MOND as the weak field limit of an extended metric theory of gravity with a matter-curvature coupling

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In this article we construct an extended relativistic $f(R)$ theory of gravity with matter-curvature couplings $F(R, \mathcal{L}_{\mathrm{matt}})$ for which its weak-field limit of approximation recovers the simplest version of MOND. We do this by (a) performing an order-of-magnitude calculation and (b) perturbing the resulting field equations of the theory to the weak-field limit. We also compute the geodesic equation of the resulting theory and show that it has an extra force, a fact that commonly appears in general matter-curvature couplings.

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I. INTRODUCTION

The nonbaryonic dark matter problem constitutes one of the most important unsolved problems in current research (cf. Refs. [1,2]). Despite the huge amount of research and its generally accepted success, the dark matter particle has never been detected. The gravitational anomaly that gives rise to the dark matter and/or energy hypothesis can also be understood as a modification of gravity at certain scales (cf. Ref. [3]) which was first discussed in the pioneering work of Milgrom [4,5], using a MOdified Newtonian Dynamics (MOND) approach. The first coherent attempt to find a relativistic version was carried out by Bekenstein [6] with a Tensor Vector Scalar (TeVeS) theory; this idea has been widely explored [7–11], but due to the extreme complexity of the theory and some clear failures, researchers have continued searching for a relativistic theory of gravity that yields MOND in its nonrelativistic, weak-field limit.

Bernal *et al.* [12] showed that MOND acceleration can be accounted for by a relativistic $f(\chi) = \chi^{3/2}$ metric theory of gravity described by the action

$$S = \frac{c^3}{16\pi G L_M^2} \int f(\chi) \sqrt{-g} \mathrm{d}^4 x + \frac{1}{c} \int \mathcal{L}_{\mathrm{matt}} \sqrt{-g} \mathrm{d}^4 x, \qquad (1)$$

where $\chi := L^2 R$, R is the Ricci scalar, $L \propto r_{\mathrm{g}}^{1/2} l^{1/2}$, where $r_{\mathrm{g}} := GM/c^2$ is the gravitational radius, $l := (GM/a_0)^{1/2}$ is the "mass-length" scale of the system and $\mathcal{L}_{\mathrm{matt}}$ is the standard matter Lagrangian, related to the energy-momentum tensor $T_{\alpha\beta}$ by

$$T_{\alpha\beta} \sqrt{-g} \delta g^{\alpha\beta} = -2\delta(\sqrt{-g} \mathcal{L}_{\mathrm{matt}}). \qquad (2)$$

The constant $a_0 \approx 1.2 \times 10^{-10}\ \mathrm{m\,s^{-2}}$ is Milgrom's acceleration constant. This proposal is consistent with the results of gravitational lensing in individual, groups and clusters of galaxies [13] and at the same second perturbation order is consistent with a parametrized post-Newtonian (PPN) description where the parameter $\gamma = 1$ [14]. Another extension of gravity was performed by Barrientos and Mendoza [15], who analyzed the action (1) using the Palatini approach; they obtained the same functional action $f(\chi) = \chi^{3/2}$ that recovers the MONDian acceleration, with a mass dependence on the coupling length L.

The problem with the action (1) is that it can only be applied in regions sufficiently far from the sources that produce the gravitational field, in order to approximate the system as a point-mass source. Carranza, Mendoza, and Torres [16] attempted to resolve this issue by considering the mass M as the causal mass for a particular observer in the cosmic flow, yielding a good description of an accelerated expansion of the Universe without the introduction of dark matter and/or energy.

Another recent exploration was carried out by Barrientos and Mendoza [17] who showed that the mass dependence on the coupling length L can be avoided by introducing derivatives of the matter Lagrangian into the action $f(\chi)$. In such a proposal the coupling constant depends exclusively on the fundamental constants c, a_0 and G, but the price to pay is in the complexity of the field equations and the theoretical inconveniences that arise due to the inclusion of matter Lagrangian derivatives.

In this article we use an extension of a metric $f(R)$ theory of gravity with matter-curvature couplings $F(R, \mathcal{L}_{\mathrm{matt}})$ following the approach of Refs. [18–22] and show that with this generalized action a relativistic theory of MOND can be constructed. The article is presented in the following manner. In Sec. II an order-of-magnitude calculation is

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performed to show that a specific $F(R,\mathcal{L}_{\rm matt})$ can reproduce MOND in its simplest form. Section III shows an exact solution for a point-mass source reproducing these results. In Sec. IV we use correct dimensional arguments to generalize an action for a $F(R,\mathcal{L}_{\rm matt})$ and show that with this it is possible to recover either MOND or Newton's gravity in the weak-field limit of the theory. Finally in Sec. V we discuss the results of the article and present our conclusions.

II. $F(R,L_{\rm matt})$ APPROACH

The lesson to learn from the action (1) is that the matter Lagrangian $\mathcal{L}_{\rm matt}$ needs to be inserted inside the gravitational action (see e.g., Ref. [3]). The idea of a nonminimal coupling between the matter and the curvature was already considered [23–26]. To do so, we can extend $f(R)$ gravity by introducing a specific $F(R,\mathcal{L}_{\rm matt})$ described by Harko and Lobo [18],

$$S=\int F(R,\mathcal{L}_{\rm matt})\sqrt{-g}\mathrm{d}^4x, \tag{3}$$

with the following field equations:

$$F_R R_{\alpha\beta}+(g_{\alpha\beta}\nabla^\mu\nabla_\mu-\nabla_\alpha\nabla_\beta)F_R -\frac{1}{2}(F-F_{\mathcal{L}_{\rm matt}})g_{\alpha\beta}=\frac{1}{2}F_{\mathcal{L}_{\rm matt}}T_{\alpha\beta}, \tag{4}$$

where $F_R:=\partial F/\partial R$ and $F_{\mathcal{L}_{\rm matt}}:=\partial F/\partial\mathcal{L}_{\rm matt}$. Note that (a) $F(R,\mathcal{L}_{\rm matt})=c^3R/16\pi G+\mathcal{L}_{\rm matt}/c$ yields standard general relativity, (b) $F(R,\mathcal{L}_{\rm matt})=f(R)/2+\mathcal{L}_{\rm matt}/c$ is standard metric $f(R)$ gravity and (c)

$$F(R,\mathcal{L}_{\rm matt})=\frac{c^3}{16\pi G}\frac{f(\chi)}{L^2}+\frac{1}{c}\mathcal{L}_{\rm matt} \tag{5}$$

is a correct generalization of Eq. (1) in which the unknown length function $L=L(\mathcal{L}_{\rm matt})$ is to be found, which together with the unknown function $f(\chi)$ must yield a correct MOND behavior in the limit of low acceleration scales $a\lessapprox a_0$.

III. MONDIAN LIMIT

Let us now show that with the assumptions made in Sec. II it is possible to obtain the basic MOND relation based on the Tully-Fisher law. To do so, let us substitute Eq. (5) into the field equations (4) and take the trace of the resulting relation:

$$f_R(\chi)R+-2f(\chi)+3L^2\nabla^\alpha\nabla_\alpha\left(\frac{f_R(\chi)}{L^2}\right)=\frac{8\pi GL^2}{c^4}T^\alpha{}_\alpha. \tag{6}$$

In order to find the correct MONDian limit equation, we follow the procedure of Bernal *et al.* [12] and so, let

$$f(\chi)=\chi^b, \quad \text{and} \quad \mathcal{L}_{\rm matt}=\rho c^2, \tag{7}$$

where we have assumed a point-mass source generating the gravitational field, and thus $\mathcal{L}_{\rm matt}$ has a dust-like form. To order of magnitude, i.e., when $R\sim r_{\rm curv}^{-2}$ (where $r_{\rm curv}$ is the radius of curvature of space) and $\nabla\sim 1/r$, it follows that the first two terms on the left-hand side of Eq. (6) are smaller than the third when $r/r_{\rm curv}\to 0$, i.e., when the equivalent acceleration a is expected to be $\lesssim a_0$.

Thus, the trace of the field equations that can be adapted to a MONDian regime of low acceleration scales is given by

$$3L^2\nabla^\alpha\nabla_\alpha\left(\frac{f_R(\chi)}{L^2}\right)=\frac{8\pi GL^2}{c^4}T^\alpha{}_\alpha. \tag{8}$$

A weak-field limit consistent with the bending of light in individual, groups and clusters of galaxies is obtained if the second perturbation order metric is given by [14]

$$\mathrm{d}s^2=\left(1+\frac{2\phi}{c^2}\right)c^2\mathrm{d}t^2-\left(1-\frac{2\phi}{c^2}\right)\mathrm{d}\boldsymbol{x}^2, \tag{9}$$

for a gravitational scalar potential ϕ and an isotropic spacetime with a PPN parameter $\gamma\approx 1$ according to observations of such MONDian systems [13]. With this, the Ricci scalar takes the form $R\approx-(2/c^2)\nabla^2\phi$, which at order of magnitude yields $R\sim a/rc^2$, for an acceleration $a=|\nabla\phi|$.

Thus, to order of magnitude, Eq. (8) yields

$$a\sim G^{1/(b-1)}\rho^{1/(b-1)}r^{(b+1)/(b-1)}c^{(2b-4)/(b-1)}L^{-2}, \tag{10}$$

and so, in order to obtain the standard MOND equation, $a=\sqrt{Ga_0M}/r\sim\sqrt{Ga_0\rho r}$. Then $b=-3$ together with $L\propto(G\rho)^{-3/8}c^{5/4}a_0^{1/4}$, which yields

$$F(R,\mathcal{L}_{\rm matt})\propto R^{-3}\mathcal{L}_{\rm matt}^3. \tag{11}$$

IV. A DIMENSIONALLY CORRECT GENERAL ACTION

Let us now consider an action motivated by Eq. (1) with the following form:

$$S=\frac{c^3}{16\pi G\alpha}\sqrt{-g}\int f(\chi,\xi)\mathrm{d}^4x+\frac{1}{c}\int\sqrt{-g}\mathcal{L}_{\rm matt}\mathrm{d}^4x, \tag{12}$$

where χ and ξ are dimensionless quantities given by

$$\xi:=\frac{\mathcal{L}_{\rm matt}}{\lambda}, \quad \text{and} \quad \chi:=\alpha R, \tag{13}$$

where α and λ are unknown "coupling" constants with dimensions of length squared and energy density respectively.

The null variations with respect to the metric yield the following field equations:

$$\alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(f - \xi f_\xi) = \left(\frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda}\right) T_{\mu\nu} - \alpha(g_{\mu\nu}\Delta - \nabla_\mu\nabla_\nu) f_\chi \tag{14}$$

with the standard definition of the energy-momentum tensor

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{\rm matt} - 2\frac{\partial\mathcal{L}_{\rm matt}}{\partial g^{\mu\nu}}, \tag{15}$$

in full agreement with Eq. (2).

The trace of Eq. (14) is given by

$$\chi f_\chi - 2(f - \xi f_\xi) + 3\alpha\Delta f_\chi = \left(\frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda}\right) T. \tag{16}$$

Since c, G and a_0 are independent fundamental constants, Buckingham's Π theorem of dimensional analysis implies that

$$\alpha = \kappa \frac{c^4}{a_0{}^2} \quad \text{and} \quad \lambda = \kappa' \frac{a_0{}^2}{G}, \tag{17}$$

where κ and κ' are pure dimensionless proportionality constants.

Following the previous approach, we can assume that

$$f(\chi, \xi) = \chi^\gamma \xi^\beta. \tag{18}$$

For the case of dust, the perturbation orders in the terms of the field equation are

$$\overbrace{\alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(f - \xi f_\xi)}^{\mathcal{O}(-2(\gamma+\beta))} + \overbrace{\alpha(g_{\mu\nu}\Delta - \nabla_\mu\nabla_\nu) f_\chi}^{\mathcal{O}(-2(\gamma+\beta+1))} = \underbrace{\frac{8\pi G\alpha}{c^4} T_{\mu\nu}}_{\mathcal{O}(2)} + \underbrace{\frac{f_\xi}{2\lambda} T_{\mu\nu}}_{\mathcal{O}(2(\gamma+\beta))}. \tag{19}$$

A. Poisson-like equation for MOND

The lowest perturbation order of the previous equation is 2 and so, the choice $\gamma = -\beta$ yields

$$(g_{\mu\nu}\Delta - \nabla_\mu\nabla_\nu) f_\chi = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{20}$$

Contracting Eq. (20) with $g^{\mu\nu}$ gives

$$3\Delta f_\chi = \frac{8\pi G}{c^4} T, \tag{21}$$

which at the lowest perturbation order for dust takes the form

$$(-2\kappa)^{\gamma-1}\kappa'^\gamma \frac{a_0{}^2}{G^{\gamma+1}} \nabla^2(\{\nabla^2\phi\}^{\gamma-1}\rho^{-\gamma}) = \frac{8\pi}{3}\rho. \tag{22}$$

To order of magnitude, this last equation implies that

$$a \approx M^{(1+\gamma)/(\gamma-1)} r^{-2(1+\gamma)/(\gamma-1)}, \tag{23}$$

and so, in order to recover a MONDian expression for the acceleration, we must have

$$\gamma = -3. \tag{24}$$

With this value, the Poisson-like equation (22) is

$$\frac{3}{8\pi} \frac{(a_0 G)^2}{(2\kappa)^4 \kappa'^3} \nabla^2(\{\nabla^2\phi\}^{-4}\rho^3) = \rho. \tag{25}$$

An analytic solution to the previous equation for the case of a point-mass source is given in the Appendix.

Note that Eq. (25) represents a nonlinear generalization of the standard Poisson equation $\nabla^2\phi \propto \rho$. A family of these nonlinear generalizations was discussed in Ref. [27], with Poisson-like equations of the form $\nabla\cdot(\mu(|\nabla\phi|)\nabla\phi) \propto \rho$ satisfying conformal invariance in all cases studied. Equation (25) does not fall into that category and as such, it differs from the standard AQUAdratic Lagrangian (AQUAL) proposal [28]. This is due to the fact that the nonlinearity of Eq. (25) applies not only to the scalar potential ϕ but also to the mass density ρ, since this last one appears inside the Laplacian operator on the left-hand side of Eq. (25).

B. Poisson's equation for Newtonian gravity

Another possible choice for Eq. (19) is $\gamma + \beta = 1$ which yields

$$\alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(f - \xi f_\xi) = \left(\frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda}\right) T_{\mu\nu}. \tag{26}$$

This lowest perturbation order choice means that

$$(g_{\mu\nu}\Delta - \nabla_\mu\nabla_\nu) f_\chi = 0. \tag{27}$$

Taking the trace of Eq. (26) for dust, a relation between the Ricci scalar and the matter density is obtained:

$$R = \left(-\frac{16\pi}{\gamma+1}(\kappa\kappa')^{1-\gamma}\right)^{1/\gamma} \frac{G}{c^2}\rho. \tag{28}$$

At the lowest perturbation order, when $R = -(2/c^2)\nabla^2\phi$, this previous equation can be constructed (with the appropriate coupling constants) to yield Newtonian gravity (Poisson's equation) for any value of $\gamma \neq -1$.

V. DISCUSSION

In this article we have shown, exactly and using an order of magnitude approach, that a $F(R, \mathcal{L}_{\rm matt})$ theory of gravity described by

$$f(\chi,\xi) = \chi^{-3}\xi^3, \qquad \chi \coloneqq \alpha R, \qquad \xi \coloneqq \mathcal{L}_{\rm matt}/\lambda, \tag{29}$$

is a good candidate for a full relativistic extension of MOND, in regions where the acceleration of test particles $\lesssim a_0$. In the weak-field limit it converges to standard MOND for a point-mass source M, with $\rho = M\delta(\mathbf{r})$ and $\mathcal{L}_{\rm matt} = \rho c^2$. It is our intention to explore this interpretation with applications to the lensing and dynamics of individual, groups and clusters of galaxies as well as cosmology. The advantage of this approach is that it is a full metric formalism and does not involve interpretations of gravity using the Palatini formalism or torsion as we have previously explored [29,30]. Furthermore, it is a correct generalization to the first attempts made by Bernal *et al.* [12].

At first sight, the action given by the Lagrangian density $R^{-3}\mathcal{L}^3_{\rm matt}$ from which we have obtained the MONDian behavior seems to diverge in the Minkowskian regime, namely when $R \to 0$. In order to show that this is not so, we proceed in the following way. Using Eqs. (17), (18), and (24), and the fact that $\gamma = -\beta$, Eq. (21) turns into

$$-\frac{9}{8\pi k^4 k'^3}\left(\frac{a_0 G}{c^6}\right)^2 \Delta(R^{-4}\mathcal{L}^3_{\rm matt}) = T, \tag{30}$$

which in the weak-field limit for a point-mass source is

$$-\frac{9}{8\pi k^4 k'^3}\left(\frac{a_0 G}{c^5}\right)^2 \nabla^2(R^{-4}\mathcal{L}^3_{\rm matt}) = M\delta(\mathbf{r}). \tag{31}$$

Using the well-known result

$$\nabla^2\left(\frac{1}{\mathbf{r}}\right) = -4\pi\delta(\mathbf{r}), \tag{32}$$

the following relation is satisfied:

$$R^{-4}\mathcal{L}^3_{\rm matt} = \frac{2\pi k^4 k'^3}{9}\left(\frac{c^5}{a_0 G}\right)^2 \frac{M}{r}. \tag{33}$$

Therefore, in the weak-field limit, this proposal has the following relation: $\mathcal{L}^3_{\rm matt} \propto R^4/r$. This implies that the Lagrangian density for the action that we are interested in converges to $R^{-3}\mathcal{L}^3_{\rm matt} \propto R/r \to 0$ as r increases.

Finally, we discuss the geodesic equation of the theory. Following a similar procedure as the one shown in Refs. [18,31], the geodesic equation is given by

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s^2} + \Gamma^\mu{}_{\nu\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = f^\mu, \tag{34}$$

where

$$f_\mu = (g_{\mu\nu} - u_\mu u_\nu)\nabla^\mu \ln\left[(16\pi\kappa\kappa' + f_\xi)\frac{\mathrm{d}\mathcal{L}_{\rm matt}}{\mathrm{d}\rho}\right]. \tag{35}$$

As expected, the usual relation $u_\mu f^\mu = 0$ is obtained. This means that the extra force is perpendicular to the four-velocity. For dust, the extra force takes the following form:

$$f_\nu = (g_{\mu\nu} - u_\mu u_\nu)\nabla^\mu \ln[16\pi\kappa\kappa' + f_\xi]. \tag{36}$$

This type of extra force has been studied and interpreted in the literature (cf. Ref. [32]) and in a very different context than the one discussed in this article to yield MOND-like accelerations in Ref. [33]. Investigations into its nature and its astrophysical consequences require further research.

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APPENDIX: POISSON-LIKE EQUATION

Let us begin by rewriting Eq. (25) as

$$K\nabla^2(\{\nabla^2\phi\}^{-4}\rho^3) = \rho, \tag{A1}$$

where for simplicity we have defined

$$K \coloneqq \frac{3}{8\pi}\frac{(a_0 G)^2}{(2\kappa)^4\kappa'^3}. \tag{A2}$$

The matter density for a point-mass source is given by

$$\rho = \frac{M}{4\pi r^2}\delta(r), \tag{A3}$$

and since the Laplacian for a spherically symmetric problem is

$$\nabla^2\psi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{d\psi}{dr}\right), \tag{A4}$$

then, Eq. (A1) turns into

$$4\pi K\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}(\{\nabla^2\phi\}^{-4}\rho^3)\right) = M\delta(r). \tag{A5}$$

Integration of the previous equation yields

$$4\pi K\frac{\mathrm{d}}{\mathrm{d}r}(\{\nabla^2\phi\}^{-4}\rho^3) = \frac{M}{r^2}, \tag{A6}$$

which another integration gives

$$4\pi K\{\nabla^2\phi\}^{-4}\rho^3 = -\frac{M}{r}. \tag{A7}$$

Using again Eqs. (A3) and (A4) and after some algebraic steps, we obtain

$$(-K)^{1/4}\left(\frac{M}{4\pi}\right)^{1/2}\left(\frac{r^3}{\delta(r)}\right)^{1/4}\delta(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right), \tag{A8}$$

which after another integration is written as

$$(-K)^{1/4}\left(\frac{M}{4\pi}\right)^{1/2}\left(\frac{r^3}{\delta(r)}\right)^{1/4}\Bigg|_0 = r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}. \tag{A9}$$

Using the fact that the acceleration $a = |\mathbf{a}| = |\nabla\phi|$ and the Dirac delta function is given by

$$\delta(r=0) = \lim_{r\to 0}\frac{1}{2\pi r}, \tag{A10}$$

the relation for the accelerations is given by

$$\left(-K\frac{M^2}{2^3\pi}\right)^{1/4}\frac{1}{r} = a. \tag{A11}$$

Substitution of the value of K given in Eq. (A2), yields

$$\left(-\frac{3}{4^5\kappa'^3\pi^2}\right)^{1/4}\frac{1}{\kappa}\frac{(a_0GM)^{1/2}}{r} = a. \tag{A12}$$

Thus, the choice $\kappa'^3 = -3/4^5\pi^2\kappa^4$ yields a MONDian acceleration $a = \sqrt{GMa_0}/r$.

[1] G. Bertone and D. Hooper, Rev. Mod. Phys. **90**, 045002 (2018).
[2] K. Freese, Int. J. Mod. Phys. D **26**, 1730012 (2017).
[3] S. Mendoza, Can. J. Phys. **93**, 217 (2015).
[4] M. Milgrom, Astrophys. J. **270**, 365 (1983).
[5] M. Milgrom, Astrophys. J. **270**, 371 (1983).
[6] J. D. Bekenstein, Phys. Rev. D **70**, 083509 (2004).
[7] J. D. Bekenstein and R. H. Sanders, Mon. Not. R. Astron. Soc. **421**, L59 (2012).
[8] R. H. Sanders, Mon. Not. R. Astron. Soc. **363**, 459 (2005).
[9] R. H. Sanders, Astrophys. J. **480**, 492 (1997).
[10] T. G. Zlosnik, P. G. Ferreira, and G. D. Starkman, Phys. Rev. D **74**, 044037 (2006).
[11] C. Skordis, Classical Quantum Gravity **26**, 143001 (2009).
[12] T. Bernal, S. Capozziello, J. C. Hidalgo, and S. Mendoza, Eur. Phys. J. C **71**, 1794 (2011).
[13] S. Mendoza, T. Bernal, X. Hernandez, J. C. Hidalgo, and L. A. Torres, Mon. Not. R. Astron. Soc. **433**, 1802 (2013).
[14] S. Mendoza and G. J. Olmo, Astrophys. Space Sci. **357**, 133 (2015).
[15] E. Barrientos and S. Mendoza, Eur. Phys. J. Plus **131**, 367 (2016).
[16] D. A. Carranza, S. Mendoza, and L. A. Torres, Eur. Phys. J. C **73**, 2282 (2013).
[17] E. Barrientos and S. Mendoza, Eur. Phys. J. Plus **132**, 361 (2017).
[18] T. Harko and F. S. N. Lobo, Eur. Phys. J. C **70**, 373 (2010).
[19] F. S. N. Lobo and T. Harko, arXiv:1211.0426.
[20] T. Harko, F. S. N. Lobo, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **84**, 024020 (2011).
[21] T. Harko, F. S. N. Lobo, and O. Minazzoli, Phys. Rev. D **87**, 047501 (2013).
[22] T. Harko, F. S. N. Lobo, G. Otalora, and E. N. Saridakis, Phys. Rev. D **89**, 124036 (2014).
[23] H. F. M. Goenner, Found. Phys. **14**, 865 (1984).
[24] S. Nojiri and S. D. Odintsov, Phys. Lett. B **599**, 137 (2004).
[25] G. Allemandi, A. Borowiec, M. Francaviglia, and S. D. Odintsov, Phys. Rev. D **72**, 063505 (2005).
[26] O. Bertolami, C. G. Böhmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007).
[27] M. Milgrom, Phys. Rev. E **56**, 1148 (1997).
[28] J. Bekenstein and M. Milgrom, Astrophys. J. **286**, 7 (1984).
[29] E. Barrientos and S. Mendoza, Eur. Phys. J. Plus **131**, 367 (2016).
[30] E. Barrientos and S. Mendoza, Eur. Phys. J. Plus **132**, 361 (2017).
[31] T. Koivisto, Classical Quantum Gravity **23**, 4289 (2006).
[32] T. Harko, Phys. Rev. D **90**, 044067 (2014).
[33] S. Vagnozzi, Classical Quantum Gravity **34**, 185006 (2017).