Coupling matter in modified Q gravity

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We present a novel theory of gravity by considering an extension of symmetric teleparallel gravity. This is done by introducing, in the framework of the metric-affine formalism, a new class of theories where the nonmetricity Q is nonminimally coupled to the matter Lagrangian. More specifically, we consider a Lagrangian of the form $L \sim f_1(Q) + f_2(Q)L_M$, where f_1 and f_2 are generic functions of Q, and L_M is the matter Lagrangian. This nonminimal coupling entails the nonconservation of the energy-momentum tensor, and consequently the appearance of an extra force. The formulation of the gravity sector in terms of the Q instead of the curvature may result in subtle improvements of the theory. In the context of nonminimal matter couplings, we are therefore motivated to explore whether the new geometrical formulation in terms of the Q, when implemented also in the matter sector, would allow more universally consistent and viable realizations of the nonminimal coupling. Furthermore, we consider several cosmological applications by presenting the evolution equations and imposing specific functional forms of the functions $f_1(Q)$ and $f_2(Q)$, such as power-law and exponential dependencies of the nonminimal couplings. Cosmological solutions are considered in two general classes of models, and found to feature accelerating expansion at late times.

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I. INTRODUCTION

The discovery of the late-time cosmic accelerated expansion [1,2] has motivated an extensive amount of research on modifications of general relativity (GR) [3–7], as a possible cause of this cosmic speed-up. A plethora of theories have been proposed in the literature, essentially based on specific approaches. For instance, one may tackle the problem with the metric formalism, which consists on setting the Levi-Civita connection and varying the action with respect to the metric, or consider the metric-affine formalism [8], where the metric and the affine connection are regarded as independent variables. Note that the metric $g_{\mu\nu}$ may be thought of as a generalization of the gravitational potential and is used to define notions such as distances, volumes and angles. On the other hand, the affine connection $\Gamma^{\mu}{}_{\alpha\beta}$ defines parallel transport and covariant derivatives.

From a mathematical point of view (but inspired by the desire of obtaining a unified field theory) the first step in going beyond Riemannian geometry was taken by Weyl [9], who extended the notion of parallel transport by considering the possibility that when vectors are transported along a closed path, their lengths, and not only their directions, change. The nonintegrability of the length was used by Weyl to find a geometric interpretation for the electromagnetic field, as well as an elegant way to unify electromagnetism and gravitation. Weyl's theory was generalized by Dirac [10], who proposed the existence of two metrics: the first is the unmeasurable metric ds_E, which changes as a result of the transformations in the standards of length, and a second metric, which is measurable, and

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which is given by the conformally invariant atomic metric ds_A. The two metrics are conformally related, so that $f(x)ds_E = ds_A$, where $f(x)$ can be taken as any function that transforms as $f(x)/\sigma(x)$ under the conformal transformation $g_{\mu\nu} \to \sigma^2 g_{\mu\nu}$. For an introduction to the Weyl-Dirac theory, see [11]. The Weyl geometry can be immediately generalized to include torsion. The corresponding geometric model is called the Weyl-Cartan geometry, and it was studied extensively from both the physical and mathematical points of view [12–15]. For a review of the basic geometric properties and of the physical applications of the Riemann-Cartan and Weyl-Cartan geometries, we refer the reader to [16].

It is a basic result in differential geometry that the general affine connection may always be decomposed into three independent components [17,18], namely,

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\lambda}{}_{\mu\nu}\} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}, \tag{1}$$

where the first term is the Levi-Civita connection of the metric $g_{\mu\nu}$, given by the standard definition

$$\{^{\lambda}{}_{\mu\nu}\} \equiv \frac{1}{2} g^{\lambda\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{2}$$

The second term $K^{\lambda}{}_{\mu\nu}$ is the contortion

$$K^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} T^{\lambda}{}_{\mu\nu} + T_{(\mu}{}^{\lambda}{}_{\nu)}, \tag{3}$$

with the torsion tensor defined as $T^{\lambda}{}_{\mu\nu} \equiv 2\Gamma^{\lambda}{}_{[\mu\nu]}$. The third term is the disformation,

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta}(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}), \tag{4}$$

which is defined in terms of the nonmetricity tensor: $Q_{\rho\mu\nu} \equiv \nabla_\rho g_{\mu\nu}$.

Thus, by making assumptions on the affine connection, one is essentially specifying a metric-affine geometry [19]. For instance, the standard formulation of GR assumes a Levi-Civita connection, which implies vanishing torsion and nonmetricity, while its teleparallel equivalent (TEGR), uses the Weitzenböck connection, implying zero curvature and nonmetricity [20]. A gravitational model in a Weyl-Cartan spacetime, in which the Weitzenböck condition of the vanishing of the sum of the curvature and torsion scalar was considered in [21]. A kinetic term for the torsion was also included in the gravitational action. The field equations of the model can be obtained from a Hilbert-Einstein type variational principle, and they lead to a complete description of the gravitational field in terms of two fields, the Weyl vector and the torsion, respectively, both defined in a curved background. The cosmological applications of the model were investigated for a particular choice of the free parameters in which the torsion vector is proportional to the Weyl vector. In particular, a de Sitter type late time evolution can be naturally obtained from the field equations of the model. The Weitzenböck condition of the exact cancellation of curvature and torsion in a Weyl-Cartan geometry was imposed into the gravitational action via a Lagrange multiplier in [22]. The dynamical variables are the spacetime metric, the Weyl vector and the torsion tensor, respectively. However, once the Weitzenböck condition is imposed on the Weyl-Cartan spacetime, the metric is not dynamical, and the gravitational dynamics and evolution is completely determined by the torsion tensor. The gravitational field equations can be obtained from a variational principle, and they explicitly depend on the Lagrange multiplier. The case of Riemann-Cartan spacetimes with zero nonmetricity that mimics the teleparallel theory of gravity was also considered.

A relatively unexplored territory consists in another equivalent formulation of GR, which is denoted the symmetric teleparallel equivalent of GR (STEGR). Here, one considers a vanishing curvature and torsion, and it is the nonmetricity tensor Q that describes the gravitational interaction. The STEGR was presented in the original brief paper [23], where the authors emphasize that the formulation brings a new perspective to bear on GR, and the gravitational interaction effects, via the nonmetricity, present a character similar to the Newtonian force and are derived from a potential, namely, the metric. However, the formulation is geometric and covariant. The topic was further explored in [24], where the STEGR was represented by the most general quadratic and parity conserving lagrangian with lagrange multipliers for vanishing torsion and curvature. It was shown that the considered lagrangian is equivalent to the Einstein-Hilbert lagrangian for certain values of the coupling coefficients. Furthermore, it was also shown that in the gravitational analogue of the Lorenz gauge [25], the field equations can be written as a system of Proca equations, which may be of interest in the study of propagation of gravitational-electromagnetic waves.

More recently, the symmetric teleparallel theories of gravity were analyzed in [26], where an exceptional class was discovered which is consistent with a vanishing affine connection. In fact, based on this remarkable property, a simpler geometrical formulation of GR was proposed that is oblivious to the affine spacetime structure, thus fundamentally depriving gravity from any inertial character. The resulting theory is described by the Einstein-Hilbert action purged from the boundary term and is more robustly underpinned by the spin-2 field theory. This construction also provides a novel starting point for modified gravity theories, and presents new and simple generalizations where analytical self-accelerating cosmological solutions arise naturally in the early and late-time universe. These topics were further explored in [27], where the linear perturbations in flat space were analyzed, and in [28,29].

A generalization of STEGR was considered in [19], where a nonminimal coupling of a scalar field to the nonmetricity invariant was introduced. The similarities and differences with analogous scalar-curvature and scalar-torsion theories were considered by discussing the field equations, role of connection, conformal transformations, relation to the $f(Q)$ theory, and cosmological applications. This recent class of scalar-nonmetricity theories was extended by considering a five-parameter quadratic non-metricity scalar and including a boundary term [30]. The equivalents for general relativity and ordinary (curvature based) scalar-tensor theories were also obtained as particular cases. These nonminimal couplings motivate us to explore modifications of STEGR by considering a coupling between nonmetricity and the matter Lagrangian, much is the spirit of the case treated in [31–33]. In fact, the nonminimal curvature-matter coupling and generalizations were extensively explored in the literature (see for instance [34–44]), and we refer the reader to [45,46] for recent reviews. The nonminimal torsion-matter coupling was also analyzed in detail [47–51] and a dynamical system analysis was developed in [52].

Thus, the aim of the present paper is to present an extension of the symmetric teleparallel gravity, by introducing a new class of theories where the nonmetricity Q is coupled nonminimally to the matter Lagrangian, in the framework of the metric-affine formalism. This work is outlined in the following manner: In Sec. II, we present and motivate the symmetric teleparallel equivalent of general relativity (STEGR). In Sec. III, we consider an extended STEGR, by coupling a general function of the nonmetricity to the matter Lagrangian. In Sec. IV, we consider some cosmological applications, and we conclude in Sec. V with a summary and some perspectives.

II. COVARIANT EINSTEIN LAGRANGIAN

In 1916, Einstein wrote down [53] a simple Lagrangian formulation for his field equations

$$L_E = g^{\mu\nu}(\{^{\alpha}{}_{\beta\mu}\}\{^{\beta}{}_{\nu\alpha}\} - \{^{\alpha}{}_{\beta\alpha}\}\{^{\beta}{}_{\mu\nu}\}), \tag{5}$$

featuring the Levi-Civita connection written here as the Christoffel symbols of the metric defined in Eq. (2). The more standard Lagrangian formulation first discovered by Hilbert in 1915, given by the metric Ricci scalar $\mathcal{R}$, contains additional terms which involve second derivatives of the metric. In fact, $\mathcal{R} = L_E + L_B$, where the boundary term (a total derivative) is

$$L_B = g^{\alpha\mu}\mathcal{D}_\alpha\{^{\nu}{}_{\mu\nu}\} - g^{\mu\nu}\mathcal{D}_\alpha\{^{\alpha}{}_{\mu\nu}\}, \tag{6}$$

where the $\mathcal{D}_\alpha$ is the covariant derivative with the connection (2). The reason why the higher-derivative formulation has become the standard one is that the Lagrangian (5) is not covariant.

This can be repaired by promoting the partial derivatives of the metric in (2) to covariant ones. We will therefore introduce an independent "Palatini connection" $\Gamma^{\alpha}{}_{\mu\nu}$, with a covariant derivative ∇_α, in order to define the tensor

$$L^{\alpha}{}_{\beta\gamma} = -\frac{1}{2}g^{\alpha\lambda}(\nabla_\gamma g_{\beta\lambda} + \nabla_\beta g_{\lambda\gamma} - \nabla_\lambda g_{\beta\gamma}), \tag{7}$$

which is nothing but the disformation (4) explicitly written. This way, the invariant

$$Q = -g^{\mu\nu}(L^{\alpha}{}_{\beta\mu}L^{\beta}{}_{\nu\alpha} - L^{\alpha}{}_{\beta\alpha}L^{\beta}{}_{\mu\nu}), \tag{8}$$

is by construction equivalent to (minus) the Einstein Lagrangian (5), when the covariant derivative reduces to the partial one, i.e.,

$$\nabla_\alpha \overset{0}{=} \partial_\alpha, \qquad Q \overset{0}{=} -L_E. \tag{9}$$

This gauge choice, denoted with the 0, was called the *coincident gauge* and shown to be consistent in the symmetric teleparallel geometry [26].

Though in this geometry the connection $\Gamma^{\alpha}{}_{\mu\nu}$ has neither curvature nor torsion, the connection (2) and its curvature still play their physical roles. Note that the Dirac Lagrangian, connected with the $\Gamma^{\alpha}{}_{\mu\nu}$ in the symmetric teleparallel geometry, filters out everything but the Christoffel symbols (2) from $\Gamma^{\alpha}{}_{\mu\nu} = \{^{\alpha}{}_{\mu\nu}\} + L^{\alpha}{}_{\mu\nu} \overset{0}{=} 0$. The Q-formulation is thus a pretty subtle improvement of GR, since (minimally coupled) fermions are still connected metrically [28], and whilst the pure gravity sector is now trivially connected, effectively nothing changes but just the higher-derivative boundary term L_B disappears from the action.

III. MATTER COUPLINGS

A. Action and field equations

More substantial distinctions arise in generalizations of the $f(Q)$ gravitational theory. In this work, we consider the action defined by two functions, given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f_1(Q) + f_2(Q)L_M\right], \tag{10}$$

where L_M is a Lagrangian function for the matter fields.

Nonminimal couplings with a function of the $\mathcal{R}$ have been considered extensively [31–33], since they predict very interesting phenomenology. Due to the higher-derivative property of the scalar $\mathcal{R}$, however, these theories are best considered as effective theories which might become problematical at certain limits. As an example, for a density of a canonical scalar field ϕ, the nonminimal coupling of the form $f_2(\mathcal{R})L_\phi$ introduces a kinetic term which does not fit into the viable Horndeski class. Let us point out, however, that such problems are likely to disappear when this coupling is formulated in the metric-affine approach because the field equations remain second order, see e.g., [54].

Therefore the proposal is to reconsider the nonminimal curvature couplings in the framework of Q gravity. Since the scalar invariant Q in Eq. (8) involves no higher derivatives, a coupling $f_2(Q)L_\phi$ results in second-order equations of motion. Thus, the form of the action (10) is inspired by the well-studied theory in the curvature formulation, wherein the two functions f_1 and f_2 have been considered to depend on the metric curvature $\mathcal{R}$ [45,46].

The motivation is to see whether the subtle improvement of the geometrical formulation, when implemented in the matter sector, would allow more universally consistent and viable realizations of the nonminimal curvature-matter coupling theories.

We define the nonmetricity tensor and its two traces as follows:

$$Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu}, \qquad Q_\alpha = Q_\alpha{}^\mu{}_\mu, \qquad \tilde{Q}_\alpha = Q^\mu{}_{\alpha\mu}. \tag{11}$$

It is also useful to introduce the superpotential

$$4P^\alpha{}_{\mu\nu} = -Q^\alpha{}_{\mu\nu} + 2Q_{(\mu}{}^\alpha{}_{\nu)} - Q^\alpha g_{\mu\nu} - \tilde{Q}^\alpha g_{\mu\nu} - \delta^\alpha_{(\mu}Q_{\nu)}, \tag{12}$$

which, by using Eq. (7), can also be written as

$$P^\alpha{}_{\mu\nu} = -\frac{1}{2}L^\alpha{}_{\mu\nu} + \frac{1}{4}(Q^\alpha - \tilde{Q}^\alpha)g_{\mu\nu} - \frac{1}{4}\delta^\alpha_{(\mu}\tilde{Q}_{\nu)}. \tag{13}$$

One can readily check that $Q = -Q_{\alpha\mu\nu}P^{\alpha\mu\nu}$ (with our sign conventions that are the same as in Ref. [26]). For notational simplicity, let us introduce the following definitions

$$f = f_1(Q) + 2f_2(Q)L_M,$$
$$F = f_1'(Q) + 2f_2'(Q)L_M, \tag{14}$$

where primes ($'$) stand for derivatives of the functions with respect to Q. We also specify the following variations

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}}, \tag{15}$$

$$H_\lambda{}^{\mu\nu} = -\frac{1}{2}\frac{\delta(\sqrt{-g}L_M)}{\delta\Gamma^\lambda{}_{\mu\nu}}, \tag{16}$$

as the energy-momentum tensor and the hypermomentum tensor density, respectively.

Varying the action (10) with respect to the metric, one obtains the gravitational field equations given by

$$\frac{2}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}FP^\alpha{}_{\mu\nu}) + \frac{1}{2}g_{\mu\nu}f_1 + F(P_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta} - 2Q_{\alpha\beta\mu}P^{\alpha\beta}{}_\nu) = -f_2T_{\mu\nu}. \tag{17}$$

When varying the action (10) with respect to the connection, there are two possibilities to impose the symmetric teleparallelism. We can either use the "inertial variation" [55] by setting the connection in its pure-gauge form in the action, or we can consider a general connection in the action but supplement it with lagrange multipliers to eliminate the curvature and torsion [29]. Either way, we now obtain

$$\nabla_\mu\nabla_\nu(\sqrt{-g}FP^{\mu\nu}{}_\alpha - f_2H_\alpha{}^{\mu\nu}) = 0. \tag{18}$$

B. Matter coupling when $L = L_E + L_M$

In the case $L = Q + L_M(g, \nabla)$, the canonical energy-momentum tensor of the metric is, by definition,

$$t^\mu{}_\nu \equiv -\frac{1}{2}\left(\frac{\partial Q}{\partial Q_{\mu\alpha\beta}}Q_{\nu\alpha\beta} + \delta^\mu_\nu Q\right) = -P^{\mu\alpha\beta}Q_{\nu\alpha\beta} - \frac{1}{2}\delta^\mu_\nu Q. \tag{19}$$

In addition, we now define the inertial hypermomentum tensor as

$$\tau^\mu{}_\nu \equiv \frac{2}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}P^{\alpha\mu}{}_\nu). \tag{20}$$

In the coincident gauge, the scalar Q in the Lagrangian reduces to the Einstein pseudotensor and the tensor (19) reduces to what is known as the Einstein canonical energy-momentum pseudotensor. The field equation is

$$\tau^\mu{}_\nu - t^\mu{}_\nu = T^\mu{}_\nu, \tag{21}$$

which is equivalent to the field equation of GR. The variation with respect to the connection yields the equation

$$\nabla_\mu(\sqrt{-g}\tau^\mu{}_\nu) = \nabla_\mu\nabla_\alpha(\sqrt{-g}H_\nu{}^{\mu\alpha}), \tag{22}$$

which together with (21) implies the covariant conservation of the energy-momentum tensor, $\mathcal{D}_\mu T^{\mu\nu} = 0$. The covariant conservation relation assumes that the right-hand side of Eq. (22) identically vanishes. Matter is minimally coupled, and the equivalence principle is obeyed. It is crucial to the gauge interpretation of the theory that the minimal coupling prescription is indeed conducted properly in the first-order framework, $\partial_\mu \to \nabla_\mu$, whereas the consistency of the Weitzenböck teleparallelism requires resorting to $\partial_\mu \to \mathcal{D}_\mu$ (see Ref. [29] for discussion, details and references).

In this paper, we study the nonminimally coupled generalizations of the theory introduced above in Sec. III A. When $f_2(Q) \neq 0$, we expect to see breaking of the equivalence principle. This is confirmed below, and we shall derive the explicit form of the ensuing "fifth force" associated with the equivalence principle violation. Such forces could potentially escape detection in the Solar system while being responsible for the dark energy or dark matter in the Universe, which requires that they are attributed with the energy scales comparable to the present Hubble rate. The recent insights into the geometric foundations of gravity [28] provide the fully covariant framework to study the subtle issue of the breaking of the equivalence principle. The energy-momentum tensor (19), which is an impossibility in the more standard description of GR that uses the (pseudo-)Riemannian surface geometry, assumes its canonical expression and the gravitational connection is generated purely by the time and the space translations, which is the natural basis for the connection of the gauge interaction sourced by energy and momentum. This understood, we explore the possible extra forces resulting from a breaking of the translation symmetry.

We would expect that the $f(Q)$ models can avoid ghosts but the issue of the strong coupling of the additional degrees of freedom remains to be clarified [29]. It is a preliminary speculation that this might be alleviated by matter coupling that also breaks the translation invariance, but this issue should be studied elsewhere.

C. The divergence of the energy-momentum tensor

To begin with, let us note that in the symmetric teleparallel geometry, the nonmetricity tensor satisfies the Bianchi identity

$$\nabla_{[\alpha}Q_{\beta]\mu\nu} = 0. \tag{23}$$

We would like to deduce the energy-momentum conservation, i.e., the metric divergence of the tensor $T_{\mu\nu}$ as defined in (15). We denote the purely Riemannian quantities with the curly symbols, and thus the metric covariant derivative with the symbol (2) is written with $\mathcal{D}_\alpha$.

To arrive at a useful form for the divergence of the energy-momentum tensor, as the first step, we raise one index in Eq. (17),

$$\frac{2}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}FP^{\alpha\mu}{}_\nu) + \frac{1}{2}\delta^\mu_\nu f_1 + FP^{\mu\alpha\beta}Q_{\nu\alpha\beta} = -f_2T^\mu{}_\nu, \tag{24}$$

which has, in fact, simplified the equation. Now recall that in symmetric teleparallel geometry the connection is the sum of the metric piece (2) and the disformation (7). Therefore, for example, for an arbitrary vector V^α, we have

$$\nabla_\mu V^\alpha = \mathcal{D}_\mu V^\alpha + L^\alpha{}_{\mu\beta}V^\beta. \tag{25}$$

Applying the same reasoning to a mixed-index tensor density $v^\mu{}_\nu$, taking into account that $L^\alpha{}_{\alpha\mu} = -\frac{1}{2}Q_\mu$, we have

$$\mathcal{D}_\mu v^\mu{}_\nu = \nabla_\mu v^\mu{}_\nu + L^\alpha{}_{\mu\nu}v^\mu{}_\alpha. \tag{26}$$

Now we can consider $v^\mu{}_\nu = \nabla_\alpha(\sqrt{-g}FP^{\alpha\mu}{}_\nu)$ such a mixed-index tensor density. Therefore, we obtain for its metric divergence

$$\mathcal{D}_\mu\nabla_\alpha(\sqrt{-g}FP^{\alpha\mu}{}_\nu) = \nabla_\alpha\nabla_\beta(f_2H_\nu{}^{\alpha\beta}) + L^\alpha{}_{\mu\nu}\nabla_\beta(\sqrt{-g}FP^{\beta\mu}{}_\alpha), \tag{27}$$

where, for the first term, we have exploited the fact that in symmetric teleparallel geometry $[\nabla_\mu, \nabla_\nu] = 0$, and then used the connection equation of motion (18). Taking the divergence of the full field equation (24) and using the above result leads to

$$\begin{aligned}&-\mathcal{D}_\mu(f_2T^\mu{}_\nu) - \frac{2}{\sqrt{-g}}\nabla_\alpha\nabla_\beta(f_2H_\nu{}^{\alpha\beta}) \\ &= L^\lambda{}_{\mu\nu}(2\nabla_\alpha - Q_\alpha)(FP^{\alpha\mu}{}_\lambda) \\ &\quad + \frac{1}{2}\partial_\nu f_1 + \mathcal{D}_\mu(FP^{\mu\alpha\beta}Q_{\nu\alpha\beta}).\end{aligned} \tag{28}$$

We can first separate the $\nabla_\alpha F$-terms by simply using the Leibniz rule on the second and the third line. The two terms we get combine to zero,

$$\begin{aligned}&2(\nabla_\beta F)P^{\beta\mu}{}_\lambda L^\lambda{}_{\mu\alpha} + (\mathcal{D}_\mu F)P^{\mu\nu\lambda}Q_{\alpha\nu\lambda} \\ &= (\nabla_\beta F)P^{\beta\mu\lambda}(2L_{\lambda\mu\alpha} + Q_{\alpha\mu\lambda}) = 0,\end{aligned} \tag{29}$$

where in the first equality, we have just regrouped the terms, and in the second equality noted that $P^{\beta\mu\lambda} = P^{\beta(\mu\lambda)}$. Thus, Eq. (28) takes the following form:

$$\begin{aligned}&-\mathcal{D}_\mu(f_2T^\mu{}_\nu) - \frac{2}{\sqrt{-g}}\nabla_\alpha\nabla_\beta(f_2H_\nu{}^{\alpha\beta}) \\ &= F(2\nabla_\beta P^{\beta\mu}{}_\lambda - P^{\beta\mu}{}_\lambda Q_\beta)L^\lambda{}_{\mu\nu} \\ &\quad + \frac{1}{2}f_{1,\nu} + F\mathcal{D}_\mu(P^{\mu\beta\lambda}Q_{\nu\beta\lambda}).\end{aligned} \tag{30}$$

Next, we rewrite the metric covariant derivative, in analogy with the expressions (25) and (26) and get

$$\begin{aligned}&-\mathcal{D}_\mu(f_2T^\mu{}_\nu) - \frac{2}{\sqrt{-g}}\nabla_\alpha\nabla_\beta(f_2H_\nu{}^{\alpha\beta}) \\ &= F(2\nabla_\beta P^{\beta\mu}{}_\lambda - P^{\beta\mu}{}_\lambda Q_\beta + P^{\mu\alpha\beta}Q_{\lambda\alpha\beta})L^\lambda{}_{\mu\nu} \\ &\quad + \frac{1}{2}(FP^{\mu\alpha\beta}Q_\mu Q_{\nu\alpha\beta} + f_{1,\nu}) + F\nabla_\mu(P^{\mu\beta\lambda}Q_{\nu\beta\lambda}).\end{aligned} \tag{31}$$

We can then easily deal with the two derivative terms. By using again the symmetry $P^{\alpha\mu\nu} = P^{\alpha(\mu\nu)}$ and the identity (23), it is not difficult to see that

$$\begin{aligned}&2(\nabla_\beta P^{\beta\mu}{}_\lambda)L^\lambda{}_{\mu\nu} + \nabla_\mu(P^{\mu\beta\lambda}Q_{\nu\beta\lambda}) \\ &= 2P^{\beta\mu\alpha}Q_{\beta\alpha\lambda}L^\lambda{}_{\mu\nu} + P^{\mu\alpha\beta}(\nabla_\nu Q_{\mu\alpha\beta}).\end{aligned} \tag{32}$$

Furthermore, by a straightforward but tedious calculation using the definition (12), one can show that

$$(\nabla_\alpha P^{\mu\nu\lambda})Q_{\mu\nu\lambda} = P^{\mu\nu\lambda}(\nabla_\alpha Q_{\mu\nu\lambda}) + (4P^{\beta\mu\gamma}Q_{\beta\gamma\lambda} + 2P^{\mu\gamma\beta}Q_{\lambda\gamma\beta} - 2P^{\beta\mu}{}_{\lambda}Q_\beta)L^\lambda{}_{\mu\alpha} - 2P^{\mu\nu\beta}Q_\mu Q_{\nu\alpha\beta}. \quad (33)$$

Using this information in Eq. (31), we then arrive at the final result as follows:

$$\begin{aligned} &-\mathcal{D}_\mu(f_2 T^\mu{}_\nu) - \frac{2}{\sqrt{-g}}\nabla_\alpha\nabla_\beta(f_2 H_\nu{}^{\alpha\beta}) \\ &= F(2P^{\beta\mu\gamma}Q_{\beta\gamma\lambda} - P^{\beta\mu}{}_{\lambda}Q_\beta + P^{\mu\alpha\beta}Q_{\lambda\alpha\beta})L^\lambda{}_{\mu\nu} \\ &\quad + \frac{1}{2}(FP^{\mu\alpha\beta}Q_\mu Q_{\nu\alpha\beta} + f_{,\alpha}) + FP^{\mu\beta\lambda}(\nabla_\nu Q_{\mu\beta\lambda}) \\ &= \frac{1}{2}f_{1,\nu} + \frac{F}{2}[(\nabla_\nu P^{\mu\beta\lambda})Q_{\mu\beta\lambda} + P^{\mu\beta\lambda}(\nabla_\nu Q_{\mu\beta\lambda})] \\ &= \frac{1}{2}f_{1,\nu} - \frac{F}{2}Q_{,\nu} = -L_M f_{2,\nu}. \end{aligned} \quad (34)$$

In the four steps above, we have substituted the results given by Eqs. (32) and (33), and then used the definitions of Q and of F, respectively.

We may write this result explicitly as

$$\begin{aligned} &\mathcal{D}_\mu T^\mu{}_\nu + \frac{2}{\sqrt{-g}}\nabla_\alpha\nabla_\beta H_\nu{}^{\alpha\beta} \\ &= -\frac{2}{\sqrt{-g}f_2}[(\nabla_\alpha\nabla_\beta f_2)H_\nu{}^{\alpha\beta} + 2f_{2,(\alpha}\nabla_{\beta)}H_\nu{}^{\alpha\beta}] \\ &\quad - (T^\mu{}_\nu - \delta^\mu_\nu L_M)\nabla_\mu \log f_2. \end{aligned} \quad (35)$$

The second line is due to the nonminimal coupling of the hypermomentum, which, perhaps interestingly, can now contribute also directly and not only via its (second) derivatives. The third line is due to the nonminimal coupling of the energy-momentum tensor. This term is second order in derivatives (assuming, of course, that the L_M does not contain higher derivatives), which confirms our optimistic expectation.

D. The energy and momentum balance equations

The expression of the divergence of the energy-momentum tensor, as given by Eq. (35), shows that due to the coupling between the nonmetricity Q and the matter fields, in the present theory the matter energy-momentum tensor is no longer conserved. Generally, in theories with nonconserved divergence of the energy-momentum tensor we can write $\mathcal{D}_\mu T^\mu{}_\nu = A_\nu$, where A_ν is a model-dependent four-vector. In order to find a physical interpretation of A_ν, we consider that the matter content of the gravitating system can be described by the energy-momentum tensor of a perfect fluid, given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \quad (36)$$

where ρ and p are the thermodynamic energy and pressure, with the four-velocity u_μ satisfying the normalization condition $u_\mu u^\mu = -1$, and the differential identity $u^\nu \mathcal{D}^\mu u_\nu = 0$, respectively. We also introduce the projection operator $h^\nu_\lambda = \delta^\nu_\lambda + u_\lambda u^\nu$, which satisfies the algebraic relation $u_\nu h^\nu_\lambda = 0$. By taking the divergence of the energy-momentum tensor given by Eq. (36), we obtain

$$(\mathcal{D}^\mu \rho + \mathcal{D}^\mu p)u_\mu u_\nu + (\rho + p)u_\nu \mathcal{D}^\mu u_\mu + (\rho + p)u_\mu \mathcal{D}^\mu u_\nu + g_{\mu\nu}\mathcal{D}^\mu p = A_\nu. \quad (37)$$

We multiply first Eq. (37) by u^ν, which immediately gives

$$\dot{\rho} + 3\mathcal{H}(\rho + p) = A_\nu u^\nu = \mathcal{S}, \quad (38)$$

where we have denoted the overdot as $\dot{} = u_\mu \mathcal{D}^\mu$, and considered the following definitions: $\mathcal{H} = (1/3)\mathcal{D}^\mu u_\mu$, and $\mathcal{S} = A_\nu u^\nu$, respectively. We multiply now Eq. (37) with the projection operator h^ν_λ, thus obtaining

$$h^\nu_\lambda[(\rho + p)\dot{u}_\nu + \mathcal{D}_\nu p] = h^\nu_\lambda A_\nu, \quad (39)$$

or, equivalently,

$$\frac{d^2x^\lambda}{ds^2} + \{^\lambda{}_{\mu\nu}\}u^\mu u^\nu = \frac{h^{\lambda\nu}}{\rho + p}(A_\nu - \mathcal{D}_\nu p) = \mathcal{F}^\lambda, \quad (40)$$

which translates as nongeodesic motion, where $\mathcal{F}^\lambda$ is an extra force arising due to the Q-matter coupling.

Equation (38) gives the energy balance equation in modified gravity with Q couplings, or, in other words, the amount of energy entering or going out from a given volume. The term $\mathcal{S}$ acts as a source for the energy creation/annihilation. The matter energy of the gravitating system is conserved only if the condition $A_\nu u^\nu \equiv 0$ is satisfied in a given spacetime volume. If $A_\nu u^\nu \neq 0$, then particles or energy transfer processes must take place in the system. One such particular physical process that could be described by an energy balance equation of type (38) is represented by particle creation that could result from the irreversible energy transfer from the gravitational field to matter [56,57]. By taking into account the explicit form of the divergence of the energy-momentum tensor from the result (35), we can decompose the energy source term as

$$\mathcal{S} = \mathcal{S}_\mathcal{T} + \mathcal{S}_\mathcal{H}, \quad (41)$$

where $\mathcal{S}_\mathcal{T}$ is defined by

$$\mathcal{S}_\mathcal{T} = (\rho + L_M)\frac{\dot{f}_2}{f_2}, \quad (42)$$

and the hypersource is given as

$$\mathcal{S}_{\mathcal{H}} = -\frac{2}{\sqrt{-g}} u^{\nu} \left[\nabla_{\alpha}\nabla_{\beta} H_{\nu}{}^{\alpha\beta} + \frac{1}{f_2}(\nabla_{\alpha}\nabla_{\beta} f_2) H_{\nu}{}^{\alpha\beta} + \frac{1}{f_2} f_{2,(\alpha}\nabla_{\beta)} H_{\nu}{}^{\alpha\beta} \right]. \tag{43}$$

Note that the energy source (42) vanishes for perfect fluids, when we adopt the Lagrangian prescription $L_M = -\rho$.

Equation (40) gives the equation of motion of massive particles in modified gravity with Q coupling. From its general form it immediately follows that the motion is not geodesic, and an extra-force with components $\mathcal{F}^{\lambda}$ exerts a supplementary force on any particle. The extra force is orthogonal to the matter four-velocity, since, due to the presence of the projection operator in its expression, we always have $\mathcal{F}^{\lambda} u_{\lambda} \equiv 0$. This result points towards the fact that the extra-force as given by Eq. (40) is physical, since it satisfies the usual condition for a "normal" force, which requires that only the components of the four-force that are orthogonal to the four-velocity of the particle can influence its trajectory. In modified gravity with Q couplings, the extra force can be written, by recalling again the result (35), as

$$\mathcal{F}^{\lambda} = -\frac{h^{\alpha\lambda}\nabla_{\alpha} p}{\rho + p} + \mathcal{F}^{\lambda}_{\mathcal{T}} + \mathcal{F}^{\lambda}_{\mathcal{H}}, \tag{44}$$

where the first term on the right-hand side is the usual general relativistic contribution of the pressure gradient, and the extra force consists of the following terms:

$$\mathcal{F}^{\lambda}_{\mathcal{T}} = (-p + L_M) h^{\lambda}_{\nu} \nabla^{\nu} \log f_2, \tag{45}$$

and the hyperforce

$$\mathcal{F}^{\lambda}_{\mathcal{H}} = -\frac{2}{\sqrt{-g}} h^{\lambda\nu} \left[\nabla_{\alpha}\nabla_{\beta} H_{\nu}{}^{\alpha\beta} + \frac{1}{f_2}(\nabla_{\alpha}\nabla_{\beta} f_2) H_{\nu}{}^{\alpha\beta} + \frac{1}{f_2} f_{2,(\alpha}\nabla_{\beta)} H_{\nu}{}^{\alpha\beta} \right], \tag{46}$$

respectively. It is interesting that the extra force (45) vanishes identically for a perfect fluid if we adopt the Lagrangian prescription $L_M = p$, in which case the source term (42) in turn would be nonvanishing.

IV. COSMOLOGICAL APPLICATION

We now explore several cosmological applications. For this purpose, consider the isotropic, homogeneous and spatially flat line element given by

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{47}$$

where we have included the lapse function $N(t)$ for generality, though in the present case we have the usual time reparametrization freedom and may impose $N = 1$ at any time. It is then convenient to define the expansion and the dilation rates as

$$H = \frac{\dot{a}}{a}, \qquad T = \frac{\dot{N}}{N}, \tag{48}$$

respectively. We shall work in the coincident gauge, and it is straightforward to obtain that $Q = 6(H/N)^2$.

We shall assume standard perfect fluid matter, whose energy-momentum tensor given by (36) is diagonal. The field equations (17) in this case imply the following two generalized Friedmann equations:

$$f_2 \rho = \frac{f_1}{2} - 6F\frac{H^2}{N^2}, \tag{49}$$

$$-f_2 p = \frac{f_1}{2} - \frac{2}{N^2}[(\dot{F} - FT)H + F(\dot{H} + 3H^2)], \tag{50}$$

respectively. It is easy to check that in the limit of standard GR, $f_1 = -Q$ and $f_2 = 1 = -F$, these reduce to the standard Friedmann equations. The equation of motion for the connection (18) is identically satisfied for the theory (10) in the background (47). The continuity equation of matter can be deduced from the above two equations (49) and (50), and is given by

$$\dot{\rho} + 3H(\rho + p) = -\frac{6f_2' H}{f_2 N^2}(\dot{H} - HT)(L_M + \rho). \tag{51}$$

This is in accordance with the general result (42). Since in the minisuperspace given by Eq. (47) setting $L_M = -\rho$, we recover the standard continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{52}$$

This is compatible with the fact that the connection equation (18) is trivialized in the isotropic and homogeneous background.

A. The cosmological evolution equations

In the following, we will adopt the gauge $N = 1$, thus working in the framework of standard Friedman-Robertson-Walker (FRW) geometry. With this choice, we have

$$Q = 6H^2, \tag{53}$$

and $T = 0$, respectively. Therefore the field equations (49) and (50) can be reformulated as

$$3H^2 = \frac{f_2}{2F}\left(-\rho + \frac{f_1}{2f_2}\right), \tag{54}$$

$$\dot{H} + 3H^2 + \frac{\dot{F}}{F}H = \frac{f_2}{2F}\left(p + \frac{f_1}{2f_2}\right). \tag{55}$$

By eliminating the term $3H^2$ between the above two equations, we obtain the following evolution equation for H

$$\dot{H} + \frac{\dot{F}}{F}H = \frac{f_2}{2F}(\rho + p). \tag{56}$$

From Eq. (54), we obtain the matter density as a function of Q in the form

$$\rho(Q) = \frac{(f_1/2f_2)[1 - 2(f_1'/f_1)Q]}{1 - (f_2'/f_2)Q}. \tag{57}$$

After adding Eqs. (55) and (56), and by introducing the effective energy density ρ_{eff} and effective pressure p_{eff} of the cosmological fluid, defined as

$$\rho_{\mathrm{eff}} = -\frac{f_2}{2F}\left(\rho - \frac{f_1}{2f_2}\right), \tag{58}$$

$$p_{\mathrm{eff}} = \frac{2\dot{F}}{F}H - \frac{f_2}{2F}\left(\rho + 2p + \frac{f_1}{2f_2}\right), \tag{59}$$

we can write the gravitational field equations in a form similar to the Friedmann equations of GR as

$$3H^2 = \rho_{\mathrm{eff}}, \tag{60}$$

$$2\dot{H} + 3H^2 = -p_{\mathrm{eff}}. \tag{61}$$

An important cosmological quantity, the deceleration parameter, defined as

$$q = \frac{d}{dt}\frac{1}{H} - 1 = -\frac{\dot{H}}{H^2} - 1, \tag{62}$$

can be obtained from Eq. (56) as

$$q = \frac{\dot{F}}{F}\frac{1}{H} - \frac{f_2}{2H^2F}(\rho + p) - 1. \tag{63}$$

Moreover, to describe cosmological evolution, and the possible transition to an accelerated phase, we also introduce the parameter w of the dark energy equation of state, defined as

$$w = \frac{p_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = \frac{-4\dot{F}H + f_2(\rho + 2p + \frac{f_1}{2f_2})}{f_2(\rho - \frac{f_1}{2f_2})}. \tag{64}$$

Alternatively, the deceleration parameter can be written as

$$q = \frac{1}{2}(1 + 3w) = 2 + \frac{3(4\dot{F}H - f_1 - 2f_2p)}{f_1 - 2f_2\rho}. \tag{65}$$

B. The de Sitter solution

As a first step in considering explicit theoretical models, we consider the problem of the existence of a de Sitter type vacuum solution of the cosmological field equations. The de Sitter solution corresponds to $\rho = p = 0$, and $H = H_0 = \text{constant}$, respectively. For a vacuum de Sitter type Universe, Eq. (56) immediately gives $\dot{F} = 0$, and $F = \text{constant} = F_0$. For the vacuum state $L_M = 0$, and therefore the definitions of f and F reduce to $f = f_1(Q)$, and $F = f_1'(Q)$.

The condition $F = \text{constant} = -F_0$ is satisfied for any Q only in the case

$$f_1(Q) = -F_0Q - 2\Lambda = -6F_0H_0^2 - 2\Lambda, \tag{66}$$

where Λ is an arbitrary constant of integration. In the vacuum de Sitter phase, both field equations (60) and (61) reduce to the algebraic form

$$3H_0^2 = \frac{6F_0H_0^2 + 2\Lambda}{4F_0}, \tag{67}$$

or equivalently

$$H_0 = \sqrt{\frac{\Lambda}{3F_0}}. \tag{68}$$

The specific case of Eq. (66) is, of course, equivalent to GR with a cosmological constant Λ with the normalization $F_0 = 1$. However, there exist vacuum de Sitter solutions in very generic cases. The combination of the field equations (60) and (61) supports consistently such solutions as long as

$$(\log f_1(Q))' = \frac{1}{12H_0^2}, \tag{69}$$

when the right-hand side is evaluated at $Q = 6H_0^2$. As one can see immediately from Eqs. (63) and (64), for the de Sitter evolution we obtain $q = -1$, and $w = -1$, respectively.

C. Cosmological models with specific forms of f_1 and f_2

In order to investigate more general cosmological models, we need to fix the functional form of the functions $f_1(Q)$ and $f_2(Q)$. Once this form is fixed *a priori*, the system of gravitational equations becomes closed, and their solutions can give a full description of the cosmological evolution. We consider two natural parametrizations, a power-law and an exponential. The functions have independent parameters. They need not have the same functional forms either as their physical role is different, but we leave the study of more general combinations elsewhere.

1. Power-law dependence of the nonminimal couplings

As a first example of cosmological models of this type we consider the case in which both f_1 and f_2 have a simple power-law dependence on Q, so that

$$f_1(Q) = AQ^{\alpha+1}, \qquad f_2(Q) = BQ^{\beta+1}, \tag{70}$$

where A, B, α and β are arbitrary constants. For the matter Lagrangian, we will adopt the expression $L_M = -\rho$. Moreover, we assume that the cosmological matter satisfies the linear barotropic equation of state with $p = (\gamma - 1)\rho$. For the function F, we obtain the expression

$$F(Q) = A(1+\alpha)Q^{\alpha} - 2B(1+\beta)Q^{\beta}\rho = A(1+\alpha)(6H^2)^{\alpha} - 2B(1+\beta)(6H^2)^{\beta}\rho. \quad (71)$$

Substituting the above expressions of f_1, f_2 and F into Eq. (54) allows us to obtain the energy density as

$$\rho = \frac{A(1+2\alpha)(6H^2)^{\alpha-\beta}}{B(2+4\beta)}. \quad (72)$$

Hence, the evolution equation of the Hubble function (56) takes the form

$$\dot{H} = -\frac{3\gamma H^2}{2(\alpha-\beta)}, \quad (73)$$

which provides the following solution

$$H(t) = \frac{2H_0(\alpha-\beta)}{2(\alpha-\beta) + 3\gamma H_0(t-t_0)}, \quad (74)$$

where $H_0 = H(t_0)$ and

$$a(t) = a_0[2(\alpha-\beta) + 3\gamma H_0(t-t_0)]^{\frac{2(\alpha-\beta)}{3\gamma}}, \quad (75)$$

respectively. This means that the Universe expands as if it was dominated by a fluid with the effective equation of state parameter

$$\gamma_{\rm eff} = \frac{\gamma}{\alpha-\beta}. \quad (76)$$

In this model, the deceleration parameter has a constant value $q = 3\gamma/2(\alpha-\beta)$ during the entire cosmological evolution (when the equation of state γ of the cosmological matter is a constant). Depending on the numerical values of α and β, a large range of cosmological behaviors can be obtained, including both accelerating and decelerating phases, with the possibility of the deceleration parameter of taking a $q \approx -1$ value. In this case, we obtain a power law type accelerating expansion of the Universe. Exact de Sitter type evolution can however only be realized with a cosmological constant $\gamma = 0$.

2. *Exponential dependence of the nonminimal couplings*

As a second example of a cosmological scenario in the framework of the matter-Q field coupling theory, we consider the case of the exponential dependencies of the functions f_1 and f_2 on the Q-field, so that

$$f_1 = Ae^{\alpha Q}, \qquad f_2 = Be^{\beta Q}, \quad (77)$$

where A, B, α and β are, once again, arbitrary constants. For the function F, we easily obtain

$$F(Q) = A\alpha e^{\alpha Q} - 2\beta Be^{\beta Q}\rho. \quad (78)$$

In the following, we assume again that $Q = 6H^2 > 0$, and $H(Q) = \sqrt{Q/6}$, $\dot{H} = \dot{Q}/2\sqrt{6}\sqrt{Q}$. Then from Eq. (54), we obtain the density of the matter as a function of Q in the form

$$\rho(Q) = \frac{A[2\alpha Q(t) - 1]e^{(\alpha-\beta)Q(t)}}{2B[2\beta Q(t) - 1]}. \quad (79)$$

By using the above representation of the density, we obtain for the function F the expression

$$F = \frac{A(\alpha-\beta)e^{\alpha Q(t)}}{1 - 2\beta Q(t)}. \quad (80)$$

The evolution of the Hubble function (56) can be obtained, in terms of Q, as the solution of the following first-order differential equation

$$\frac{dQ}{dt} = \sqrt{\frac{3}{2}}\frac{\gamma\sqrt{Q}(1-2\alpha Q)(1-2\beta Q)}{(\alpha-\beta)(1+2(\alpha+\beta)Q - 4\alpha\beta Q^2)}. \quad (81)$$

The general solution of Eq. (81) is given by

$$t(Q) - t_0 = \frac{2}{\gamma}\sqrt{\frac{2}{3}}\{\sqrt{2\alpha}[\tanh^{-1}(\sqrt{2\alpha Q}) - \tanh^{-1}(\sqrt{2\alpha Q_0})] + \sqrt{2\beta}[\tanh^{-1}(\sqrt{2\beta Q_0}) - \tanh^{-1}(\sqrt{2\beta Q})] - (\alpha-\beta)(\sqrt{Q} - \sqrt{Q_0})\}, \quad (82)$$

where we have used the initial condition $Q(t_0) = Q_0$. Hence, we have obtained the general solution of the field equations in a parametric form, with Q taken as the parameter.

The evolution of the scale factor can be obtained from the equation

$$\frac{1}{a}\frac{da}{dQ} = \sqrt{\frac{Q}{6}}\frac{dt}{dQ}, \quad (83)$$

and is given by

$$a(Q) = a_0\frac{(1-2\beta Q)^2}{(1-2\alpha Q)^2}e^{-(\alpha-\beta)Q/3\gamma}, \quad (84)$$

where a_0 is an arbitrary constant of integration.

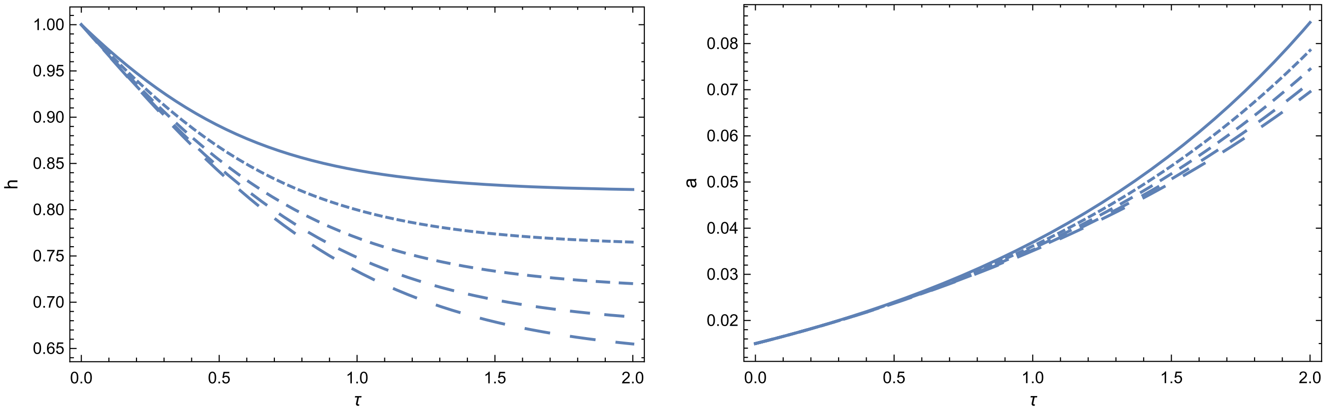

FIG. 1. Specific case of the exponential dependence of the nonminimal couplings. Variation as function of the dimensionless time τ of the Hubble function h (left figure) and of the scale factor a (right figure) for a pressureless Universe with $\gamma = 1$ for $\tilde{\beta} = 0.054$, and different values of $\tilde{\alpha}$: $\tilde{\alpha} = 0.124$ (solid curve), $\tilde{\alpha} = 0.144$ (dotted curve), $\tilde{\alpha} = 0.164$ (short dashed curve), $\tilde{\alpha} = 0.184$ (dashed curve), and $\tilde{\alpha} = 0.204$ (long dashed curve). For Q_0, we have adopted the initial value $Q_0 = 6$, and $a(0) = 0.015$. We refer the reader to the text for more details.

The deceleration parameter can be obtained as

$$q(Q) = -\frac{3}{2}\frac{\gamma(1-2\alpha Q)(1-2\beta Q)}{(\alpha-\beta)Q(1+2(\alpha+\beta)Q-4\alpha\beta Q^2)} - 1. \tag{85}$$

In order to obtain a dimensionless form of the cosmological evolution equations, we introduce a set of dimensionless variables $(h, \tau, r, \tilde{\alpha}, \tilde{\beta}, \tilde{Q})$, defined as

$$H = H_0 h, \qquad t = \frac{\tau}{H_0}, \qquad \rho = 3H_0^2 r,$$
$$\alpha = \frac{\tilde{\alpha}}{H_0^2}, \qquad \beta = \frac{\tilde{\beta}}{H_0^2}, \qquad Q = H_0^2 \tilde{Q}, \tag{86}$$

where H_0 is a fixed value of the Hubble function, which may correspond, for example, to the end of inflation, or to the present age of the Universe. For the ratio of the constants A and B, we obtain the expression

$$\frac{A}{B} = \frac{6H_0^2(2\beta Q_0 - 1)}{2(2\alpha Q_0 - 1)} e^{-(\alpha-\beta)Q_0}, \tag{87}$$

where $Q_0 = Q(\tau_0)$. For $A/B > 0$, the condition of the positivity of the matter energy density imposes the constraints $\alpha > 1/12$ and $\beta > 1/6$ on the model parameters α and β. All the dimensionless expressions of the time evolution, Hubble function, energy density and deceleration parameter can be simply obtained from the dimensional form by simply substituting the initial variables with the dimensionless ones. Hence we will not write down the explicit form of the dimensionless representation of the basic cosmological evolution equations, and of their solutions.

The variations of the Hubble function, scale factor, matter energy density, and deceleration parameter are represented, for different values of the model parameters $\tilde{\alpha}$ and $\tilde{\beta}$ in Figs. 1 and 2, respectively.

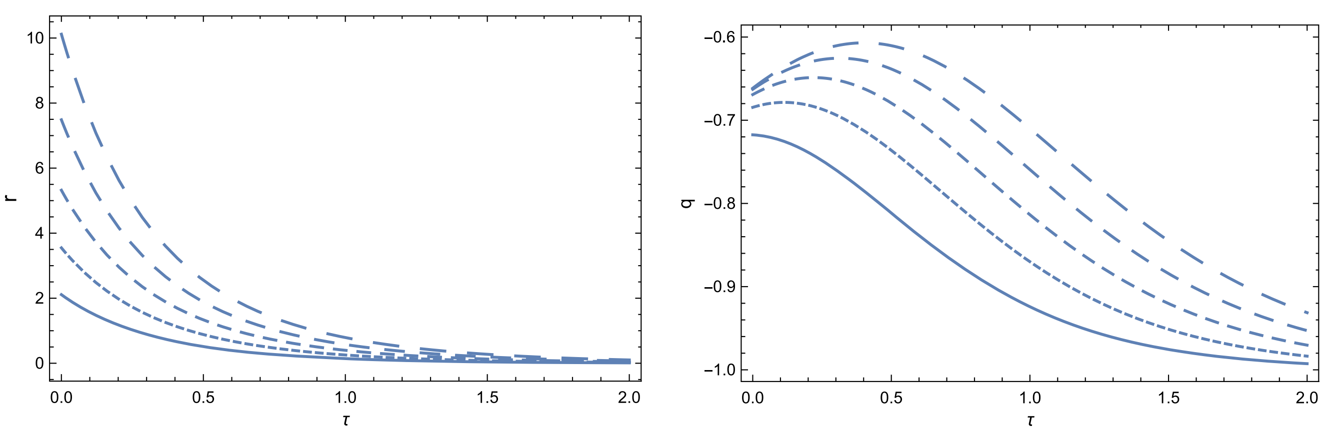

FIG. 2. Specific case of the exponential dependence of the nonminimal couplings. Variation as function of the dimensionless time τ of the energy density of the matter r (left figure) and of the deceleration parameter q (right figure) for a pressureless Universe with $\gamma = 1$ for $\tilde{\beta} = 0.054$, and different values of $\tilde{\alpha}$: $\tilde{\alpha} = 0.124$ (solid curve), $\tilde{\alpha} = 0.144$ (dotted curve), $\tilde{\alpha} = 0.164$ (short dashed curve), $\tilde{\alpha} = 0.184$ (dashed curve), and $\tilde{\alpha} = 0.204$ (long dashed curve). For Q_0, we have adopted the initial value $Q_0 = 6$, and $a(0) = 0.015$. See the text for more details.

As one can see from Fig. 1 the Hubble function is a monotonically decreasing function of the time, indicating an expansionary evolution of the Universe. In the large time limit, the rate of time variation h is slow, and it shows a significant dependence on the numerical values of the model parameters $\tilde{\alpha}$ and $\tilde{\beta}$. The scale factor is a monotonically increasing function of time, and in the late stages of the cosmological evolution, it shows a relatively weak dependence on $\tilde{\alpha}$ and $\tilde{\beta}$. The energy density of the matter, presented in the left panel of Fig. 2, monotonically decreases in time, and in the large time limit it tends to zero in a way almost independent on $\tilde{\alpha}$ and $\tilde{\beta}$. However, the early time evolution is significantly influenced by the model parameters. The evolution of the Universe begins in an accelerating state, with the deceleration parameter q, shown in the right panel of Fig. 2, taking negative initial values of the order of $q \approx -0.70$. Then the Universe begins to accelerate, with q, showing a complex dynamics, decreasing in time. In the large time limit, the Universe reaches the exponentially accelerating de Sitter phase with $q = -1$, a result which is independent on the model parameters.

We obtain accelerating solutions, which imply the breaking of the strong energy condition. Super-accelerating solutions were not found, so there is no evidence of the null energy condition violation.

V. SUMMARY AND FUTURE OUTLOOK

In this work, we have explored an extension of the symmetric teleparallel gravity, by considering a new class of theories where the nonmetricity Q is coupled nonminimally to the matter Lagrangian, in the framework of the metric-affine formalism. As in the standard curvature-matter couplings, this nonminimal Q-matter coupling entails the nonconservation of the energy-momentum tensor, and consequently the appearance of an extra force. We have verified whether the subtle improvement of the geometrical formulation, when implemented in the matter sector, would allow more universally consistent and viable realizations of the nonminimal curvature-matter coupling theories. Furthermore, we have also analyzed several cosmological applications.

As a first step in this direction we have obtained the generalized Friedmann equations describing the cosmological evolution in flat FRW type geometry. The coupling between matter and the Q field introduces two types of corrections. The first is the presence of a term of the form $f_2/2F$ multiplying the components of the energy-momentum tensor (energy density and pressure) in both Friedmann equations. Secondly, an additive term of the form $f_1/4F$ also appears in the generalized Friedmann equations. The basic equations describing the cosmological dynamics can then be reformulated in terms of an effective energy density and pressure, which both depend on the standard components of the energy-momentum tensor, and on the functions $f_i(Q)$, $i = 1, 2$, and on $F(Q, \rho)$. In the vacuum case $\rho = p = 0$, the deceleration parameter takes the form $q = -1 + 12\dot{F}H/f_1$, showing that, depending on the mathematical forms of the coupling functions, a large number of cosmological evolutionary scenarios can be obtained. Generally, we have shown explicitly that for late times, the Universe attains an exponentially accelerating de Sitter phase.

We have also considered two explicit classes of cosmological models obtained by choosing some specific functional forms for the functions $f_1(Q)$ and $f_2(Q)$, corresponding to power law and exponential forms of the couplings. In the case of the power law dependence of $f_i(Q)$, $i = 1, 2$, the field equations can be solved exactly, leading to a power-law dependence of the scale factor. The deceleration coefficient is constant, but by an appropriate choice of the parameters accelerating evolutions can be easily obtained. In the case of the exponential dependence of the couplings, the overall cosmological dynamics of the Universe is very complex, and the relevant results can be obtained only by numerically integrating the evolution equation. The results are strongly dependent on the numerical values of the model parameters. For the specific range of cosmological parameters, we have considered that the Universe is born in an accelerated phase, and in the large time limit it reaches the de Sitter phase, which acts as an attractor for the generalized Friedmann equations. This solution may represent an alternative to the standard inflationary scenario [58,59], in which the de Sitter phase is triggered by the presence of some cosmological scalar fields. Of course, the present model is also valid when instead of ordinary matter one considers scalar fields. Considering inflation in Q-coupling gravity in the presence of scalar fields may give a new perspective on the physical, geometrical and cosmological processes that may have played a dominant role in the very early evolution of the Universe.

Thus, in summary, we have established the theoretical consistency and motivations on these extensions of $f(Q)$ family of theories. Furthermore, we considered cosmological applications, in which the presented approach provides gravitational alternatives to dark energy. As future avenues of research, one should aim in characterizing the phenomenology predicted by these theories with a nonmetricity-matter coupling, in order to find constraints arising from observations. The study of these phenomena may also provide some specific signatures and effects, which could distinguish and discriminate between the various theories of modified gravity. We also propose to use a background metric to analyze the dynamic system for specific nonmetricity-matter coupling models, and use the data of SNIa, BAO, CMB shift parameter to obtain restrictions for the respective models, and explore in detail the analysis of structure formation. Another topic that needs to be addressed is the analysis of the post-Newtonian formalism applied to this nonminimal extension of $f(Q)$ gravity, in order to pass the local gravity constraints. Work along these lines are presently underway, and are to be presented in the near future.

The nonminimally coupled theory violates the equivalence principle, which is interesting to study in the new covariant framework of Q gravity that makes possible the canonical energy-momentum tensor and thus the localization of gravitational energy. Theoretical investigations into the nonminimally coupled Q gravity can be carried out further into the physical implications to the violations of the energy conditions and the possible violations of usual thermodynamical relations.

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