BTZ dilatonic black holes coupled to Maxwell and Born-Infeld electrodynamics

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Motivated by the string theory corrections in the low-energy limit of both gauge and gravity sides, we consider three-dimensional black holes in the presence of dilatonic gravity and the Born-Infeld nonlinear electromagnetic field. We find that geometric behavior of the solutions is similar to the behavior of the hyperscaling violation metric, asymptotically. We also investigate thermodynamics of the solutions and show that the generalization to dilatonic gravity introduces novel properties into thermodynamics of the black holes which were absent in the Einstein gravity. Furthermore, we explore the possibility of tuning out part of the dilatonic effects using the Born-Infeld generalization.

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I. INTRODUCTION

One of our main interests in considering a dilaton field is the fact that the low-energy limit of string theory precisely involves a massless scalar dilaton field. This, in turn, has motivated the scientific community to study dilaton gravity from different viewpoints. The scalar dilaton field has significant impact on the casual structure as well as on the thermodynamic features of the charged black holes. In fact, the presence of the dilaton field also affects the structure of spacetime geometry. The impact is sometimes effective concerning the asymptotic behavior of dilaton solutions. In particular, it was proved that in the presence of one or two Liouville-type dilaton potentials, black hole spacetimes are neither asymptotically flat nor (anti)–de Sitter [1–3]. Nevertheless, in the case of three Liouville type dilaton potentials, it is possible to construct dilatonic black hole solutions in the background of (anti)–de Sitter (A)dS spacetime [4,5]. Also, the coupling of a dilaton field with other gauge fields may have profound effects on the resulting solutions [6–8]. Dilaton fields can also be relevant to the construction of black holes with rather unconventional asymptotes. For example, charged Lifshitz black holes with an arbitrary dynamical exponent can be sustained by the presence of at least two dilaton scalar fields [9]. The extension of this dilatonic model with nonlinear electrodynamics was considered in Refs. [10–13]. Recently, studies on neutron stars in the context of dilaton gravity [14] as well as black holes in dilaton gravity's rainbow [15,16] have been done.

In the present work, we will focus on three-dimensional dilaton gravity. Our motivations come from the fact that the discovery of the three-dimensional black hole (BTZ) [17] and lower-dimensional gravity has gained a lot of interest in the last two decades [18–26]. Indeed, the reasons for studying three-dimensional gravity theories are multiple. For example, the near horizon geometry of three-dimensional solutions can serve as a worthwhile model to investigate some conceptual questions about the AdS/CFT correspondence [27]. Moreover, the BTZ solution is a ground that offers many facets to explore. For example, the study of the BTZ black hole has improved our knowledge on gravitational systems and their interactions in three dimensions [27]. It also opens up the possible existence of the gravitational Aharonov-Bohm effect due to the noncommutative BTZ black holes [28]. The existence of specific relations between these black holes and effective action in string theory [29,30] is the motivation. Concerning the black hole solutions that are BTZ-like, the current literatures contains many of them. For example, the existence of BTZ black holes/wormholes in the presence of nonlinear electrodynamics has been investigated in [31–33] and in higher dimensions in [34,35]. In addition, exact BTZ-like solutions were shown to arise in massive gravity [36], dilatonic gravity [37–39], gravity's rainbow [40,41], new massive gravity [42,43], Lifshitz gravity [44], and massive gravity's rainbow [45] (see also Refs. [46–52]

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for more details). Moreover, thermal aspects have been explored, where the existence of a phase transition between the BTZ black hole and thermal AdS space is possible [53]. It is worth mentioning that the three-dimensional BTZ solution is also interesting from a quantum point of view [54–59].

Here, we consider dilatonic BTZ black holes coupled with linear and nonlinear Born-Infeld electrodynamics. In addition to what we said for our motivations in the previous paragraphs, we are going to investigate the effects of nonlinearity on the properties of the solutions. Although classical electrodynamics is well organized with the Maxwell equations (accompanying to Lorentz force), some of their shortcomings motivate one to consider nonlinear theory. One of the old successful theories of nonlinear electrodynamics is the so-called Born-Infeld theory [60]. This Abelian theory enjoys most of the Maxwell properties and also its related electric field of a pointlike charge is regular everywhere. The foundations of this theory became firmly established when it is realized that one can obtain its Lagrangian from a class of the low-energy limit of string theory [61–66]. Hoffmann employed this type of nonlinear theory in context of Einstein gravity [67]. Then, different types of black holes in the presence of this electrodynamics have been studied in Refs. [68–82]. Based on the mentioned motivation, we discuss dilatonic black holes with Maxwell and Born-Infeld theories in two separated sections.

II. BTZ BLACK HOLE SOLUTIONS IN DILATON-MAXWELL GRAVITY

Here, we consider the three-dimensional action given by

$$I=-\frac{1}{16\pi}\int_M d^3x\sqrt{-g}[R-4(\nabla\Phi)^2-V(\Phi)+L(h,\Phi)], \tag{1}$$

where Φ is the dilaton field, $V(\Phi)$ is a scalar potential, R denotes the scalar curvature and

$$L(h,\Phi)=-e^{-4\alpha\Phi}h. \tag{2}$$

In the last expression, h stands for the Maxwell invariant $h=h_{\mu\nu}h^{\mu\nu}$ where $h_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and α represents the dilaton coupling (dimensionless) parameter. In order to avoid the vanishing electromagnetic Lagrangian, we adjust the dilaton field parameter in which the $\alpha\Phi$ term is finite everywhere. The variation of the action (1) with respect to the metric tensor, the dilaton field (Φ) and the gauge field (A_μ), yields

$$R_{\mu\nu}=4\left[\partial_\mu\Phi\partial_\nu\Phi+\frac{1}{4}g_{\mu\nu}V(\Phi)\right]+2e^{-4\alpha\Phi}\left(h_{\mu\eta}h_\nu^\eta-\frac{1}{2}g_{\mu\nu}h_{\lambda\eta}h^{\lambda\eta}\right), \tag{3}$$

$$\nabla^2\Phi=\frac{1}{8}\frac{\partial V(\Phi)}{\partial\Phi}+\frac{\alpha}{2}e^{-4\alpha\Phi}h_{\lambda\eta}h^{\lambda\eta}, \tag{4}$$

$$0=\partial_\mu(\sqrt{-g}e^{-4\alpha\Phi}h^{\mu\nu}). \tag{5}$$

Since we are interested in electrically charged static solution, we consider the following ansatz,

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2R^2(r)d\varphi^2, \tag{6}$$

where $f(r)$ and $R(r)$ are two metric functions. Our study being dedicated to black holes with a radial electric field, the suitable choice of gauge potential is given by $A_\mu=\delta_\mu^t A_0(r)$, where A_0 denotes the electric potential. As usual the direct integration of the Maxwell equation (5) permits to express the electric field $E(r)$ as

$$E(r)=\frac{qe^{4\alpha\Phi}}{rR(r)}, \tag{7}$$

where q is an integration constant which is related to the electric charge. Now, the Einstein field equations (3) can be rearranged as

$$eq_{tt}:\ \frac{1}{2}\left[f''(r)+\left(\frac{1}{r}+\frac{R'(r)}{R(r)}\right)f'(r)\right]+V(\Phi)=0, \tag{8}$$

$$eq_{rr}:\ eq_{tt}+\left[\frac{R''(r)}{R(r)}+\frac{2R'(r)}{rR(r)}+4\Phi'^2(r)\right]f(r)=0, \tag{9}$$

$$eq_{\theta\theta}:2E^2(r)e^{-4\alpha\Phi(r)}+\left[\frac{R''(r)}{R(r)}+\frac{2R'(r)}{rR(r)}\right]f(r)+\left[\frac{1}{r}+\frac{R'(r)}{R(r)}\right]f'(r)+V(\Phi)=0. \tag{10}$$

It is then easy to see that the substraction of Eq. (8) with Eq. (9) yields the following constraint:

$$\frac{R''(r)}{R(r)}+\frac{2R'(r)}{rR(r)}+4\Phi'^2(r)=0. \tag{11}$$

In order to transform this equation into a differential equation for the dilaton field, we use the following judicious ansatz [83] for the metric function,

$$R(r)=e^{2\alpha\Phi(r)}. \tag{12}$$

This in turn implies that the dilaton scalar field can be determined to be

$$\Phi(r)=\frac{\gamma}{2\alpha}\ln\left(\frac{b}{r}\right), \tag{13}$$

where b is an arbitrary nonzero constant with length dimension. It is also notable that the function $\Phi(r)$ ($\Phi(r) \in (-\infty, +\infty)$) is a decreasing function of r ($r \in (0, +\infty)$) with one real root at $r = b$. For convenience, we have defined

$$\gamma = \frac{\alpha^2}{(\alpha^2 + 1)}. \tag{14}$$

Using Eqs. (7), (12)–(14), the electrical field is given as

$$E(r) = \frac{q}{r}\left(\frac{b}{r}\right)^{\frac{\alpha^2}{(\alpha^2+1)}}, \tag{15}$$

considering that the electrical field vanishes at infinity ($r \to \infty$), any arbitrary value of α may satisfy the above equation.

In order to find analytic solutions, the Liouville-type dilation potential is chosen as

$$V(\Phi) = 2\Lambda e^{4\alpha\Phi}, \tag{16}$$

where Λ is a free parameter that plays the role of a cosmological constant. Note that this kind of potential have been used in the context of Friedman-Robertson-Walker scalar field cosmology [85] as well as in the case of Maxwell-dilaton black holes [86,87]. Finally, the remaining metric function is found to be

$$f(r) = \frac{2q^2(\alpha^2+1)^2}{\alpha^2} - mr^{\gamma} + \frac{2\Lambda r^2(\alpha^2+1)^2}{\alpha^2-2}\left(\frac{b}{r}\right)^{2\gamma}, \tag{17}$$

where m is an integration constant related to the mass of black holes.

Before continuing our study, it is interesting to rewrite the metric solution (6) in the "standard form" by defining the radial coordinate $\rho = rR(r)$

$$ds^2 = -F(\rho)dt^2 + \frac{b^{\frac{2\gamma}{\gamma-1}}}{(1-\gamma)^2}\frac{d\rho^2}{\rho^{\frac{2\gamma}{\gamma-1}}F(\rho)} + \rho^2 d\varphi^2,$$

with

$$F(\rho) = \frac{2q^2(\alpha^2+1)^2}{\alpha^2} - m\rho^{\frac{\gamma}{1-\gamma}} b^{\frac{-\gamma^2}{1-\gamma}} + \frac{2\Lambda(\alpha^2+1)^2}{\alpha^2-2}\rho^2.$$

It is interesting to note that for $\alpha \to 0$ with $q/\alpha \to 0$, the solution reduces to the uncharged static BTZ black hole. Asymptotically for $\rho \gg 1$, the metric behaves as

$$ds^2 \sim -\rho^2 dt^2 + \frac{d\rho^2}{\rho^{2(1-\alpha^2)}} + \rho^2 d\varphi^2, \qquad \alpha < \sqrt{2},$$

or

$$ds^2 \sim -\rho^{\alpha^2} dt^2 + \rho^{\alpha^2} d\rho^2 + \rho^2 d\varphi^2, \qquad \alpha > \sqrt{2}.$$

In both cases, the asymptotic behavior is similar to the one of the hyperscaling violation metric

$$\begin{aligned} ds^2 &= \frac{1}{r^{2\theta}}\left[-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\varphi^2\right] \\ &\sim -\rho^{\frac{2(z-\theta)}{1-\theta}}dt^2 + \frac{d\rho^2}{\rho^{\frac{2}{1-\theta}}} + \rho^2 d\varphi^2, \end{aligned}$$

where z is the Lifshitz dynamical exponent and θ is the hyperscaling violating parameter. More precisely, for $\alpha < \sqrt{2}$, the asymptotic metric corresponds to an hyperscaling violation metric with $z = 1$ and $\theta = \alpha^2/(\alpha^2 - 1)$, and for $\alpha > \sqrt{2}$, this corresponds to $z = 2/\alpha^2$ and $\theta = (2 + \alpha^2)/\alpha^2$.

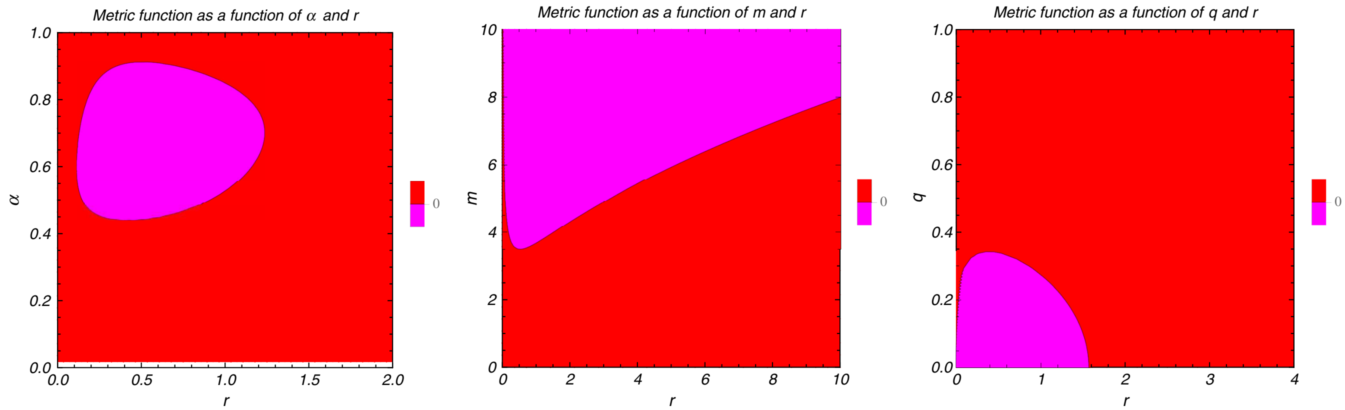
The charged dilatonic BTZ black holes are different from the charged BTZ black holes in Einstein gravity. The existence of the dilaton may exchange the role of the mass with the charge and vice et versa. Indeed, in the Einstein-Maxwell gravity, the mass is associated to the constant term of the metric function, while the charge term appears in the structural metric function with a function depending on the radial coordinate. As one can note from the expression (17), in the dilatonic case, this is exactly the opposite that occurs. In addition, it is worth mentioning that the obtained charged dilatonic BTZ solution does not reduce to the charged BTZ black hole solution in the absence of dilaton field. It is expected and comes from the difference between polynomial functions and logarithmic one. This behavior is the same as the comparison between higher-dimensional charged black holes and the three-dimensional case.

We are looking for the curvature singularity, in order to confirm the black hole interpretation of the solutions. For this purpose, we calculate the Ricci and Kretschmann scalars as

$$R = \frac{4q^2}{r^2} - \frac{m\gamma}{r^{(\alpha^2+2)/(\alpha^2+1)}} + \frac{4\Lambda(2\alpha^2-3)}{\alpha^2-2}\left(\frac{b}{r}\right)^{2\gamma}, \tag{18}$$

$$\begin{aligned} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= \frac{16q^4}{r^4} + \frac{3\gamma^2 m^2}{r^{2(\alpha^2+2)/(\alpha^2+1)}} \\ &\quad + \frac{32(\alpha^4 - 2\alpha^2 + \frac{3}{2})\Lambda^2}{(\alpha^2-2)^2}\left(\frac{b}{r}\right)^{4\gamma} \\ &\quad - \frac{32q^2(\alpha^2-1)\Lambda b^{2\gamma}}{(\alpha^2-2)r^{2(2\alpha^2+1)/(\alpha^2+1)}} - \frac{8m\gamma q^2}{r^{(3\alpha^2+4)/(\alpha^2+1)}} \\ &\quad - \frac{8\gamma m(2\alpha^2-1)\Lambda b^{2\gamma}}{(\alpha^2-2)r^{(3\alpha^2+2)/(\alpha^2+1)}}. \end{aligned} \tag{19}$$

Calculations show that for finite values of the radial coordinate, the Ricci and Kretschmann scalars are finite. Also, for very small and very large values of r, we have



(a) $\alpha = 1$ and $q = 0.4$ (b) $m = 3$ and $q = 0.4$ (c) $\alpha = 1$ and $m = 3$

FIG. 1. Variation of the $f(r)$ as a function of different parameters for $b = 0.3$ and $\Lambda = -1$.

$$\lim_{r\to 0^+} R = \infty,$$
$$\lim_{r\to 0^+} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \infty, \tag{20}$$

$$\lim_{r\to\infty} R \propto \frac{4\Lambda(2\alpha^2-3)}{\alpha^2-2}\left(\frac{b}{r}\right)^{2\gamma},$$
$$\lim_{r\to\infty} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \propto \frac{32(\alpha^4-2\alpha^2+\frac{3}{2})\Lambda^2}{(\alpha^2-2)^2}\left(\frac{b}{r}\right)^{4\gamma}. \tag{21}$$

The above Eq. (20) confirms that there is an essential singularity located at $r = 0$, and Eq. (21) for $\alpha = 0$, the asymptotic behavior of solutions is (A)dS ($\lim_{r\to\infty} R \propto 6\Lambda$ and $\lim_{r\to\infty} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \propto 12\Lambda^2$), while for nonzero α, the asymptotic behavior of solutions is not that of (A)dS. It is noteworthy that for $\alpha \to \infty$, the asymptotic behaviors of Ricci and Kretschmann scalars are as $\lim_{r\to\infty} R = 0$ and $\lim_{r\to\infty} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 0$. In other word, the effect of curvature singularity at infinity vanishes. Actually, the mentioned theory becomes quantum mechanically strongly coupled at some finite radius r, but this effect vanishes at infinity ($r \to \infty$).

As for the possible roots of metric function, we investigate different cases. In the absence of electric charge, root of the metric function is given as

$$r(f(r)=0)|_{q=0} = 2^{\frac{\alpha^2+1}{\alpha^2-2}}\left(\frac{(\alpha^2+1)^2\Lambda b^{\frac{2\alpha^2}{\alpha^2+1}}}{(\alpha^2-2)m}\right)^{\frac{\alpha^2+1}{\alpha^2-2}}, \tag{22}$$

Evidently, for $\alpha = \sqrt{2}$, the root will be divergent everywhere indicating that dilatonic parameter could not attain this value. The root of metric function is positive if $\alpha > \sqrt{2}$ and $\Lambda > 0$, or for $\alpha < \sqrt{2}$ and $\Lambda < 0$.

In the absence of geometrical mass, root of the metric function is obtained as

$$r(f(r)=0)|_{m=0} = \left(-\frac{i\sqrt{\alpha^2-2}\,qb^{\frac{1}{\alpha^2+1}-1}}{\alpha\sqrt{\Lambda}}\right)^{\alpha^2+1}. \tag{23}$$

In this case, the existence of real valued positive root is given by one of the following cases: (i) If $\Lambda < 0$ and $\alpha > \sqrt{2}$, and (ii) $\Lambda > 0$ and $0 < \alpha < \sqrt{\mathbf{2}}$. These two conditions actually are guidelines to the possibility of black hole solutions in AdS and dS cases.

In the absence of both geometrical mass and electric charge, the metric function will be without root. As for the general case, it was not possible to obtain the root of metric function analytically. Therefore, we use numerical approach. For more details regarding the behavior of the metric function, we plot $f(r)$ versus r and other parameters in Fig. 1. As one can see, this solution may contain real positive roots, and therefore, the singularity can be covered with an event horizon and interpreted as a black hole.

A. Thermodynamic properties

In this section, we are going to calculate the thermodynamic and the conserved quantities of the solutions and then examine the first law of thermodynamics.

In order to calculate the Hawking temperature, we employ the definition of surface gravity. In doing so, we have that

$$T = \frac{1}{2\pi}\sqrt{-\frac{1}{2}(\nabla_\mu\chi_\nu)(\nabla^\mu\chi^\nu)}, \tag{24}$$

where $\chi = \partial/\partial t$ is the Killing vector. The Hawking temperature for the this black hole can be written as

$$T_+ = -\frac{(\alpha^2+1)}{2\pi r_+}\left[q^2 + \Lambda r_+^2\left(\frac{b}{r_+}\right)^{2\gamma}\right], \tag{25}$$

where r_+ is the event horizon of black hole which is the largest real root of metric function, that is $f(r = r_+) = 0$.

In addition, the electric potential U is defined by the gauge potential in the following form [88]

$$U = A_\mu \chi^\mu|_{r\to\text{reference}} - A_\mu \chi^\mu|_{r=r_+}, \tag{26}$$

and for our solutions, we obtain

$$U = \frac{q}{\gamma}\left(\frac{b}{r_+}\right)^\gamma. \tag{27}$$

Also, one can use the area law of entropy in Einstein gravity to obtain the entropy of this black hole. Based on this law, the black hole's entropy equals to one-quarter of horizon area [89–91]. Therefore, the entropy is

$$S = \frac{\pi r_+}{2}\left(\frac{b}{r_+}\right)^\gamma. \tag{28}$$

In order to obtain the electric charge of the black holes, one can calculate the flux of the electromagnetic field at infinity. The electric charge is obtained as

$$Q = \frac{q}{2}. \tag{29}$$

Regarding the timelike Killing vector ($\xi = \partial/\partial t$), one can show that the finite mass can be obtained as

$$M = \frac{m}{8}(1-\gamma)b^\gamma. \tag{30}$$

By evaluating the metric function on the largest root of the solution, one is able to extract the geometrical mass (m) and insert into the total mass (30). This leads to the following relation:

$$M = \frac{(\alpha^2+1)\Lambda r_+^2}{4(\alpha^2-2)}\left(\frac{b}{r_+}\right)^{3\gamma} + \frac{q^2(\alpha^2+1)}{4\alpha^2}\left(\frac{b}{r_+}\right)^\gamma. \tag{31}$$

Using the relations obtained for the entropy (28) and the total electric charge (29), one is able to find a Smarr-like formula as

$$M(S,Q) = \frac{(\alpha^2+1)\Lambda(\frac{2S}{\pi})^{2(\alpha^2+1)}b^{-2\alpha^2}}{4(\alpha^2-2)}\left(\frac{\pi b}{2S}\right)^{3\alpha^2} + \frac{Q^2(\alpha^2+1)}{\alpha^2}\left(\frac{\pi b}{2S}\right)^{\alpha^2}. \tag{32}$$

Now, we can check the validity of first law of thermodynamics. In order to achieve this task, we note that

$$dM(S,Q) = \left(\frac{\partial M(S,Q)}{\partial S}\right)_Q dS + \left(\frac{\partial M(S,Q)}{\partial Q}\right)_S dQ, \tag{33}$$

and it is a matter of check to show that following equalities hold

$$T = \left(\frac{\partial M}{\partial S}\right)_Q \quad \& \quad U = \left(\frac{\partial M}{\partial Q}\right)_S. \tag{34}$$

The above relations confirm that the first law of thermodynamics is valid, namely

$$dM = TdS + UdQ. \tag{35}$$

B. Thermodynamic behavior

Here, the main goal in this section is to specify the effects on the coupling constants of the problem on the thermodynamic behavior of the solution, particularly for the mass, the temperature and the heat capacity.

1. Mass/internal energy

The mass of the black holes is usually interpreted as the internal energy of the system. Nevertheless, in presence of a cosmological constant, the mass can also be viewed as an enthalpy with the cosmological constant playing the role of the thermodynamic pressure. Here, we do not consider such a possibility and instead regard the mass as the internal energy.

First of all, the mass as defined in (31) requires the coupling constant $\alpha \neq \sqrt{2}$. Now, since the dilatonic parameter $b > 0$, the q^2-part of the mass expression is always positive. On the other hand, it is known that the constant Λ plays the role of a cosmological constant in the dilaton gravity. Therefore, it could be negative (for the AdS case) or positive (for the dS case). Considering these two options, one finds that the Λ contribution of the mass expression is positive provided that

$$\begin{aligned} \Lambda > 0 \quad &\& \quad \alpha > \sqrt{2}, \\ \Lambda < 0 \quad &\& \quad \alpha < \sqrt{2}, \end{aligned} \tag{36}$$

under these conditions, to absence of roots and the positivity of the internal energy are ensured. In contrast, a negative value of the Λ-contribution of the mass expression can yield the existence of a root and a region of negativity for the internal energy. In such case, the root of the internal energy is obtained as

$$r_+|_{M=0} = \left(-\frac{q^2(\alpha^2-2)}{\Lambda\alpha^2}\right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}. \tag{37}$$

Evidently, the root of the internal energy is a decreasing function of Λ, while it is an increasing function of the electric charge. In order to elaborate our results, we have plotted a series of diagrams (see Fig. 2). Finally, considering positive

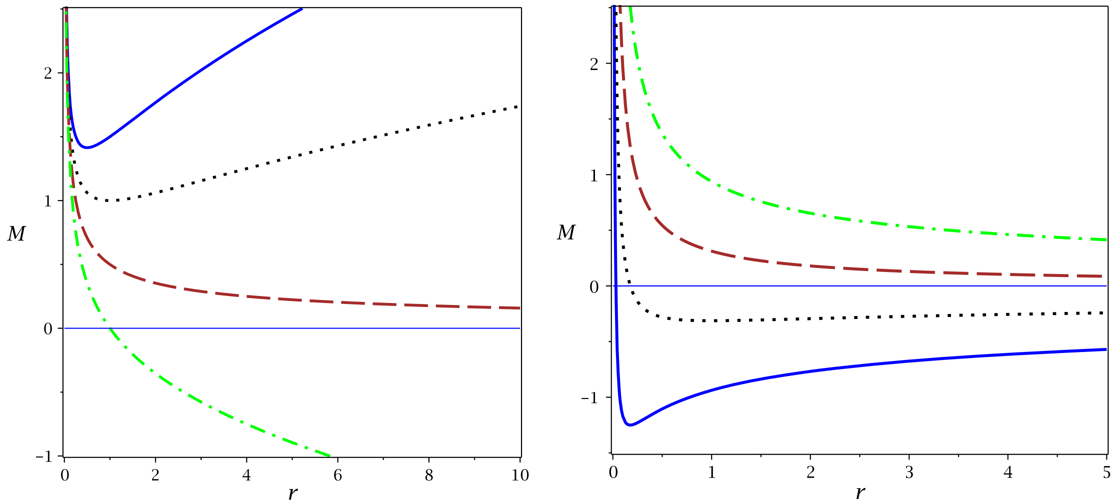


FIG. 2. M versus r_+, for $b = 1$, $q = 1$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line) and $\Lambda = 1$ (dashed-dotted line). Left panels: $\alpha = 1$; Right panels: $\alpha = 2$.

internal energy as a condition for having black holes, one can conclude that physical black hole solutions are present in range of $0 < r_+ < r_+|_{M=0}$. Later, we will add some other restrictions which are imposed by temperature and heat capacity to complete our picture for the solutions to describe physical black holes.

2. Temperature

In classical thermodynamics of black holes, one of the conditions for having physical solutions is the positivity of the temperature. This highlights the importance of the roots of temperature. It is a matter of calculation to show that for these black holes, we have the following root for temperature,

$$r_+|_{T=0} = \left(-\frac{q^2}{\Lambda}\right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}. \tag{38}$$

Here, the root of the temperature is a decreasing function of Λ, while it is an increasing function of the electric charge. Considering the possibility of having both positive and negative values for Λ, the positive valued root only exists for $\Lambda < 0$. On the other hand, for positive values of the cosmological constant ($\Lambda > 0$), the temperature will be negative. This indicates that in the classical thermodynamics of black holes, physical solutions exist only for $\Lambda < 0$ with the following condition,

$$\Lambda < -\frac{q^2}{r_+^2}\left(\frac{b}{r_+}\right)^{-2\gamma}.$$

Let us summarize what we have found by studying the temperature (in classical thermodynamics of black holes). First of all, physical solutions only exist for negative values of Λ. Also, we found an upper limit on the values of Λ which is obtained by the condition of having positive temperature. Condition that bridges the values of the electric charge, q with Λ. For completeness, we also present the following diagrams for the temperature in terms of r_+ (see Fig. 3). In the case of the absence of the root, the temperature is negative valued everywhere. On the other hand, in the presence of the root, the positive valued temperature only exists for $r_+|_{T=0} < r_+$. By suitable choices of different parameters, the temperature could also acquire an extremum (maximum). It is a matter of calculation to show that this extremum is obtained as

$$r_+|_{T=T_{\mathrm{Maximum}}} = \left(-\frac{q^2(\alpha^2+1)}{\Lambda(\alpha^2-1)}\right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}. \tag{39}$$

It is notable that, $r_+|_{T=T_{\mathrm{Maximum}}}$ is positive when Λ and α satisfy the following condition,

$$\begin{aligned} &\Lambda > 0 \quad \& \quad \alpha < 1, \\ &\Lambda < 0 \quad \& \quad \alpha > 1. \end{aligned} \tag{40}$$

Later, we will show that this extremum coincides with the divergencies of the heat capacity.

3. Heat capacity

The study of the heat capacity is important from two perspectives. First, their discontinuities represent thermodynamic phase transition critical points. Second, the sign of the heat capacity determines whether the system is thermally stable or unstable.

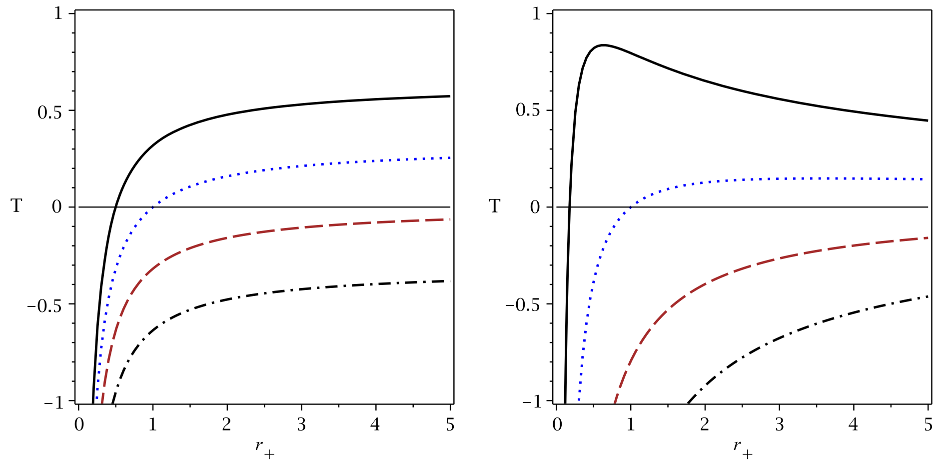


FIG. 3. T versus r_+, for $b=1$, $q=1$, $\Lambda=-2$ (continuous line), $\Lambda=-1$ (dotted line), $\Lambda=0$ (dashed line) and $\Lambda=1$ (dashed-dotted line). Left panels: $\alpha=1$; Right panels: $\alpha=2$.

The heat capacity is given by

$$C_Q = \frac{T}{\left(\frac{\partial^2 M}{\partial S^2}\right)_Q} = T\left(\frac{\partial S}{\partial T}\right)_Q = T\frac{\left(\frac{\partial S}{\partial r_+}\right)_Q}{\left(\frac{\partial T}{\partial r_+}\right)_Q}, \tag{41}$$

where by using Eqs. (25) and (28), this expression becomes

$$C_Q = -\frac{\pi r_+ \left(\frac{b}{r_+}\right)^\gamma [q^2 + \Lambda r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma}](\alpha^2+1)}{2q^2(3\alpha^2-1) - 2(\alpha^2-1)\Lambda r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma}}. \tag{42}$$

It is a matter of calculation to show that the root and divergence points of the heat capacity are given, respectively, by

$$r_+(C_Q=0) = \left(-\frac{q^2}{\Lambda}\right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}, \tag{43}$$

$$r_+(C_Q\to\infty) = \left(-\frac{q^2(\alpha^2+1)}{\Lambda(\alpha^2-1)}\right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}. \tag{44}$$

Evidently, both the heat capacity and the temperature share the same roots. Therefore, the arguments given for the root of the temperature stand for the heat capacity as well. In addition, the extremum for the temperature (39) and divergence point of the heat capacity are same. In order to have a positive valued divergence point for the heat capacity, the following conditions should be satisfied,

$$\begin{aligned} &\Lambda > 0 \quad \& \quad \alpha < 1, \\ &\Lambda < 0 \quad \& \quad \alpha > 1. \end{aligned} \tag{45}$$

In the previous section, we have shown that only for negative values of the Λ do our solutions enjoy a positive temperature. Consequently, for negative values of Λ and $\alpha > 1$, our solutions will develop a phase transition in their thermodynamic structure. In order to have a positive heat capacity which implies stable solutions, the denominator and numerator of the heat capacity must be of the same sign, that is,

$$q + \Lambda r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma} < 0 \quad \&$$

$$2q^2(\alpha^2+1) + 2(\alpha^2-1)\Lambda r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma} > 0,$$

or

$$q + \Lambda r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma} > 0 \quad \&$$

$$2q^2(\alpha^2+1) + 2(\alpha^2-1)\Lambda r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma} < 0.$$

In order to have a better picture of the behavior of the heat capacity, we have plotted various diagrams (see Fig. 4). It is explicitly shown that there are three possible cases for the heat capacity: (i) The heat capacity has no root and is negative everywhere, (ii) the heat capacity has only one root, in which after that, the heat capacity is positive and the solutions are stable, (iii) finally, the heat capacity enjoys one root and one divergence point in its structure. In this case, before the root and after the divergency, the heat capacity is negative and solutions are thermally unstable, whereas only between the root and divergence point is the system thermally stable. To end this section, we would like to add a comment. Previously, it was shown that the heat capacity in the context of BTZ-Λ-Maxwell theory enjoys the existence of the root only for negative values of Λ, whereas the divergence point was only observed for positive values of the Λ [92]. Here, we see that the generalization to dilaton gravity has a significant effect

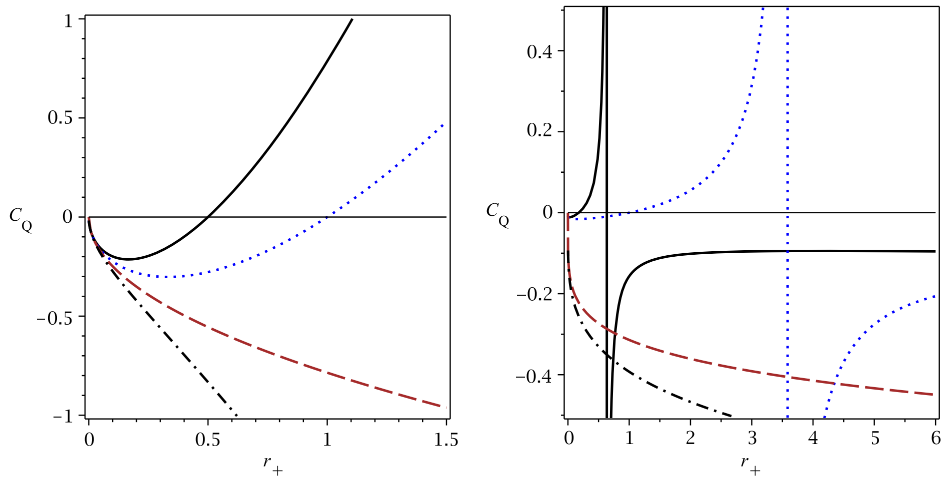


FIG. 4. C_Q versus r_+, for $b = 1$, $q = 1$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line) and $\Lambda = 1$ (dashed-dotted line). Left panels: $\alpha = 1$; Right panels: $\alpha = 2$.

on such behavior. The negative values of Λ can enjoy both root and divergency in their structure, while the branch of the positive Λ was ruled out due to the absence of positive valued temperature. Thermodynamically speaking, the generalization to dilaton gravity provides black holes with more complexity in their thermodynamic structure on the level of the introduction of a new phase transition point which was absent in the previous case. This highlights the differences between these two theories.

In order to have physical black holes, we have to consider negative values of the Λ for the temperature of black holes ($\Lambda < 0$). On the other hand, according to the obtained limit for positive internal energy [Eq. (36)], the existence of the event horizon for the obtained black holes [Eq. (22)] and also this fact that the heat capacity must be positive, the valid limitations for physical black holes are $\Lambda < 0$ and $1 < \alpha < \sqrt{2}$. Also, by using the obtained valid limitations for the cosmological constant and the dilaton parameter, numerical calculation of Ricci and Kretschmann scalars show that these quantities have maximum value when α is near to $\sqrt{2}(\alpha \to \sqrt{2})$. In other words, by the growth of dilaton parameter from 1 to $\sqrt{2}$, and also by considering finite radius, the curvature invariants increase.

III. BTZ BLACK HOLE SOLUTIONS IN DILATON-BORN-INFELD GRAVITY

We now turn in the derivation of the dilatonic-BI-BTZ black holes where the Lagrangian of the BI-dilaton part is given by

$$L(h,\Phi) = 4\beta^2 e^{4\alpha\Phi}\left(1 - \sqrt{1 + \frac{e^{-8\alpha\Phi}h}{2\beta^2}}\right), \tag{46}$$

where β is the BI parameter. It is notable that, in the limit $\beta \to \infty$, the Lagrangian BI, reduces to the standard Maxwell field coupled to a dilaton field as $L(h,\Phi) = -e^{4\alpha\Phi}h$. Varying the action (1) with respect to the metric tensor $g_{\mu\nu}$, the dilaton field Φ and the gauge field A_μ, we obtain the following field equations

$$R_{\mu\nu} = 4\left[\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{4}g_{\mu\nu}V(\Phi)\right] - 4e^{-4\alpha\Phi}\partial_Y L(Y)h_{\mu\eta}h_\nu^\eta + 4\beta^2 e^{4\alpha\Phi}[2Y\partial_Y L(Y) - L(Y)]g_{\mu\nu}, \tag{47}$$

$$\nabla^2\Phi = \frac{1}{8}\frac{\partial V(\Phi)}{\partial\Phi} + 2\alpha\beta^2 e^{4\alpha\Phi}[2Y\partial_Y L(Y) - L(Y)], \tag{48}$$

$$0 = \partial_\mu(\sqrt{-g}e^{-4\alpha\Phi}\partial_Y L(Y)h^{\mu\nu}), \tag{49}$$

where $L(Y) = 1 - \sqrt{1+Y}$ and Y is given as

$$Y = \frac{e^{-8\alpha\Phi}h}{2\beta^2}. \tag{50}$$

In order to derive the black hole solutions of this system of equations, we use the static ansatz metric as defined in Eq. (6) with a purely electrical field $A_\mu = \delta_\mu^0 A(r)$. Using the Eqs. (49) and (6), we have

$$4\beta^2\left[\alpha r E(r)R(r)\Phi'(r) - \frac{(rE(r)R(r))'}{4}\right]e^{8\alpha\Phi(r)} + [rR'(r) + R(r)]E^3(r) = 0, \tag{51}$$

where we can obtain the electric field as

$$E(r) = \frac{dA(r)}{dr} = \frac{qe^{4\alpha\Phi}}{rR(r)\sqrt{1+\Gamma}}, \tag{52}$$

where we have defined $\Gamma = \frac{q^2}{r^2\beta^2 R^2(r)}$. For latter convenience, we chose a Liouville-type dilation potential defined by $V(\Phi) = 2\Lambda e^{4\alpha\Phi}$ with the ansatz $R(r) = e^{2\alpha\Phi(r)}$. Using Eq. (52), the electrical field is

$$E(r) = \frac{q}{r}\frac{\left(\frac{b}{r}\right)^{\frac{2\alpha^2}{(\alpha^2+1)}}}{\sqrt{\frac{q^2}{r^2\beta^2} + \left(\frac{b}{r}\right)^{\frac{2\alpha^2}{(\alpha^2+1)}}}}, \tag{53}$$

according to the fact that the electrical field vanishes at infinity ($r \to \infty$), so $\beta \geq 0$ and $\alpha \geq 0$.

After some algebraic calculations, we obtain the following differential equations

$$\frac{\alpha^2 f(r) - r(1+\alpha^2)f'(r)}{r^2(1+\alpha^2)^2} + 4\left(\beta^2 - \frac{\Lambda}{2}\right)\left(\frac{b}{r}\right)^{2\gamma} - 4\beta^2\left(\frac{b}{r}\right)^{2\gamma}\sqrt{1 + \frac{q^2}{r^2\beta^2\left(\frac{b}{r}\right)^{2\gamma}}} = 0, \tag{54}$$

$$2(e^{2\alpha\Phi(r)})' + r(e^{2\alpha\Phi(r)})'' + 4r(\Phi'(r))^2 e^{2\alpha\Phi(r)} = 0. \tag{55}$$

We are now in a position to obtain exact solutions. Considering Eqs. (54) and (55), we can obtain the general solutions as

$$f(r) = \frac{2(\alpha^2+1)^2(\Lambda - 2\beta^2)r^2}{\alpha^2 - 2}\left(\frac{b}{r}\right)^{2\gamma} - mr^\gamma + \frac{4\beta^2(\alpha^2+1)^2}{\alpha^2-2}\left(\frac{b}{r}\right)^{2\gamma} r^2 H_1 + \frac{4q^2(\alpha^2+1)^2}{\alpha^2}H_2, \tag{56}$$

$$\Phi(r) = \frac{\gamma}{2\alpha}\ln\left(\frac{b}{r}\right), \tag{57}$$

in which H_1 and H_2 are the following hypergeometric functions

$$H_1 = {}_2F_1\left(\left[\frac{1}{2}, \frac{\alpha^2-2}{2}\right], \left[\frac{\alpha^2}{2}\right], -\Gamma\right),$$
$$H_2 = {}_2F_1\left(\left[\frac{1}{2}, \frac{\alpha^2}{2}\right], \left[\frac{\alpha^2+2}{2}\right], -\Gamma\right).$$

It is notable that, in the absence of a BI field ($\beta \to \infty$), the solutions (56) reduce to the charged dilatonic BTZ black hole solutions [see Eq. (17)].

Calculation of the Kretschmann scalar shows that it is finite for nonzero r and its behavior for very small and very large values of r can be reported as

$$\lim_{r\to 0^+} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \infty, \tag{58}$$

$$\lim_{r\to\infty} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \propto \frac{32\Lambda^2(\alpha^4 - 2\alpha^2 + \frac{3}{2})}{(\alpha^2-2)^2}\left(\frac{b}{r}\right)^{4\gamma}. \tag{59}$$

Equation (58) confirms that there is an essential singularity located at $r = 0$, while Eq. (59) shows that in the presence of the dilaton field ($\alpha \neq 0$), the asymptotic behavior of the solutions is not that of (A)dS. It is notable that, in the absence of dilaton field ($\alpha = 0$), the asymptotic behavior of the solutions is (A)dS. Similar to Maxwell case, for $\alpha \to \infty$, the asymptotic behaviors of Ricci and Kretschmann scalars are as $\lim_{r\to\infty} R = 0$ and $\lim_{r\to\infty} R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 0$. Indeed the effects of curvature singularity at infinity vanish.

Let us now discuss possibility of the existence of root. In the absence of electric charge, the obtained root for metric function will be identical to that obtained in Eq. (22). In the absence of the geometrical mass, the existence of root is only determined by sign of the first term in Eq. (56). For the metric function to have root, this term must be negative and nonzero; therefore, $\Lambda \neq 2\beta^2$. The existence of the root then is limited to the satisfaction of the one set of following conditions: (i) $\Lambda > 2\beta^2$ and $\alpha^2 < 2$, and (ii) $\Lambda < 2\beta^2$ and $\alpha^2 > 2$. In the absence of electric charge and geometrical mass, the metric function will be without any root.

Due to the complexity of the obtained metric function, it is not possible to obtain its roots analytically. Therefore, we use a numerical method. We plot the obtained metric function Eq. (56) in Fig. 5. This figure shows that the metric function may contain real positive roots, and, thus, the curvature singularity can be covered with an event horizon and interpreted as a black hole.

A. Thermodynamic properties

We can obtain the Hawking temperature by using Eq. (24) for the this black hole as

$$T = \frac{(\alpha^2+1)r_+}{2\pi}\left(\frac{b}{r_+}\right)^{2\gamma}[2\beta^2(1 - H_{1_+}) - \Lambda] + \frac{q^2(\alpha^2+1)}{\alpha^2\pi r_+}\left[H_3 - \alpha^2 H_{2_+} + \frac{\alpha^2\Gamma_+ H_4}{(\alpha^2+2)}\right]. \tag{60}$$

Notice that $H_{1_+} = H_1|_{r=r_+}$ and $H_{2_+} = H_2|_{r=r_+}$ and also that H_3 and H_4 are given by

$$H_3 = {}_2F_1\left(\left[\frac{3}{2}, \frac{\alpha^2}{2}\right], \left[\frac{\alpha^2+2}{2}\right], -\Gamma_+\right), \tag{61}$$

$$H_4 = {}_2F_1\left(\left[\frac{3}{2}, \frac{\alpha^2+2}{2}\right], \left[\frac{\alpha^2+4}{2}\right], -\Gamma_+\right), \tag{62}$$

where $\Gamma_+ = \Gamma|_{r=r_+}$. The electric potential U is obtained,

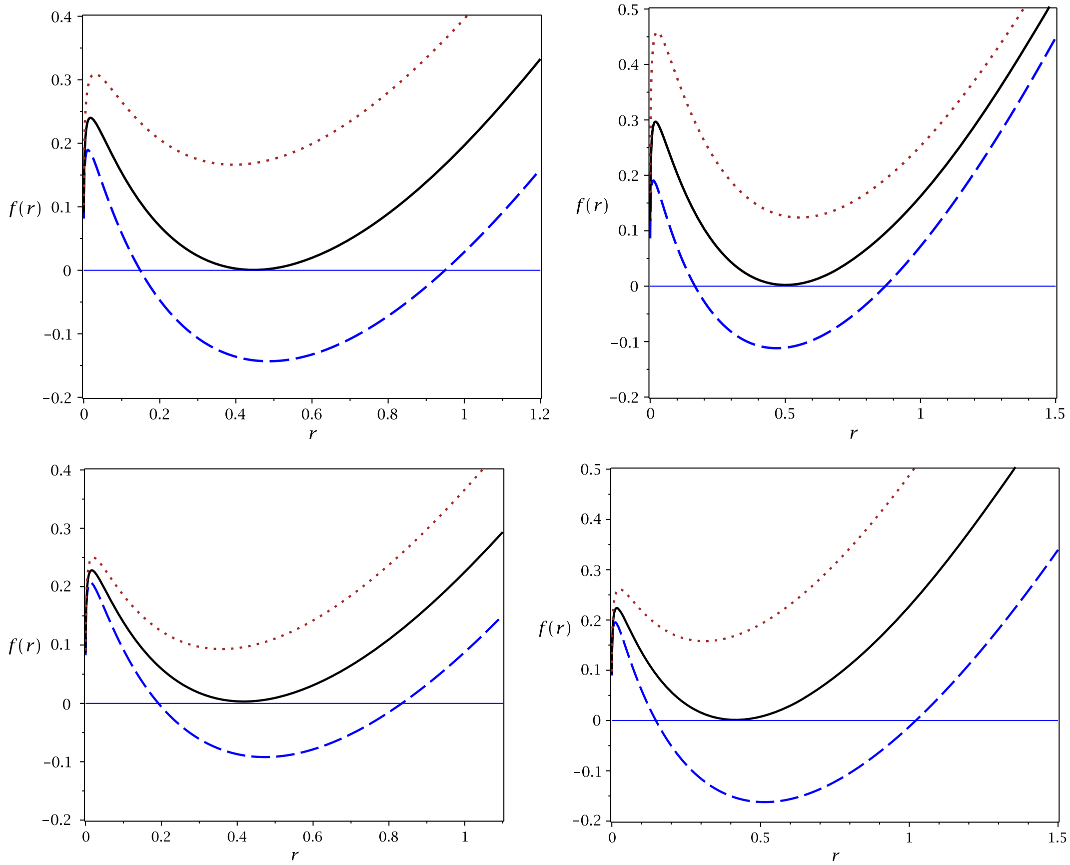


FIG. 5. $f(r)$ versus r, for $b = 0.3$, $\Lambda = -1$. Left top panel: for $m = 4$, $\alpha = 1$, $\beta = 0.5$, $q = 0.490$ (dashed line), $q = 0.518$ (continuous line) and $q = 0.550$ (dotted line). Right top panel: for $m = 4$, $\alpha = 1$, $q = 0.5$, $\beta = 0.480$ (dashed line), $\beta = 0.615$ (continuous line) and $\beta = 0.850$ (dotted line). Right bottom panel: for $m = 4$, $q = 0.5$, $\beta = 0.5$, $\alpha = 1.000$ (dashed line), $\alpha = 1.021$ (continuous line) and $\alpha = 1.040$ (dotted line). Left bottom panel: for $q = 0.5$, $\beta = 0.5$, $\alpha = 1$, $m = 4.10$ (dashed line), $m = 3.86$ (continuous line) and $m = 3.60$ (dotted line).

$$U = \frac{q}{\gamma}\left(\frac{b}{r_+}\right)^{\gamma} H_{2_+}. \tag{63}$$

Using the area law of entropy in Einstein gravity, and also calculating the flux of electromagnetic field at infinity, we can obtain the entropy and the electric charge of this black hole as

$$S = \frac{\pi r_+}{2}\left(\frac{b}{r_+}\right)^{\gamma}, \tag{64}$$

$$Q = \frac{q}{2}. \tag{65}$$

According to the mentioned method for calculation of the total finite mass of the metric presented in Eq. (6), we can obtain the total mass as

$$M = \frac{m}{8}(1-\gamma)b^{\gamma}, \tag{66}$$

which does not depend on the nonlinearity and on the electromagnetic field directly since both the nonlinearity and the electromagnetic field vanish for $r \to \infty$. Following the steps of the pervious sections, the total mass of the black hole solution is obtained by evaluating the metric function on its largest root,

$$M(r_+) = \frac{(\alpha^2+1)}{2(\alpha^2-2)}\left(\frac{b}{r_+}\right)^{\gamma}\left\{\left(\frac{\Lambda}{2}-\beta^2\right)r_+^2 + \beta^2\left[r_+^2\left(\frac{b}{r_+}\right)^{2\gamma}H_{1_+} + \frac{q^2(\alpha^2-2)H_{2_+}}{\alpha^2\beta^2}\right]\right\}, \tag{67}$$

yielding as well to a Smarr-like formula given by

$$M(S,Q) = \frac{(\alpha^2+1)}{2(\alpha^2-2)}\left(\frac{\pi b}{2S}\right)^{\alpha^2}\left\{\left(\frac{\Lambda}{2}-\beta^2\right)\frac{(\frac{2S}{\pi})^{2(\alpha^2+1)}}{b^{2\alpha^2}} + \beta^2\left[\left(\frac{2S}{\pi}\right)^2 H_{1_{SQ}} + \frac{4Q^2(\alpha^2-2)H_{2_{SQ}}}{\alpha^2\beta^2}\right]\right\}, \tag{68}$$

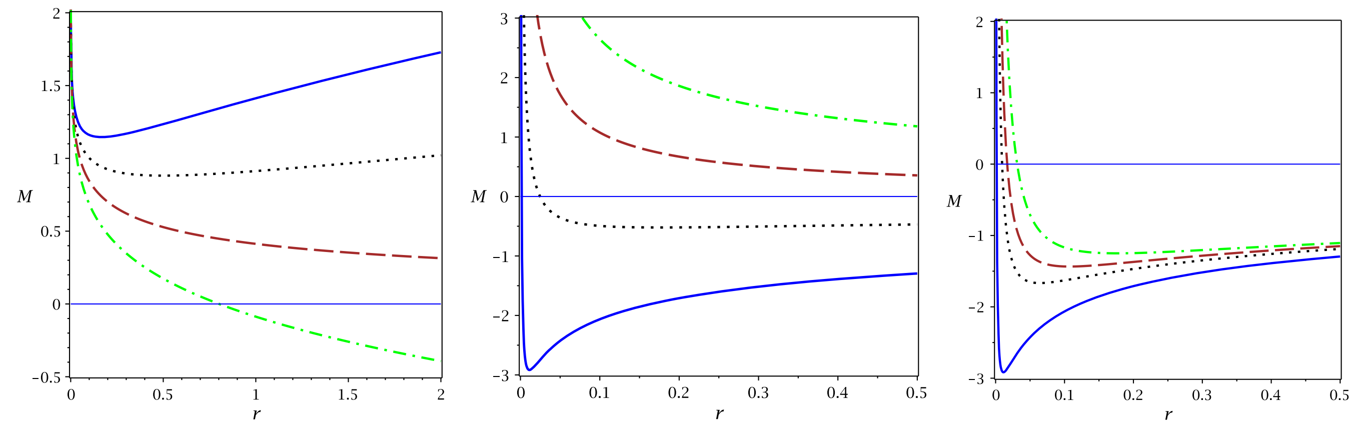


FIG. 6. M versus r_+, for $b = 1$, $q = 1$, $\beta = 0.5$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line), and $\Lambda = 1$ (dashed-dotted line). Left panels: $\alpha = 1$; Middle panels: $\alpha = 2$. Right panels: for $\Lambda = -2$, $\alpha = 2$, $\beta = 0.5$ (continuous line), $\beta = 1$ (dotted line), $\beta = 1.5$ (dashed line), and the Maxwell case: $\beta \to \infty$ (dashed-dotted line).

where $H_{1_{SQ}} = H_{1_+}|_{r_+ = b(\frac{2S}{\pi b})^{\alpha^2+1}, q=2Q}$ and $H_{2_{SQ}} = H_{2_+}|_{r_+ = b(\frac{2S}{\pi b})^{\alpha^2+1}, q=2Q}$. As a direct consequence, the first law of thermodynamics holds,

$$dM = TdS + UdQ, \tag{69}$$

with

$$dM(S,Q) = \left(\frac{\partial M(S,Q)}{\partial S}\right)_Q dS + \left(\frac{\partial M(S,Q)}{\partial Q}\right)_S dQ, \tag{70}$$

and

$$T = \left(\frac{\partial M}{\partial S}\right)_Q \quad \& \quad U = \left(\frac{\partial M}{\partial Q}\right)_S. \tag{71}$$

B. Thermodynamic behavior

In this section, we would like to stress how the Born-Infeld theory can modify the thermodynamic behavior of the black holes. Our main motivation is to distinguish the differences between the Maxwell and Born-Infeld theories in the thermodynamic context of charged BTZ-dilatonic black holes.

1. Mass/internal energy

As in the Maxwell case, the condition $\alpha \neq \sqrt{2}$ must also be taken into consideration. This is not surprising since this condition is only originated from the dilatonic part of the action. Interestingly enough, one can see from the expression of the mass that for

$$\Lambda = 2\beta^2, \tag{72}$$

the effects of the Λ term are canceled by the nonlinearity term and, since the other two terms (q and other β terms) are positive valued, the mass is positive everywhere without any root for this case. This is one of the most important contributions of the generalization to the BI field which is not seen in the context of the Maxwell case. Due to the complexity of the mass relation, it is not possible to obtain the root of the mass analytically. Therefore, we employ some numerical method and plot the following diagrams (see left and middle panels of Fig. 6). It could be seen that similar behaviors to those in Maxwell case are observed here too. This means that the mass of these black holes could enjoy the existences of root and two regions of positivity and negativity or it could be positive valued everywhere. In the case of the existence of root, the mass is only positive valued before the root. The place of this root is a function of the nonlinearity parameter (see right panel of Fig. 6). The term "$\alpha^2 - 2$" is present in all terms except q-term. Therefore, we can separate the effects of different terms (except the q-term) in the mass into two categories: $\alpha > \sqrt{2}$ and $\alpha < \sqrt{2}$. We give the details for $\alpha > \sqrt{2}$ since the opposite holds for the case $\alpha < \sqrt{2}$ (except for the q-term). For $\alpha > \sqrt{2}$, the q-term and one of the β-terms (the one which is coupled with the electric charge) have positive contributions on the total value of the mass. Whereas, the other β-term has always negative contribution. The effects of the Λ-term depends on the choices of Λ itself. For negative Λ, the effect of this term is toward decreasing mass while the opposite is observed for positive Λ. The existence of a root for positive Λ depends on the following condition:

$$\Lambda < 2\beta^2.$$

If the mentioned condition is satisfied, it is possible to find a root for the obtained mass, otherwise, the mass is always positive valued without any root. The situation for a negative Λ term is different and depends on the following condition

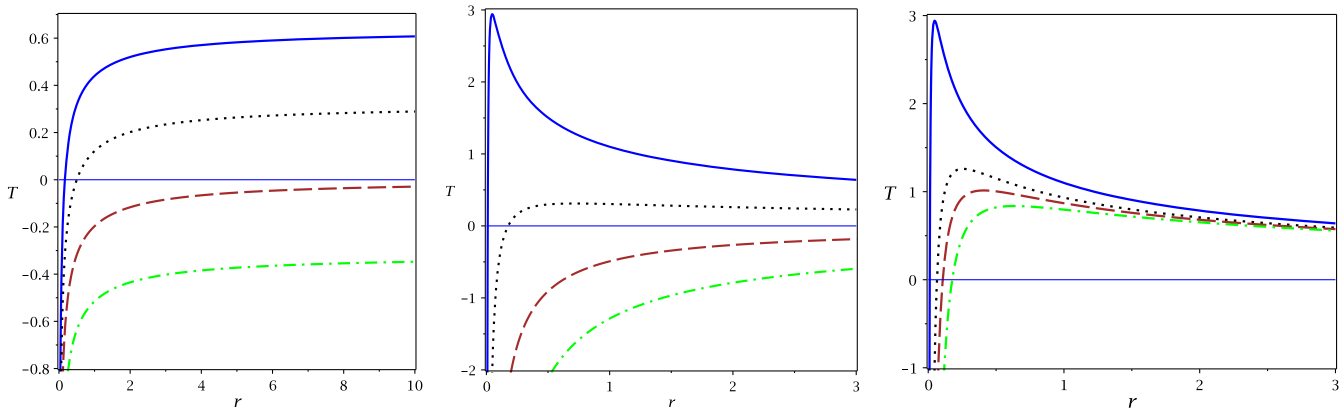


FIG. 7. T versus r_+, for $b = 1$, $q = 1$, $\beta = 0.5$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line), and $\Lambda = 1$ (dashed-dotted line). Left panels: $\alpha = 1$; Middle panels: $\alpha = 2$. Right panels: for $\Lambda = -2$, $\alpha = 2$, $\beta = 0.5$ (continuous line), $\beta = 1$ (dotted line), $\beta = 1.5$ (dashed line), and the Maxwell case: $\beta \to \infty$ (dashed-dotted line).

$$\left(\frac{\Lambda}{2} - \beta^2\right) r_+^2 > \beta^2 \left[r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma} H_{1_+} + \frac{q^2(\alpha^2 - 2)H_{2_+}}{\alpha^2 \beta^2} \right],$$

which highly depends on choices of the nonlinearity parameter, β.

Here, we see that the effects of the nonlinearity parameter, hence the BI generalization on the properties of the mass, are significant. The presence of β provided an extra degree of freedom, and because of that, the mass of the solutions may have different behaviors. In the next sections, we will give further details regarding this matter.

2. Temperature

Temperature for these black holes contains several terms: (i) Three q terms in which, two of them have positive contributions on the values of temperature while the other one has negative effect, (ii) two β terms, one of which contains an electric charge parameter, while the other one has only a dilatonic parameter. The earlier β term has negative effect on the values of temperature while the later one has opposite (positive) effect, and (iii) a Λ term for which, if Λ is negative, the effects are toward increasing the temperature and the opposite exists for the positive values of Λ. Here, too, the factors of Λ and one of the β terms are the same and by tuning the nonlinearity parameter properly ($\Lambda = 2\beta^2$), the effects of the Λ term could be canceled. This highlights one of the important effects of the generalization to BI theory.

Obtaining the root and extremum point of temperature is not possible analytically. As before, we will use some numerical method to plot the following diagrams (see left and middle panels of Fig. 7). Evidently, depending on the choices of different parameters, (i) the temperature could be completely negative which in turn implies that the solutions are not physical, (ii) the temperature could have one root and, before it, the temperature is negative valued, or (iii) the temperature could have one root and one maximum in which the maximum is located after the root and only after the root the temperature is positive. The places of the root and extremum depend on the choices of the nonlinearity parameter (see right panel of Fig. 7). The generalization to the nonlinear electromagnetic field provided us with extra terms in the temperature which eventually modified the root, the regions of negativity or positivity, and the extremum of temperature. In addition, this generalization results into one more degree of freedom which could be used to tune out the effects of some part of the dilaton gravity. This option was not possible in the context of Maxwell theory.

3. Heat capacity

Our final study in this section is devoted to the heat capacity of the nonlinearly charged solutions. By taking a closer look at Eq. (41), one can see that the heat capacity contains temperature and the derivations of entropy and temperature with respect to the horizon radius. The conditions regarding the roots and the positivity or negativity of the temperature were given in the last section. The derivation of the entropy with respect to the horizon radius does not produce any singular point, and it does not contain terms that could contribute to the positivity or negativity of the heat capacity. Therefore, we focus our attention on $(\frac{\partial T}{\partial r_+})_Q$ given by

$$\left(\frac{\partial T}{\partial r_+}\right)_Q = \frac{(\alpha^2 - 1)(\frac{\Lambda}{2} - \beta^2)(\frac{b}{r_+})^{2\gamma}}{\pi} + \frac{(\alpha^2 - 1)\beta^2 (\frac{b}{r_+})^{2\gamma}}{\pi} H_{1_+} - \frac{(2\alpha^2 - 1)H_{3_+} - \alpha^2(\alpha^2 + 1)H_{2_+}}{\pi \alpha^2 r_+^2} \tag{73}$$

$$+ \frac{(2\alpha^2 + 3)H_4 - 3H_5}{\pi \beta^2 (\alpha^2 + 2) r_+^4 (\frac{b}{r_+})^{2\gamma}} + \frac{3q^6 H_6}{\pi \beta^4 (\alpha^2 + 4) r_+^6 (\frac{b}{r_+})^{4\gamma}}, \tag{74}$$

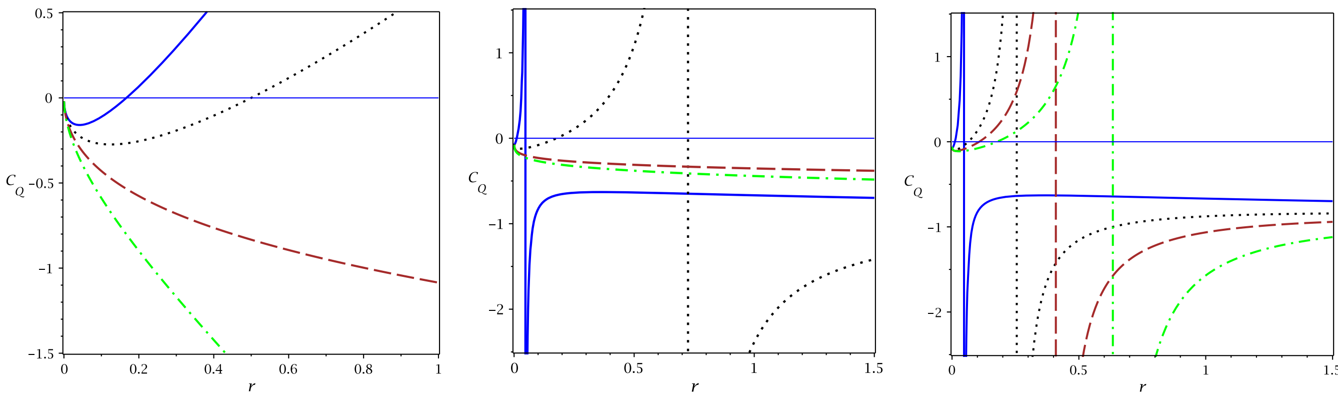


FIG. 8. C_Q versus r_+, for $b = 1$, $q = 1$, $\beta = 0.5$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line), and $\Lambda = 1$ (dashed-dotted line). Left panels: $\alpha = 1$; Right panels: $\alpha = 2$. Right panels: for $\Lambda = -2$, $\alpha = 2$, $\beta = 0.5$ (continuous line), $\beta = 1$ (dotted line), $\beta = 1.5$ (dashed line), and the Maxwell case: $\beta \to \infty$ (dashed-dotted line).

where H_5 and H_6 are defined as follows:

$$H_5 = {}_2F_1\left(\left[\frac{5}{2}, \frac{\alpha^2+2}{2}\right], \left[\frac{\alpha^2+4}{2}\right], -\Gamma_+\right), \tag{75}$$

$$H_6 = {}_2F_1\left(\left[\frac{5}{2}, \frac{\alpha^2+4}{2}\right], \left[\frac{\alpha^2+6}{2}\right], -\Gamma_+\right). \tag{76}$$

Similar to other quantities, here, we are able to tune out the effects of the Λ term by suitable choices of the nonlinearity parameter. In other words, it is possible to cancel out the effects of Λ in the heat capacity of BI solutions by choosing $\Lambda = 2\beta^2$. Once more, we point it out that such possibility is present in BI generalization while it is not seen in the linear Maxwell theory. The presence of the nonlinear electromagnetic field provided a complicated system of terms in the heat capacity. Unfortunately, such complication does not allow us to extract divergence points of the heat capacity analytically. We have plotted the following diagrams (see left and middle panels of Fig. 8). Depending on the choices of different parameters, one of the following cases would take place for the heat capacity: (i) Two states of stable and unstable which are separated by a root. In the previous section, it was shown that the root of the temperature, hence the heat capacity, is a function of nonlinearity parameter. (ii) The heat capacity could be negative everywhere without any root or divergency. In this case, the solutions are unstable but according to the results of previous section, the temperature is also negative which indicates that the solutions are not physical. (iii) The heat capacity could enjoy one root and one divergency. The divergency points out the existence of a phase transition. Before the root, the heat capacity is also negative. Therefore, the only physically stable solutions exist between the root and divergency of the heat capacity. The plotted diagram for the variation of the nonlinearity parameter (see right panel of Fig. 8), β, shows that the location of divergency is a function of this parameter. This indicates that the region in which physical stable solutions exist depends on the choice of β. This highlights another significant effect of the nonlinearity parameter.

IV. CONCLUSION

The paper at hand regarded BTZ black holes in the presence of two generalizations which are motivated by string theory: dilaton gravity and a Born-Infeld nonlinear electromagnetic field.

First, the solutions in the presence of dilaton gravity were extracted and their thermodynamic properties were studied. It was shown that, in comparison to Einstein gravity, here, the mass of these black holes could enjoy the existence of a root and a region of negative mass/internal energy. In addition, specific limits for the dilaton parameter and Λ were obtained, and it was shown that, thermodynamically speaking, only for a specific region of the dilaton parameter physical solutions exist. Therefore, although generalization to dilaton gravity provided us with new properties for the solutions, at the same time, it imposed specific limits on them as well. In other words, introducing new properties into solutions by dilatonic generalization was obtained at the cost of harder restrictions on the solutions.

Next, Born-Infeld generalization was implied to the action. It was shown that this generalization provides the possibility of canceling a part of dilatonic contribution by suitable choices of parameters. In addition, it was shown that although some of the Maxwell conditions for having thermodynamically physical solutions stand for this case as well, the general behavior of the solutions including phase transition point, region of stable solutions, conditions of having physical solutions were modified due to the contributions of BI theory.

The study conducted here could be employed to investigate aspects such as superconductor properties, holographical principles and entropy spectrum. Specially, it is interesting to study the central charges of $1+1$ theories and understand the effects of the dilatonic gravity and BI generalization in this context. In addition, as it is known, we can discuss dyon solutions of our dilatonic setup with two horizons. We address these subjects in the forthcoming works.

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APPENDIX: POSSIBILITY OF THE STRONG COUPLING

The runaway type behavior of the dilaton implies that the theory becomes quantum mechanically strongly coupled at some finite radius r, as can be seen for example by the growth of the curvature invariants, or the growth of the effective electric charge. This can invalidate all semiclassical calculations on top of the corresponding solutions.

Considering Eq. (2), one can find that the coefficient of the kinetic term of the electromagnetic Lagrangian goes to zero for $r \to 0$ on the specific solution (13). Although we work in the case that $\alpha\Phi(r)$ is finite everywhere, it is interesting to investigate the possible existence of strong coupling of the effective action. As an example, one may regard a non-vanishing current-current correlator case for dynamical charges in which the induced electromagnetic Lagrangian term could overpower the leading contribution. In order to check that such issue is not important in our case, one may compare the size of the horizon to the radius at which the effective charge $\frac{qe^{4\alpha\Phi(r)}}{R(r)}$ goes to one. Using Eqs. (12) and (13), one can obtain

$$\mathcal{Q}_{\mathrm{eff}} = \frac{qe^{4\alpha\Phi(r)}}{R(r)} = q\left(\frac{b}{r}\right)^{\frac{\alpha^2}{1+\alpha^2}}, \tag{A1}$$

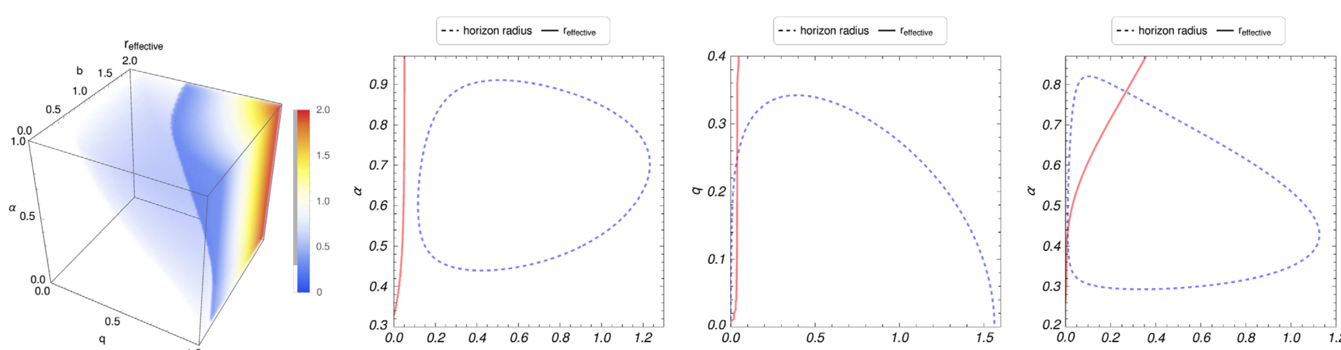
or equivalently, we find that

$$r = b\left(\frac{q}{\mathcal{Q}_{\mathrm{eff}}}\right)^{\frac{1+\alpha^2}{\alpha^2}}.$$

In order to find the effective radius, we suppose that the effective charge goes to one, $\mathcal{Q}_{\mathrm{eff}} = 1$, yielding

$$r_{\mathrm{eff}} = bq^{\frac{1+\alpha^2}{\alpha^2}}.$$

The effects of different parameters on the effective radius are given in Fig. 9a showing that it is an increasing function of the electric charge and dilaton parameters. That being said, the important issue is that this effective radius is smaller than horizon radius. To address this issue, we employ the numerical method and plots diagrams in Figs. 9b and 9c. Evidently, for considered values, the effective radius is smaller than horizon radius or at least the largest horizon radius. Therefore, for considered values for different parameters, the effective radius is indeed within acceptable regime. Though for considered values this is true, we should highlight a few matters: the metric function contains extra parameters such as the cosmological constant, geometrical mass and nonlinearity parameter which are absent in Eq. (A1). This indicates that these parameters could considerably change the horizon radius in a way that it becomes smaller than effective radius. An example of such case could be seen in Fig. 9d. As one can see, such cases usually happens when b parameters becomes significantly large. Therefore, by keeping the value of such parameter to small ones, it is possible to avoid situations where effective radius becomes larger than horizon radius. This is actually what is done through our calculations.



(a) (b) (c) (d)

FIG. 9. Horizon and effective radius for the Maxwell case. (a) Variation of r_{eff} as a function electric charge and dilaton parameters. (b) Variations of r_{eff} and horizon radius as a function of α for $m = 3$, $\Lambda = -1$, $b = 0.3$ and $q = 0.4$. (c) Variations of r_{eff} and horizon radius as a function of q for $m = 3$, $\Lambda = -1$, $b = 0.3$ and $\alpha = 1$. (d) Variation of r_{eff} and horizon radius as a function of α for $\Lambda = -1$, $q = 0.4$, $b = 3$ and $m = 5$.

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