Imperfect fluid description of modified gravities

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The Brans-Dicke-like field of scalar-tensor gravity can be described as an imperfect fluid in an approach in which the field equations are regarded as effective Einstein equations. After completing this approach, we recover, as a special case, the known effective fluid for a scalar coupled nonminimally to the Ricci curvature, and we describe the imperfect fluid equivalent of $f(R)$ gravity. A symmetry of electrovacuum Brans-Dicke gravity is translated into a symmetry of the corresponding effective fluid. The discussion is valid for any spacetime geometry.

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I. INTRODUCTION

The 1998 discovery of the present acceleration of the Universe with type Ia supernovae requires a theoretical explanation. The standard cosmological model based on general relativity (GR), i.e., the Λ cold dark matter model, requires either an incredibly small cosmological constant Λ or a dark energy with very negative pressure introduced completely *ad hoc* [1]. As an alternative to dark energy, many researchers have turned to modifying gravity on cosmological scales. This option is far from unrealistic because GR has been tested only in a small range of regimes [2]. While many such possibilities exist [3], the class of $f(R)$ theories of gravity [4] seems by far the most popular. Although this class is nothing but old scalar-tensor gravity [5,6] in disguise, many of its features, related or unrelated to cosmology, were only understood in the last decade [7].

Independent motivation for modifying Einstein's theory of gravity comes from high-energy physics. Virtually every attempt to quantize GR introduces deviations from this theory in the form of extra degrees of freedom, higher order derivatives in the field equations, higher powers of the curvature in the action, or nonlocal terms. For example, the low-energy limit of the simplest string theory, the bosonic string theory, yields Brans-Dicke gravity [5] with Brans-Dicke coupling parameter $\omega = -1$ [8].

In many areas of research, especially in cosmology and in models of neutron star and white dwarf interiors, it is common to use fluids as the matter source of the Einstein equations. In alternative theories of gravity, the more complicated field equations are often recast as effective Einstein equations by moving geometric terms other than those entering the Einstein tensor $G_{ab} \equiv R_{ab} - g_{ab}R/2$ (where R_{ab} and $R \equiv g^{ab}R_{ab}$ are the Ricci tensor and the Ricci scalar, respectively) to the right-hand side and by regarding them as an effective energy-momentum tensor. This approach has proven very useful in reducing problems of alternative gravity to known problems of GR (e.g., Ref. [9]). But how should one interpret the right-hand side of these field equations? In this approach, it makes sense to ask whether this effective stress-energy tensor can be formally regarded as a fluid, given that a fluid is used so often as the matter source in relativistic physics. It is not obvious that this will work, given the stress-energy tensor's origin and the fact that it is merely an *effective* stress-energy tensor. However, it turns out to be true [10]. In Einstein's theory, an effective perfect fluid description can be given for a canonical, minimally coupled scalar field ϕ [11–15], and this fact is well known for special spacetimes, such as the Friedmann-Lemaître-Robertson-Walker (FLRW) spaces used in cosmology. Roughly speaking, fixing the scalar field potential $V(\phi)$ corresponds to prescribing the equation of state of the fluid, but this is not a one-to-one correspondence [12,16–19]. The effective fluid description of more general theories containing a scalar field, such as k-essence and special cases of Horndeski gravity, has been worked out in detail with respect to cosmological perturbations or to general spacetimes [18–21]. In the case of general spacetimes, the description of a theory containing a scalar (to which we restrict ourselves) as an effective fluid should not be taken for granted. A very interesting nonstandard scenario is the one of Ref. [22] in which scalar field-fluid elements move along timelike geodesics but the pressure is not vanishing, thanks to a second scalar

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field acting as a Lagrange multiplier. Another possibility is the higher derivative mimetic dark matter scenario containing an effective imperfect fluid which is generated by a scalar field [23]. Similar to our paper, this effective fluid has an energy flow, but contrary to our case, it also has vorticity [23].

Here, we focus on relatively simple scalar field theories, for which the effective fluid description is not yet complete. Among all possible alternatives to GR, it is natural to first consider scalar-tensor gravity [5,6], which only adds a (usually massive) scalar degree of freedom, the Brans-Dicke-like scalar ϕ, to the two massless, spin-2 polarizations contained in the metric tensor g_{ab} and familiar from GR. The correspondence between the effective stress-energy tensor of ϕ and a fluid has been worked out explicitly, first for the case of a nonminimally coupled scalar field [11] and then for general scalar-tensor gravity [10]. In general, the corresponding fluid is an imperfect fluid, contrary to the case of a minimally coupled scalar, which can always be described as a perfect fluid when the scalar field gradient is timelike. However, in special spacetimes endowed with symmetries, it may be possible to recover the perfect fluid behavior also for nonminimally coupled scalars [24,25].

Here, we extend and complete the correspondence between an (imperfect) effective fluid and the Brans-Dicke-like field, we show how a symmetry of Brans-Dicke gravity translates into a symmetry of this fluid, and we apply the discussion to $f(R)$ gravity. We do not restrict to special situations such as cosmology or black holes, and our discussion is valid for general spacetime geometries.

Scalar-tensor gravity is described by the Jordan frame action (we follow the notation of Ref. [26], and we use units in which Newton's constant G and the speed of light c are unity)

$$S_{\mathrm{ST}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] + S^{(m)}, \tag{1.1}$$

where $\phi > 0$ is the Brans-Dicke scalar (approximately equivalent to the inverse of the effective gravitational coupling strength), the function $\omega(\phi)$ (which was a strictly constant parameter in the original Brans-Dicke theory [5]) is the "Brans-Dicke coupling," $V(\phi)$ is a scalar field potential (absent in the original Brans-Dicke theory), whereas $S^{(m)} = \int d^4x\sqrt{-g}\mathcal{L}^{(m)}$ describes the matter sector. Since our task here regards only the gravitational sector, we will not need to specify this ordinary matter.

The (Jordan frame) field equations obtained by varying the action (1.1) with respect to the inverse metric g^{ab} and to the scalar ϕ are [5,6]

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi}{\phi}T^{(m)}_{ab} + \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}, \tag{1.2}$$

$$\Box\phi = \frac{1}{2\omega+3}\left(\frac{8\pi T^{(m)}}{\phi} + \phi\frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi}\nabla^c\phi\nabla_c\phi\right), \tag{1.3}$$

where $T^{(m)} \equiv g^{ab}T^{(m)}_{ab}$ is the trace of the matter stress-energy tensor $T^{(m)}_{ab}$. The matter energy-momentum tensor and the effective stress-energy tensor of the scalar ϕ are covariantly conserved separately. Let us proceed to examine the effective fluid description of these field equations and their implications.

II. KINEMATICS OF THE SCALAR FIELD FLUID

In this section, we identify the kinematic quantities which describe the effective fluid associated with the Brans-Dicke-like scalar field. The ϕ-fluid correspondence is possible when the gradient $\nabla^a\phi$ is timelike; then, one can introduce the fluid 4-velocity

$$u^a = \frac{\nabla^a\phi}{\sqrt{-\nabla^e\phi\nabla_e\phi}}, \tag{2.1}$$

which is clearly normalized, $u^c u_c = -1$. This timelike vector field determines the $3+1$ splitting of spacetime into the three-dimensional space "seen" by the comoving observers of the fluid and their time direction u^a. This 3-space is endowed with the Riemannian metric

$$h_{ab} \equiv g_{ab} + u_a u_b, \tag{2.2}$$

while $h_a{}^b$ is the usual projection operator on this 3-space and satisfies

$$h_{ab}u^a = h_{ab}u^b = 0, \tag{2.3}$$

$$h^a{}_b h^b{}_c = h^a{}_c, \qquad h^a{}_a = 3. \tag{2.4}$$

The fluid 4-acceleration

$$\dot{u}^a \equiv u^b\nabla_b u^a \tag{2.5}$$

is orthogonal to the 4-velocity, $\dot{u}^c u_c = 0$.

The (double) projection of the velocity gradient onto the 3-space orthogonal to u^c is the purely spatial tensor

$$V_{ab} \equiv h_a{}^c h_b{}^d \nabla_d u_c, \tag{2.6}$$

which is decomposed into its symmetric and antisymmetric parts, while the symmetric part is further decomposed into its trace-free and pure trace parts,

$$V_{ab} = \theta_{ab} + \omega_{ab} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab}, \tag{2.7}$$

where the expansion tensor $\theta_{ab} = V_{(ab)}$ is the symmetric part of V_{ab}, $\theta \equiv \theta^c{}_c = \nabla^c u_c$ is its trace, the vorticity tensor $\omega_{ab} = V_{[ab]}$ is its antisymmetric part, and the shear tensor

$$\sigma_{ab} \equiv \theta_{ab} - \frac{\theta}{3} h_{ab} \tag{2.8}$$

is the trace-free part of θ_{ab}. Like h_{ab} and V_{ab}, expansion, vorticity, and shear are purely spatial tensors,

$$\theta_{ab} u^a = \theta_{ab} u^b = \omega_{ab} u^a = \omega_{ab} u^b = \sigma_{ab} u^a = \sigma_{ab} u^b = 0, \tag{2.9}$$

and $\sigma^a{}_a = \omega^a{}_a = 0$ by definition. The shear scalar σ and the vorticity scalar ω (not to be confused with the Brans-Dicke coupling) are defined by

$$\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab}, \tag{2.10}$$

$$\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^{ab}, \tag{2.11}$$

and they are both non-negative. In general, we have [27]

$$\nabla_b u_a = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} - \dot{u}_a u_b = V_{ab} - \dot{u}_a u_b. \tag{2.12}$$

The projection of this equation onto the time direction produces $\dot{u}_a$, while the projection onto the 3-space orthogonal to u^a gives V_{ab}.

Let us specialize these general definitions [26,27] to our particular case. Contrary to the effective stress-energy tensor, the corresponding kinematic quantities for the effective fluid were not given in Ref. [10], which only discussed the equivalence of a Brans-Dicke field with a fluid.

The definition (2.1) of u^a gives

$$h_{ab} = g_{ab} - \frac{\nabla_a \phi \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \tag{2.13}$$

and the velocity gradient

$$\nabla_b u_a = \frac{1}{\sqrt{-\nabla^e \phi \nabla_e \phi}} \left(\nabla_a \nabla_b \phi - \frac{\nabla_a \phi \nabla^c \phi \nabla_b \nabla_c \phi}{\nabla^e \phi \nabla_e \phi} \right). \tag{2.14}$$

The acceleration, its norm $\dot{u}^a \dot{u}_a$, and its divergence $\nabla_a \dot{u}^a$ are

$$\dot{u}_a = (-\nabla^e \phi \nabla_e \phi)^{-2} \nabla^b \phi [(-\nabla^e \phi \nabla_e \phi) \nabla_a \nabla_b \phi + \nabla^c \phi \nabla_b \nabla_c \phi \nabla_a \phi], \tag{2.15}$$

$$\dot{u}^a \dot{u}_a = (-\nabla^e \phi \nabla_e \phi)^{-3} [-\nabla^e \phi \nabla_e \phi \nabla_b \phi \nabla^d \phi \nabla^b \nabla^a \phi \nabla_d \nabla_a \phi + (\nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi)^2], \tag{2.16}$$

$$\begin{aligned} \nabla_a \dot{u}^a = {} & (-\nabla^e \phi \nabla_e \phi)^{-2} [-\nabla^e \phi \nabla_e \phi \nabla^b \phi \Box (\nabla_b \phi) + \nabla^c \phi \nabla^a \phi \nabla^b \phi \nabla_b \nabla_a \nabla_c \phi] \\ & + (-\nabla^e \phi \nabla_e \phi)^{-3} [(\nabla^e \phi \nabla_e \phi)^2 \nabla^a \nabla^b \phi \nabla_a \nabla_b \phi - \nabla^e \phi \nabla_e \phi \nabla^b \phi \nabla^c \phi \nabla_b \nabla_c \phi \Box \phi \\ & - 4 (\nabla^e \phi \nabla_e \phi) \nabla^c \phi \nabla_b \phi \nabla_a \nabla_c \phi \nabla^b \nabla^a \phi + 4 (\nabla^a \phi \nabla^b \phi \nabla_b \nabla_a \phi)^2]. \end{aligned} \tag{2.17}$$

Using Eqs. (2.15) and (2.1), it is straightforward to check explicitly that $\dot{u}_c u^c = 0$. The timelike worldlines of the fluid elements, with 4-tangents u^a, are geodesics if and only if $\dot{u}_a = 0$, or

$$\nabla^e \phi \nabla_{[e} \phi \nabla_{a]} \nabla_b \phi \nabla^b \phi = 0 \tag{2.18}$$

and the tensor V_{ab} defined by Eq. (2.6) reduces to

$$\begin{aligned} V_{ab} = {} & \frac{\nabla_a \nabla_b \phi}{(-\nabla^e \phi \nabla_e \phi)^{1/2}} \\ & + \frac{(\nabla_a \phi \nabla_b \nabla_c \phi + \nabla_b \phi \nabla_a \nabla_c \phi) \nabla^c \phi}{(-\nabla^e \phi \nabla_e \phi)^{3/2}} \\ & + \frac{\nabla_d \nabla_c \phi \nabla^c \phi \nabla^d \phi}{(-\nabla^e \phi \nabla_e \phi)^{5/2}} \nabla_a \phi \nabla_b \phi. \end{aligned} \tag{2.19}$$

The vorticity tensor $\omega_{ab} \equiv V_{[ab]}$ vanishes identically, together with the vorticity scalar because the fluid 4-velocity u^c originates from a gradient (this fact is, of course, consistent with the general statement that $\omega = 0$ if and only if $\omega_{ab} = 0$ [27]). Then, we have, for the ϕ-fluid,

$$V_{ab} = \theta_{ab}, \qquad \nabla_b u_a = \theta_{ab} - \dot{u}_a u_b, \tag{2.20}$$

and the vector field u^a is hypersurface-orthogonal, and the line element can be diagonalized in an appropriate coordinate system. There exists a family of three-dimensional hypersurfaces Σ with Riemannian metric h_{ab} which are orthogonal to the 4-velocity field u^a and coincide with the 3-spaces seen by observers comoving with the fluid, who have 4-velocity u^a [26,27].

Since u^a and $\dot{u}^a$ are orthogonal, it is clear from Eq. (2.12) that the expansion scalar reduces to the divergence

$$\theta = \nabla_a u^a = \frac{\Box\phi}{(-\nabla^e\phi\nabla_e\phi)^{1/2}} + \frac{\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi}{(-\nabla^e\phi\nabla_e\phi)^{3/2}}. \tag{2.21}$$

The shear tensor is

$$\sigma_{ab} = (-\nabla^e\phi\nabla_e\phi)^{-3/2}\Big[-(\nabla^e\phi\nabla_e\phi)\nabla_a\nabla_b\phi - \frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\Box\phi - \frac{1}{3}\left(g_{ab} + \frac{2\nabla_a\phi\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right)\nabla_c\nabla_d\phi\nabla^d\phi\nabla^c\phi + (\nabla_a\phi\nabla_c\nabla_b\phi + \nabla_b\phi\nabla_c\nabla_a\phi)\nabla^c\phi\Big], \tag{2.22}$$

while the shear scalar reads

$$\sigma = \left(\frac{1}{2}\sigma^{ab}\sigma_{ab}\right)^{1/2} = (-\nabla^e\phi\nabla_e\phi)^{-3/2}\Big\{\frac{1}{2}(\nabla^e\phi\nabla_e\phi)^2\Big[\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi - \frac{1}{3}(\Box\phi)^2\Big] + \frac{1}{3}(\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi)^2 - (\nabla^e\phi\nabla_e\phi)\left(\nabla_a\nabla_b\phi\nabla^b\nabla_c\phi - \frac{1}{3}\Box\phi\nabla_a\nabla_c\phi\right)\nabla^a\phi\nabla^c\phi\Big\}^{1/2}. \tag{2.23}$$

Since $\sigma^2 \geq 0$, Eq. (2.23) yields the inequality

$$-\nabla^e\phi\nabla_e\phi\Big[\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi - \frac{1}{3}(\Box\phi)^2\Big] + 2\left(\nabla_a\nabla_b\phi\nabla^b\nabla_c\phi - \frac{1}{3}\Box\phi\nabla_a\nabla_c\phi\right)\nabla^a\phi\nabla^c\phi - \frac{2(\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi)^2}{3\nabla^e\phi\nabla_e\phi} \geq 0. \tag{2.24}$$

Remember that $\sigma^2 = 0$ if and only if $\sigma_{ab} = 0$ [27]. In many applications involving the focusing or defocusing of the fluid worldlines, it is essential to know the sign of the expansion, which is given by Eq. (2.21):

$$\theta \geq 0 \Leftrightarrow \nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi - \nabla^e\phi\nabla_e\phi\Box\phi \geq 0. \tag{2.25}$$

III. SCALAR FIELD EFFECTIVE STRESS-ENERGY TENSOR

In scalar-tensor gravity, the effective stress-energy tensor of the Brans-Dicke-like field is given by

$$8\pi T_{ab}^{(\phi)} = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi) - \frac{V}{2\phi}g_{ab}. \tag{3.1}$$

This effective stress-energy tensor and the matter stress-energy tensor are covariantly conserved separately,

$$\nabla^b T_{ab}^{(m)} = 0, \qquad \nabla^b T_{ab}^{(\phi)} = 0. \tag{3.2}$$

It was shown in Ref. [10] that the effective energy-momentum tensor $T_{ab}^{(\phi)}$ can be written as the stress-energy tensor of an imperfect fluid. The latter admits the decomposition

$$T_{ab} = \rho u_a u_b + q_a u_b + q_b u_a + \Pi_{ab}, \tag{3.3}$$

where

$$\rho = T_{ab}u^a u^b, \tag{3.4}$$

$$q_a = -T_{cd}u^c h_a{}^d, \tag{3.5}$$

$$\Pi_{ab} \equiv P h_{ab} + \pi_{ab} = T_{cd}h_a{}^c h_b{}^d, \tag{3.6}$$

$$P = \frac{1}{3}g^{ab}\Pi_{ab} = \frac{1}{3}h^{ab}T_{ab}, \tag{3.7}$$

$$\pi_{ab} = \Pi_{ab} - P h_{ab} \tag{3.8}$$

are, respectively, the effective energy density, heat flux density, stress tensor,[1] isotropic pressure, and anisotropic stresses (the trace-free part π_{ab} of the stress tensor Π_{ab}) in the comoving frame. In this frame, by definition, the fluid elements are at rest, and the heat flux density, which is the only energy flow, is purely spatial,

$$q_c u^c = 0, \tag{3.9}$$

while

[1] The right-hand side in the last equality of (3.6) points to an incorrect sign in Eq. (9) of Ref. [10], but this incorrect sign was not used in the rest of this reference.

$$\Pi_{ab}u^b = \pi_{ab}u^b = \Pi_{ab}u^a = \pi_{ab}u^a = 0, \quad \pi^a{}_a = 0. \tag{3.10}$$

The covariant conservation of $T_{ab}^{(\phi)}$ can be projected along the time direction u^a and on the 3-space with Riemannian metric h_{ab}, which yields [27]

$$u^a\nabla_a\rho^{(\phi)} + (P^{(\phi)} + \rho^{(\phi)})\theta + \Pi^{ab}\sigma_{ab} + \nabla^a q_a + q^a\dot{u}_a = 0, \tag{3.11}$$

$$(P^{(\phi)} + \rho^{(\phi)})\dot{u}_a + h_a{}^c(\nabla_c P^{(\phi)} + \nabla^b\Pi_{cb} + u^e\nabla_e q_c) + \left(\omega_a{}^b + \sigma_a{}^b + \frac{4}{3}\theta h_a{}^b\right)q_b = 0. \tag{3.12}$$

Since we used the fluid 4-velocity u^c and the corresponding 3-metric h_{ab} to project, the imperfect fluid quantities defined in this way are those in the comoving frame. When calculated explicitly, they read [10]

$$8\pi\rho^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi} + \frac{1}{\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{3.13}$$

$$8\pi q_a^{(\phi)} = \frac{\nabla^c\phi\nabla^d\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{3/2}}(\nabla_d\phi\nabla_c\nabla_a\phi - \nabla_a\phi\nabla_c\nabla_d\phi) \tag{3.14}$$

$$= -\frac{\nabla^c\phi\nabla_a\nabla_c\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{1/2}} - \frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d\phi}{\phi(-\nabla^e\phi\nabla_e\phi)^{3/2}}\nabla_a\phi, \tag{3.15}$$

$$8\pi\Pi_{ab}^{(\phi)} = (-\nabla^e\phi\nabla_e\phi)^{-1}\left[\left(-\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^e\phi\nabla_e\phi) - \frac{\nabla^d\phi}{\phi}\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\right] \tag{3.16}$$

$$= \left(-\frac{\omega}{2\phi^2}\nabla^c\phi\nabla_c\phi - \frac{\Box\phi}{\phi} - \frac{V}{2\phi}\right)h_{ab} + \frac{1}{\phi}h_a{}^c h_b{}^d\nabla_c\nabla_d\phi, \tag{3.17}$$

$$8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} - \frac{1}{3\phi}\left(2\Box\phi + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right), \tag{3.18}$$

$$8\pi\pi_{ab}^{(\phi)} = \frac{1}{\phi\nabla^e\phi\nabla_e\phi}\left[\frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\left(\Box\phi - \frac{\nabla^c\phi\nabla^d\phi\nabla_d\nabla_c\phi}{\nabla^e\phi\nabla_e\phi}\right) + \nabla^d\phi\left(\nabla_d\phi\nabla_a\nabla_b\phi - \nabla_b\phi\nabla_a\nabla_d\phi - \nabla_a\phi\nabla_d\nabla_b\phi + \frac{\nabla_a\phi\nabla_b\phi\nabla^c\phi\nabla_c\nabla_d\phi}{\nabla^e\phi\nabla_e\phi}\right)\right], \tag{3.19}$$

$$8\pi T^{(\phi)} \equiv 8\pi g^{ab}T_{ab}^{(\phi)} = -\frac{\omega}{\phi^2}\nabla^c\phi\nabla_c\phi - \frac{3\Box\phi}{\phi} - \frac{2V}{\phi}. \tag{3.20}$$

Apart from different notations, Eqs. (3.13)–(3.20) agree with the corresponding expressions of Ref. [10]. It is straightforward to check explicitly that q^a, Π^{ab}, π^{ab} are purely spatial and that the trace of the effective stress-energy tensor of ϕ coincides, as it should, with $-\rho^{(\phi)} + 3P^{(\phi)}$, with $\rho^{(\phi)}$ and $P^{(\phi)}$ given by Eqs. (3.13) and (3.18). Furthermore, by comparing Eqs. (3.14) and (2.15), one obtains

$$q_a^{(\phi)} = -\frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\dot{u}_a, \tag{3.21}$$

making obvious the fact that this vector is purely spatial. What is more, it is easy to see that in general the heat flux density $q_a^{(\phi)}$ and the anisotropic stresses $\pi_{ab}^{(\phi)}$ do not vanish and that the effective fluid is necessarily an imperfect one. The relation (3.21), which seems to have gone unnoticed in the literature thus far, has a physical consequence in Eckart's first order thermodynamics [28] (which is notoriously plagued by noncausality and instability but is still widely used as an approximation). In this theory, the heat flux density is related to the temperature $\mathcal{T}$ by the generalized Fourier law [28]

$$q_a = -K(h_{ab}\nabla^b \mathcal{T} + \mathcal{T}\dot{u}_a), \tag{3.22}$$

where K is the thermal conductivity. The comparison of Eqs. (3.21) and (3.22) leads to the result that, in the comoving frame, the spatial temperature gradient vanishes and the heat flow is then due purely to the inertia of energy described by the acceleration term in Eq. (3.22). Moreover, the product of the thermal conductivity and the temperature of the effective fluid is

$$K\mathcal{T} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}, \tag{3.23}$$

which is positive definite.

An alternative approach consists of trading temperature with chemical potential, assigning zero temperature and entropy but nonzero chemical potential to the effective fluid. This approach is pioneered in Ref. [19] (note the similarity between our Eqs. (3.21) and (3.35) of Ref. [18]).

It is sometimes convenient to replace the d'Alembertian $\Box\phi$ in the expressions of $\rho^{(\phi)}$ and $P^{(\phi)}$ with its value obtained from the field equation (1.3). For reference, we provide the corresponding expressions

$$\begin{aligned} 8\pi\rho^{(\phi)} = & -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi + \frac{V}{2\phi}\left(\frac{2\omega-1}{2\omega+3}\right) \\ & + \frac{1}{\phi}\left[\frac{1}{2\omega+3}\left(\phi\frac{dV}{d\phi} - \nabla^e\phi\nabla_e\phi\frac{d\omega}{d\phi}\right)\right. \\ & \left. - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right], \end{aligned} \tag{3.24}$$

$$\begin{aligned} 8\pi P^{(\phi)} = & -\frac{\omega}{2\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{6\phi}\frac{(6\omega+1)}{(2\omega+3)} \\ & - \frac{1}{3\phi}\left[\frac{2}{2\omega+3}\left(\phi\frac{dV}{d\phi} - \nabla^e\phi\nabla_e\phi\frac{d\omega}{d\phi}\right)\right. \\ & \left. + \frac{\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi}{\nabla^e\phi\nabla_e\phi}\right]. \end{aligned} \tag{3.25}$$

It is now straightforward to check that the imperfect fluid stress-energy tensor (3.3) is reproduced by $T_{ab}^{(\phi)}$. In fact, adding Eqs. (3.13)–(3.19) with the appropriate coefficients, one obtains

$$\rho^{(\phi)}u_a u_b + q_a^{(\phi)}u_b + q_b^{(\phi)}u_a + \Pi_{ab}^{(\phi)} = T_{ab}^{(\phi)}. \tag{3.26}$$

Being built by hand out of geometric or gravitational terms, the effective fluid stress-energy tensor $T_{ab}^{(\phi)}$, in general, does not satisfy any energy condition because of the presence of second derivatives of ϕ. The weak energy condition $T_{ab}t^a t^b \geq 0$ for all timelike vectors t^a [26] becomes, for $T_{ab}^{(\phi)}$ and the fluid 4-velocity,

$$-\frac{\omega}{2\phi}\nabla^e\phi\nabla_e\phi + \frac{V}{2} + \Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi} \geq 0. \tag{3.27}$$

The strong energy condition $(T_{ab} - Tg_{ab}/2)t^a t^b \geq 0$ for all timelike vectors t^a [26] reads, when applied to u^a and to $T_{ab}^{(\phi)}$,

$$\begin{aligned} \left(T_{ab}^{(\phi)} - \frac{1}{2}T^{(\phi)}g_{ab}\right)u^a u^b = & \frac{1}{2}(\rho^{(\phi)} + 3P^{(\phi)}) \\ = & -\frac{\omega}{\phi^2}\nabla^e\phi\nabla_e\phi - \frac{V}{2\phi} \\ & + \frac{1}{\phi}\left[-\frac{1}{2}\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right] \geq 0. \end{aligned} \tag{3.28}$$

However, as noted above, it is not physically meaningful to impose these energy conditions on $T_{ab}^{(\phi)}$. Moreover, the energy conditions reported involve density and stresses *in the comoving frame* because we imposed that the observer coincides with the fluid 4-velocity (i.e., $t^a = u^a$). The energy conditions of an imperfect fluid with respect to arbitrary timelike observers are discussed in Refs. [29,30].

IV. SPECIAL CASE: THE NONMINIMALLY COUPLED SCALAR FIELD

The correspondence between a Brans-Dicke-like scalar ϕ and an imperfect fluid was studied in Ref. [11], for general spacetimes, in the case of a scalar field coupling nonminimally to the Ricci curvature. Such a nonminimal coupling appears when quantizing a canonical, minimally coupled test scalar field in a curved space [31] and also in the context of radiation problems [32–34] (see also Refs. [35–39]). The nonminimal coupling of the scalar has been studied extensively during inflation of the early Universe (see Ref. [40] and references therein). When the scalar is allowed to gravitate, one has, for all practical purposes, a scalar-tensor theory [41,42]. We now show that a known representation of the theory of a nonminimally coupled scalar field as an imperfect fluid [11] is contained, as a special case, in our formulas. The action for a nonminimally coupled scalar ϕ is

$$S_{\text{NMC}} = \int d^4x\sqrt{-g}\left[\left(\frac{1}{8\pi} - \xi\phi^2\right)\frac{R}{2} - \frac{1}{2}\nabla^e\phi\nabla_e\phi - V(\phi)\right], \tag{4.1}$$

where ξ is the dimensionless coupling constant (with $\xi = 1/6$ corresponding to conformal coupling [26,31]), the value of which depends on the nature of the scalar and can often be determined as a running coupling going to an infrared fixed point under a renormalization group flow [35,43]. Our general scalar-tensor action *in vacuo* is instead

$$S_{\rm ST} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\psi R - \frac{\omega(\psi)}{\psi}\nabla^c\psi\nabla_c\psi - V(\psi)\right]. \tag{4.2}$$

In order to establish the connection between these two actions, it is sufficient to write the particular form of the function $\omega(\psi)$ that corresponds to the Brans-Dicke-like representation of the action (4.1). Identifying the first term in each action gives

$$\psi = 1 - 8\pi\xi\phi^2. \tag{4.3}$$

Contrary to the Brans-Dicke-like field ϕ, the nonminimally coupled scalar ϕ is not restricted to being positive. However, since $\psi > 0$, for $\xi > 0$, the scalar ϕ must satisfy $|\phi| < \phi_c \equiv 1/\sqrt{8\pi\xi}$, while all values of ϕ are admissible if $\xi < 0$. By using

$$\phi = \pm\sqrt{\frac{1-\psi}{8\pi\xi}}, \tag{4.4}$$

$$\nabla_e\phi = \mp\frac{\nabla_e\psi}{\sqrt{32\pi\xi(1-\psi)}}, \tag{4.5}$$

$$\nabla_a\nabla_b(\phi^2) = -\frac{1}{8\pi\xi}\nabla_a\nabla_b\psi, \tag{4.6}$$

one finds easily

$$\omega(\psi) = \frac{\psi}{4\xi(1-\psi)}. \tag{4.7}$$

Vice versa, it is

$$\nabla_e\psi = -16\pi\xi\phi\nabla_e\phi, \tag{4.8}$$

$$\nabla_a\nabla_b\psi = -8\pi\xi\nabla_a\nabla_b(\phi^2), \tag{4.9}$$

$$\omega(\psi(\phi)) = \frac{1-8\pi\xi\phi^2}{32\pi\xi^2\phi^2}. \tag{4.10}$$

These transformation properties allow us to recover the theory described by the action (4.1), the effective fluid representation of which is discussed in Ref. [11], as a special case of the general theory described by the scalar-tensor action (4.2). This limit was not derived in Ref. [10].

The effective stress-energy tensor (3.1) of the Brans-Dicke-like scalar field ψ becomes, in terms of the new field ϕ,

$$T_{ab}^{(\psi)} = (1-8\pi\xi\phi^2)^{-1}\left\{\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^e\phi\nabla_e\phi - \frac{U(\phi)}{2}g_{ab} - \xi[\nabla_a\nabla_b(\phi^2) - g_{ab}\Box(\phi^2)]\right\}, \tag{4.11}$$

where

$$U(\phi) = \frac{V[\psi(\phi)]}{8\pi}. \tag{4.12}$$

As already shown, the energy-momentum tensor (4.11) assumes the form of an imperfect fluid energy-momentum tensor. The effective energy density is

$$\rho^{(\psi)} = (1-8\pi\xi\phi^2)^{-1}\left\{-\frac{1}{2}\nabla^e\phi\nabla_e\phi + \frac{U(\phi)}{2} + \xi\left[\frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b(\phi^2)}{\nabla^e\phi\nabla_e\phi} - \Box(\phi^2)\right]\right\}, \tag{4.13}$$

while the effective heat flux density is

$$q_a^{(\psi)} = \xi(1-8\pi\xi\phi^2)^{-1}\frac{\nabla^c\phi\nabla^d\phi}{(-\nabla^e\phi\nabla_e\phi)^{3/2}}[\nabla_d\phi\nabla_a\nabla_c(\phi^2) - \nabla_a\phi\nabla_c\nabla_d(\phi^2)]. \tag{4.14}$$

The effective pressure reads

$$P^{(\psi)} = (1-8\pi\xi\phi^2)^{-1}\left\{-\frac{1}{2}\nabla^e\phi\nabla_e\phi - \frac{U(\phi)}{2} + \frac{\xi}{3}\left[2\Box(\phi^2) + \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right]\right\}, \tag{4.15}$$

while the stress tensor and the anisotropic stresses are

$$\Pi_{ab}^{(\psi)} = (1-8\pi\xi\phi^2)^{-1}\left\{\left[-\frac{1}{2}\nabla^e\phi\nabla_e\phi - \frac{U(\phi)}{2}\right]h_{ab} + \xi[h_{ab}\Box(\phi^2) - h_a{}^c h_b{}^d\nabla_c\nabla_d(\phi^2)]\right\} \tag{4.16}$$

and

$$\pi_{ab}^{(\psi)} = -\frac{\xi(1-8\pi\xi\phi^2)^{-1}}{\nabla^e\phi\nabla_e\phi}\left\{\frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^e\phi\nabla_e\phi)\left[\Box(\phi^2) - \frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right] + \nabla^d\phi\left[\nabla_d\phi\nabla_a\nabla_b(\phi^2) - \nabla_b\phi\nabla_a\nabla_d(\phi^2) + \frac{\nabla^a\phi\nabla_b\phi\nabla_c\nabla_d(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right]\right\}, \quad (4.17)$$

respectively. Finally, the trace of the stress-energy tensor is

$$T^{(\psi)} = (1-8\pi\xi\phi^2)^{-1}[-\nabla^e\phi\nabla_e\phi - 2U(\phi) + 3\xi\Box(\phi^2)]. \quad (4.18)$$

Equations (4.11)–(4.18) reproduce the corresponding effective fluid quantities of Ref. [11] after accounting for the different notations.

A. Minimally coupled scalar field

By setting the coupling constant ξ to zero, the scalar ϕ decouples from the Ricci curvature and assumes the ordinary nongravitational form considered in GR. In the limit $\xi \to 0$, Eqs. (4.11)–(4.18) yield the effective fluid quantities

$$T_{ab}^{(0)} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^e\phi\nabla_e\phi - \frac{U(\phi)}{2}g_{ab}, \quad (4.19)$$

$$\rho^{(0)} = -\frac{1}{2}\nabla^e\phi\nabla_e\phi + \frac{U(\phi)}{2}, \quad (4.20)$$

$$P^{(0)} = -\frac{1}{2}\nabla^e\phi\nabla_e\phi - \frac{U(\phi)}{2}, \quad (4.21)$$

$$\Pi_{ab}^{(0)} = \left[-\frac{1}{2}\nabla^e\phi\nabla_e\phi - \frac{U(\phi)}{2}\right]h_{ab}, \quad (4.22)$$

$$T^{(0)} = -\nabla^e\phi\nabla_e\phi - 2U(\phi), \quad (4.23)$$

while the heat flux density $q_a^{(0)}$ and the anisotropic stresses $\pi_{ab}^{(0)}$ vanish identically, giving the energy-momentum tensor of the minimally coupled scalar field the structure of a perfect fluid, as is well known [11–15]. One can also write

$$T_{ab}^{(0)} = \nabla_a\phi\nabla_b\phi - \mathcal{L}^{(0)}g_{ab}, \quad (4.24)$$

where

$$\mathcal{L}^{(0)} = \frac{1}{2}\nabla^c\phi\nabla_c\phi - U(\phi) \quad (4.25)$$

is the minimally coupled scalar field Lagrangian density. Furthermore, it is

$$\mathcal{L}^{(0)} = P^{(0)}. \quad (4.26)$$

In the effective fluid approach to scalar field theories, this equation is significant because it is consistent with the fact, well known in relativistic and nonrelativistic fluid dynamics, that equivalent Lagrangian densities for a perfect fluid are $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$ [44–46]. These two Lagrangians become inequivalent if the fluid couples to another component of the matter sector of the theory, as discussed in the recent Ref. [47]. When ϕ couples to the Ricci curvature (i.e., $\xi \neq 0$), instead, the equivalent effective fluid is no longer a perfect fluid, and Eq. (4.26) no longer holds (therefore, it is meaningless to discuss the equivalence of P and $-\rho$ as Lagrangians).

For the minimally coupled scalar, we have also

$$\rho^{(0)} = \mathcal{L}^{(0)} + 2U(\phi). \quad (4.27)$$

An equation of state for this perfect fluid is specified by giving two relations: $\rho = \rho(\mathcal{L}^{(0)}, U)$, $P = P(\mathcal{L}^{(0)}, U)$ [12].

V. FLUID SYMMETRY FOR ELECTROVACUUM BRANS-DICKE GRAVITY

Consider now the vacuum or electrovacuum Brans-Dicke theory with $\omega = \text{const}$. The Brans-Dicke action (1.1) with constant ω is invariant in form under the one-parameter group of symmetries [48]

$$g_{ab} \to \tilde{g}_{ab} = \phi^{2\alpha}g_{ab}, \quad (5.1)$$

$$\phi \to \tilde{\phi} = \phi^{1-2\alpha}, \qquad \alpha \neq 0, 1/2, \quad (5.2)$$

provided that the Brans-Dicke parameter ω and the scalar potential $V(\phi)$ are replaced by

$$\tilde{\omega}(\omega, \alpha) = \frac{\omega + 6\alpha(1-\alpha)}{(1-2\alpha)^2}, \quad (5.3)$$

$$\tilde{V}(\tilde{\phi}) = \tilde{\phi}^{\frac{-4\alpha}{1-2\alpha}}V(\tilde{\phi}^{\frac{1}{1-2\alpha}}). \quad (5.4)$$

This one-parameter symmetry group is used to generate new solutions from known ones [49] and to study the limit to GR of Brans-Dicke gravity [48]. Assuming that $\nabla^c\phi$ is timelike, under the transformation (5.1) and (5.2), the fluid 4-velocity is mapped to

$$u_c \to \tilde{u}_c \equiv \frac{\tilde{\nabla}_c\tilde{\phi}}{\sqrt{-\tilde{g}^{cd}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}_d\tilde{\phi}}} = \phi^\alpha u_c, \quad (5.5)$$

$$u^c \to \tilde{u}^c = \phi^{-\alpha}u^c. \quad (5.6)$$

Since $\phi > 0$, the transformation property (5.5) preserves the timelike character of the 4-velocity, and since u^c and $\tilde{u}^c$ are normalized with respect to different metrics, it also preserves the normalization of the 4-velocity,

$$\tilde{g}^{ab}\tilde{u}_a\tilde{u}_b = g^{ab}u_a u_b = -1. \tag{5.7}$$

The fluid quantities $\tilde{\rho}^{(\tilde{\phi})}$, $\tilde{P}^{(\tilde{\phi})}$, $\tilde{q}_a^{(\tilde{\phi})}$, $\tilde{\Pi}_{ab}^{(\tilde{\phi})}$, and $\tilde{\pi}_{ab}^{(\tilde{\phi})}$ are given in terms of $\tilde{\phi}$ and its derivatives by the analog of Eqs. (3.13)–(3.19), obtained by replacing nontilded with tilded quantities. In terms of the "original" ϕ-fluid, the effective energy-momentum tensor of the "new" $\tilde{\phi}$-fluid is

$$\tilde{T}_{ab}^{(\tilde{\phi})} = T_{ab}^{(\phi)} + \frac{\alpha}{4\pi\phi}\left[\frac{(1+\alpha)}{\phi}\nabla_a\phi\nabla_b\phi + \frac{(\alpha-2)}{2\phi}\nabla^e\phi\nabla_e\phi g_{ab} - (\nabla_a\nabla_b\phi - g_{ab}\Box\phi)\right]. \tag{5.8}$$

We then proceed to find the new fluid quantities in the comoving frame expressed in terms of the original ones. The effective energy density is

$$\tilde{\rho}^{(\tilde{\phi})} = \phi^{-2\alpha}\left[\rho^{(\phi)} - \frac{3\alpha^2}{8\pi\phi^2}\nabla^e\phi\nabla_e\phi - \frac{\alpha}{4\pi\phi}\left(\Box\phi - \frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right)\right]. \tag{5.9}$$

Using Eq. (3.13), one can rewrite it as

$$\tilde{\rho}^{(\tilde{\phi})} = \phi^{-2\alpha}\left[(1-2\alpha)\rho^{(\phi)} - \frac{\alpha(3\alpha+\omega)}{8\pi\phi^2}\nabla^e\phi\nabla_e\phi + \frac{\alpha V}{8\pi\phi}\right]. \tag{5.10}$$

The effective heat flux density is

$$\tilde{q}_a^{(\tilde{\phi})} = \phi^{-\alpha}\left\{q_a^{(\phi)} + \frac{\alpha}{4\pi\phi\sqrt{-\nabla^e\phi\nabla_e\phi}}\left[\nabla_a\nabla_c\phi\nabla^c\phi - \frac{(\nabla_c\nabla_d\phi\nabla^c\phi\nabla^d\phi)}{\nabla^e\phi\nabla_e\phi}\nabla_a\phi\right]\right\}, \tag{5.11}$$

and using Eq. (3.14), this turns into

$$\tilde{q}_a^{(\tilde{\phi})} = (1-2\alpha)\phi^{-\alpha}q_a^{(\phi)}. \tag{5.12}$$

The effective pressure is

$$\tilde{P}^{(\tilde{\phi})} = \phi^{-2\alpha}\left\{P^{(\phi)} + \frac{\alpha}{4\pi\phi}\left[\frac{(\alpha-2)}{2\phi}\nabla^e\phi\nabla_e\phi + \frac{2}{3}\Box\phi + \frac{\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi}{3\nabla^e\nabla_e\phi}\right]\right\}, \tag{5.13}$$

which, using Eq. (3.18), becomes

$$\tilde{P}^{(\tilde{\phi})} = \phi^{-2\alpha}\left[(1-2\alpha)P^{(\phi)} + \frac{\alpha(\alpha-\omega-2)}{8\pi\phi^2}\nabla^e\phi\nabla_e\phi - \frac{\alpha V}{8\pi\phi}\right]. \tag{5.14}$$

The effective spatial stress tensor is computed as

$$\tilde{\Pi}_{ab}^{(\tilde{\phi})} = \Pi_{ab}^{(\phi)} + \frac{\alpha}{4\pi\phi}\left[\frac{(\alpha-2)}{2\phi}\nabla^e\phi\nabla_e\phi + \Box\phi\right]h_{ab} - \frac{\alpha}{4\pi\phi}\left[\nabla_a\nabla_b\phi - \frac{\nabla^c\phi(\nabla_a\nabla_c\phi\nabla_b\phi + \nabla_b\nabla_c\phi\nabla_a\phi)}{\nabla^e\phi\nabla_e\phi} + \frac{(\nabla_c\nabla_d\phi\nabla^c\phi\nabla^d\phi)}{(\nabla^e\phi\nabla_e\phi)^2}\nabla_a\phi\nabla_b\phi\right] \tag{5.15}$$

$$= \Pi_{ab}^{(\phi)} + \frac{\alpha}{4\pi\phi}\left[\frac{(\alpha-2)}{2\phi}\nabla^e\phi\nabla_e\phi + \Box\phi\right]h_{ab} - \frac{\alpha}{4\pi\phi}\nabla_c\nabla_d\phi h_a{}^c h_b{}^d. \tag{5.16}$$

The use of Eq. (3.17) in Eq. (5.16) then gives

$$\tilde{\Pi}_{ab}^{(\tilde{\phi})} = (1-2\alpha)\Pi_{ab}^{(\phi)} + \frac{\alpha}{8\pi\phi}\left[\frac{(\alpha-\omega-2)}{\phi}\nabla^e\phi\nabla_e\phi - V\right]h_{ab}. \tag{5.17}$$

The effective anisotropic stresses are simply

$$\tilde{\pi}_{ab}^{(\tilde{\phi})} = (1-2\alpha)\pi_{ab}^{(\phi)} \tag{5.18}$$

in terms of those associated with the ϕ-fluid.

A symmetry transformation (5.1), (5.2) with $\alpha > 1/2$ reverses the sign of the heat flux density and of the anisotropic stresses. For example, in spherical symmetry, a (radial) ingoing energy flow will be changed into an outgoing flow by such a transformation. This fact is of some interest in the context of inhomogeneous universes describing black holes embedded in cosmological backgrounds [50].

VI. EFFECTIVE FLUID DESCRIPTION OF $f(R)$ GRAVITY

$f(R)$ gravity, which is an extremely popular class of theories used to explain the present acceleration of the Universe without dark energy [4], is described by the action

$$S_{f(R)} = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R) + S^{(m)}, \tag{6.1}$$

where $f(R)$ is a nonlinear function of the Ricci scalar and, as usual, $S^{(m)}$ is the action of ordinary matter. It is well known that the gravitational action $S_{f(R)}$ is equivalent to that of a Brans-Dicke theory with Brans-Dicke field

$\phi = f'(R)$, Brans-Dicke coupling $\omega = 0$, and scalar field potential [7]

$$V(\phi) = Rf'(R) - f(R)|_{R=R(\phi)}, \tag{6.2}$$

where R is now a function of the scalar field $\phi = f'(R)$ and a prime denotes differentiation with respect to the Ricci scalar R. In general, the relation $R = R(\phi)$ cannot be inverted explicitly to obtain an explicit function $V(\phi)$.

The fourth order vacuum field equations are

$$f'(R)R_{ab} - \frac{f(R)}{2} g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab}\Box f'(R) \tag{6.3}$$

and can be written as the effective Einstein equations [7]

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}^{(\mathrm{eff})}, \tag{6.4}$$

where

$$T_{ab}^{(\mathrm{eff})} = \frac{1}{8\pi f'(R)} \left[\nabla_a \nabla_b f'(R) - g_{ab}\Box f'(R) + \frac{f(R) - Rf'(R)}{2} g_{ab} \right]. \tag{6.5}$$

We need the transformation properties

$$\nabla^a f' = f'' \nabla^a R, \tag{6.6}$$

$$\nabla_a \nabla_b f' = f'' \nabla_a \nabla_b R + f''' \nabla_a R \nabla_b R, \tag{6.7}$$

$$\Box f' = f'' \Box R + f''' \nabla^e R \nabla_e R. \tag{6.8}$$

We have $\phi = f'(R) > 0$ in order for the graviton to carry positive kinetic energy, while we require $f''(R) > 0$ to avoid the notorious Dolgov-Kawasaki instability [51,52]. As a result, the condition that $\nabla_a \phi$ be timelike implies that $\nabla_a R$ is also timelike, and the definition (2.1) of the effective fluid 4-velocity yields

$$u_a = \frac{\nabla_a R}{\sqrt{-\nabla^e R \nabla_e R}}. \tag{6.9}$$

The effective fluid quantities are computed using Eqs. (6.6)–(6.9) in the expressions of the effective scalar-tensor fluid, obtaining

$$\rho^{(\mathrm{eff})} = \frac{1}{8\pi f'} \left\{ f'' \left[\Box R - \frac{\nabla_a \nabla_b R \nabla^a R \nabla^b R}{\nabla^e R \nabla_e R} \right] + \frac{Rf' - f}{2} \right\}, \tag{6.10}$$

$$q_a^{(\mathrm{eff})} = \frac{f''}{8\pi f' \sqrt{-\nabla^e R \nabla_e R}} \left[\frac{(\nabla_c \nabla_d R \nabla^c R \nabla^d R)}{\nabla^e R \nabla_e R} \nabla_a R - \nabla_a \nabla_c R \nabla^c R \right], \tag{6.11}$$

$$\Pi_{ab}^{(\mathrm{eff})} = h_a^c h_b^d \nabla_c \nabla_d f'(R) - \left(\Box f'(R) + \frac{Rf' - f}{2} \right) h_{ab} \tag{6.12}$$

$$= \frac{1}{8\pi f'} \left\{ f'' \left[\nabla_a \nabla_b R + \frac{(\nabla_a R \nabla_b \nabla_c R + \nabla_b R \nabla_a \nabla_c R)\nabla^c R}{-\nabla^e R \nabla_e R} + \frac{(\nabla_c \nabla_d R \nabla^c R \nabla^d R) \nabla_a R \nabla_b R}{(\nabla^e R \nabla_e R)^2} \right] - \left(f'' \Box R + f''' \nabla^e R \nabla_e R + \frac{Rf' - f}{2} \right) h_{ab} \right\}, \tag{6.13}$$

$$P^{(\mathrm{eff})} = \frac{1}{8\pi f'} \left[-\frac{f''}{3} \left(2\Box R + \frac{\nabla_a \nabla_b R \nabla^a R \nabla^b R}{\nabla^e R \nabla_e R} \right) - \left(f''' \nabla^e R \nabla_e R + \frac{Rf' - f}{2} \right) \right], \tag{6.14}$$

$$\pi_{ab}^{(\mathrm{eff})} = \frac{f''}{8\pi f'} \left[\nabla_a \nabla_b R - \frac{(\nabla_a R \nabla_b \nabla_c R + \nabla_b R \nabla_a \nabla_c R)\nabla^c R}{\nabla^e R \nabla_e R} + \frac{(\nabla_c \nabla_d R \nabla^c R \nabla^d R)}{(\nabla^e R \nabla_e R)^2} \nabla_a R \nabla_b R + \frac{1}{3} \left(\frac{\nabla_c \nabla_d R \nabla^c R \nabla^d R}{\nabla^e R \nabla_e R} - \Box R \right) h_{ab} \right]. \tag{6.15}$$

Finally, the trace of the effective stress-energy tensor is

$$T^{(\mathrm{eff})} = -\rho^{(\mathrm{eff})} + 3P^{(\mathrm{eff})} = \frac{1}{8\pi f'} [-3(f'' \Box R + f''' \nabla^e R \nabla_e R) + 2(f - Rf')]. \tag{6.16}$$

It is thus demonstrated that, in general, the terms generated by a nonlinear function $f(R)$ in the gravitational Lagrangian are equivalent to an imperfect fluid when writing the field equations as the effective Einstein equations (6.4). In special geometries, this imperfect fluid reduces to a perfect fluid dubbed "curvature fluid" [53–55]. This is the case for the FLRW geometry [53–55], for the Lorentzian version of a Hawking wormhole [25], and for a Witten bubble spacetime solution [25].

VII. DISCUSSION AND CONCLUSIONS

The fluid equivalent of the Brans-Dicke-like scalar field ϕ of scalar-tensor gravity in the Jordan frame has been worked out in detail, completing and extending the work of Ref. [10]. The field equations (1.2) and (1.3) can be regarded as effective Einstein equations, and the terms originating from ϕ and its derivatives, relegated to the right-hand side, can always be interpreted as an effective fluid. Contrary to the case of a canonical scalar field minimally coupled to the curvature, which is definitely a matter field of nongravitational nature and is equivalent to a perfect fluid, the effective fluid corresponding to a Brans-Dicke-like field is an imperfect one (except for special circumstances in highly symmetric geometries—but we refer to the general situation here). As expected, since the effective fluid is generated by a purely scalar degree of freedom, it is irrotational. Dissipation in fluids is important, and it is related to the stability of star models [56], while dissipative fluids are also the subject of a vast literature related to the AdS/CFT correspondence [57]. The discussion and formulas presented here should find applications when modified gravity is discussed in conjunction with these areas of research.

Contrary to what we have done here, in principle, one could have started out with the Einstein frame representation of scalar-tensor gravity, obtained by the conformal transformation and nonlinear field redefinition $(g_{ab}, \phi) \to (\tilde{g}_{ab}, \tilde{\phi})$ with

$$\tilde{g}_{ab} = \phi g_{ab}, \tag{7.1}$$

$$d\tilde{\phi} = \sqrt{\frac{|2\omega(\phi)+3|}{16\pi G}}\frac{d\phi}{\phi}, \tag{7.2}$$

which is completely different from the symmetry (5.1) and (5.2) discussed in Sec. V. The Einstein frame version of the scalar-tensor action (1.1) is

$$S_{\rm ST} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}^{(m)}[\phi^{-1}\tilde{g}_{cd}, \psi^{(m)}]}{\phi^2(\tilde{\phi})}\right], \tag{7.3}$$

where

$$U(\tilde{\phi}) = \left.\frac{V(\phi)}{\phi^2}\right|_{\phi=\phi(\tilde{\phi})} \tag{7.4}$$

and $\psi^{(m)}$ collectively denotes the matter fields. The Lagrangian density of the new scalar $\tilde{\phi}$ has canonical form save for the fact that it couples explicitly to all other forms of matter except conformally invariant matter [41,42]. If one considers (electro)vacuum scalar-tensor gravity, the Einstein frame scalar is equivalent to a perfect fluid. In this case (but not in the general case of nonconformal matter), the transformation from the Jordan to Einstein frame changes an equivalent imperfect fluid into a perfect one, and vice versa. (The behavior of more general perfect and imperfect fluids under conformal transformations is discussed in Ref. [58]).

Finally, let us discuss the implications of the work presented here for the different, long-standing problem of finding a Lagrangian description of a dissipative imperfect fluid. It is notoriously difficult to give a Lagrangian or Hamiltonian description of dissipative systems [59], except for simplistic models of friction in point particle mechanics [60]. The problem is even more difficult in fluid mechanics [59]. By reversing our original problem of finding an effective fluid description of a scalar field theory, we are able to provide a very limited answer: an irrotational imperfect fluid with an energy-momentum tensor (3.3) can be given a scalar field description and, therefore, a full Lagrangian description, if its irrotational 4-velocity field can be written in the form (2.1) for a suitable scalar ϕ. Needless to say, this is an extremely restrictive condition which makes the answer useless for most practical purposes because, in general, one cannot integrate Eq. (2.1) to determine the velocity potential ϕ, but this condition has been usefully implemented using a Lagrange multiplier (a second scalar field) in more complicated theories [22,23]. A much simpler problem occurs by restricting oneself to specific spacetimes with a high degree of symmetry. If the scalar field ϕ is forced to depend on just one of the four coordinates, which can happen only in highly symmetric spaces because ϕ is a matter source, then Eq. (2.1) simplifies considerably. This is the case, e.g., of spatially homogeneous and isotropic FLRW cosmology. In a FLRW space with line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{7.5}$$

the gravitating scalar field depends only on time, $\phi = \phi(t)$, and then the fluid 4-velocity simplifies to $u_\mu = -\delta_{0\mu}$. Even in this case, however, integrating Eq. (2.21) with $\theta = 3H \equiv 3\dot{a}/a$ to determine $\phi(t)$ is not a trivial task because this equation is nonlinear for a general potential $V(\phi)$. The conclusion is that the scalar field-fluid correspondence does not allow for significant progress in the problem of the

Lagrangian description of dissipative fluids. An alternative approach to dissipation, obtained by resorting to a modification of gravity different from the scalar-tensor prescription and doing away with dark energy, is discussed in Ref. [61]. Apart from the approach to a Lagrangian description of dissipation, the correspondence between Brans-Dicke-like scalar field and fluid is now clarified in the important situations where a scalar field appears in the modeling of cosmology and stellar interiors in the context of modified [scalar-tensor and $f(R)$] gravity.

For a last remark, we mention that we presented a general formalism without committing to any specific geometry. Two applications to specific spacetime geometries would be particularly interesting: the first is the case of perturbed Friedmann-Lemaître-Robertson-Walker universes, and the second is the perturbation of black hole spacetimes. In the first case, perturbations of universes filled with an imperfect fluid have been studied in the literature (see Ref. [62] for a review). This description may be adapted to scalar-tensor gravity, with the addition of a matter fluid to the picture. This application of the imperfect fluid formalism presented here necessarily involves many details and will be presented elsewhere. In the second case, perturbations of black holes in scalar-tensor gravity have been studied, but perturbations in the presence of an imperfect fluid in general relativity are less clear and will also require a separate analysis. Likewise, the imperfect fluid corresponding to modified gravity constitutes a form of nonadiabatic dark energy quite different from the standard dark energy models, which would give rise to nonadiabatic perturbations. Nonadiabatic dark energy has been considered in various works (e.g., Ref. [63]), but the detailed relations with the present imperfect fluid formalism are still missing and will be explored elsewhere.

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