -\frac{1}{2}, \qquad m = \pm\left\{\frac{1}{2}, \frac{3}{2}, \ldots\right\} \tag{3.19}$$

and

$$j = -1, \qquad m = \pm\{1, 2, \ldots\}. \tag{3.20}$$

Note that there are also continuous (principal and exceptional) representations of primaries of the $c = 1$ string theory [50], see also a brief review of discrete and continues spectra in [1]. In particular, continuous representation correspond to non-normalizable states in the Liouville direction. Moreover, in [1] we suggested an interpretation of these non-normalizable states: they corresponds to decaying modes of normalizable 4D states. We also confirm this interpretation showing that spectra of continuous states start from thresholds given by masses (3.24) and (3.27) of 4D states (see below). Still we believe that the relation between discrete and continuous states needs future clarification.

C. Scalar and spin-2 states

Four-dimensional spin-0 and spin-2 states were found in [1] using vertex operators [(3.11)]. The 4D scalar vertices V^S in the $(-1,-1)$ picture have the form [21]

$$V^{S,L}_{j,m} \times V^{S,R}_{j,-m}(p_\mu) = e^{-\varphi_L - \varphi_R} e^{ip_\mu x^\mu} V^L_{j,m} \times V^R_{j,-m}, \tag{3.21}$$

where superscript S stands for scalar, $\varphi_{L,R}$ represents the bosonized ghost in the left and right-moving sectors, while p_μ is the 4D momentum of the string state.

The condition for the state (3.21) to be physical is

$$\frac{1}{2} + \frac{p_\mu p^\mu}{8\pi T} + m^2 - j(j+1) = 1, \tag{3.22}$$

where $1/2$ comes from the ghost and we used (3.14). We note that the conformal dimension of the ghost operator $\exp(q\varphi)$ is equal to $-(q + q^2/2)$, where q is the picture number.

The GSO projection restricts the integer $2m$ for the operator in (3.21) to be odd [21,51],[11]

$$m = \frac{1}{2} + \mathbb{Z}. \tag{3.23}$$

For half-integer m we have only one possibility $j = -\frac{1}{2}$; see (3.19). This determines the masses of the 4D scalars,

$$\frac{(M^S_m)^2}{8\pi T} = -\frac{p_\mu p^\mu}{8\pi T} = m^2 - \frac{1}{4}, \tag{3.24}$$

where the Minkowski 4D metric with the diagonal entries $(-1, 1, 1, 1)$ is used.

In particular, the state with $m = \pm 1/2$ is the massless baryon b, associated with deformations of the conifold complex structure [1], while states with $m = \pm(3/2, 5/2, \ldots)$ are massive 4D scalars.

At the next level we consider 4D spin-2 states. The corresponding vertex operators are given by

$$\begin{aligned}(V^L_{j,m} \times V^R_{j,-m}(p_\mu))^{\text{spin-2}} \\ = \xi_{\mu\nu} \psi^\mu_L \psi^\nu_R e^{-\varphi_L - \varphi_R} e^{ip_\mu x^\mu} V^L_{j,m} \times V^R_{j,-m},\end{aligned} \tag{3.25}$$

where $\psi^\mu_{L,R}$ are the world-sheet superpartners to 4D coordinates x^μ, while $\xi_{\mu\nu}$ is the polarization tensor.

The condition for these states to be physical takes the form

$$\frac{p_\mu p^\mu}{8\pi T} + m^2 - j(j+1) = 0. \tag{3.26}$$

The Gliozzi–Scherk–Olive (GSO) projection selects now $2m$ to be even, $|m| = 0, 1, 2, \ldots$ [21]; thus we are left with only one allowed value of j, $j = -1$ in (3.20). Moreover, the value $m = 0$ is excluded. This leads to the following expression for the masses of spin-2 states:

$$(M^{\text{spin-2}}_m)^2 = 8\pi T m^2, \qquad |m| = 1, 2, \ldots. \tag{3.27}$$

We see that all spin-2 states are massive. This confirms the result in [19] that no massless 4D graviton appears in our theory. It also matches the fact that our boundary theory, 4D $\mathcal{N} = 2$ QCD, is defined in flat space without gravity.

To determine baryonic charge of these states we note that $\mathrm{U}(1)_B$ transformation of b in the Liouville interaction (3.8) is compensated by a shift of Y. The baryonic charge of b is 2; see (2.16). Below we use the following convention: upon splitting Y into left and right-moving parts $Y = Y_L + Y_R$ we define that only Y_L is shifted under $\mathrm{U}(1)_B$ transformation,

$$b \to e^{2i\theta} b, \qquad Y_L \to Y_L + 2\sqrt{2}\theta, \qquad Y_R \to Y_R. \tag{3.28}$$

This gives for the baryon charge of the vertex operator (3.11)

$$Q_B = 4m. \tag{3.29}$$

[11] We demonstrate this in the next section.

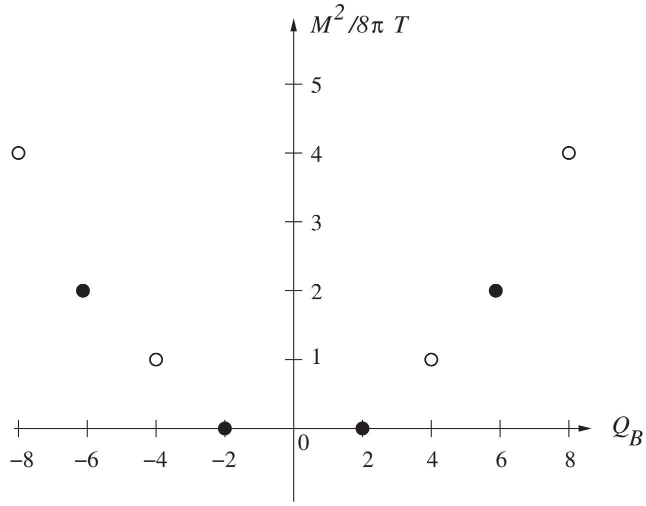

FIG. 2. Spectrum of spin-0 and spin-2 states as a function of the baryonic charge. Closed and open circles denote spin-0 and spin-2 states, respectively.

We see that the momentum m in the compact Y direction is in fact the baryon charge of a string state. All states we found above are baryons. Their masses as a function of the baryon charge are shown in Fig. 2.

The momentum m in the compact dimension is also related to the R-charge. On the world sheet we can introduce the left and right R-charges separately. Normalizing charge of θ^+, namely, $R_L^{(2)}(\theta^+)=1$, we see that Y should be shifted under the $R_L^{(2)}$ symmetry to make invariant the Liouville interaction (3.8).

This gives

$$R_L^{(2)}(V^L_{j,m}) = -2m \tag{3.30}$$

for the $R_L^{(2)}$ charge of the vertex (3.11), which is the bottom component of the world-sheet supermultiplet. The $R_R^{(2)}$ charge in the right-moving sector is defined similarly. Here superscript (2) denotes the world-sheet R-charge.

As was discussed above, the massless baryon b corresponds to $j=-1/2$, $m=\pm 1/2$. Thus, the associated vertex $V_{j,m}$ has $R_L^{(2)}=\pm 1$ and conformal dimension $\Delta = 1/2$; see (3.14). Therefore it satisfies the relation

$$\Delta = \frac{|R_L^{(2)}|}{2} \tag{3.31}$$

as expected for the bottom component of a chiral primary operator, which defines the short representation of supersymmetry algebra (and similar relation in the right-moving sector). In 4D theory b is a component of a short $\mathcal{N}=2$ BPS multiplet, namely hypermultiplet.

IV. MASSLESS HYPERMULTIPLET

The remainder of this paper is devoted to the study of the supermultiplet structure of the 4D string states described in the previous sections. Our strategy is as follows: we explicitly construct 4D supercharges and use them to generate all components of a given multiplet starting from a scalar or spin-2 representative shown in (3.21) or (3.25). We generate supermultiplets originating from the lowest states with $j=-1/2$, $m=\pm(1/2,3/2)$ and $j=-1$, $m=\pm 1$. In this section we start with the massless baryon b.

A. 4D supercharges

First we bosonize world-sheet fermions ψ_μ, ψ_ϕ, and ψ_Y, the superpartners of x_μ, the Liouville field ϕ, and the compact scalar Y, respectively. Following the standard rule we divide them into pairs

$$\psi_k = \frac{1}{\sqrt{2}}(\psi_{2k-1} - i\psi_{2k}),$$
$$\bar\psi_k = \frac{1}{\sqrt{2}}(\bar\psi_{2k-1} + i\bar\psi_{2k}), \qquad k=1,2, \tag{4.1}$$

$$\psi = \frac{1}{\sqrt{2}}(\psi_\phi - i\psi_Y), \qquad \bar\psi = \frac{1}{\sqrt{2}}(\bar\psi_\phi + i\bar\psi_Y), \tag{4.2}$$

and define

$$\psi_k\bar\psi_k = i\partial\_H_k \qquad (\text{no summation}), \qquad \psi\bar\psi = i\partial\_H, \tag{4.3}$$

where the bosons H_k and H have the standard propagators

$$\langle H_k(z), H_l(0)\rangle = -\delta_{kl}\log z, \qquad \langle H(z), H(0)\rangle = -\log z \tag{4.4}$$

and

$$\psi_k \sim e^{iH_k}, \qquad \psi \sim e^{iH}. \tag{4.5}$$

The above formulas are written for the left-moving sector. In the right-moving sector bosonization is similar with the replacement $z\to\bar z$ and $\partial_z \to \partial_{\bar z}$.

As usual, we define spinors in terms of scalars H. Namely,

$$S_\alpha = e^{\sum_k is_k H_k}, \qquad \bar S_{\dot\alpha} = e^{\sum_k i\bar s_k H_k} \tag{4.6}$$

are 4D spinors, $\alpha = 1,2$, $\dot\alpha = 1,2$. Moreover,

$$S = e^{i\frac{H}{2}}, \qquad \bar S = e^{-i\frac{H}{2}} \tag{4.7}$$

are spinors associated with "extra" dimensions ϕ and Y. Here $s_k = \pm\frac{1}{2}$, $k=1,2$ and the choices of the allowed values of s_k are restricted by the GSO projection; see below.

Supercharges for noncritical string are defined in [51]. In our case four 4D $\mathcal{N}=1$ supercharges

$$Q_\alpha = \frac{1}{2\pi i}\frac{\bar{b}}{|b|}\int dz e^{-\frac{\varphi}{2}} S_\alpha S \exp\left(\frac{i}{\sqrt{2}}Y\right),$$
$$\bar{Q}_{\dot\alpha} = \frac{1}{2\pi i}\frac{b}{|b|}\int dz e^{-\frac{\varphi}{2}} \bar{S}_{\dot\alpha}\bar{S} \exp\left(-\frac{i}{\sqrt{2}}Y\right) \quad (4.8)$$

act in the left-moving sector, where we used the $(-\frac{1}{2})$ picture. We have to multiply these supercharges in the left-moving sector by the phase factors $\bar{b}/|b|$ and $b/|b|$ to make them neutral with respect to baryonic $U(1)_B$. Other four supercharges of $\mathcal{N}=2$ 4D supersymmetry are given by similar formulas and act in the right-moving sector. The action of the supercharge on a vertex is understood as an integral around the location of the vertex on the world sheet.

Supercharges (4.8) satisfy 4D space-time supersymmetry algebra

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2P_\mu \sigma^\mu, \quad (4.9)$$

while all other anticommutators vanish. Note that P_μ is the 4D momentum operator; the anticommutator (4.9) does not produce translation in the Liouville direction.

The GSO projection is the requirement of locality of a given vertex operator with respect to the supercharges (4.8).

Let us start with Q_α with $s_k^{(0)} = (1/2, 1/2)$. Then mutual locality of the supercharges (4.8) selects polarizations

$$s_k = \pm\left(\frac{1}{2}, \frac{1}{2}\right), \qquad \bar{s}_k = \pm\left(\frac{1}{2}, -\frac{1}{2}\right) \quad (4.10)$$

associated with four supercharges Q_α and $\bar{Q}_{\dot\alpha}$.

As an example, let us check the GSO selection rule (3.23) for 10D "tachyon" vertices (3.21). We have

$$\langle Q_\alpha, V^{S,L}_{jm}(w)\rangle \sim \int dz\{(z-w)^{-(\frac{1}{2}-m)} + \cdots\}, \quad (4.11)$$

where dots stand for less singular operator product expansion (OPE) terms and $1/2$ comes from the ghost φ. We see that locality requirement selects half-integer m as shown in (3.23). Note that an important feature of the supercharges (4.8) is the dependence on momentum m in the compact direction Y. Without this dependence all 10D tachyon vertices (3.21) would be projected out as it happens for critical strings. Note also that none of the states (3.21) are tachyonic in 4D.

Now we can introduce 4D space-time R-charges. We normalize them as follows:

$$R^{(4)} = R^{(4)}_L + R^{(4)}_R, \qquad R^{(4)}_L(Q_\alpha) = -1, \qquad R^{(4)}_L(\bar{Q}_{\dot\alpha}) = 1, \quad (4.12)$$

and use the same normalizations for $R^{(4)}_R$. This definition ensures that for a given vertex operator we have

$$R^{(4)}_L = -2m_L, \qquad R^{(4)}_R = -2m_R. \quad (4.13)$$

Note that the scalars H are not shifted upon $R^{(4)}$ rotations, so the world-sheet fermions ψ_k, ψ do not have $R^{(4)}$ charges. This is in contrast with the action of the world-sheet $R^{(2)}$ symmetry.

B. Fermion vertex

To generate a fermion vertex for the b state we apply supercharges (4.8) to the left-moving part of the vertex (3.21) with $j=-1/2$ and $m=\pm 1/2$. To get the fermion vertex in the standard $(-1/2)$ picture we have to convert the vertex (3.21) from the (-1) to (0) picture. This is done in Appendix A using the BRST operator. The left-moving part of the scalar vertex (3.21) in the (0) picture has the form

$$V^{(0)}_{j,m}(p_\mu) = \left[\sqrt{2}(j\psi_\phi + im\psi_Y) + \frac{i}{\sqrt{4\pi T}}p_\mu\psi^\mu\right] \times e^{ip_\mu x^\mu + \sqrt{2}j\phi + i\sqrt{2}mY}, \quad (4.14)$$

where we skip the subscripts L.

Let us start with $j=-1/2$ and $m=1/2$. The vertex (4.14) reduces to

$$V^{(0)}_{-\frac{1}{2},m=\frac{1}{2}}(p_\mu) = \left[-\psi + \frac{i}{\sqrt{4\pi T}}p_\mu\psi^\mu\right] e^{ip_\mu x^\mu - \frac{\phi}{\sqrt{2}} + i\frac{Y}{\sqrt{2}}}. \quad (4.15)$$

Applying the supercharge Q_α we find that the correlation function does not contain pole contribution and hence gives 0. On the other hand $\bar{Q}_{\dot\alpha}$ produces the following fermion vertex,

$$\bar{V}^{(-\frac{1}{2})}_{\dot\alpha} = \langle \bar{Q}_{\dot\alpha}, V^{(0)}_{-\frac{1}{2},m=\frac{1}{2}}(p_\mu)\rangle \sim e^{-\frac{\varphi}{2}}\left[-\bar{S}_{\dot\alpha}S + \frac{ip_\mu}{\sqrt{4\pi T}}(\bar\sigma_\mu)_{\dot\alpha\alpha}S^\alpha\bar{S}\right] e^{ip_\mu x^\mu - \frac{\phi}{\sqrt{2}}}, \quad (4.16)$$

where we used

$$\langle\psi(z), \bar{S}(w)\rangle \sim \frac{1}{\sqrt{(z-w)}}S,$$
$$\langle e^{\frac{iY(z)}{\sqrt{2}}}, e^{\frac{-iY(w)}{\sqrt{2}}}\rangle \sim \frac{1}{\sqrt{(z-w)}},$$
$$\langle\psi_\mu(z), \bar{S}(w)_{\dot\alpha}\rangle \sim \frac{1}{\sqrt{(z-w)}}(\bar\sigma_\mu)_{\dot\alpha\alpha}S^\alpha. \quad (4.17)$$

Note that the momentum m along the compact direction is 0 for the fermion vertex (4.16).

As a check we can calculate the conformal dimension of the vertex (4.16). The condition for this vertex to be physical is

$$\frac{3}{8}+\frac{3}{8}+\frac{p_\mu p^\mu}{8\pi T}-j(j+1)=1, \tag{4.18}$$

where the first and the second contributions come from the ghost φ and the scalars H_k and H, respectively. We see that for $j=-1/2$ this state is massless, as expected.

By the same token, for $m=-1/2$ we consider the action of the supercharges on the vertex in (4.15) with $\psi\to\bar{\psi}$ and $m=-1/2$. Only the action of Q_α gives nontrivial fermion vertex. We get

$$V^{\alpha,(-\frac{1}{2})}=\langle Q^\alpha, V^{(0)}_{-\frac{1}{2},m=-\frac{1}{2}}(p_\mu)\rangle \sim e^{-\frac{\varphi}{2}}\left[-S^\alpha\bar{S}+\frac{ip_\mu}{\sqrt{4\pi T}}(\sigma_\mu)^{\alpha\dot{\alpha}}\bar{S}_{\dot{\alpha}}S\right]e^{ip_\mu x^\mu-\frac{\phi}{\sqrt{2}}}. \tag{4.19}$$

To conclude this subsection we note that if we apply supercharges to the fermion vertices (4.16) and (4.19) we do not generate new states. For example, acting on (4.16) with Q_α gives (the left-moving part of) the scalar vertex (3.21),

$$\langle Q_\alpha, \bar{V}^{(-\frac{1}{2})}_{\dot{\alpha}}\rangle \sim \frac{p_\mu}{\sqrt{4\pi T}}(\bar{\sigma}_\mu)_{\dot{\alpha}\alpha}V^{S,L}_{-\frac{1}{2},m=\frac{1}{2}} \tag{4.20}$$

in the picture (-1). This result is in full accord with supersymmetry algebra (4.9). Acting with $Q_{\dot{\alpha}}$ produces the scalar vertex (3.21) with $m=-1/2$,

$$\langle Q_{\dot{\alpha}}, \bar{V}^{(-\frac{1}{2})}_{\dot{\beta}}\rangle \sim \varepsilon_{\dot{\alpha}\dot{\beta}}V^{S,L}_{-\frac{1}{2},m=-\frac{1}{2}}. \tag{4.21}$$

C. Building the hypermultiplet

In this section we use the bosonic and fermionic vertices obtained above to construct a hypermultiplet of the massless b states. For simplicity in this section and below we consider only bosonic components of supermultiplets. As was already mentioned, in the case of type IIA superstring we should consider the states with $m_R=-m_L\equiv -m$. We prove this statement below, in this and the subsequent subsections.

In the Neveu–Schwarz/Neveu–Schwarz (NS-NS) sector we have one complex (or two real) scalars (3.21),

$$b=V^{S,L}_{j=-\frac{1}{2},m}\times V^{S,R}_{j=-\frac{1}{2},-m} \tag{4.22}$$

associated with $m=\pm 1/2$.

Since for the scalar states the momentum m is opposite in the left- and right-moving sectors, for the R-R states we get the product of fermion vertices (4.16) and (4.19), namely,

$$V_{\dot{\alpha}\alpha}=\bar{V}^L_{\dot{\alpha}}\times V^R_\alpha, \qquad \bar{V}_{\alpha\dot{\alpha}}=V^L_\alpha\times\bar{V}^R_{\dot{\alpha}}. \tag{4.23}$$

The vertices above define a complex vector C^μ via

$$V_{\dot{\alpha}\alpha}=(\bar{\sigma}_\mu)_{\dot{\alpha}\alpha}C^\mu. \tag{4.24}$$

However, as is usual for the massless R-R string states, the number of physical d.o.f. reduces because the fermion vertices (4.16) and (4.19) satisfy the massless Dirac equations, which translate into the Bianchi identity for the associated form. For 1-form (vector) we have

$$\partial_\mu C_\nu-\partial_\nu C_\mu=0, \tag{4.25}$$

which ensures that the complex vector reduces to a complex scalar,

$$C_\mu=\partial_\mu\tilde{b}. \tag{4.26}$$

Altogether we have two complex scalars, b and $\tilde{b}$, which form the bosonic part of the hypermultiplet. As was already mentioned, deformations of the complex structure of a Calabi-Yau manifold give a massless hypermultiplet for type IIA theory and a massless vector multiplet for type IIB theory. The derivation above shows that our choice $m_R=-m_L$ corresponds to type IIA string.

We stress again that this massless hypermultiplet is a short BPS representation of $\mathcal{N}=2$ supersymmetry algebra in 4D and is characterized by the nonzero baryonic charge $Q_B(b)=\pm 2$.

Let us also note that the four-dimensional space-time $R^{(4)}$ charge of the vertex operator (4.22) vanishes due to cancellation between left- and right-moving sectors; see (4.13). For the vertex (4.23) it is also 0 since both m_L and m_R are 0. Thus we conclude that b and $\tilde{b}$ have the vanishing $R^{(4)}$ charge, as expected for the scalar components of a hypermultiplet.

D. What would we get for type IIB superstring?

Our superstring is of type IIA. This is fixed by derivation of our string theory as a description of non-Abelian vortex in 4D $\mathcal{N}=2$ SQCD; see [19]. In this subsection we "forget" for a short while about this and consider superstring theory on the manifold (3.1) on its own right. Then, as usual in string theory, we have two options for a closed string: type IIA and type IIB. We show below that the type IIB option corresponds to the choice $m_R=m_L$.

For this choice the massless state with $j=-1/2$ is described as follows. In the NS-NS sector we have one complex scalar,

$$a=V^{S,L}_{j=-\frac{1}{2},m}\times V^{S,R}_{j=-\frac{1}{2},m}, \tag{4.27}$$

associated with $m=\pm 1/2$. In the R-R sector we now obtain

$$V_{\alpha\beta} = V^L_\alpha \times V^R_\beta, \qquad \bar{V}_{\dot\alpha\dot\beta} = \bar{V}^L_{\dot\alpha} \times \bar{V}^R_{\dot\beta}. \tag{4.28}$$

Expanding the complex vertex $V_{\alpha\beta}$ in the basis of σ matrices

$$V^\alpha_\beta = F\delta^\beta_\alpha + (\sigma_\mu\bar\sigma_\nu)^\alpha_\beta C^{\mu\nu} \tag{4.29}$$

we get a complex scalar F and a complex 2-form $C_{\mu\nu}$ that can be expressed in terms of a real 2-form, $C_{\mu\nu} = F_{\mu\nu} - iF^*_{\mu\nu}$, where $F_{\mu\nu}$ is real and $F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$. The Dirac equations for the fermion vertices (4.16) and (4.19) imply that F is a constant, while $F_{\mu\nu}$ satisfies the Bianchi identity. This ensures that $F_{\mu\nu}$ can be constructed in terms of a real vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.30}$$

We see that we get a massless $\mathcal{N}=2$ BPS vector multiplet with the bosonic components given by the complex scalar a and the gauge potential A_μ. This is what we expect from deformation of the complex structure of a Calabi-Yau manifold for type IIB string.

Let us note that R charges also match since the $R^{(4)}$ charge of a in (4.27) is $R^{(4)} = \pm 2$ [see (4.13)] while the $R^{(4)}$ charge of (4.28) and A_μ are 0 as expected.

However, if we try to interpret this $\mathcal{N}=2$ vector multiplet as a state of the non-Abelian vortex in $\mathcal{N}=2$ SQCD we get an inconsistency. To see this one can observe that our state has nonzero baryonic charge that cannot be associated with a gauge multiplet. This confirms our conclusion that the string theory for our non-Abelian vortex-string is of IIA type.

V. EXCITED STATE WITH $j = -1/2$

Below we consider the supermultiplet structure of the lowest massive states given by the vertex operators (3.21) and (3.25). In this section we start with the first excited state of the scalar vertex (3.21) with $j = -1/2$ and $m = \pm 3/2$. The mass of this state is

$$\frac{(M_{j=-\frac{1}{2},m=\pm 3/2})^2}{8\pi T} = 2; \tag{5.1}$$

see (3.24).

A. Action of supercharges

The left-moving part of the vertex operator in the (0) picture is given by (4.14). For $m = 3/2$ we obtain

$$V^{(0)}_{-\frac{1}{2},\frac{3}{2}}(p_\mu) = \left[-(2\psi - \bar\psi) + \frac{i}{\sqrt{4\pi T}}p_\mu\psi^\mu\right]e^{ip_\mu x^\mu - \frac{\phi}{\sqrt{2}} + i\frac{3}{\sqrt{2}}Y}. \tag{5.2}$$

In much the same way as for the b state, the supercharge Q acting on the vertex above gives 0 while the supercharge $\bar Q$ produces the following fermion vertex in the picture $(-\frac{1}{2})$:

$$\bar V^{(-\frac{1}{2})}_{\dot\alpha} = \langle \bar Q_{\dot\alpha}, V^{(0)}_{-\frac{1}{2},m=\frac{3}{2}}(p_\mu)\rangle \sim e^{-\frac{\varphi}{2}}\left[-2\bar S_{\dot\alpha}S + \frac{ip_\mu}{\sqrt{4\pi T}}(\bar\sigma_\mu)_{\dot\alpha\alpha}S^\alpha\bar S\right](\partial_- Y + \psi_\phi\psi_Y)e^{ip_\mu x^\mu - \frac{\phi}{\sqrt{2}} + i\sqrt{2}Y}. \tag{5.3}$$

Note that the momentum m along the compact dimension is

$$m = 1$$

for this vertex. It is easy to check that the mass of this fermion is given by (5.1).

In a similar manner, for $m = -3/2$ we use the bosonic vertex (5.2) with $\psi \to \bar\psi$ and $m = -3/2$. Action of supercharge Q gives the following fermion vertex:

$$V^{\alpha,(-\frac{1}{2})} = \langle Q^\alpha, V^{(0)}_{-\frac{1}{2},m=-\frac{1}{2}}(p_\mu)\rangle \sim e^{-\frac{\varphi}{2}}\left[-2S^\alpha\bar S + \frac{ip_\mu}{\sqrt{4\pi T}}(\sigma_\mu)^{\alpha\dot\alpha}\bar S_{\dot\alpha}S\right](\partial_- Y + \psi_\phi\psi_Y)e^{ip_\mu x^\mu - \frac{\phi}{\sqrt{2}} - i\sqrt{2}Y}, \tag{5.4}$$

with $m = -1$.

Now let us apply the supercharges to the fermion vertices (5.3) and (5.4). Action of Q on (5.3) does not produce new states, while $\bar Q$ gives

$$\langle \bar Q_{\dot\alpha}, \bar V^{(-\frac{1}{2})}_{\dot\beta}\rangle \sim \varepsilon_{\dot\alpha\dot\beta}V^{S,\mathrm{excited}}_{m=\frac{1}{2}}, \tag{5.5}$$

where the new excited scalar vertex in the picture (-1) with $m = 1/2$ has the form

$$V^{S,\mathrm{excited}}_{m=\frac{1}{2}} = \left[-2\partial^2_- Y + \frac{ip_\mu}{\sqrt{\pi T}}\psi^\mu\bar\psi\partial_- Y\right]e^{-\varphi}e^{ip_\mu x^\mu - \frac{\phi}{\sqrt{2}} + i\frac{Y}{\sqrt{2}}}. \tag{5.6}$$

The mass of this state is still given by (5.1). Action of supercharge Q on the fermion vertex (5.4) produces the conjugated scalar with $m = -1/2$.

B. Building the massive vector supermultiplet

Now we can use the vertices obtained in the previous subsection to construct supermultiplets at the level (5.1). We have two scalar vertices with $m = \pm 3/2$ and $m = \pm 1/2$, see the left-hand side of (3.21) and (5.6). Using these vertices we can construct the scalar states in the NS-NS sector. Namely, we have one complex scalar,

$$V^{S,L}_{j=-\frac{1}{2},m=\pm\frac{3}{2}} \times V^{S,R}_{j=-\frac{1}{2},m=\mp\frac{3}{2}}, \tag{5.7}$$

formed by the $m=\pm 3/2$ vertices and one complex scalar,

$$V^{S,\mathrm{excited},L}_{j=-\frac{1}{2},m=\pm\frac{1}{2}} \times V^{S,\mathrm{excited},R}_{j=-\frac{1}{2},m=\mp\frac{1}{2}}, \tag{5.8}$$

formed by the $m=\pm 1/2$ vertices (5.6).

Moreover, we have also another two complex scalars,

$$V^{S,L}_{j=-\frac{1}{2},m=\pm\frac{3}{2}} \times V^{S,\mathrm{excited},R}_{j=-\frac{1}{2},m=\mp\frac{1}{2}} \tag{5.9}$$

and

$$V^{S,\mathrm{excited},L}_{j=-\frac{1}{2},m=\pm\frac{1}{2}} \times V^{S,R}_{j=-\frac{1}{2},m=\mp\frac{3}{2}} \tag{5.10}$$

formed by products of two different vertices. Altogether in the NS-NS sector we observe four complex scalars.

In the R-R sector we have

$$V^{\mathrm{excited}}_{\dot\alpha\alpha} = \bar V^{L}_{\dot\alpha} \times V^{R}_{\alpha}, \qquad \bar V^{\mathrm{excited}}_{\alpha\dot\alpha} = V^{L}_{\alpha} \times \bar V^{R}_{\dot\alpha}, \tag{5.11}$$

where now the fermion vertices are given by (5.3) and (5.4). Expanding these vertices in the basis of σ matrices,

$$V^{\mathrm{excited}}_{\dot\alpha\alpha} = (\bar\sigma_\mu)_{\dot\alpha\alpha} B^\mu + (\bar\sigma_\mu \sigma_\nu \bar\sigma_\rho)_{\dot\alpha\alpha} B^{\mu\nu\rho}, \tag{5.12}$$

we arrive at the complex vector field B^μ and the complex 3-form $B^{\mu\nu\rho}$.

In four dimensions the massive 3-form is dual to a massive scalar [52].[12] Generically the rules of dualizing can be summarized as follows [52]. In D dimensions massless p-forms have

$$c^p_{D-2} = \frac{(D-2)!}{p!(D-2-p)!} \tag{5.13}$$

physical d.o.f. Therefore, the rule of dualizing of the massless p-form is

$$p \to (D-2-p). \tag{5.14}$$

In particular, the 3-form in 4D has no d.o.f.

For the massive p forms we have

$$c^p_{D-1} = \frac{(D-1)!}{p!(D-1-p)!} \tag{5.15}$$

physical d.o.f. The rule of dualizing now becomes

$$p \to (D-1-p). \tag{5.16}$$

Thus the massive 3-form in 4D is dual to a massive scalar. Explicitly the duality relation can be written as [52,53]

$$B_{\mu\nu\rho} \sim \varepsilon_{\mu\nu\rho\lambda} \partial^\lambda c. \tag{5.17}$$

We conclude in the R-R sector we obtained one complex scalar c and the complex vector B^μ. Altogether the bosonic part of the supermultiplet with mass (5.1) contains five scalars and a vector, all complex. This is exactly the bosonic content of two real $\mathcal{N}=2$ long massive vector multiplets, each containing five scalars and a vector; see Appendix B,

$$(\mathcal{N}=2)_{\mathrm{vector}} = 1_{\mathrm{vector}} + 5_{\mathrm{scalar}}. \tag{5.18}$$

Let us note that the $\mathcal{N}=2$ massive vector multiplet can be realized as a result of Higgsing of a $U(1)$ massless gauge multiplet containing gauge field and a complex scalar (two real scalars) by VEVs of a hypermultiplet that contains four real scalars. After Higgsing, one scalar is "eaten" by the Higgs mechanism, so we are left with massive vector field and five scalars. The number of d.o.f. in this massive $\mathcal{N}=2$ long vector multiplet is $8=3+5$, where 3 comes from the massive vector.

Summarizing this section we present 4D R charges of the vector multiplet components. Because of cancellation of the R charges of the left and right-moving sectors, the $R^{(4)}$ charges of the R-R states (5.11) and two scalars (5.7), (5.8) of the NS-NS sector vanish; see (4.13). The R-charges of two scalars (5.9) and (5.10) are nonzero, $R^{(4)}=\pm 2$. These are exactly the R-charges of a massive $\mathcal{N}=2$ vector multiplet. This can be easily understood in terms of Higgsing of the massless gauge multiplet by hypermultiplet VEVs. The gauge field and scalars from the hypermultiplet have the zero R charge while the R charges of two scalar superpartners of the gauge field in the massless vector multiplet are indeed characterized by $R^{(4)}=\pm 2$; cf. Sec. IV D.

VI. THE LOWEST $j=-1$ MULTIPLET

In this section we consider the lowest spin-2 supermultiplet produced by the vertex operator (3.25). The mass of the state with $j=-1$ and $m=\pm 1$ is

$$\frac{(M_{j=-1,m=\pm 1})^2}{8\pi T} = 1; \tag{6.1}$$

see (3.27).

We see below that the spin-2 state (3.25) is the highest component of this supermultiplet. To simplify our discussion it is easier to start from a scalar component of this supermultiplet replacing the world-sheet fermions $\psi^{L,R}_\mu$ by

[12] We did not include the 3-form in the expansion (4.24) because in the massless case it contains no physical d.o.f.; see below.

$\psi_\phi^{L,R}$ and $\psi_Y^{L,R}$. Thus, in the left-moving sector we start from the scalar vertex, which, in the picture (-1), has the form

$$V^{(-1)}_{j=-1,m=1} = \psi e^{-\varphi} e^{ip_\mu x^\mu - \sqrt{2}\phi + i\sqrt{2}mY}, \tag{6.2}$$

where we skip the superscripts L, while ψ is given by (4.2) and $m = 1$. For $m = -1$ we use a similar vertex with replacement $\psi \to \bar{\psi}$. The conformal dimension of this vertex is the same as that of the vertex in (3.25), so we have a scalar state with mass (6.1).

A. Action of supercharges

To convert this vertex operator into the picture (0) we use the BRST operator; see Appendix A. Then in the picture (0) we have

$$V^{(0)}_{j=-1,m=1} = \left[\frac{1}{\sqrt{2}}(\partial_-\phi - i\partial_-Y) + \frac{ip_\mu}{\sqrt{4\pi T}}\psi^\mu\psi\right] \times e^{-\varphi} e^{ip_\mu x^\mu - \sqrt{2}\phi + i\sqrt{2}mY} \tag{6.3}$$

for $m = 1$ and a similar vertex with $\psi \to \bar{\psi}$ for $m = -1$.

Now, let us apply the supercharges to generate the fermion vertices. Q acts trivially on (6.3), while $\bar{Q}$ produces the following fermion vertex in the picture $(-1/2)$:

$$\bar{V}^{(-\frac{1}{2})}_{\dot{\alpha}} = \langle \bar{Q}_{\dot{\alpha}}, V^{(0)}_{-1,m=1}(p_\mu)\rangle \sim e^{-\frac{\varphi}{2}}\left[\bar{S}_{\dot{\alpha}}\bar{S} + \frac{p_\mu}{\sqrt{4\pi T}}(\bar{\sigma}_\mu)_{\dot{\alpha}\alpha}S^\alpha S\right](\partial_-Y + \psi_\phi\psi_Y)e^{ip_\mu x^\mu - \sqrt{2}\phi + i\frac{Y}{\sqrt{2}}}. \tag{6.4}$$

This fermion vertex has $m = 1/2$.

In a similar manner applying supercharge Q to the scalar vertex $V^{(0)}_{j=-1,m=-1}$ we get a fermion vertex with $m = -1/2$,

$$(V^\alpha)^{(-\frac{1}{2})} = \langle Q^\alpha, V^{(0)}_{-1,m=-1}(p_\mu)\rangle \sim e^{-\frac{\varphi}{2}}\left[S^\alpha S + \frac{p_\mu}{\sqrt{4\pi T}}(\sigma_\mu)^{\alpha\dot{\alpha}}\bar{S}_{\dot{\alpha}}\bar{S}\right](\partial_-Y + \psi_\phi\psi_Y)e^{ip_\mu x^\mu - \sqrt{2}\phi - i\frac{Y}{\sqrt{2}}}. \tag{6.5}$$

In order to generate new bosonic vertex operators with the same mass (6.1) we apply supercharges to the fermion vertices above. Supercharge Q acting on (6.4) gives the following bosonic vertices in the picture (-1):

$$\langle Q^\alpha, \bar{V}^{\dot{\alpha}}\rangle \sim \sigma^{\alpha\dot{\alpha}}_\mu\left(\psi^\mu + \frac{p^\mu}{\sqrt{4\pi T}}\psi\right)e^{-\varphi}e^{ip_\mu x^\mu - \sqrt{2}\phi + i\sqrt{2}Y} = \sigma^{\alpha\dot{\alpha}}_\mu\left(V^\mu_{j=-1,m=1} + \frac{p^\mu}{\sqrt{4\pi T}}V^{(-1)}_{j=-1,m=1}\right), \tag{6.6}$$

where $V^{(-1)}_{j=-1,m=1}$ is the scalar vertex (6.2), while

$$V^\mu_{j=-1,m=1} = \psi^\mu e^{-\varphi}e^{ip_\mu x^\mu - \sqrt{2}\phi + i\sqrt{2}mY} \tag{6.7}$$

is a new vector vertex operator with $m = 1$. We recognize it as a left-moving part of the spin-2 vertex (3.25). As was mentioned above, we obtained it by applying the supercharges to the scalar vertex (6.2). In a similar way we can generate the complex-conjugated vector $V^\mu_{j=-1,m=-1}$ with $m = -1$ if we apply the supercharge $\bar{Q}$ to the fermion vertex (6.5).

We can also apply the supercharge $\bar{Q}$ to the fermion vertex (6.4). This gives

$$\langle Q^{\dot{\alpha}}, \bar{V}_{\dot{\beta}}\rangle \sim \delta^{\dot{\alpha}}_{\dot{\beta}} V_{j=-1,m=0}, \tag{6.8}$$

where

$$V^{(-1)}_{j=-1,m=0} = \left(\bar{\psi} + \frac{p^\mu}{\sqrt{4\pi T}}\psi_\mu\right)\partial_-Y e^{-\varphi}e^{ip_\mu x^\mu - \sqrt{2}\phi} \tag{6.9}$$

is a new scalar vertex with $m = 0$ and mass (6.1). Similarly, the action of Q on the fermion vertex (6.5) gives a complex-conjugated scalar vertex with the replacement $\bar{\psi} \to \psi$.

Finally, instead of the scalar vertex (6.2) we can start from another scalar vertex,

$$\tilde{V}^{(-1)}_{j=-1,m=1} = \bar{\psi}e^{-\varphi}e^{ip_\mu x^\mu - \sqrt{2}\phi + i\sqrt{2}mY}. \tag{6.10}$$

Note that this vertex is different from the one complex conjugated to (6.2) because here we take $m = 1$. The conjugated to (6.10) expression is obtained by replacing $\bar{\psi} \to \psi$ and taking $m = -1$.

Following the same steps as above in the case of the vertex (6.2) one can show that the action of supercharges on the scalar vertex (6.10) produces the same states that we already obtained from (6.2).

Summarizing, in the bosonic left-moving sector for the $j = -1$ multiplet we find a complex vector vertex (6.7) and three complex scalar vertices,

$$V^{(-1)}_{j=-1,m=\pm 1}, \qquad \tilde{V}^{(-1)}_{j=-1,m=\pm 1}, \qquad V^{(-1)}_{j=-1,m=0}, \tag{6.11}$$

given by (6.2), (6.10), and (6.9), respectively.

B. Building the spin-2 multiplet

Now we use bosonic and fermionic vertex operators from the previous subsection to construct the supermultiplet with $j = -1$ and mass (6.1). Let us start with the R-R sector. In much the same way as for the excited state in Sec. V B we arrive at

$$V^{j=-1}_{\dot{\alpha}\alpha} = \bar{V}^L_{\dot{\alpha}}\left(m=\frac{1}{2}\right) \times V^R_{\alpha}\left(m=-\frac{1}{2}\right),$$
$$\bar{V}^{j=-1}_{\alpha\dot{\alpha}} = V^L_{\alpha}\left(m=-\frac{1}{2}\right) \times \bar{V}^R_{\dot{\alpha}}\left(m=\frac{1}{2}\right), \qquad (6.12)$$

where the fermion vertices are given by (6.4) and (6.5). Expanding $V^{j=-1}_{\dot{\alpha}\alpha}$ and $\bar{V}^{j=-1}_{\alpha\dot{\alpha}}$ as in (5.12) we get a complex vector and a complex 3-form. As was discussed in Sec. V B, the massive 3-form dualizes into a massive scalar. Thus in the R-R sector we get one complex vector and one complex scalar.

Now we pass to the NS-NS sector. The scalar vertices (6.11) give $3\times 3=9$ scalars of the form

$$V^L_i(m\geq 0)\times V^R_j(m\leq 0), \qquad (6.13)$$

where $V_i(m)$, $i=1, 2, 3$, are given by (6.2), (6.10), and (6.9), respectively. Changing the sign of m together with the replacement $\psi\to\bar{\psi}$ gives nine complex-conjugated scalars in addition to those in (6.13).

Combining the vector vertex (6.7) with three scalar vertices (6.11) provides us with six vectors of the form

$$(V^{\mu}_{j=-1,m=1})^L \times V^R_j(m\leq 0),$$
$$V^L_i(m\geq 0)\times (V^{\mu}_{j=-1,m=-1})^R, \qquad i=1,2,3. \qquad (6.14)$$

Again changing the sign of m together with the replacement $\psi\to\bar{\psi}$ gives six complex-conjugated vectors to those.

Finally we can combine two vector vertices (6.7) to produce a tensor,

$$(V^{\mu}_{j=-1,m=1})^L \times (V^{\nu}_{j=-1,m=-1})^R. \qquad (6.15)$$

Changing the sign of m gives a complex-conjugated tensor. In 4D a massive vector has $(D-1)=3$ physical d.o.f. Therefore for the tensor state (6.15) we get

$$3\times 3 = 9 = 5+3+1 \Rightarrow 1_{\text{spin-2}} + 1_{\text{vector}} + 1_{\text{scalar}} \qquad (6.16)$$

massive d.o.f., where we show the expansion of the massive tensor into irreducible representations of $\mathrm{SO}(D-1=3)$. Thus, from the complex tensor (6.15) we obtain one spin-2 state, one vector, and one scalar, all of them complex.

Combining all bosonic states together we get

$$1_{\text{spin-2}} + 8_{\text{vector}} + 11_{\text{scalar}}, \qquad (6.17)$$

where we show the numbers of states with the given spin.

How do they split into 4D $\mathcal{N}=2$ supermultiplets? The long $\mathcal{N}=2$ spin-2 multiplet contains [54]

$$(\mathcal{N}=2)_{\text{spin-2}} = 1_{\text{spin-2}} + 6_{\text{vector}} + 1_{\text{scalar}} \qquad (6.18)$$

bosonic spin states while the long $\mathcal{N}=2$ vector multiplet has

$$(\mathcal{N}=2)_{\text{vector}} = 1_{\text{vector}} + 5_{\text{scalar}} \qquad (6.19)$$

bosonic spin states; see Appendix B and Eq. (5.18).

We conclude that $j=-1$ states with mass (6.1) form

$$(j=-1)\text{states} = 1\times(\mathcal{N}=2)_{\text{spin-2}} + 2\times(\mathcal{N}=2)_{\text{vector}} \qquad (6.20)$$

(one spin-2 and two vector) $\mathcal{N}=2$ long (non-BPS) supermultiplets, all complex.

VII. REGGE TRAJECTORIES

In this section we show that all states we discussed in this paper (shown in Fig. 2) are the lowest states of the corresponding linear Regge trajectories. To construct these Regge trajectories we multiply the vertex operators (3.21) or (3.25) by derivatives of flat 4D coordinates. For example, for the scalar vertices (3.21) we construct a family of vertices

$$\prod_{i=1}^{n} \partial_- x^{\mu_i}\partial_+ x^{\nu_i} e^{-\varphi_L-\varphi_R} e^{ip_\mu x^\mu} V^{S,L}_{j=-\frac{1}{2},m} \times V^{S,R}_{j=-\frac{1}{2},-m}, \qquad (7.1)$$

where n is $n=0,1,2,\ldots$. The hadronic states associated with these vertices have at most spin $2n$. Their mass is

$$\frac{(M_{j=-\frac{1}{2},m})^2}{8\pi T} = m^2 - \frac{1}{4} + n, \qquad n=0,1,2,\ldots \qquad (7.2)$$

We see that mass squared for these states depends linearly on the spin. This linear Regge dependence appears because we use the flat 4D part of the string σ model to construct the Regge trajectories.

A similar construction can be developed for vertices (3.25). Masses of these states are

$$\frac{(M_{j=-1,m})^2}{8\pi T} = m^2 + n, \qquad n=0,1,2,\ldots \qquad (7.3)$$

We have the same linear dependence with the same slope.

VIII. CONCLUSIONS

In [2] we observed that a vortex string supported in $\mathcal{N}=2$ SQCD is critical provided the following conditions are met:

(i) The gauge group of the model considered is U(2).
(ii) The number of flavor hypermultiplets is $N_f = 2N = 4$.

The 4D theory under consideration is not conformal because of the Fayet-Iliopoulos parameter $\xi\neq 0$. However, the gauge coupling β function vanishes; the Fayet-Iliopoulos parameter does not run either.

In addition to four translational zero modes this string exhibits three orientational and three size zero modes. Their geometry is described by a noncompact six-dimensional Calabi-Yau manifold, the so-called resolved conifold Y_6. The target space takes the form $\mathbb{R}^4 \times Y_6$. The emergence of six extra zero modes on the string under consideration makes the target-space model conformal; the overall Virasoro central charge (including the ghost contribution) vanishes. Thus, this string is critical. The phenomenon we observed could be called a reverse holography.

The next question that was natural to address was the quantization of this closed critical string and the derivation of the hadronic spectrum. The present paper completes the work started in [1,18,19]. We calculated the masses of the massive spin-0 and spin-2 states and constructed the 4D supermultiplets to which they belong. Our formulas match the previous result for the massless states.

The massive supermultiplets are shown to be long (non-BPS saturated). We also prove that the above states are the lowest states on the corresponding Regge trajectories, which are linear and parallel.

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APPENDIX A: BRST OPERATOR AND VERTICES IN THE PICTURE (0)

To convert the vertex operator in the picture (-1) into picture (0) we use the BRST operator as follows [55]:

$$V^{(0)} = \langle Q_{\mathrm{BRST}}, \zeta V^{(-1)} \rangle, \tag{A1}$$

where

$$Q_{\mathrm{BRST}} = \frac{1}{2\pi i} \int dz \left[cT^m + \gamma G^m + \frac{1}{2}(cT^{gh} + \gamma G^{gh}) \right]. \tag{A2}$$

Here c and γ are the ghosts of fermionic (b, c) and bosonic (β, γ) systems, respectively, while T^m, G^m and T^{gh}, G^{gh} are the energy momentum tensor and the supercurrent for matter and ghosts. Below we need the explicit expression for the matter supercurrent,

$$G^m = i(\psi^\mu \partial\_x_\mu + \psi_\phi \partial\_\phi + \psi_Y \partial\_Y). \tag{A3}$$

The ghost system (β, γ) can be expressed in terms of fermions η, ζ,

$$\gamma = e^{\varphi}\eta, \qquad \beta = e^{-\varphi}\partial\_\zeta, \tag{A4}$$

where the propagator of η, ζ is normalized as

$$\langle \eta(z), \zeta(0) \rangle = \frac{1}{z}. \tag{A5}$$

To convert the left-moving part of the scalar vertex (3.21) in the picture (-1) into the picture (0) we use the rule (A1). We arrive at the expression (4.14) with the help of (A3).

Similarly, for the $j = -1$ vertex (6.2) we again use (A3) to obtain the vertex operator (6.3) in the picture (0).

APPENDIX B: LONG $\mathcal{N} = 2$ VECTOR AND SPIN-2 MULTIPLETS IN 4D

In this appendix we briefly review construction of $\mathcal{N} = 2$ long massive supermultiplets in four dimensions. For massive states in the rest frame supersymmetry generators $Q^{\alpha f}$ and $\bar{Q}_{f\dot{\alpha}}$ can be viewed as annihilation and creation operators, where $f = 1$, 2 is the index of two $\mathcal{N} = 1$ supersymmetries that constitute $\mathcal{N} = 2$. Assuming that the annihilation operators $Q^{\alpha f}$ produce 0 upon acting on a "ground state" $|a\rangle$ we can generate all states of a given supermultiplet applying to $|a\rangle$ the creation operators $\bar{Q}_{f\dot{\alpha}}$.

For simplicity we consider only the bosonic states in a multiplet. Assuming that $|a\rangle$ is a bosonic state we have six possibilities,

$$\{\bar{Q}_{1\dot{1}}\bar{Q}_{2\dot{1}}, \bar{Q}_{1\dot{1}}\bar{Q}_{1\dot{2}}, \bar{Q}_{1\dot{1}}\bar{Q}_{2\dot{2}}, \bar{Q}_{2\dot{1}}\bar{Q}_{1\dot{2}}, \bar{Q}_{2\dot{1}}\bar{Q}_{2\dot{2}}, \bar{Q}_{1\dot{2}}\bar{Q}_{2\dot{2}}\} \times |a\rangle \tag{B1}$$

at level 2 and only one possibility,

$$\bar{Q}_{1\dot{1}}\bar{Q}_{2\dot{1}}\bar{Q}_{1\dot{2}}\bar{Q}_{2\dot{2}} \times |a\rangle, \tag{B2}$$

at level 4.

First let us construct the long $\mathcal{N} = 2$ massive vector supermultiplet. In this case we choose $|a\rangle$ to be a scalar

TABLE I. Structure of the vector multiplet. We show the numbers of states with the given J_z produced by the action of supercharges at each level and their sum.

J_z	Level 0	Level 2	Level 4	Sum
1	0	1	0	1
0	1	4	1	6
−1	0	1	0	1

TABLE II. Spin-2 multiplet.

J_z	Level 0	Level 2	Level 4	Sum
2	0	1	0	1
1	1	$4+1=5$	1	7
0	1	$4+1+1=6$	1	8
−1	1	$4+1=5$	1	7
−2	0	1	0	1

with spin $J = 0$. The construction is shown in Table I where J_z is the z-projection of spin and level 0 denotes the state $|a\rangle$ itself. Here we used the fact that, say, in Eq. (B1) the product $\bar{Q}_{1\dot{1}}\bar{Q}_{2\dot{1}}$ acting on $|a\rangle$ increases J_z by 1, four product operators of the type $\bar{Q}_{f\dot{1}}\bar{Q}_{g\dot{2}}$ $(f, g = 1, 2)$ do not change J_z, while the product $\bar{Q}_{1\dot{2}}\bar{Q}_{2\dot{2}}$ reduces J_z by 1.

Overall we observe one state with $J_z = 1$, one state with $J_z = -1$, and 6 states with $J_z = 0$. This gives the decomposition (5.18).

Now let us pass to the spin-2 supermultiplet. To this end we take $|a\rangle$ to be a vector state with spin $J = 1$. The resulting structure is shown in Table II. Here, say, $4 + 1 + 1 = 6$ means that at level 2 four states are generated from the state at level 0 in the same row, while two other states come from states at level 0 in the upper or the lower neighboring rows. The last column gives decomposition (6.18).

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