

Note on orbit space of G membranes

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The motion of test membranes on which the group G of isometries of a spacetime M acts has been considered in general settings. It has been shown that the configuration of Nambu-Goto membranes is described by the Nambu-Goto membranes in the quotient manifold M/G with an appropriate projected metric if (i) G is Abelian, (ii) G is semisimple and compact, or (iii) the orthogonal distribution of the orbit of G is integrable, but in general not. It has also been shown that a similar result holds when the membranes couple with scalar maps or differential form fields.

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I. INTRODUCTION

Extended objects in cosmology such as cosmic strings and membranes (domain walls) naturally arise as topological defects associated with various symmetry-breaking phenomena in quantum field theory. They play an important role in the scenario of the structure formation of the Universe.

The description of motion of extended objects such as strings and membranes in a spacetime with Killing vector fields simplifies when they respect the spacetime symmetry.

This kind of simplification has long been known in the minimal surface theory [1] as the cohomogeneity technique. In general relativity, Frolov *et al.* [2] find that the configuration of stationary Nambu-Goto strings in a stationary spacetime is determined by the geodesic equation in a certain Riemannian 3-manifold. Their reasoning is based on the observation that when the stationary string ansatz is substituted into the Nambu-Goto action, the action reduces to the geodesic action via the dimensional reduction.

This technique is widely applied to the construction of stationary string solutions in various background spacetimes [3–8]. It is also useful to find dynamical string solutions in spacetimes with a Killing vector field [9–13].

A similar idea for this dimensional reduction also works for $f + 1$ -dimensional Nambu-Goto membranes when the spacetime has f pairwise commuting Killing vector fields and the membrane respects this symmetry [14].

The authors of Ref. [15] pointed out that the dimensional reduction at the action level also occurs when the f Killing vector fields are noncommuting. With this observation, they claim that the $f + 1$ -dimensional Nambu-Goto membranes respecting the f -dimensional non-Abelian group of isometry of the spacetime could be reduced to the geodesic motion in the quotient manifold.

However, it is of course not the correct procedure to put a trial solution directly into the action. Hence, we would like to confirm whether the above claim is correct or not.

In the following, we consider Nambu-Goto membranes of general dimensions in spacetime with a non-Abelian group of isometries, assuming that the membranes respect the spacetime symmetry. We show that the resultant equation of motion for the membranes is almost that for lower-dimensional Nambu-Goto membranes, but with extra force terms. This force term becomes zero for the Abelian case, or the semisimple and compact cases, but it does not in general. In particular, the claim in Ref. [15] is not the case.

The organization of this paper is as follows. In Sec. II, the mathematical settings are described. In Sec. III, the Nambu-Goto membranes with spacetime symmetries are considered, and their general equation of motion is derived. In Sec. IV, similar consideration on the membranes coupled with scalar maps is made. In Sec. V, membranes coupled with a differential form field are treated. In Sec. VI, several remarks are made.

II. ISOMETRIC ACTIONS ON WORLD SHEETS

Let (M, g) be an m -dimensional spacetime, which is a differentiable manifold M endowed with a spacetime metric g of the signature $(-, +, \dots, +)$. Let G be an f -dimensional connected Lie subgroup of the full isometry group of M .

The (left) G -action on M is a group homomorphism

$$G \rightarrow \text{Diff}(M); g \mapsto f_g$$

of G into the group of diffeomorphisms on M , such that

$$G \times M \rightarrow M; (g, x) \mapsto f_g(x)$$

is differentiable.

We assume that the G -action on M is free, which means that f_g does not have a fixed point on M for every nonidentity element g of G , or in other words, M admits

f linearly independent Killing vector fields as the infinitesimal generators of G .

We also require that the G -action on M be proper, i.e., the map

$$G \times M \rightarrow M \times M; (g, x) \mapsto (f_g(x), x)$$

is proper, which means that the preimage of any compact set is compact.

For each element x in M , the set

$$Gx = \{f_g(x) | g \in G\}$$

is called the orbit of x . The set of these orbits is called the orbit space, and it is denoted by M/G .

Under the free and proper action of G , it is guaranteed that (a) each orbit Gx is an embedded closed submanifold of M , (b) Gx is diffeomorphic with G , and (c) the orbit space M/G naturally acquires a differentiable structure.

In the rest of this section, we give the general form of the Lorentzian metric g on M . The construction goes along similar lines to that of homogeneous universes [16].

We first determine the geometry of the orbit Gx . We assume that each orbit is non-null, so that the induced metric on Gx is a nondegenerate Riemannian or Lorentzian metric.

Let $\{y^i\}_{i=1,2,\dots,f}$ be a local coordinate system on Gx . Since f linearly independent Killing vector fields $\{\xi_I\}_{I=1,2,\dots,f}$ generating a G -action are tangent to Gx , these can be written as $\xi_I = \xi_I^i \partial_i$ with this coordinate basis.

Since these Killing vector fields generate a left action of G on Gx , they are identified with the right invariant vector fields on G . According to the general theory of Lie groups, the right invariant vector fields on G are subject to the commuting relation

$$[\xi_I, \xi_J] = f_{IJ}^K \xi_K,$$

where f_{IJ}^K 's are the structure constants for the Lie algebra \mathfrak{g} of G .

A left invariant vector field σ^i on Gx is a tangent vector field invariant under the G -action, characterized by the equation

$$\mathcal{L}_{\xi_I} \sigma^i = 0,$$

where \mathcal{L}_{ξ_I} denotes the Lie derivative. This equation admits f linearly independent solutions, which we denote by $\{\sigma_I^i\}_{I=1,2,\dots,f}$. By taking a linear combination, it is always possible to find the basis $\{\sigma_I^i\}_{I=1,2,\dots,f}$ of the left invariant vector fields, such that

$$[\sigma_I, \sigma_J] = f_{IJ}^K \sigma_K$$

holds.

The dual basis of 1-forms $\{\sigma_I^J\}_{I=1,2,\dots,f}$, characterized by

$$\sigma_I^k \sigma_k^J = \delta_I^J,$$

consists of left invariant 1-forms. These satisfy

$$\begin{aligned} \mathcal{L}_{\xi_I} \sigma_k^J &= 0, \\ d\sigma^K &= -\frac{1}{2} f_{IJ}^K \sigma^I \wedge \sigma^J. \end{aligned}$$

The induced metric on Gx can be written in terms of this left invariant basis as

$$g_{ij} = \phi_{IJ} \sigma_i^J \sigma_j^I, \quad (1)$$

with entries ϕ_{IJ} of the nondegenerate symmetric matrix. Since g_{ij} is invariant under the G -action, i.e.,

$$\mathcal{L}_{\xi_I} g_{ij} = 0$$

should be required, the coefficients ϕ_{IJ} are constants over Gx . Equation (1) gives the general form of the metric on the orbit.

Now, we can write the spacetime metric g in the present setting. Since the orbit space M/G is the differentiable manifold, it has a local coordinate system, which we denote by $\{z^\mu\}_{\mu=1,2,\dots,b}$, where $b = m - f$. The spacetime metric could in general be written as

$$\begin{aligned} g &= \phi_{IJ}(z^\alpha) (\sigma^I(y^k) - w^I_\mu(y^k, z^\lambda) dz^\mu) (\sigma^J(y^k) \\ &\quad - w^J_\nu(y^k, z^\lambda) dz^\nu) + h_{\mu\nu}(y^k, z^\lambda) dz^\mu dz^\nu, \end{aligned}$$

so that it induces $g_{ij} = \phi_{IJ} \sigma_i^I \sigma_j^J$ on each orbit by setting $z^\mu = \text{const}$. Since the spacetime metric g is invariant under the G -action, it is subject to the Killing equation

$$\mathcal{L}_{\xi_I} g = 0.$$

Then, it is required that

$$w^I_\nu = w^I_\nu(z^k), \quad h_{\mu\nu} = h_{\mu\nu}(z^k)$$

hold. This gives the general local form of the spacetime metric g .

III. NAMBU-GOTO G MEMBRANES

Let us consider the motion of extended objects in spacetimes equipped with the isometric G -action. It is generally expected that the equation of motion simplifies when the extended object also respects the isometry. The simplification typically occurs in the form of the dimensional reduction; i.e., the equation of motion reduces to that for the objects (e.g., particles, strings, or membranes) in the orbit space M/G . In this section, we study the Nambu-Goto

membranes as a basic example of extended objects in the relativistic mechanics.

In general, a relativistic membrane is described as a timelike immersion $i: W \rightarrow M$ of a differentiable manifold W , called a world sheet, into the spacetime M . Let $\{x^a\}_{a=1,2,\dots,m}$ and $\{s^A\}_{A=1,2,\dots,w}$ be a local coordinate system on M and W , respectively. The immersion i is locally described as

$$x^a = X^a(s^1, \dots, s^w),$$

in terms of the m scalar functions X^a on the world sheet W . Then, the Lorentzian metric

$$G_{AB} = g_{ab} X^a_{,A} X^b_{,B}$$

is locally induced on W .

The Nambu-Goto action for the relativistic membrane is given by

$$S[X^a] = -\tau \int_W ds^1 \dots ds^w \sqrt{|G|},$$

where τ is a constant identified with the tension of the membrane, and $G = \det G_{AB}$. The Euler-Lagrange equation becomes

$$K^a := D_C D^C X^a + \Gamma_{bc}^a G^{bc} = 0,$$

where D_C denotes the covariant derivative with respect to G_{AB} , Γ_{bc}^a the Christoffel symbol for g_{ab} , and

$$\mathcal{G}^{ab} = G^{AB} (D_A X^a) D_B X^b$$

has been defined, where G^{AB} denotes the inverse matrix of G_{AB} .

The vector field K^a on W has a simple geometrical meaning. It is the mean curvature vector

$$K^a = G^{BC} K_{BC}^a,$$

which is the trace of the extrinsic curvature vector

$$K_{BC}^a = D_B D_C X^a + \Gamma_{bc}^a (D_B X^b) D_C X^c.$$

The extrinsic curvature vector is defined as follows: let U^A and V^A be tangent vector fields on W , and let U^a and V^a be smoothly extended vector fields of $i_* U$ and $i_* V$, respectively, to a neighborhood \mathcal{W} of W . For $x \in W$, the orthogonal decomposition $T_x M = T_x W \oplus N_x W$ of $(\nabla_U \mathcal{V})_x$ is written as

$$\nabla_U \mathcal{V} = D_U V + K(U, V).$$

Then, $K: T_x W \times T_x W \rightarrow N_x W$ is defined by this equation.

We assume that the group of isometry G acts freely and properly on M . Then, as we see in Sec. II, the spacetime metric can be written locally as

$$g = \phi_{IJ} (\sigma^I - w^I_\mu dz^\mu) (\sigma^J - w^J_\nu dz^\nu) + h_{\mu\nu} dz^\mu dz^\nu,$$

where $\sigma^I = \sigma^I(y^k)$ constitutes a left invariant basis of 1-forms on the orbit Gx , and

$$\phi_{IJ} = \phi_{IJ}(z^\lambda), \quad w^I_\mu = w^I_\mu(z^\lambda), \quad h_{\mu\nu} = h_{\mu\nu}(z^\lambda)$$

are the scalar, vector and metric tensor fields on the orbit space M/G .

Let the membrane respect this isometry, so that the image of the world sheet W is G -invariant, i.e., invariant under the action of G . The general form of such G membranes can be written as

$$\begin{aligned} X^i &= \alpha^i \quad (i = 1, \dots, f) \\ X^\mu &= X^\mu(\beta^{A'}) \quad (\mu = f + 1, \dots, m) \end{aligned}$$

in terms of the world sheet coordinates

$$\{s^A\} = \{\alpha^i; \beta^{A'}\}_{i=1,\dots,f, A'=f+1,\dots,w}$$

This is identified with the immersion of the world sheet orbit space W/G into the spacetime orbit space M/G , characterized by

$$X^\mu = X^\mu(\beta^{A'}).$$

Thus, a G membrane can be regarded as a membrane in the orbit space M/G .

Although the following calculations are most efficiently executed via the Cartan's structure equations for connection forms, we show the results of the direct coordinate calculations for the reader's convenience. In the following calculations, indices are raised and lowered, respectively, by ϕ^{IJ} , ϕ_{IJ} , λ^{ij} , λ_{ij} , $h^{\mu\nu}$, and $h_{\mu\nu}$, where ϕ^{IJ} denotes the entries of the inverse matrix of ϕ_{IJ} , λ_{ij} is defined by

$$\lambda_{ij} = \phi_{IJ} \sigma^I_i \sigma^J_j,$$

λ^{ij} is its inverse, and $h^{\mu\nu}$ is the inverse of $h_{\mu\nu}$. The covariant derivative compatible with $h_{\mu\nu}$ is denoted by the semicolon.

The components of the spacetime metric g and its inverse g^{-1} are given by

$$\begin{aligned} g_{ij} &= \lambda_{ij}, & g_{i\mu} &= -w_{i\mu}, & g_{\mu\nu} &= h_{\mu\nu} + w_{K\mu} w^K_\nu, \\ g^{ij} &= \lambda^{ij} + \sigma^I_i \sigma^J_j w^I_\lambda w^{J\lambda}, & g^{i\mu} &= \sigma_K^i w^{K\mu}, & g^{\mu\nu} &= h^{\mu\nu}. \end{aligned}$$

The Christoffel symbols are computed as

$$\Gamma_{jk}^i = \sigma_I^i \sigma^J_{(j,k)} + \sigma_I^i \sigma^J_j \sigma^K_k \left[f^I_{(JK)} + f_{L(JK)} w^L_\mu w^{\mu} \right. \\ \left. - \frac{1}{2} \phi_{JK,\mu} w^{\mu} \right], \quad (2)$$

$$\Gamma_{j\lambda}^i = \sigma_I^i \sigma^J_j \left[\frac{1}{2} \phi_{JK,\lambda} \phi^{IK} + \frac{1}{2} \phi_{JK,\rho} w^{I\rho} w^K_\lambda \right. \\ \left. - \frac{1}{2} f^I_{JK} w^K_\lambda - \frac{1}{2} f_{LJK} w^{I\rho} w^L_\rho w^K_\lambda - w^{I\rho} \phi_{JK} w^K_{[\rho;\lambda]} \right], \quad (3)$$

$$\Gamma_{\nu\lambda}^i = \sigma_K^i \left[-\phi^{IK} \phi_{IJ,(\nu} w^J_{\lambda)} - w^K_{(\nu,\lambda)} + w^K_\mu {}^h \Gamma_{\nu\lambda}^\mu \right. \\ \left. + w^{K\mu} \left(-\frac{1}{2} \phi_{IJ,\mu} w^I_\nu w^J_\lambda + w_{I\nu} w^I_{[\mu;\lambda]} + w_{I\lambda} w^I_{[\mu;\nu]} \right) \right], \quad (4)$$

$$\Gamma_{jk}^\mu = \sigma^J_j \sigma^K_k \left(-\frac{1}{2} \phi_{JK}{}^\mu + f_{I(JK)} w^{I\mu} \right), \quad (5)$$

$$\Gamma_{\nu k}^\mu = \sigma^K_k \left(\frac{1}{2} \phi_{JK}{}^\mu w^J_\nu + \phi_{JK} h^{\mu\lambda} w^J_{[\nu;\lambda]} - \frac{1}{2} f_{IKJ} w^{I\mu} w^J_\nu \right), \quad (6)$$

$$\Gamma_{\nu\lambda}^\mu = {}^h \Gamma_{\nu\lambda}^\mu - \frac{1}{2} \phi_{IJ}{}^\mu w^I_\nu w^J_\lambda + w_{I(\nu} w^{I\mu}_{;\lambda)} - w_{I(\nu} w^I_{\lambda)}{}^\mu, \quad (7)$$

where ${}^h \Gamma_{\nu\lambda}^\mu$ denotes the Christoffel symbol with respect to $h_{\mu\nu}$. Note that we raise or lower the indices I, J, \dots with ϕ_{IJ} , ϕ^{IJ} , but not with the Killing metric on the Lie algebra \mathfrak{g} , so that, e.g., f_{IJK} may not be totally antisymmetric under the permutation of the indices.

The induced metric G_{AB} and its inverse G^{AB} on the world sheet W become

$$G_{ij} = \lambda_{ij}, \quad G_{iB'} = -\lambda_{ij} C^j_{B'}, \\ G_{A'B'} = G'_{A'B'} + \lambda_{ij} C^i_{A'} C^j_{B'}, \\ G^{ij} = \lambda^{ij} + C^i_{A'} C^j_{B'} G'^{A'B'}, \\ G^{iB'} = C^i_{A'} G'^{A'B'}, \quad G^{A'B'} = G'^{A'B'},$$

where

$$C^j_{B'} = \sigma_J^j w^J_\lambda D_{B'} X^\lambda, \\ G'_{A'B'} = h_{\mu\nu} (D_{A'} X^\mu) D_{B'} X^\nu$$

have been defined, and $G'^{A'B'}$ denotes the inverse of $G'_{A'B'}$. This $G'_{A'B'}$ gives the induced metric on the quotient world sheet W/G as the membrane immersed in $(M/G, h)$.

The spacetime component $\mathcal{G}^{\mu\nu}$ of G^{AB} is calculated as

$$\mathcal{G}^{ij} = \sigma_I^i \sigma_J^j (\phi^{IJ} + w^I_\mu w^J_\nu \mathcal{G}'^{\mu\nu}), \\ \mathcal{G}^{i\nu} = \sigma_I^i w^I_\mu \mathcal{G}'^{\mu\nu}, \\ \mathcal{G}^{\mu\nu} = \mathcal{G}'^{\mu\nu},$$

where

$$\mathcal{G}'^{\mu\nu} = G'^{A'B'} (D'_{A'} X^\mu) D'_{B'} X^\nu$$

has been defined, which is the spacetime component of $G'^{A'B'}$.

In order to derive the equation of motion for G membranes, we need the expression for the extrinsic curvature vector:

$$K^a_{BC} = D_B D_C X^a + \Gamma^a_{bc} (D_B X^b) D_C X^c \\ = X^a_{,BC} - {}^G \Gamma^A_{BC} X^a_{,A} + \Gamma^a_{bc} X^b_{,B} X^c_{,C},$$

where ${}^G \Gamma^A_{BC}$ denotes the Christoffel symbol with respect to G_{AB} .

Noting that

$${}^G \Gamma^A_{BC} = G^{AD} X^a_{,D} g_{ad} (X^d_{,BC} + \Gamma^d_{bc} X^b_{,B} X^c_{,C}),$$

we have another expression for the extrinsic curvature vector:

$$K^a_{AB} = (\delta^a_d - \mathcal{G}^a_d) (X^d_{,AB} + \Gamma^d_{bc} X^b_{,A} X^c_{,B}).$$

The direct computations show

$$K^{\mu}_{ij} = N'^{\mu\nu} \left(f_{K(IJ)} w^K_\nu - \frac{1}{2} \phi_{IJ,\nu} \right) \sigma^I_i \sigma^J_j, \quad (8)$$

$$K^{\mu}_{iB'} = \frac{1}{2} N'^{\mu\nu} (f_{IJK} w^J_\nu w^K_\lambda + \phi_{IJ,\nu} w^J_\lambda - 2\phi_{IJ} w^I_{[\nu;\lambda]}) \\ \times \sigma^I_i D'_{B'} X^\lambda, \quad (9)$$

$$K^{\mu}_{A'B'} = K'^{\mu}_{A'B'} \\ + N'^{\mu\nu} \left(-\frac{1}{2} \phi_{IJ,\nu} w^I_\lambda w^J_\rho + 2\phi_{IJ} w^I_{[\nu;\lambda]} w^J_\rho \right) \\ \times (D'_{A'} X^\lambda) D'_{B'} X^\rho, \quad (10)$$

$$K^i_{AB} = \sigma_I^i w^I_\mu K^{\mu}_{AB}, \quad (11)$$

where $D'_{A'}$ denotes the covariant derivative compatible with $G'_{A'B'}$; $N'^{\mu\nu}$ the projection onto the normal space to W/G in M/G , defined as

$$N'^{\mu\nu} = h^{\mu\nu} - \mathcal{G}'^{\mu\nu};$$

and

$$K'^{\mu}_{A'B'} = D'_{A'} D'_{B'} X^{\mu} + {}^h \Gamma^{\mu}_{\nu\lambda} (D'_{A'} X^{\nu}) D'_{B'} X^{\lambda}$$

is the extrinsic curvature vector of W/G relative to $(M/G, h)$.

Then, the equation of motion is calculated as

$$K^{\mu} = K'^{\mu} + N'^{\mu\nu} \left(f_{IJ}{}^J w^I{}_{\nu} - \frac{1}{2} \phi^{-1} \phi_{,\nu} \right) = 0,$$

where $K'^{\mu} = G'^{A'B'} K'^{\mu}_{A'B'}$ is the mean curvature vector of W/G relative to $(M/G, h)$, and we abbreviate as $\phi = \det \phi_{IJ}$. The remaining equation $K^i = 0$ does not give further restriction since

$$K^i = \sigma_I{}^i w^I{}_{\mu} K^{\mu}$$

holds.

This resembles the equation of motion for Nambu-Goto membranes, but with the extra force term. We can partially reduce the force term via the conformal transformation

$$h_{\mu\nu} = |\phi|^{-1/\tilde{w}} \tilde{h}_{\mu\nu},$$

$$G'_{A'B'} = |\phi|^{-1/\tilde{w}} \tilde{G}_{A'B'},$$

where $\tilde{w} = w - f$. The inverses of $\tilde{h}_{\mu\nu}$ and $\tilde{G}_{A'B'}$ are, respectively, written as $\tilde{h}^{\mu\nu}$ and $\tilde{G}^{A'B'}$.

The extrinsic curvature vector of W/G relative to $(M/G, \tilde{h})$ is written as

$$\tilde{K}^{\mu}_{A'B'} = \tilde{D}_{A'} \tilde{D}_{B'} X^{\mu} + \tilde{\Gamma}^{\mu}_{\nu\lambda} (\tilde{D}_{A'} X^{\nu}) \tilde{D}_{B'} X^{\lambda},$$

where $\tilde{D}_{A'}$ denotes the covariant derivative with respect to the conformally transformed world sheet metric $\tilde{G}_{A'B'}$ and $\tilde{\Gamma}^{\mu}_{\nu\lambda}$ the Christoffel symbol with respect to $\tilde{G}_{A'B'}$. Here and in what follows, the indices for conformally transformed quantities are raised or lowered in terms of $\tilde{G}^{A'B'}$, $\tilde{G}_{A'B'}$, $\tilde{h}^{\mu\nu}$, and $\tilde{h}_{\mu\nu}$.

The extrinsic curvature vector undergoes the conformal transformation as

$$\tilde{K}^{\mu}_{A'B'} = K'^{\mu}_{A'B'} - \frac{1}{2\tilde{w}} \phi^{-1} \phi_{,\nu} \tilde{N}^{\mu\nu} \tilde{G}_{A'B'},$$

where the projection tensor $\tilde{N}^{\mu\nu}$ has been defined as

$$\tilde{N}^{\mu\nu} = |\phi|^{-1/\tilde{w}} N'^{\mu\nu}.$$

Finally the equation of motion for G membranes becomes

$$\tilde{K}^{\mu} = \tilde{N}^{\mu\nu} f_{IJ}{}^J w^I{}_{\nu}, \quad (12)$$

where

$$\tilde{K}^{\mu} = \tilde{G}^{A'B'} \tilde{K}^{\mu}_{A'B'} = \tilde{D}_{C'} \tilde{D}^{C'} X^{\mu} + \tilde{\Gamma}^{\mu}_{\nu\lambda} (\tilde{D}_{C'} X^{\nu}) \tilde{D}^{C'} X^{\lambda}$$

is the mean curvature vector of W/G relative to the orbit space $(M/G, \tilde{h})$. In this way, the force term generally appears at the right-hand side of the reduced equation of motion (12).

In Ref. [15], the G -invariant Nambu-Goto membranes are considered in the case of $\tilde{w} = 1$, and it is argued that the equation of motion reduces to the geodesic equation in the conformally transformed orbit space $(W/G, \tilde{h})$, which is based on the dimensional reduction at the action level,

$$S = -\tau \int_W d^w s \sqrt{|G|}$$

$$= -\tau \int_G d^f \alpha |\det \sigma^I{}_i| \int_{W/G} d\beta^1 \sqrt{|\tilde{G}_{11}|}$$

$$\propto \int_{W/G} d\beta^1 \sqrt{|\tilde{G}_{11}|}.$$

The last expression gives the geodesic action. However it turns out that it generally does not produce a correct equation of motion due to the presence of the force term, as we have explicitly shown.

In certain special cases, the force term becomes zero so that the configuration of the G -invariant membranes corresponds to the extremal surface in the orbit space $(W/G, \tilde{h})$ or to the geodesic when $\tilde{w} = 1$. They include when

- (i) G is Abelian: All the structure constants $f_{IJ}{}^K$ become zero. This includes the case when the orbit Gx is one dimensional.
- (ii) G is semisimple and compact: The Jacobi identity for the structure constants implies that $f_{IJ}{}^J = 0$ automatically holds.
- (iii) Orthogonal distributions of G -orbits are integrable: When the orbit Gx is everywhere orthogonal to the orbit space, $w^I{}_i$ becomes identically zero.

IV. G MEMBRANES COUPLED TO SCALAR MAP

It would be natural to ask whether a reduction mechanism similar to that shown in the previous section works in the presence of the external fields. As a simple case, we here consider the membranes coupled to a single complex scalar field without U_1 gauge couplings.

Assume that there is a complex scalar field $\psi: W \rightarrow \mathbb{C}$ on the membrane. We consider the following model for the membrane coupled to a scalar map:

$$S[X^a, \psi] = S_{\text{NG}}[X^a] + S_{\psi}[X^a, \psi],$$

$$S_{\text{NG}}[X^a] = -\tau \int_W ds^1 \dots ds^w \sqrt{|G|},$$

$$S_{\psi}[X^a, \psi] = -\kappa \int_W ds^1 \dots ds^w \sqrt{|G|} [G^{AB} (D_A \psi^*) D_B \psi + U(\psi^* \psi)].$$

The first variation of this action with respect to X^a gives the equation of motion for the membrane

$$\frac{1}{\sqrt{|G|}} g^{ab} \frac{\delta S}{\delta X^b} = \frac{1}{\sqrt{|G|}} \partial_A [\sqrt{|G|} (\tau G^{AB} - T^{AB}) D_B X^a]$$

$$+ \Gamma^a{}_{bc} [\tau \mathcal{G}^{bc} - T^{BC} (D_B X^b) D_C X^c] = 0, \quad (13)$$

where the stress-energy tensor

$$T^{AB} = \kappa[2(D^A\psi^*)D^B\psi - (D_C\psi^*)(D^C\psi)G^{AB} - UG^{AB}]$$

on the world sheet W has been defined.

The first variation with respect to ψ^* gives the wave equation

$$\frac{1}{\kappa\sqrt{|G|}}\frac{\delta S}{\delta\psi^*} = D_C D^C\psi - U'\psi = 0 \quad (14)$$

for ψ . This implies the local conservation law for the energy

$$D_A T^{AB} = 0.$$

Using this equation, Eq. (13) reduces to

$$\tau K^a - T^{AB} K_{AB}^a = 0, \quad (15)$$

in terms of the extrinsic curvature vector.

Here we assume the G -invariant configuration for the metric

$$g = \phi_{IJ}(\sigma^I - w^I_\mu dz^\mu)(\sigma^J - w^J_\nu dz^\nu) + |\phi|^{-1/\tilde{w}} \tilde{h}_{\mu\nu} dz^\mu dz^\nu,$$

as in the previous section, and for the membrane and the scalar field on it:

$$X^i = \alpha^i, \quad X^\mu = X^\mu(\beta^A), \quad \psi = \psi(\beta^A).$$

Then, Eq. (15) reduces to

$$\begin{aligned} \tau K^\mu - T^{AB} K_{AB}^\mu &= |\phi|^{1/\tilde{w}} \left\{ \tau \tilde{K}^\mu - \tilde{T}^{A'B'} \tilde{K}_{A'B'}^\mu \right. \\ &+ \tilde{N}^{\mu\nu} \left[\tau f_{IJ} J w^I_\nu + |\phi|^{1/\tilde{w}} (\tilde{D}_{C'}\psi^*)(\tilde{D}^{C'}\psi) \right. \\ &\left. \left. \times \left(f_{IJ} J w^I_\nu - \frac{1}{\tilde{w}} \phi^{-1} \phi_{,\nu} \right) \right] \right\} = 0, \end{aligned}$$

and

$$\tau K^i - T^{AB} K_{AB}^i = \sigma_I^i w^I_\mu (\tau K^\mu - T^{AB} K_{AB}^\mu) = 0,$$

where the reduced stress-energy tensor is defined by

$$\tilde{T}^{A'B'} = \kappa \{ |\phi|^{1/\tilde{w}} [2(\tilde{D}^{(A'}\psi^*)\tilde{D}^{B')}\psi - (\tilde{D}_{C'}\psi^*)(\tilde{D}^{C'}\psi)\tilde{G}^{A'B'}] - U\tilde{G}^{A'B'} \}.$$

On the other hand, the wave equation (14) becomes

$$\tilde{D}_{C'}(|\phi|^{1/\tilde{w}}\tilde{D}^{C'}\psi) - U'\psi = 0. \quad (16)$$

In summary, the equation of motion reduces to

$$\begin{aligned} \tau \tilde{K}^\mu - \tilde{T}^{A'B'} \tilde{K}_{A'B'}^\mu &= \tilde{N}^{\mu\nu} \left[\tau f_{IJ} J w^I_\nu + |\phi|^{1/\tilde{w}} (\tilde{D}_{C'}\psi^*)(\tilde{D}^{C'}\psi) \right. \\ &\left. \times \left(f_{IJ} J w^I_\nu + \frac{1}{\tilde{w}} \phi^{-1} \phi_{,\nu} \right) \right]. \quad (17) \end{aligned}$$

Except for the term with the factor $f_{IJ} J w^I_\nu$, Eqs. (15) and (17) are derived from the action

$$\begin{aligned} \tilde{S}[X^\mu, \psi] &= - \int_{W/G} d\beta^1 \dots d\beta^{\tilde{w}} \sqrt{|\tilde{G}|} \{ \tau + \kappa[|\phi|^{1/\tilde{w}} \\ &\times (\tilde{D}_{C'}\psi^*)\tilde{D}^{C'}\psi + U(\psi^*\psi)] \}, \quad (18) \end{aligned}$$

obtained via the naive dimensional reduction.

V. COUPLING TO DIFFERENTIAL FORM FIELD

As another model for matter coupling, we consider a background differential form field ω , which is a w -form field on M .

The simplest model would be given by

$$\begin{aligned} S[X^a] &= S_{\text{NG}}[X^a] + S_\omega[X^a], \\ S_{\text{NG}}[X^a] &= -\tau \int_W ds^1 \dots ds^w \sqrt{|G|}, \\ S_\omega[X^a] &= -\lambda \int_W i^* \omega \\ &= \frac{\lambda}{w!} \int_W ds^1 \dots ds^w \omega_{a_1 \dots a_w} (D_{A_1} X^{a_1}) \\ &\quad \dots (D_{A_w} X^{a_w}) \epsilon^{A_1 \dots A_w}, \end{aligned}$$

where $\epsilon^{A_1 \dots A_w}$ denotes the w -index Levi-Civita symbol on W such that $\epsilon^{12 \dots w} = -1$.

The first variation of the action is calculated as

$$\begin{aligned} \frac{\delta S}{\delta X^a} &= \tau \sqrt{|G|} g_{ab} K^b + \frac{\lambda(w+1)}{w!} \omega_{[a_1 \dots a_w, a]} (D_{A_1} X^{a_1}) \\ &\quad \dots (D_{A_w} X^{a_w}) \epsilon^{A_1 \dots A_w}. \end{aligned}$$

Thus the equation of motion for the membrane becomes

$$\begin{aligned} \tau K^a &= -\frac{\lambda(w+1)}{w! \sqrt{|G|}} g^{ab} \omega_{[a_1 \dots a_w, b]} (D_{A_1} X^{a_1}) \\ &\quad \dots (D_{A_w} X^{a_w}) \epsilon^{A_1 \dots A_w}. \quad (19) \end{aligned}$$

We assume that both the membrane and the background w -form field are G -invariant. The general form of the G -invariant w -form is

$$(\text{left invariant } p\text{-form on } Gx) \wedge [(w-p)\text{-form on } M/G]$$

or their linear combination. Among these, only the $p = f$ case results in the reduction of the system to the membrane equation in M/G . Hence we choose

$$\omega = \sigma^1 \wedge \sigma^2 \wedge \dots \wedge \sigma^f \wedge \tilde{\omega},$$

where $\tilde{\omega}$ is the \tilde{w} -form on M/G . In terms of coordinate components, we assume that

$$X^i = \alpha^i, \quad X^\mu = X^\mu(\beta^{A'}),$$

$$\omega_{a_1 \dots a_w} = \frac{w!}{\tilde{w}!} \sigma^1_{[a_1} \dots \sigma^f_{a_f} \tilde{\omega}(x^\mu)_{a_{f+1} \dots a_{f+\tilde{w}}]}.$$

Then, Eq. (19) reduces to the equation for the membrane in M/G as

$$\tau \tilde{K}^\mu = (-1)^{s+t} \frac{\lambda(\tilde{w}+1)}{\tilde{w}! \sqrt{|\tilde{G}|}} \tilde{h}^{\mu\nu} \tilde{\omega}_{[\mu_1 \dots \mu_{\tilde{w}}, \nu]} (D_{A'_1} X^{\mu_1})$$

$$\dots (D_{A'_w} X^{\mu_w}) e^{A'_1 \dots A'_w} + \tau \tilde{N}^{\mu\nu} f_{JI}^J W^I_\nu. \quad (20)$$

The factor $(-1)^s$ in the first term on the rhs is $(+1)$ if $\tilde{G}_{A'B'}$ has Riemannian signature and (-1) if Lorentzian, and the factor $(-1)^t$ denotes the signature of $\det \sigma^I_j$. The \tilde{w} -index Levi-Civita symbol on W/G has been normalized such that $e^{f+1, f+2, \dots, f+\tilde{w}} = (-1)^s$.

Except for the final force term, Eq. (20) has the same form as Eq. (19), which is derived from the naive reduced action

$$\tilde{S}[X^\mu, \tilde{\omega}] = -\tau \int_{W/G} d\beta^1 \dots d\beta^{\tilde{w}} \sqrt{|\tilde{G}|} - (-1)^t \lambda \int_{W/G} i^* \tilde{\omega}. \quad (21)$$

VI. CONCLUDING REMARKS

We have considered in general settings the motion of test membranes on which the group G of spacetime isometries acts. We have found that the configuration of Nambu-Goto membranes is described by the Nambu-Goto membranes in a quotient manifold with the appropriate projected metric, if at least one of the following conditions holds: (i) G is Abelian, (ii) G is semisimple and compact, or (iii) the orthogonal distribution of the orbit of G is integrable. We have also obtained similar results for the membranes coupled with the scalar maps or the differential form fields.

At the same time, it should be emphasized that the usual dimensional reduction procedure at the action level is not always justified. This is because the variational principle for dimensionally reduced action does not incorporate the variation of membranes with inhomogeneous variation with respect to the G -orbits.

Nevertheless, the correct equation of motion for G membranes derived here is only slightly different from the naive equation of motion by force terms written with local geometrical quantities. Hence, our formalism would be useful when we seek for more general string/membrane solutions in spacetimes with isometries and when we classify such solutions.

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- [1] W.-Y. Hsiang and H. B. Lawson, Jr., *J. Diff. Geom.* **5**, 1 (1971).
 - [2] V. P. Frolov, V. D. Skarzhinsky, A. I. Zelnikov, and O. Heinrich, *Phys. Lett. B* **224**, 255 (1989).
 - [3] B. Carter and V. P. Frolov, *Classical Quantum Gravity* **6**, 569 (1989).
 - [4] H.-J. Schmidt and U. Semmelmann, *Astron. Nachr.* **310**, 103 (1989).
 - [5] M. Christensen, V. P. Frolov, and A. L. Larsen, *Phys. Rev. D* **58**, 085008 (1998).
 - [6] V. P. Frolov, A. L. Larsen, and M. Christensen, *Phys. Rev. D* **59**, 125008 (1999).
 - [7] V. P. Frolov and K. A. Stevens, *Phys. Rev. D* **70**, 044035 (2004).
 - [8] H. Ahmedov and A. N. Aliev, *Phys. Rev. D* **78**, 064023 (2008).
 - [9] H. Ishihara and H. Kozaki, *Phys. Rev. D* **72**, 061701(R) (2005).
 - [10] T. Koike, H. Kozaki, and H. Ishihara, *Phys. Rev. D* **77**, 125003 (2008).
 - [11] H. Kozaki, T. Koike, and H. Ishihara, *Classical Quantum Gravity* **27**, 105006 (2010).
 - [12] T. Igata and H. Ishihara, *Phys. Rev. D* **81**, 044024 (2010).
 - [13] T. Igata and H. Ishihara, *Phys. Rev. D* **82**, 044014 (2010).
 - [14] D. Kubiznák and V. P. Frolov, *J. High Energy Phys.* **02** (2008) 007.
 - [15] H. Kozaki, T. Koike, and H. Ishihara, *Phys. Rev. D* **91**, 025007 (2015).
 - [16] G. F. R. Ellis and M. A. H. MacCallum, *Commun. Math. Phys.* **12**, 108 (1969).