Special Finslerian generalization of the Reissner-Nordström spacetime

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We have obtained a Finslerian Reissner-Nordström solution where it is asymptotic to a Finsler spacetime with a constant flag curvature while $r \to \infty$. The covariant derivative of a modified Einstein tensor in a Finslerian gravitational field equation for this solution is conserved. The symmetry of the special Finslerian Reissner-Nordström spacetime, namely, Finsler spacetime with a constant flag curvature, has been investigated. It admits four independent Killing vectors. The Finslerian Reissner-Nordström solution differs from the Reissner-Nordström metric only in two-dimensional subspace, and our solution requires that its two-dimensional subspace has a constant flag curvature. We have obtained the eigenfunction of the Finslerian Laplacian operator of the "Finslerian sphere," namely, a special subspace with a positive constant flag curvature. The eigenfunction is of the form $\bar{Y}_l^m = Y_l^m + \epsilon^2 (C_{l+2}^m Y_{l+2}^m + C_{l-2}^m Y_{l-2}^m)$ in powers of the Finslerian parameter ϵ, where C_{l+2}^m and C_{l-2}^m are constant. However, the eigenvalue depends on both l and m. The eigenvalues corresponding to Y_1^0 remain the same with the Riemannian Laplacian operator and the eigenvalues corresponding to $Y_1^{\pm 1}$ are different. This fact just reflects the symmetry of the Finslerian sphere, which admits a z-axis rotational symmetry and breaks other symmetry of the Riemannian sphere. The eigenfunction of the Finslerian Laplacian operator implies that monopolar and dipolar terms of the multipole expansion of gravitational potential are unchanged and other multipole terms are changed.

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I. INTRODUCTION

A black hole is a specific region of spacetime that has such a strong enough gravity that even light cannot escape from it. Black hole physics has been discussed intensively by physicists. The research on black hole physics originates from the exact solution of Einstein's gravitational field equation. In four-dimensional spacetime, Schwarzschild and Kerr solutions are exact solutions of the Einstein vacuum field equation, which correspond to the spherical symmetry and axis symmetry, respectively, and Reissner-Nordström spacetime is a solution that corresponds to the gravitational field generated by a charged and spherical symmetric gravitational source. The general form of the solutions are called Kerr-Newmann spacetime, which corresponds to the gravitational field generated by a charged and axis symmetric gravitational source [1]. Inspired from Kerr-Newmann spacetime, the no-hair theorem of black holes was formulated by physicists [2]. It states that a black hole only has three properties, namely, mass, electric charge, and spin. The above solutions or spacetimes are asymptotically flat. Schwarzschild–de Sitter spacetime is a solution with cosmological horizons, and it is asymptotic to de Sitter spacetime [3]. Based on Kerr-Newmann spactime, Hawking and Bekenstein *et al.* involved the concept of entropy for black holes [4], constructed a theory of black hole thermodynamics, and then proposed four laws of black hole thermodynamics [5]. Hawking radiation [6], as an important complement of black hole thermodynamics, exhibits the quantum features of a black hole.

Three physical processes could lead to the formation of a black hole. One is the gravitational collapse of a heavy star [7]. The second is the gravitational collapse of the primordial overdensity in the early Universe [8–12]. The third is high-energy collisions [13]. The merge of a black hole binary could generate gravitational waves, such phenomena has been observed by the Advanced LIGO detectors, i.e., GW150914, GW151226, GW170104, and GW170814 [14–17].

It is interesting and important to search more exact solutions of gravitational field equations in four-dimensional spacetime, and black holes that corresponded to these solutions are expected to be tested by gravitational wave detectors in the near future. Finsler geometry [18] is a natural generalization of Riemannian geometry. The basic feature of Finsler geometry is that its length element does not have a quadratic restriction. Generally, the Finslerian extension of a given Riemannian spacetime has less symmetry than the Riemannian spacetime [19,20]. A typical example of Finsler spacetime, i.e., Randers spacetime [21], breaks rotational symmetry and induces a parity violation. By this basic feature of Finsler geometry, Finsler geometry is used to describe a violation of Lorentzian

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invariance [22–25] and study the anisotropy of our Universe [26,27].

We have suggested an anisotropic inflation model in which the background space is taken to be a Randers space. This anisotropic inflation model could account for the power asymmetry of the cosmic microwave background (CMB) [28]. In Finsler geometry, there is no unique extension of Riemannian geometric objects, such as the connections and curvature [18]. This feature is also involved in searching the gravitational field equation in Finsler spacetime [29–33]. We have proposed the gravitational field equation in Finsler spacetime and found a non-Riemannian exact solution [34]. This solution, or metric, is none other than the Schwarzschild metric, except for the change from the Riemann sphere to the "Finslerian sphere," and an interior solution for the Finslerian Schwarzschild metric also exists. We have proved that the covariant derivative of the Finslerian gravitational field equation for the metric is conserved. It is interesting to search a general solution for our Finslerian gravitational field equation and test if the black hole corresponding to the solution possesses the three properties, namely, mass, electric charge, and whether or not there is spin. The spherical harmonics are an eigenfunction of the Laplacian operator for the Riemann sphere, which plays an important role in modern physics. Since our Finslerian Schwarzschild solution admits a Finslerian sphere, it is worth investigating the eigenfunction of the Laplacian operator for a Finslerian sphere. It is expected that the symmetry of the Finslerian sphere has a direct influence on its eigenfunction.

This paper is organized as follows. In Sec. II, we give a brief introduction to the Reissner-Nordström metric. Then, we present the Finslerian Reissner-Nordström solution in Finsler spacetime. We discuss the symmetry of the Finslerian Reissner-Nordström metric at the end of Sec. II. In Sec. III, we present the Finslerian Laplacian operator of the Finslerian sphere, and give the eigenfunction and corresponding eigenvalue of the Finslerian Laplacian operator. Conclusions and remarks are given in Sec. IV.

II. EXACT SOLUTION OF GRAVITATIONAL FIELD EQUATION IN FINSLER SPACETIME

A. Brief introduction to the Reissner-Nordström metric

In general relativity, the Reissner-Nordström spacetime is given as follows

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_k^2, \tag{1}$$

where $f = k - \frac{2GM}{r} - br^2 + \frac{4\pi G Q^2}{r^2}$, and the metric $d\Omega_k^2$ denotes the two-dimensional metric with constant sectional curvature k. Usually, after a reparametrization, we can set k to be 1, 0, -1. The Reissner-Nordström is a solution of the Einstein field equation, i.e.,

$$Ric_{\mu\nu} - g_{\mu\nu} S/2 = 8\pi G T_{\mu\nu}, \tag{2}$$

where $Ric_{\mu\nu}$ is the Ricci tensor and $S = g^{\mu\nu} Ric_{\mu\nu}$ is the scalar curvature. The energy-momentum tensor of the Reissner-Nordström spacetime is given as

$$T_{\mu\nu} = T^{em}_{\mu\nu} + T^{c}_{\mu\nu}, \tag{3}$$

where $T^{em}_{\mu\nu} = \frac{Q^2}{2r^4} \mathrm{diag}\{f, -f, r^2 g^{\omega}_{ij}\}$ (g^{ω}_{ij} is the metric of $d\Omega_k^2$) denotes the energy-momentum tensor of the electromagnetic field and $T^c_{\mu\nu} = -3b g_{\mu\nu}/8\pi G$. The Reissner-Nordström spacetime will reduce to four-dimensional spacetime with a constant curvature, namely, the de Sitter spacetime, if $Q = M = 0$ and $k = 1$.

B. Finslerian Reissner-Nordström solution

Instead of defining an inner product structure over the tangent bundle in Riemann geometry, Finsler geometry is based on the so-called Finsler structure F with the property $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, where $x \in M$ represents position and $y \equiv \frac{dx}{d\tau}$ represents velocity. The Finslerian metric is given as [18]

$$g_{\mu\nu} \equiv \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} \left(\frac{1}{2} F^2\right). \tag{4}$$

In the Ref. [18], the Finslerian structure is positive definite. In physics, the Finsler structure F is not positive definite at every point of the Finsler manifold. A positive, zero, or negative F corresponds to timelike, null, or spacelike curves, respectively. Recently, Javaloyes and Sanch have presented a well-defined definition on the Finsler structures with Finsler metrics of Lorentzian signature [35]. Throughout this paper, all Finsler formula are valid for discussing the Finsler structures with Finsler metrics of Lorentzian signature. One can apply the approach used in Weinberg's book[36] for discussing the null particles. It uses a nonvanishing Finsler structure L to derive all formula and setting $F = EL$ where $E = 0$ for discussing the null particles.

Rutz has suggested that Finslerian vacuum gravitational field equation is vanish of the Ricci scalar[31]. In our previous research [34], we have obtained a solution of the Finslerian vacuum field equation which was suggested by Rutz. It is given as

$$F^2 = -\left(1 - \frac{2GM}{r}\right) y^t y^t + \left(1 - \frac{2GM}{r}\right)^{-1} y^r y^r + r^2 \bar{F}^2, \tag{5}$$

where $\bar{F}$ is a two dimensional Finsler space with positive constant flag curvature. The Ricci scalar is given as

$$Ric \equiv R^\mu{}_\mu = \frac{1}{F^2}\left(2\frac{\partial G^\mu}{\partial x^\mu} - y^\lambda \frac{\partial^2 G^\mu}{\partial x^\lambda \partial y^\mu} + 2G^\lambda \frac{\partial^2 G^\mu}{\partial y^\lambda \partial y^\mu} - \frac{\partial G^\mu}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\mu}\right), \tag{6}$$

where

$$G^{\mu} = \frac{1}{4} g^{\mu\nu} \left(\frac{\partial^2 F^2}{\partial x^{\lambda} \partial y^{\nu}} y^{\lambda} - \frac{\partial F^2}{\partial x^{\nu}} \right) \tag{7}$$

is called geodesic spray coefficients.

Now, we propose an ansatz that the Finsler structure is of the form

$$F^2 = -f(r) y^t y^t + f(r)^{-1} y^r y^r + r^2 \bar{F}^2(\theta, \varphi, y^{\theta}, y^{\varphi}). \tag{8}$$

Throughout the paper, the index labeled by Greek alphabet denote the index of four-dimensional spacetime F, and the index labeled by Latin alphabet denote the index of two-dimensional subspace $\bar{F}$. Plugging the Finsler structure (8) into the formula (7), we obtain that

$$G^t = \frac{f'}{2f} y^t y^r, \tag{9}$$

$$G^r = -\frac{f'}{4f} y^r y^r + \frac{f f'}{4} y^t y^t - \frac{r}{2A} \bar{F}^2, \tag{10}$$

$$G^{\theta} = \frac{1}{r} y^{\theta} y^r + \bar{G}^{\theta}, \tag{11}$$

$$G^{\varphi} = \frac{1}{r} y^{\varphi} y^r + \bar{G}^{\varphi}, \tag{12}$$

where the prime denotes the derivative with respect to r, and the $\bar{G}$ is the geodesic spray coefficients derived by $\bar{F}$. Plugging the geodesic coefficients (9)–(12) into the formula of Ricci scalar (6), we obtain that

$$F^2 Ric = \left[\frac{f f''}{2} + \frac{f f'}{r} \right] y^t y^t + \left[-\frac{f''}{2f} - \frac{f'}{rf} \right] y^r y^r + [\bar{R}ic - f - r f'] \bar{F}^2 \tag{13}$$

where $\bar{R}ic$ denotes the Ricci scalar of Finsler structure $\bar{F}$. We have used the property of homogenous function $H(\lambda y) = \lambda^n H(y)$, i.e., $y^{\mu} \frac{\partial H(y)}{\partial y^{\mu}} = n H(y)$, to derive the geodesic spray coefficients G^{μ} and Ricci scalar. By equation (13), we obtain the solution of $Ric = 0$. It is of the form

$$\bar{R}ic = k, \tag{14}$$

$$f_S = k - 2GM/r, \tag{15}$$

where $k = \pm 1, 0$. And the solution of constant Ricci scalar ($Ric = 3b$) is given as

$$\bar{R}ic = k, \tag{16}$$

$$f_{Sd} = k - 2GM/r - b r^2. \tag{17}$$

A Finsler spacetime with constant flag curvature K must have constant Ricci scalar $(n-1)K$ (n is the dimension of the spacetime). However, the reverse statement is not true. Now, we test whether the solution (17) is corresponded to the Finsler spacetime with constant flag curvature or not. A Finsler spacetime with constant flag curvature K is equivalent to its predecessor of flag curvature possess the following form [18]

$$F^2 R^{\mu}_{\nu} = K \left(F^2 \delta^{\mu}_{\nu} - \frac{y^{\mu}}{2} \frac{\partial F^2}{\partial y^{\nu}} \right), \tag{18}$$

where $F^2 R^{\mu}_{\nu}$ is defined as

$$F^2 R^{\mu}{}_{\nu} = 2 \frac{\partial G^{\mu}}{\partial x^{\nu}} - y^{\lambda} \frac{\partial^2 G^{\mu}}{\partial x^{\lambda} \partial y^{\nu}} + 2 G^{\lambda} \frac{\partial^2 G^{\mu}}{\partial y^{\lambda} \partial y^{\nu}} - \frac{\partial G^{\mu}}{\partial y^{\lambda}} \frac{\partial G^{\lambda}}{\partial y^{\nu}}. \tag{19}$$

Plugging the ansazt into the formula (19), after tedious calculation, we obtain

$$F^2 R^t_t = -\frac{f''}{2f} y^r y^r - \frac{r f'}{2} \bar{F}^2, \tag{20}$$

$$F^2 R^t_r = \frac{f''}{2f} y^t y^r, \tag{21}$$

$$F^2 R^t_i = \frac{r f'}{4} y^t \frac{\partial \bar{F}^2}{\partial y^i}, \tag{22}$$

$$F^2 R^r_r = \frac{f f''}{2} y^t y^t - \frac{r f'}{2} \bar{F}^2, \tag{23}$$

$$F^2 R^r_i = \frac{r f'}{4} y^r \frac{\partial \bar{F}^2}{\partial y^i}, \tag{24}$$

$$F^2 R^i_j = \bar{F}^2 \bar{R}^i_j + \left(\frac{f f'}{2r} y^t y^t - \frac{f'}{2rf} y^r y^r - f \bar{F}^2 \right) \delta^i_j + \frac{f}{2} y^i \frac{\partial \bar{F}^2}{\partial y^j}. \tag{25}$$

Plugging the solution (17) into the above formula (20)–(25), it is obvious from the formula (18) that Finsler metric (8) with the solution (17) does not have constant flag curvature if the parameters $M \neq 0$ and $b \neq 0$. However, it is Einstein metric since its Ricci scalar is constant. By making use of the property $R_{\mu\nu} = R_{\nu\mu}$ and noticing the $\bar{F}$ is a Finsler structure with constant Ricci scalar, we obtain that the special case of the solution (17) with parameter $M = 0$ and $b \neq 0$ is corresponded to Finsler spacetime with constant flag curvature, i.e., it satisfies the following relation

$$F^2 R^{\mu}_{\nu} = b \left(F^2 \delta^{\mu}_{\nu} - \frac{y^{\mu}}{2} \frac{\partial F^2}{\partial y^{\nu}} \right). \tag{26}$$

It means that the flag curvature is constant and equals to b.

Pfeifer *et al.* have studied the electromagnetic field in Finsler spacetime [37]. However, no specific solution of electromagnetic field equation, such as static electric field, is discussed. Now, we study the solution of Finslerian gravity with "electric charge." Analogy to the Riemannian Reissner-Nordström metric, we suggest that the Finsler metric has the form of the ansatz (8) with $f_{RN} = k - \frac{2GM}{r} + \frac{4\pi_F GQ^2}{r^2}$. Then, by formula (13), its Ricci scalar is of the form

$$F^2 Ric = \frac{4\pi_F GQ^2}{r^4}(f y^t y^t - f^{-1} y^r y^r + r^2 \bar{F}^2). \quad (27)$$

In Finslerian gravity, there are various generalizations of Einstein's gravitational field equations. In general, there are three types of generalizations of Einstein's gravitational field equations. Pfeifer *et al.* [33] have constructed gravitational dynamics for Finsler spacetimes in terms of an action integral on the sphere bundle. Miron *et al.* [30] have constructed Finslerian gravitational field equations from the second Bianchi identities in Finsler geometry. Vacaru *et al.* [32] have constructed Finslerian gravitational field equations by generalizing the Einstein gravitational field equation in terms of Finslerian geometrical quantities, i.e., replacing the Riemannian Ricci tensor and scalar curvature with the Finslerian Ricci tensor and scalar curvature. However, these Finslerian gravitational field equations are not equivalent to one another. At present, it is still an open debate about which Finslerian generalizations of gravitational field equations are physically relevant.

In Ref. [34], we have presented a Finslerian gravitational field equation. A Finslerian Schwarzschild spacetime (5) is the solution of the Finslerian vacuum gravitational field equation, and the interior solution of the Finslerian Schwarzschild spacetime is derived from the Finslerian gravitational field equation. In the Finslerian Schwarzschild spacetime, its geodesic equation returns to its counterpart in Newtonian gravity in the weak-field approximation. We proved that the Finslerian covariant derivative of the Finslerian gravitational field equation for the ansatz metric (8) is conserved. It should be noticed that our Finslerian gravitational field equation is valid for a special Finslerian spacetime (8). Because of the good properties of our Finslerian gravitational field equation, we expected that a general Finslerian gravitational field equation should involve our Finslerian gravitational field equation as its special case.

The specific form of our Finslerian gravitational field equation is given as [34]

$$G^\mu_\nu = 8\pi_F G T^\mu_\nu, \quad (28)$$

where the modified Einstein tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} \equiv Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S, \quad (29)$$

and $4\pi_F$ denotes the volume of $\bar{F}$. Here, the Ricci tensor we used is first introduced by Akbar-Zadeh[38]

$$Ric_{\mu\nu} = \frac{\partial^2(\frac{1}{2}F^2 Ric)}{\partial y^\mu \partial y^\nu}, \quad (30)$$

and the scalar curvature in Finslerian geometry is given as $S = g^{\mu\nu} Ric_{\mu\nu}$. Then, by making use of the equations (27), (29), (30), we obtain the nonvanishing components of the Einstein tensor

$$Ric_{\mu\nu} = G_{\mu\nu} = \frac{4\pi_F GQ^2}{r^4}\mathrm{diag}\{f, -f^{-1}, r^2 \bar{g}_{ij}\}. \quad (31)$$

One should notice that the scalar curvature S vanishes. It means that the trace of the energy-momentum tensor vanishes, and such a fact implies that the particle is massless in physics. Plugging the result of the Einstein tensor (31) into the field equation (28), we obtain the energy-momentum tensor

$$T_{\mu\nu} = \frac{Q^2}{2r^4}\mathrm{diag}\{f, -f^{-1}, r^2 \bar{g}_{\mathrm{ij}}\}. \quad (32)$$

In Ref. [34], we proved that the covariant derivative of the modified Einstein tensor is conserved in the Finsler spacetime (8), i.e., $G^\mu_{\nu|\mu} = 0$, where "|" denotes the covariant derivative. Since the Finslerian Schwarzschild-de Sitter solution $f = f_{Sd}$ and the Finslerian Reissner-Nordström solution $f = f_{RN}$ both possess the same form as the Finslerian Schwarzschild solution (8). Thus, following the same process of Ref. [34], one can find that the modified Einstein tensor and the energy-momentum tensor corresponding to the solution are also conserved. The energy-momentum tensor of the general electromagnetic field in Finsler spacetime should reduce to the energy-momentum tensor given above (32) if the Finsler spacetime reduces to the Finslerian Reissner-Nordström solution, i.e., the ansatz metric (8) with $f = f_{RN}$.

It should be noted that we do not generate the energy-momentum tensor (32) from a theory of electromagnetic fields in Finsler spacetime. However, the energy-momentum tensor of the Finslerian Reissner-Nordström spacetime will inspire us to construct a theory of electromagnetic fields in Finsler spacetime. This work will be done in our future research.

C. Symmetry of Finslerian Reissner-Nordström solution

In Riemannian geometry, spaces with a constant sectional curvature are equivalent, and all of these spaces have $n(n+1)/2$ independent Killing vectors. However, the number of independent Killing vectors of an n-dimensional non-Riemannian Finsler spacetime should be no more than $\frac{n(n-1)}{2} + 1$, $n \neq 4$ & $n \geq 3$ [19], and a four-dimensional non-Riemannian Finsler spacetime has no more than eight independent Killing vectors [39]. In general, the Finslerian extension of a given Riemannian spacetime has less symmetry than the Riemannian spacetime. For example, the Finslerian Schwarzschild spacetime $f = f_S$ only has two independent Killing vectors [34]. The Kerr spacetime has two independent Killing vectors. Thus, the Finslerian extension of Kerr spacetime should only have an independent Killing vector of one or zero. Furthermore, if we require the Finslerian extensional spacetime to be static, then an arbitrary static Finsler spacetime should be a Finslerian extension of Kerr spacetime. This fact implies that it is hard to find a Finslerian extension of Kerr spacetime. In Ref. [34], we have shown that the Finsler spacetime with the specific form (8) could form a horizon at $f = 0$. Therefore, at present, one can find from the Finslerian Reissner-Nordström solution that a Finslerian black hole has two properties, namely, mass and electric charge.

The Killing equation $K_V(F)$ in Finsler spacetime is of the form [20]

$$K_V(F) \equiv V^\mu \frac{\partial F}{\partial x^\mu} + y^\nu \frac{\partial V^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu} = 0. \tag{33}$$

Plugging the formula (4) into the Killing equation (33), we obtain

$$V^\mu \frac{\partial g_{\alpha\beta}}{\partial x^\mu} + g_{\alpha\lambda}\frac{\partial V^\lambda}{\partial x^\beta} + g_{\lambda\beta}\frac{\partial V^\lambda}{\partial x^\alpha} + y^\nu \frac{\partial V^\mu}{\partial x^\nu}\frac{\partial g_{\alpha\beta}}{\partial y^\mu} = 0. \tag{34}$$

The left side of Killing equation (34) is just the Lie derivative of the Finsler metric $g_{\alpha\beta}$ [40]. From the equation (34), we investigate the symmetry of the Finsler spacetime (8) with a constant flag curvature, i.e., $f = f_d = k - br^2$. It should be noted that the Killing equation (34) differs from Riemannian Killing equation in $y^\nu \frac{\partial V^\mu}{\partial x^\nu}\frac{\partial g_{\alpha\beta}}{\partial y^\mu}$, and the only component g_{ij} of the Finsler metric of the Finsler spacetime (8) has y dependence. Also, Finsler spacetime (8) with $f = f_d$ reduces to Riemannian spacetime with a constant sectional curvature if $\bar{F}$ reduces to a Riemannian surface with a constant sectional curvature. Therefore, there are three independent Killing vectors that have index t and r only, and these Killing vectors only depend on coordinates t and r. We have also shown in Ref. [34] that a Finslerian sphere with a constant positive flag curvature admits one Killing vector $V^\varphi = C^\varphi$, where C^φ is a constant. It is obvious that $V^\varphi = C^\varphi$ is the Killing vector of the Finsler spacetime (8) with a constant flag curvature. Finally, we conclude that the Finsler spacetime

$$F_d^2 = -(1 - br^2)y^t y^t + (1 - br^2)^{-1} y^r y^r + r^2 F_{FS}^2 \tag{35}$$

admits four independent Killing vectors. The specific form of the Finslerian sphere will be given in the next section. We have shown in Ref. [20] that a four-dimensional, projectively flat Randers spacetime with a constant flag curvature admits six independent Killing vectors. Thus, Finsler spacetime (35) and projectively flat Randers spacetime with a constant flag curvature are not equivalent; namely, after a coordinate transformation, one can change into other. This fact is quite different from the one in Riemannian geometry. Since each Riemannian spacetime with a constant sectional curvature is equivalent.

III. FINSLERIAN LAPLACIAN OPERATOR ON FINSLERIAN SPHERE

The eigenfunction of the Laplacian operator of a Riemannian 2-sphere is spherical harmonics. Since our Finslerian Reissner-Nordström solution in Finsler spacetime differs from Reissner-Nordström metric only in $\bar{F}$, the Laplacian operator of the Finslerian surface with a positive constant flag curvature is worth investigating. In the discussion of the above section, we have shown that Finsler spaces with a constant flag curvature may not be equivalent to each other. Thus, we adopt a specific form of the Finsler surface with a positive constant flag curvature to study its Laplacian operator, i.e., a two-dimensional Randers-Finsler space with a constant positive flag curvature [41]

$$F_{\rm FS} = \frac{\sqrt{(1-\epsilon^2\sin^2\theta)y^\theta y^\theta + \sin^2\theta y^\varphi y^\varphi}}{1-\epsilon^2\sin^2\theta} - \frac{\epsilon\sin^2\theta y^\varphi}{1-\epsilon^2\sin^2\theta}, \tag{36}$$

where $0 \le \epsilon < 1$. We call it a Finslerian sphere.

In Riemannian geometry, the Laplacian operator can be defined in several different ways [42], and these Laplacian operators are equivalent to each other. However, the Finslerian extension of these definitions will lead to different Laplacian operators. Various Finslerian Laplacian operators are respectively defined by Bao and Lackey [43], Shen [44], Barthelme [45]. In this paper, we will adopt the Finslerian Laplacian operator defined by Barthelme. The Finslerian Laplacian operator for the Finslerian sphere (36) is of the form [45]

$$\begin{aligned}\Delta_{FS} = {}& \frac{2(1-\epsilon^2\sin^2\theta)^{3/2}}{\sin^2\theta(1+\sqrt{1-\epsilon^2\sin^2\theta})}\frac{\partial^2}{\partial\varphi^2} \\ &+ \frac{2(1-\epsilon^2\sin^2\theta)}{1+\sqrt{1-\epsilon^2\sin^2\theta}}\frac{\partial^2}{\partial\theta^2} \\ &+ \frac{2\cos\theta(\epsilon^2\sin^2\theta+\sqrt{1-\epsilon^2\sin^2\theta})}{\sin\theta(1+\sqrt{1-\epsilon^2\sin^2\theta})}\frac{\partial}{\partial\theta}.\end{aligned} \tag{37}$$

While $\epsilon = 0$, the Finslerian Laplacian operator (37) reduces to the Riemanian Laplacian operator for a 2-sphere, and its eigenfunction is just spherical harmonics $Y_{lm}(\theta, \varphi)$, which satisfy

$$\Delta_{FS}|_{\epsilon=0} Y_l^m = -l(l+1) Y_l^m. \quad (38)$$

By using Eq. (38), one can find that Y_0^0, Y_1^0, $Y_1^{\pm 1}$ are eigenfunctions of the Finslerian Laplacian operator Δ_{FS}, and they satisfy the following equations

$$\Delta_{FS} Y_0^0 = 0, \qquad \Delta_{FS} Y_1^0 = -2 Y_1^0,$$
$$\Delta_{FS} Y_1^{\pm 1} = (-2 + 2\epsilon^2) Y_1^{\pm 1}. \quad (39)$$

It is obvious that the eigenvalues corresponding to Y_0^0, Y_1^0 remain the same with a Riemannian Laplacian operator and the eigenvalues corresponding to $Y_1^{\pm 1}$ are different. This fact just reflects the symmetry of the Finslerian sphere. As for the symmetry of the Finslerian sphere, we showed that z-axis rotational symmetry, which corresponds to Killing vector $V^\varphi = C^\varphi$, is preserved and other symmetry of the Riemanian 2-sphere is broken [34].

The symmetry of the Finslerian sphere may account for some specific physical phenomena, such as the power asymmetry of CMB [46], and such phenomena can be treated as a perturbation of standard physical theories. Thus, we expand the Finslerian Laplacian operator Δ_{FS} in powers of ϵ. To the first order in ϵ^2, the Finslerian Laplacian operator (37) is given as

$$\Delta_{FS} = \frac{4 - 5\epsilon^2 \sin^2\theta}{4\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} + \left(1 - \frac{3}{4}\epsilon^2 \sin^2\theta\right) \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\left(1 + \frac{3}{4}\epsilon^2 \sin^2\theta\right)\frac{\partial}{\partial\theta}. \quad (40)$$

By making use of the recurrence formula of spherical harmonics, to the first order in ϵ^2, we obtain the eigenfunction of the Finslerian Laplacian operator (40)

$$\bar{Y}_l^m = Y_l^m + \epsilon^2 (C_{l+2}^m Y_{l+2}^m + C_{l-2}^m Y_{l-2}^m), \quad (41)$$

where

$$C_{l+2}^m = -\frac{3l(l-1)}{8(2l+3)^2} \sqrt{\frac{(l+m+1)(l-m+1)(l+m+2)(l-m+2)}{(2l+1)(2l+5)}}, \quad (42)$$

$$C_{l-2}^m = \frac{3(l+1)(l+2)}{8(2l-1)^2} \sqrt{\frac{(l+m)(l-m)(l+m-1)(l-m-1)}{(2l+1)(2l-3)}}, \quad (43)$$

and the corresponding eigenvalue of the Finslerian Laplacian operator is given as

$$\lambda = -l(l+1) + \epsilon^2 \left(\frac{3(l-1)l(l+1)(l+2)}{2(2l-1)(2l+3)} + \frac{m^2(14l^3 + 21l^2 + 19l + 6)}{2(2l+1)(2l-1)(2l+3)}\right). \quad (44)$$

The spatial geometry of the Finslerian Schwarzschild metric is

$$F_{3d}^2 = y^r y^r + r^2 \bar{F}_{FS}^2, \quad (45)$$

while $M = 0$. Since $\bar{F}_{FS}$ does not depend on r, thus, following the definition of Finslerian Laplacian operator [45], we find that the Finslerian Laplace equation for the three-dimensional space (45) is of the form

$$\Delta W = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) + \Delta_{FS} W = 0. \quad (46)$$

The solution of the Finslerian Laplace equation (46) is of the form

$$W = (A r^{n_1} + B r^{n_2}) \bar{Y}_l^m, \quad (47)$$

where A and B are constants that depend on the boundary condition of the Finslerian Laplace equation, and

$$n_1 = \frac{-1 + \sqrt{1 - 4\lambda}}{2}, \quad (48)$$

$$n_2 = \frac{-1 - \sqrt{1 - 4\lambda}}{2}. \quad (49)$$

Two facts for the solution (47) should be noticed. One is the index n_1 and n_2 depend not only on l but also on m. Another fact is that the monopolar and dipolar term of the multipole expansion of gravitational potential is unchanged in a Finsler spacetime with a Finslerian sphere. It is consistent with our previous research given in Ref. [34] where we showed that the gravitational potential for the Schwarzschild-like spacetime is the same as the Newtonian gravity in a weak field approximation. However, due to the

Finslerian modification of the eigenvalue λ (44), other multipole terms are changed.

IV. DISCUSSIONS AND CONCLUSIONS

Besides the definition of the Ricci tensor, introduced by Akbar-Zadeh, Shen introduced another definition [47]

$$Ric_{\mu\nu} = (R_{\mu}{}^{\lambda}{}_{\lambda\nu} + R_{\nu}{}^{\lambda}{}_{\lambda\mu})/2, \tag{50}$$

where $R_{\mu}{}^{\lambda}{}_{\lambda\nu}$ denotes the Riemann curvature tensor of the Berwald connection. The two definitions are equivalent if the Finsler spacetime has a constant flag curvature. However, one can check that the two definitions are not equivalent for the ansatz metric with $f_S = k - \frac{2GM}{r}$, i.e., the Ricci flat case. Since the Ricci tensor introduced by Akbar-Zadeh corresponds to the Finslerian gravitational field equation (28) with exact solutions, such as $f = f_S$, $f = f_{Sd}$ and $f = f_{RN}$, the Ricci tensor introduced by Akbar-Zadeh is therefore preferred in physics.

In Finsler geometry, there are two types of volume forms; namely, the Busemann-Hausdorff volume form and Holmes-Thompson volume form [44]. We have shown in Ref. [34] that the volume of the Finslerian sphere in terms of the Busemann-Hausdorff volume form is 4π. The Finslerian Laplacian operator for the Finslerian sphere (37) is defined on a fiber bundle, and its definition is related to the Holmes-Thompson volume form [45]. The volume of the Finslerian sphere in terms of the Holmes-Thompson volume form is given as

$$\mathrm{Vol}_{FS} = \int \frac{\sin\theta}{(1-\epsilon^2\sin^2\theta)^{3/2}} d\theta \wedge d\varphi = \frac{4\pi}{1-\epsilon^2}. \tag{51}$$

The two definitions of volume form will slightly alter the Finslerian gravitation field equation (28), for the term $8\pi_F$ is double the surface volume of Finsler space $\bar{F}$. In general relativity, black hole entropy depends on the surface volume of the black hole. Thus, studying black hole thermodynamics in Finsler spacetime will help us find which volume form is preferred in physics. It will be discussed in our future work.

In this paper, we have obtained the Finslerian Reissner-Nordström solution where it is asymptotic to a Finsler spacetime with a constant flag curvature while $r \to \infty$ (35). The covariant derivative of a modified Einstein tensor in a Finslerian gravitational field equation for this solution is conserved. It should be noted that the covariant derivative in Finsler geometry is directly dependent on the connection. In Finsler geometry, there are types of connections, such as the Chern connection, the Cartan connection, and the Berwald connection [18]. In this paper, the covariant derivative we used is defined by the Chern connection, which is the same as in Ref. [34]. The symmetry of the special Finslerian Reissner-Nordström spacetime (35) has been investigated. It admits four independent Killing vectors. The Finslerian Reissner-Nordström solution only differs from the Reissner-Nordström metric in two-dimensional subspace $\bar{F}$. The Finslerian Reissner-Nordström could form horizons at $f_{RN} = 0$. At present, we can conclude that the Finslerian black hole has at least two properties, namely, mass and electric charge. Our solutions show that two-dimensional subspace $\bar{F}$ has a constant flag curvature. The eigenfunction of the Laplacian operator of the Riemannian 2-sphere is spherical harmonics. We have obtained the eigenfunction of the Finslerian Laplacian operator, introduced by Barthelme [45], of the Finslerian sphere (36). The eigenfunction (41) is just a combination of spherical harmonics in powers of the Finslerian parameter ϵ. However, the eigenvalue depends on both l and m. The eigenvalues corresponding to Y_1^0 remain the same with the Riemannian Laplacian operator, and the eigenvalues corresponding to $Y_1^{\pm1}$ are different. This fact just reflects the symmetry of the Finslerian sphere, which admits a Killing vector $V^{\varphi} = C^{\varphi}$ and breaks other symmetry of the Riemannian sphere. The eigenfunction (41) of the Finslerian Laplacian operator implies that monopolar and dipolar terms of a multipole expansion of gravitational potential are unchanged and other multipole terms are changed.

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