Astrophysical gravitational waves in conformal gravity

Chiara Caprini*

Laboratoire Astroparticule et Cosmologie, CNRS UMR 7164, Université Paris-Diderot, 10 rue Alice Domon et Léonie Duquet, 75013 Paris, France

Patric Hölscher† and Dominik J. Schwarz‡

Fakultät für Physik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

(Received 24 April 2018; published 2 October 2018)

We investigate the gravitational radiation from binary systems in conformal gravity (CG) and massive conformal gravity (MCG). CG might explain observed galaxy rotation curves without dark matter, and both models are of interest in the context of quantum gravity. Here we show that gravitational radiation emitted by compact binaries allows us to strongly constrain both models. We work in Weyl gauge, which fixes the rescaling invariance of the models, and derive the linearized fourth-order equation of motion for the metric, which describes massless and massive modes of propagation. In the limit of a large graviton mass, MCG reduces to general relativity (GR), whereas CG does not. Coordinates are fixed by Teyssandier gauge to show that for a conserved energy-momentum tensor the gravitational radiation is due to the time-dependent quadrupole moment of a nonrelativistic source, and we derive the gravitational energy-momentum tensor for both models. We apply our findings to the case of close binaries on circular orbits, which have been used to indirectly infer the existence of gravitational radiation prior to the direct observation of gravitational waves. As an example, we analyze the binary system PSR J1012 + 5307, chosen for its small eccentricity. When fixing the graviton mass in CG such that observed galaxy rotation curves could be explained without dark matter, the gravitational radiation from a binary system is much smaller than in GR. Thus in CG one cannot explain the orbital decay of binary systems via gravitational radiation, and replace dark matter simultaneously. In MCG with small masses of the graviton, again one cannot reproduce the orbit of binaries by the emission of gravitational waves. On the other hand, for large graviton masses, the orbital period of compact binaries is in agreement with the data, as MCG reduces to GR in this limit.

DOI: 10.1103/PhysRevD.98.084002

I. INTRODUCTION

The standard model of gravity, general relativity (GR), is tested very well. The equivalence principle has been probed for a large region of the relevant parameter space and GR passes all solar system tests (see, e.g., [1]). Also the orbits of relativistic compact binaries show no deviation from the GR prediction and provide indirect evidence for the existence of gravitational waves [2,3].

In 2015, the aLIGO interferometers recorded the first direct observation of gravitational waves, as they observed the very last moment of a binary black-hole merger [4,5]. So far, five binary black-hole mergers have been reported by the LIGO/VIRGO Collaboration [6–8]. Very recently, the aLIGO and VIRGO interferometers detected a gravitational wave signal from the merger of two neutron stars (GW170817) with follow-up measurements across the electromagnetic spectrum coming from GRB 170817A [9–11]. Strong constraints on the speed of gravitational waves follow from the difference in arrival times of the gravitational and the electromagnetic signals, which in turn allows one to constrain modified models of gravity [12–19].

However, GR also faces shortcomings. We do not understand how to combine quantum physics with the principles of GR. In the ultraviolet regime, GR does not lead to a renormalizable model and therefore is non-predictive. In the infrared regime, cosmological and astrophysical observations interpreted in the context of GR imply the existence of dark energy and dark matter, and the observed smallness of the cosmological constant is not understood.

This motivates our study of modified models of gravity that change the gravitational field equations, instead of introducing a dark sector. Most physical models employ second-order equations of motion, which ensures that the theory is free from Ostrogradski instabilities [20,21]. However, higher-derivative theories, albeit suffering from ghost instabilities, can improve renormalization issues.

*caprini@apc.in2p3.fr
†patric.hoelscher@physik.uni-bielefeld.de
‡dschwarz@physik.uni-bielefeld.de

In this work we consider a unique higher-order derivative theory of gravity: a conformal model that reduces either to conformal gravity (CG) or to massive conformal gravity (MCG). The difference between those models is encoded in a parameter ϵ, with $\epsilon = -1$ corresponding to CG and $\epsilon = +1$ to MCG (see Sec. II for details). These models are not only invariant under general coordinate transformations, but also under Weyl rescaling of the metric and the matter fields. The purpose of this work is to study gravitational waves in conformal models of gravity.

Conformal models of gravity have been considered for the first time shortly after the introduction of GR, especially CG by Weyl and Bach [22,23]. The approach of Weyl has been dropped briefly after its publication because of non-integrability. On the other hand, the theory of Bach has been the precursor of CG, introduced by Mannheim and Kazanas [24–30]. More recently 't Hooft, who considered a nonperturbative approach of the path-integral formalism for quantum gravity, found connections between GR and CG. Terms of the same form as in the Weyl action appear as the only divergent term after a dimensional regularization [31–34]. Maldacena considered CG as a possible UV completion of GR by using specific boundary conditions, which separate out the Einstein-Hilbert solutions from the larger set of solutions in CG [35].

The early approach to CG has been discarded. First, the fourth-order structure of the theory made it mathematically uncomfortable. Second, from an experimental point of view there was no need to modify GR. Last but not least, the theory did not allow for bare mass terms in the matter action and our every day experience is strongly against the concept of scaling invariance.

It is now clear that masses in particle physics arise dynamically. In that light CG has been revived by Mannheim and Kazanas in 1989, and masses arise only after a spontaneous breaking of the conformal symmetry [29]. Besides that, Mannheim and Kazanas got some remarkable results, which made CG interesting again. CG was demonstrated to be renormalizable [30,34], and they solved the field equations for a static and spherically symmetric system in the Newtonian limit. They found a modified Newtonian potential which contains a term that grows linearly with distance [28,36–38]. This modified potential makes it possible to fit rotation curves of a huge class of galaxies [39–41].

It was shown that CG contains viable cosmological solutions, which fit the Hubble diagram and solve the singularity and cosmological constant problems [25,29,42–45]. However, in [46] it has been argued that the Λ cold dark matter model is favored by data from gamma-ray bursts and quasars. Besides checking the Hubble diagram, much work is left to be done. There is no analysis of the cosmic microwave background yet. Primordial nucleosynthesis has been analyzed in conformal models of gravity [47,48], and it seems that there is a tension with the deuterium and lithium abundances, the latter also being at odds with the cosmological standard model. But most importantly, structure formation has not been investigated in any detail.

Several authors claimed that light deflection is problematic in CG, but possibly there is a way out [38,49–56]. The work of Perlick and Xu [57] represented the major criticism on CG for a long time. They have shown that pure CG without a Weyl invariant energy-momentum tensor of matter is ruled out and contributed to fundamental advances in the understanding of conformally invariant theories. For a detailed discussion of this work, see also [58].

Previous investigations of gravitational waves in CG presented first steps such as the linearization of the equations of motion and the calculation of the gravitational energy-momentum tensor in pure CG [59–61].

The present work goes well beyond these first studies. We include matter, we discuss in depth the choice of gauge, and we derive the formalism for analyzing the gravitational radiation by binary systems in CG and MCG.

Until recently, there were only indirect measurements of gravitational radiation from pulsar binary systems; see, e.g., [3]. Gravitational radiation was indirectly detected through the measurement of the decreasing orbital period of the system. The recent direct detection, as already discussed above, and especially the observation of the merger of two neutron stars open new possibilities to test GR and its alternatives. Several models of modified gravity such as $f(R)$ gravity, Horndeski's theories, vector theories, or bimetric theories have been tested and constrained [12–19].

Here we study linearized gravity for nonrelativistic binaries, and thus we can compare our findings to systems long before the merger. This allows us to demonstrate that CG, when fixing the free parameters to explain galaxy rotation curves, cannot at the same time reproduce the gravitational radiation from binaries (observed indirectly via their orbital period decay). For MCG, on the other hand, there is a region of parameter space that is in concordance with observations.

Similar analyses study generalizations of CG and MCG [62–64]. In these works an incomplete gravitational energy-momentum tensor has been used to calculate the radiated energy from a binary system. It is assumed that the expression for the radiated energy is approximately the same as in GR, and hence the result differs significantly from ours. As we show, there are important additional contributions.

Section II gives an introduction to CG and MCG with the basic assumptions and equations. In Sec. III we show how to obtain the linearized field equations for the gravitational field and obtain their general solutions. We calculate the decay of the orbital period of coalescing binaries in the early inspiraling phase for CG and MCG in Sec. IV, and in Sec. V we derive the gravitational energy-momentum tensor in CG and MCG. In Sec. VI we evaluate the radiated energy from a binary system, and in the last section we conclude and summarize our findings.

For the Weyl and Riemann tensor we use definitions and sign conventions of Weinberg [65]; see Appendix A. We use natural units in which $c = \hbar = 1$, unless stated otherwise. Greek letters denote spacetime indices (0…3) and latin letters are spatial indices (1…3).

II. CONFORMAL GRAVITY

Conformal and massive conformal gravity are based on a Weyl invariant action. The spacetime metric $g_{\mu\nu}$ is rescaled by a Weyl transformation (conformal transformation) according to

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x), \tag{1}$$

where $\Omega > 0$ is a real and smooth function called the conformal factor and x denotes the spacetime coordinates. To model gravity, the Einstein-Hilbert action is replaced by the Weyl action I_W and the action for the Universe is given by

$$I = I_W + I_M = -\alpha_g \int d^4x \sqrt{-g} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} + I_M \tag{2}$$

$$= -\alpha_g \int d^4x \sqrt{-g} \left[2\left(R_{\mu\kappa} R^{\mu\kappa} - \frac{1}{3} R^2 \right) + L_L \right] + I_M, \tag{3}$$

where I_M is the matter action. α_g is a dimensionless coupling constant, $g = \det(g_{\mu\nu})$, and $C_{\lambda\mu\nu\kappa}$, $R_{\mu\nu}$, and R are the Weyl and Ricci tensors and the Ricci scalar, defined in Appendix A. To obtain expression (3) the Gauss-Bonnet term (Lanczos Lagrangian), which is a total derivative in four spacetime dimensions, has been used [66],

$$\sqrt{-g} L_L = \sqrt{-g} (R_{\lambda\mu\nu\kappa} R^{\lambda\mu\nu\kappa} - 4 R^{\mu\nu} R_{\mu\nu} + R^2), \tag{4}$$

where $R_{\lambda\mu\nu\kappa}$ denotes the Riemann tensor. Hence, it does not contribute to the field equations and can be discarded. Let us note that it is forbidden to introduce a cosmological constant term in Eq. (2), because of the Weyl symmetry.

The Weyl tensor has some outstanding properties. It is the traceless part of the Riemann tensor

$$g^{\mu\kappa} C^{\lambda}_{\mu\nu\kappa} = 0, \tag{5}$$

and under the transformation (1) it behaves as

$$C^{\lambda}_{\mu\nu\kappa}(x) \to C^{\lambda}_{\mu\nu\kappa}(x), \tag{6}$$

$$C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa} \to \Omega^{-4} C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa}. \tag{7}$$

Variation of the action (3) with respect to $g_{\mu\nu}$ leads to the equation for the gravitational field [23],

$$4\alpha_g W^{\mu\nu} = 4\alpha_g [2 C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu}_M, \tag{8}$$

where

$$W^{\mu\nu} = -\frac{1}{6} g^{\mu\nu} R^{;\beta}{}_{;\beta} + R^{\mu\nu;\beta}{}_{;\beta} - R^{\mu\beta;\nu}{}_{;\beta} - R^{\nu\beta;\mu}{}_{;\beta} - 2R^{\mu\beta} R^{\nu}{}_{\beta} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + \frac{2}{3} R^{;\mu;\nu} + \frac{2}{3} R R^{\mu\nu} - \frac{1}{6} g^{\mu\nu} R^2 \tag{9}$$

is the Bach tensor and

$$T^{\mu\nu}_M \equiv \frac{2}{(-g)^{1/2}} \frac{\delta I_M}{\delta g_{\mu\nu}} \tag{10}$$

is the matter energy-momentum tensor.

The matter energy-momentum tensor should also be Weyl invariant. Then the most general local matter action for a generic scalar and spinor field coupled conformally to gravity is [25]

$$I_M = -\int d^4x \sqrt{-g} \Big[\epsilon \left(-\frac{S^{;\mu} S_{;\mu}}{2} + \frac{S^2 R}{12} \right) + \lambda S^4 + i\bar{\psi}\gamma^{\mu}(x)[\partial_\mu + \Gamma_\mu(x)]\psi - \xi S \bar{\psi}\psi \Big]. \tag{11}$$

$S(x)$ represents a self-interacting scalar field and $\psi(x)$ is a generic spin-½ fermion field. ξ and λ are dimensionless coupling constants, $\gamma^{\mu}(x)$ are the vierbein-dependent Dirac-gamma matrices, $\bar{\psi} = \psi^{\dagger}\gamma^0$, and $\Gamma_\mu(x)$ is the fermion spin connection [67]. To be invariant under local Weyl transformations the matter fields have to transform as $S(x) \to \Omega^{-1}(x) S(x)$, $\psi(x) \to \Omega^{-3/2}(x)\psi(x)$, and $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$. The exponent of the conformal factor is called conformal weight.

In (11) we introduce the parameter ϵ, which can assume values of -1 or $+1$. In the first case, the theory corresponds to CG, while in the second it corresponds to MCG [68–70], as will become clear later. Note that only the combination of the two terms in parentheses is Weyl invariant.

For $\epsilon R < 0$ and $\lambda > 0$ the potential $V(S) = \epsilon S^2 R/12 + \lambda S^4$ can lead to a spontaneous breaking of Weyl symmetry.

The matter action in (11) serves as a toy model to investigate the gravitational radiation in CG and MCG. Since the standard model of particle physics is locally conformally invariant before the spontaneous symmetry breaking of the gauge symmetry $SU(3) \times SU(2) \times U(1)$ (i.e., before mass generation), one can add the complete standard model of particles physics, up to modifications of the Higgs sector. The complex Higgs doublet $H(x)$ has to be coupled conformally; thus a term $\propto H^{\dagger} H R$ with a conformal coupling coefficient has to be added to the kinetic term of the Higgs and the Higgs potential becomes $a S^2 H^{\dagger} H + b (H^{\dagger} H)^2$, where a and b are constants that have to be fixed appropriately to generate the Higgs mass and self-coupling after fixing the Weyl gauge $S = S_0$. For details see [71].

We find the field equations for the scalar and fermion fields

$$\epsilon\left(-S^{;\mu}_{;\mu} - \frac{1}{6}SR\right) - 4\lambda S^3 + \xi\bar{\psi}\psi = 0, \tag{12}$$

$$i\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi - \xi S\psi = 0. \tag{13}$$

Variation of the action (11) with respect to $g_{\mu\nu}$ and using the equation of motion (13) leads to the matter energy-momentum tensor

$$T^M_{\mu\nu} = T^f_{\mu\nu} + \epsilon\left[-\frac{2S_{,\mu}S_{,\nu}}{3} + \frac{g_{\mu\nu}S^{,\alpha}S_{,\alpha}}{6} + \frac{SS_{,\mu;\nu}}{3} - \frac{g_{\mu\nu}SS^{,\alpha}{}_{;\alpha}}{3} + \frac{1}{6}S^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\right] - g_{\mu\nu}\lambda S^4, \tag{14}$$

where

$$T^f_{\mu\nu} \equiv \frac{1}{2}[i\bar{\psi}\gamma_\mu(x)[\partial_\nu + \Gamma_\nu(x)]\psi + (\mu \leftrightarrow \nu)] \tag{15}$$

is the energy-momentum tensor of the fermion.

Since the action I given in Eq. (2) is invariant under a Weyl transformation, it is always possible to choose a frame in which the scalar field is constant,

$$S(x) \to S'(x) = \Omega^{-1}(x)S(x) = S_0 = \text{const.}, \tag{16}$$

with $\Omega(x) = S(x)/S_0$. This is called the Higgs or unitary gauge [68,72]. Because of this choice all terms with derivatives on S vanish.

Up to now, CG and MCG are still unrelated to GR. However, these theories were proposed in order to overcome the usual GR problems and, at the same time, to reproduce its very accurate predictions on solar system distance scales. Hence, they must be arranged in a way that makes the connection and the differences to GR explicit. This is possible because S_0 is not a free parameter of the theory: therefore, physics is unchanged for any choice of the configuration of the scalar field. For this reason we introduce the relations

$$8\pi\tilde{G} \equiv \frac{6}{S_0^2}, \tag{17}$$

$$\Lambda \equiv 6\lambda S_0^2, \tag{18}$$

where $\tilde{G}$ denotes an effective Newton's constant. For $\alpha_g \to 0$, $\tilde{G} = G$, where G is Newton's constant, and interpreting Λ as the cosmological constant, this leads exactly to the GR field equations for the metric. As we will see in the following, in all cases of interest we will set $\tilde{G} = G$, Newton's constant.

Since the scalar field $S(x)$ can always be fixed to a constant by choosing a specific Weyl gauge, it is just an auxiliary field and does not represent a dynamical degree of freedom (d.o.f.) [71,73]. Therefore, we do not need to worry about its stability properties. We nevertheless discuss them in Appendix C, where we follow the analysis of [74].

In this gauge (fixing the Weyl invariance), there is a constant mass for the fermions given by $m_f = \xi S_0$. Since we know from experiments that fermions have masses, one should choose $\xi S_0 > 0$. Consequently, (12) and (13) become

$$-\frac{\epsilon R + 4\Lambda}{8\pi\tilde{G}} + m_f\bar{\psi}\psi = 0, \tag{19}$$

$$T_f - m_f\bar{\psi}\psi = 0, \tag{20}$$

where T_f denotes the trace of the fermion energy-momentum tensor. These two equations can be combined to

$$\epsilon R + 4\Lambda = 8\pi\tilde{G}T_f. \tag{21}$$

With the energy-momentum tensor introduced above, the equation for the gravitational field becomes [75,76]

$$4\alpha_g W_{\mu\nu} = T^f_{\mu\nu} + \frac{1}{8\pi\tilde{G}}[\epsilon G_{\mu\nu} - g_{\mu\nu}\Lambda], \tag{22}$$

where $G_{\mu\nu}$ denotes the Einstein tensor. Note that the fermion energy-momentum tensor is covariantly conserved,

$$T^{\mu\nu}_{f\ ;\nu} = 0, \tag{23}$$

due to the Bianchi identities for the Bach and Einstein tensors.

Before we continue to discuss solutions of the field equation, we observe that it is convenient to introduce a "graviton mass" m_g via

$$m_g^2 \equiv \frac{1}{32\pi\tilde{G}\alpha_g}. \tag{24}$$

Besides having the dimensions of a mass, at this point it is not obvious that m_g does indeed play the role of a mass for the graviton. This will become clear in the next section. We can then write

$$-\epsilon G_{\mu\nu} + g_{\mu\nu}\Lambda + \frac{1}{m_g^2}W_{\mu\nu} = 8\pi\tilde{G}T^f_{\mu\nu}, \tag{25}$$

and observe that in the limit $m_g \to \infty$ ($\alpha_g \to 0$), the Einstein equations are recovered for $\epsilon = +1$ or equivalently I_W vanishes in (3). This is the case of MCG, which therefore differs from most theories with a massive spin-2 field, for which GR is recovered in the limit of $m_g \to 0$. In the next section it will become clear that CG and MCG contain two spin-2 fields, one that is massless as in GR, and one that has the mass m_g. Hence, the small mass limit does not reproduce GR, but provides a theory with two effectively massless spin-2 fields. This distinguishes CG and MCG from other theories with a massive graviton, such as the de Rham–Gabadaze–Tolley massive gravity or bigravity, for which one would expect to recover GR in the massless limit. Note that, in these theories, taking the massless limit is problematic, because the metric does not reduce to a single massless spin-2 field as there is an additional scalar d.o.f. left [77]: this is called the van Dam–Veltman–Zakharov discontinuity [78,79]. For MCG it was shown that the theory is continuous in the massless limit [80]. The same result also holds for CG. On the other hand, CG ($\epsilon = -1$) does not contain GR as a limit, because in the massless limit Eq. (25) provides a different sign than in GR. Note that the trace of (25) reproduces Eq. (21).

For conformally flat spacetimes, $W_{\mu\nu} = 0$, and thus, independently of the value of m_g the solutions agree with those of GR for MCG (but not for CG, where the relative sign between the Einstein and the energy-momentum tensors is reversed). In particular, MGC leaves the isotropic and homogeneous Friedmann-Lemaître models untouched.

III. WEAK GRAVITATIONAL FIELD IN TEYSSANDIER GAUGE

A. Equation of motion

Let us now turn to the study of gravitational waves in CG and MCG. In the following we drop, for simplicity, the cosmological constant ($\Lambda = 0$) and linearize around flat Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small metric perturbation. For consistency we have to assume that the energy-momentum tensor vanishes at zeroth order.

The second term of the Bach tensor in Eq. (8) is at least of second order in $h_{\mu\nu}$; hence we only need to consider the first term

$$C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} = \partial_\kappa \partial_\lambda C^{\mu\lambda\nu\kappa}_{(1)} + \mathcal{O}(h^2), \tag{26}$$

where $\ldots_{(1)}$ denotes terms of first order in $h_{\mu\nu}$. This term can be rewritten as

$$\partial_\kappa \partial_\lambda C^{\mu\lambda\nu\kappa}_{(1)} = \frac{1}{2}\Box R^{\mu\nu}_{(1)} - \frac{1}{12}\eta^{\mu\nu}\Box R_{(1)} - \frac{1}{6}\partial^\mu\partial^\nu R_{(1)}, \tag{27}$$

where we have used the Bianchi identities

$$\partial_\kappa \partial_\lambda R^{\lambda\mu\nu\kappa}_{(1)} = \Box R^{\mu\nu}_{(1)} - \partial_\lambda \partial^\mu R^{\lambda\nu}_{(1)}, \tag{28}$$

$$\partial_\lambda R^{\lambda\mu}_{(1)} = \frac{1}{2}\partial^\mu R_{(1)}. \tag{29}$$

The d'Alembert operator is defined as $\Box \equiv \partial_\mu \partial^\mu$. This leads us to the linearized field equations for the metric

$$-\epsilon\left(R^{\mu\nu}_{(1)} - \frac{1}{2}\eta^{\mu\nu}R_{(1)}\right) + \frac{1}{m_g^2}\left(\Box R^{\mu\nu}_{(1)} - \frac{1}{6}\eta^{\mu\nu}\Box R_{(1)} - \frac{1}{3}\partial^\mu\partial^\nu R_{(1)}\right) = 8\pi\tilde{G}T^{\mu\nu}_{f(1)}. \tag{30}$$

The linearized energy-momentum tensor satisfies

$$\partial_\mu T^{\mu\nu}_{f(1)} = 0. \tag{31}$$

From now on, all quantities are of first order and we write $T^f_{\mu\nu} = T_{\mu\nu}$.

Using (21) and the expressions from Appendix A we can rewrite (30) as

$$m_g^{-2}(\Box - \epsilon m_g^2)\left(\frac{1}{2}\Box h_{\mu\nu} - \frac{1}{6}\eta_{\mu\nu}R\right) + \frac{1}{2}(\partial_\mu Z_\nu + \partial_\nu Z_\mu) = 8\pi\tilde{G}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right), \tag{32}$$

where $Z_\mu \equiv -m_g^{-2}[(\Box - \epsilon m_g^2)\partial_\rho \bar{h}^\rho_\mu + (1/3)\partial_\mu R]$ and

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{33}$$

is the trace-reversed metric perturbation.

It turns out to be convenient to choose the gauge condition

$$Z_\mu = 0. \tag{34}$$

This is called the Teyssandier gauge [81] (see also Appendix B). Then Eq. (32) simplifies to

$$\frac{1}{m_g^2}(\Box - \epsilon m_g^2)\left(\Box h_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}R\right) = 16\pi\tilde{G}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right). \tag{35}$$

Before we proceed to solve Eq. (35), let us analyze the forms this equation can take in different limits of CG and MCG. We denote with L the typical variation scale of the metric perturbation and with d the typical size of the source.

To analyze the various limits of this theory it is useful to rewrite Eq. (35) to

$$[\Box^2 - \epsilon m_g^2 \Box]h_{\mu\nu} = 16\pi\tilde{G}m_g^2\bar{T}_{\mu\nu}, \tag{36}$$

where

$$\bar{T}_{\mu\nu} = (T_{\mu\nu} - 1/2\eta_{\mu\nu}T) + \epsilon/(6m_g^2)\eta_{\mu\nu}\Box T. \tag{37}$$

TABLE I. Equation (36) for different limits of CG and MCG.

	CG ($\epsilon = -1$)	MCG ($\epsilon = +1$)	Remarks
$m_g L \ll 1, m_g d \ll 1$	$3\Box^2 h^{\mu\nu} = -8\pi\tilde{G}\eta^{\mu\nu}\Box T$	$3\Box^2 h^{\mu\nu} = 8\pi\tilde{G}\eta^{\mu\nu}\Box T$	Light m_g
$m_g L \ll 1, m_g d \gg 1$	$\Box^2 \bar{h}^{\mu\nu} = 16\pi\tilde{G}m_g^2 T^{\mu\nu}$	$\Box^2 \bar{h}^{\mu\nu} = 16\pi\tilde{G}m_g^2 T^{\mu\nu}$	Irrelevant for $L > d$
$m_g L \gg 1, m_g d \ll 1$	$3\Box h^{\mu\nu} = -8\pi\tilde{G}m_g^{-2}\eta^{\mu\nu}\Box T$	$-3\Box h^{\mu\nu} = 8\pi\tilde{G}m_g^{-2}\eta^{\mu\nu}\Box T$	Intermediate m_g
$m_g L \gg 1, m_g d \gg 1$	$\Box \bar{h}^{\mu\nu} = 16\pi\tilde{G}T^{\mu\nu}$	$-\Box \bar{h}^{\mu\nu} = 16\pi\tilde{G}T^{\mu\nu} \Leftrightarrow$ GR	Heavy m_g

By writing (36) approximately as $[L^{-4} - \epsilon m_g^2 L^{-2}]h \sim \tilde{G}m_g^2 T + \epsilon\tilde{G}d^{-2}T$, it appears that there are four relevant cases. If $Lm_g \ll 1$, one recovers the limits of CG and MCG without the GR part, since the term with higher derivatives dominates the left-hand side. If instead $Lm_g \gg 1$, the wave equation is of second order. Depending on the relation among m_g and d, different terms dominate the right-hand side. Note that, if both $Lm_g \gg 1$ and $dm_g \gg 1$, MCG reduces to GR, while CG provides the same equation as in GR, but with a flip of the sign. The limiting cases are summarized in Table I.

B. Gravitational wave propagator

The solution to the inhomogeneous Eq. (36) is given by

$$h_{\mu\nu} = 16\pi\tilde{G}\int d^4x'\mathcal{G}(x - x')\bar{T}_{\mu\nu}(x'). \tag{38}$$

Green's function $\mathcal{G}(x)$ is defined by

$$(\Box - \epsilon m_g^2)\Box\mathcal{G}(x - x') = m_g^2\delta^4(x - x'). \tag{39}$$

For the Fourier transformed Green's function one finds

$$\tilde{\mathcal{G}}(k) = \frac{m_g^2}{(\omega^2 - k^2 - \epsilon m_g^2)(\omega^2 - k^2)}. \tag{40}$$

This can be rewritten as

$$\tilde{\mathcal{G}}(k) = \epsilon\left[-\frac{1}{(\omega^2 - k^2)} + \frac{1}{(\omega^2 - k^2 - \epsilon m_g^2)}\right], \tag{41}$$

where $k^2 \equiv \mathbf{k}^2$. In the propagator for the spin-2 metric perturbation $h_{\mu\nu}$ above, either the massless term ($\epsilon = -1$) or the massive term ($\epsilon = +1$) comes with the wrong sign: the so-called Weyl ghost (see, e.g., [82]). Note that the spin-2 ghost excitation around the Minkowski vacuum is present independently of ϵ [83]. However, CG and MCG have different stability properties and relations to GR.

For CG ($\epsilon = -1$) we have demonstrated previously (cf. Table I) that there is no limit leading to the action or equations of GR, since the sign of the Einstein-Hilbert term in the matter action is opposite to GR (the Newtonian limit of this theory is studied in Sec. IV). As a consequence, the massless part of the propagator has the wrong sign, representing a ghost instability. Additionally, the massive part of the gravitational wave represents a tachyon; i.e., it travels faster than the speed of light. The ghost instability in CG has been widely discussed by Mannheim and Bender [84–91]: they analyzed in a toy model the Pais-Uhlenbeck fourth-order oscillator [92], which was believed to suffer from ghost instabilities, too, and in a series of papers they claimed that this is not the case and that an explicit quantization and construction of the Hilbert space is necessary in order to judge whether a theory suffers from instabilities or not [93].

The case of MCG is different, since it has the correct sign for the Einstein-Hilbert term and thus it includes GR as a limiting case (cf. Table I). The Newtonian gravitational potential can be recovered (the Newtonian limit of MCG is also studied in Sec. IV and Appendix D) and the massless excitation represents a healthy graviton traveling at the speed of light. In this case the wrong sign in the propagator appears for the massive graviton, which is, however, subluminal and does not propagate at all in the GR limit.

Note that for both theories the shortcoming of the appearance of the Weyl ghost comes along with the major advantage of being better behaved in the UV limit, i.e., being renormalizable [95] and hence viable theories of quantum gravity [30,34,96]. See also [97] for a similar theory presenting the Weyl ghost.

C. Massive and massless mode

Let us proceed to solve Eq. (35) now. It is possible to reduce the order of the wave equation by splitting the metric perturbation

$$h_{\mu\nu} = \epsilon(H_{\mu\nu} + \Psi_{\mu\nu}), \tag{42}$$

where $H_{\mu\nu}$ and $\Psi_{\mu\nu}$ are symmetric tensors, and making the ansatz

$$\Psi_{\mu\nu} = \frac{1}{m_g^2}\left(\Box h_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}R\right). \tag{43}$$

Then, Eq. (35) turns into the equation of motion for a massive mode

$$(\Box - \epsilon m_g^2)\Psi_{\mu\nu} = 16\pi\tilde{G}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right). \tag{44}$$

We now use (44), eliminate the term $m_g^2\Psi_{\mu\nu}$ by means of (43), and replace the Ricci scalar by means of (21). Finally, we use (42) and (33) to arrive at a massless equation of motion that looks familiar,

$$\Box\bar{H}_{\mu\nu} = -16\pi\tilde{G}T_{\mu\nu}, \tag{45}$$

where $\bar{H}_{\mu\nu}$ is the trace-reversed massless mode. In the last step we exploit the gauge condition (34). Using (45), (44), and (21) we find

$$Z_\mu = -\partial_\rho\bar{H}^\rho_\mu = 0, \tag{46}$$

the condition for the massless mode to be transverse. But there is one more condition that is fixed in the Teyssandier gauge. From the expression for the Ricci scalar, condition (46), the trace of (45), and (21) it follows that

$$\partial_\rho\partial_\sigma\Psi^{\rho\sigma} = \Box\Psi. \tag{47}$$

Defining $\hat{\Psi}_{\mu\nu} \equiv \Psi_{\mu\nu} - \eta_{\mu\nu}\Psi$, Eqs. (44) and (47) are equivalent to

$$(\Box - \epsilon m_g^2)\hat{\Psi}_{\mu\nu} = 16\pi\tilde{G}T_{\mu\nu}, \qquad \partial_\rho\partial_\sigma\hat{\Psi}^{\rho\sigma} = 0. \tag{48}$$

Hence, the total metric perturbation $h_{\mu\nu}$ is decomposed into a transverse massless mode $H_{\mu\nu}$ and a massive mode $\Psi_{\mu\nu}$.

It is interesting to note that in the limit $m_g \to 0$, Eq. (48) becomes a massless wave equation, differing only by a sign from Eq. (45). Writing (42) as

$$h_{\mu\nu} = \epsilon\left(\bar{H}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{H} + \hat{\Psi}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\hat{\Psi}\right), \tag{49}$$

in this limit and under the assumption that the traces of both modes vanish, which is indeed the case for the coordinate gauge that we will use in Sec. IV B, the total metric perturbation vanishes, too. This demonstrates that $m_g \to 0$ is not the GR limit. One recovers GR in the limit $m_g \to \infty$, which makes the massive spin-2 field nondynamical.

In the homogeneous case, Eqs. (45) and (48) take the form

$$\Box\bar{H}_{\mu\nu} = 0, \tag{50}$$

$$(\Box - \epsilon m_g^2)\hat{\Psi}_{\mu\nu} = 0. \tag{51}$$

The solutions to (50) and (51) are a massless plane wave and a massive plane wave

$$\bar{H}_{\mu\nu} = a_{\mu\nu}e^{ik_\rho x^\rho}, \qquad k_\rho k^\rho = 0, \tag{52}$$

$$\hat{\Psi}_{\mu\nu} = b_{\mu\nu}e^{il_\rho x^\rho}, \qquad l_\rho l^\rho = -\epsilon m_g^2, \tag{53}$$

where $a_{\mu\nu}$ and $b_{\mu\nu}$ are constant and symmetric. Depending on the values of ϵ, the wave vector l^ρ is timelike or spacelike, corresponding to a wave that travels slower than the speed of light for MCG ($\epsilon = +1$) and a tachyon that is faster than the speed of light for CG ($\epsilon = -1$). For more details, see Appendix C.

In the next subsections, we derive the solutions of Eqs. (45) and (48).

D. Solution with a source: Massive part

In the following, we only analyze the massive wave equation (48), since the massless part is known from GR. The most convenient way to analyze the inhomogeneous solutions is to keep real space while switching to ω dependence. We define

$$\hat{\Psi}_{\mu\nu} = 16\pi\tilde{G}\int d^4x'\mathcal{G}(x - x')T_{\mu\nu}(x'), \tag{54}$$

with frequency-domain Green's function

$$\begin{aligned}\mathcal{G}(\omega, \mathbf{x} - \mathbf{x}') &= \frac{1}{(2\pi)^3}\int d^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\omega^2 - k^2 - \epsilon m_g^2}\\ &= \frac{-i}{2(2\pi)^2|\mathbf{x} - \mathbf{x}'|}\int_{-\infty}^{\infty} dk\frac{k}{\omega^2 - k^2 - \epsilon m_g^2}\\ &\quad\times(e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|}),\end{aligned} \tag{55}$$

where we have integrated over the angles and extended the k integral to $-\infty$ to find the last expression. The poles of the integrand are at

$$k = \pm\sqrt{\omega^2 - \epsilon m_g^2}. \tag{56}$$

In MCG with $\epsilon = +1$ we have to distinguish two cases, $\omega^2 > m_g^2$ and $\omega^2 < m_g^2$, while CG with $\epsilon = -1$ always leads to a positive radicand.

1. Propagator for small graviton mass

For CG and MCG with a small graviton mass ($m_g^2 < \omega^2$) the radicand is positive, so by finding the residues of these poles we get

$$\mathcal{G}(\omega, \mathbf{x} - \mathbf{x}') = -\frac{e^{ik_{\omega,\epsilon}|\mathbf{x}-\mathbf{x}'|}\theta(\omega - m_g) + \text{c.c.}\theta(-\omega - m_g)}{4\pi|\mathbf{x} - \mathbf{x}'|}, \tag{57}$$

where $k_{\omega,\epsilon} \equiv \sqrt{\omega^2 - \epsilon m_g^2}$ and c.c. is the complex conjugate of the exponential function. In the far zone approximation ($r \gg |\mathbf{x}'|$) we get $|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}'\cdot\mathbf{n} + O(d^2/r)$, where r denotes the distance between the observer and the source and $\mathbf{n}$ the spatial unit vector pointing from the source to the observer. Keeping only the first order yields

$$\mathcal{G}(\omega, \mathbf{x}-\mathbf{x}') = -\frac{e^{ik_{\omega,\epsilon}(r-\mathbf{x}'\cdot\mathbf{n})}\theta(\omega - m_g) + \text{c.c.}\theta(-\omega - m_g)}{4\pi r}. \tag{58}$$

Note that this result also holds for CG with a large graviton mass ($m_g^2 > \omega^2$). However, we do not consider this case in this work, because the reason for proposing CG was that it can fit galaxy rotation curves without dark matter in the small mass case. Furthermore, it is not obvious that the case of a large graviton mass exhibits a valid Newtonian limit (the gravitational potential oscillates).

2. Propagator for large graviton mass

For MCG with a large graviton mass ($m_g^2 > \omega^2$) the radicand is negative, and thus $k = \pm i\sqrt{m_g^2 - \omega^2}$. Green's function becomes

$$\begin{aligned}\mathcal{G}(\omega, \mathbf{x}-\mathbf{x}') &= -\frac{e^{-k_{\omega,>}|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\theta(m_g - |\omega|) \\ &= -\frac{e^{-k_{\omega,>}(r-\mathbf{x}'\cdot\mathbf{n})}}{4\pi r}\theta(m_g - |\omega|),\end{aligned} \tag{59}$$

where $k_{\omega,>} \equiv \sqrt{m_g^2 - \omega^2}$. In the second line the far zone approximation has been applied.

Let us remark that (54) together with (58) and (59) is valid for relativistic and nonrelativistic sources.

IV. GRAVITATIONAL WAVES FROM A BINARY SYSTEM

We now consider binary systems with masses m_1 and m_2 on circular orbits moving at a speed small compared to the speed of light. This means we can treat the source in the nonrelativistic and weak field limits. Hence, we can neglect contributions of the gravitational potential and the kinetic energy to the energy-momentum tensor $T_{\mu\nu}$ in Eq. (54). In general, these approximations do not hold true for binaries consisting of compact objects like neutron stars. However, for binary systems in the inspiraling phase of their evolution, where the objects are still far apart, these assumptions are adequate for analyzing the gravitational radiation behavior. Moreover, here we do not consider the backreaction on the binaries' motion due to its gravitational wave emission.

In particular, we look at the binary system PSR J1012 + 5307 [98–100], which is a neutron star–white dwarf system in quasicircular motion; cf. Table II. The orbital frequency of this system is given by

$$\omega_s \approx 1.3 \times 10^{-20}\ \text{eV} \approx 1.9 \times 10^{-5}\ \text{Hz}. \tag{60}$$

The system is picked for its small eccentricity of the orbit, such that we can apply the results of our study of circular orbits. Its orbital speed is of order 10^{-5} c, which justifies the low-velocity approximation. The orbital period P of the binary system PSR J1012 + 5307 and its time derivative $\dot{P}$ have been derived from data collected over 15 years and are in excellent agreement with the assumption that its decay of the orbital period is due to gravitational radiation as predicted by GR.

TABLE II. Orbital data for the binary system PSR J1012 + 5307 [98–100] consisting of a neutron star and a white dwarf in quasicircular motion. The semimajor axis is given in light seconds (ls).

Period P (days)	0.60467271355(3)
Period derivative (observed) $\dot{P}_{\text{obs}}$	$5.0(1.4) \times 10^{-14}$
Period derivative (intrinsic) $\dot{P}_{\text{intr}}$	$-1.5(1.5) \times 10^{-14}$
Mass ratio q	10.5(5)
Neutron star mass $m_1(M_\odot)$	1.64(22)
White dwarf mass $m_2(M_\odot)$	0.16(2)
Eccentricity $e(10^{-6})$	1.2(3)
Projected semimajor axis a	0.581872(2) ls
Distance r	(840 ± 90) pc

A. Newtonian limit and Kepler's third law

In general, the analysis of the gravitational wave emission proceeds as follows. The first step is to calculate the decay of the orbital period $\dot{P}/P$ ($P = 2\pi/\omega_s$) via Kepler's third law for two objects of mass m_1 and m_2 in the Newtonian limit for a circular orbit in the center of mass frame, where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. Since the gravitational potential is modified in CG and MCG, we have to rederive Kepler's third law in these theories. In general, we can write

$$\frac{\dot{P}}{P} = \frac{\dot{R}}{2R} - \frac{\dot{V}'}{2V'}, \tag{61}$$

where $V'(R) = \mu^{-1}\partial_R E_{\text{pot}}(R)$ is the derivative of the gravitational potential V with respect to the distance between the objects R and E_{pot} the gravitational potential energy.

Note that in GR it is assumed that the total decrease of the orbital period occurs due to the emission of energy in gravitational radiation. The result of this chapter is that the theories we investigate in this work should predict the same amount of energy that is radiated by gravitational waves (within the precision of the measurements) as GR in order to explain the decrease of the orbital period of binary systems without using any other mechanism than gravitational wave emission.

1. Conformal gravity

In [101] it has been claimed that in CG ($\epsilon = -1$) with a small graviton mass the line element in a static, spherically symmetric geometry exterior to a source of one solar mass with a nonvanishing scalar field S_0 can be written in the form

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \tag{62}$$

where $B(r) = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - kr^2$. Here β, γ, and k are constants of integration and are used to fit galaxy rotation curves. k has an influence on the outer parts of galaxies, but is much smaller than β and γ, and we can neglect the k term in the following. Also terms proportional to $\beta\gamma \ll 1$ are negligible on the distance scale that corresponds to our binary system. For a source of one solar mass $M_\odot$, the parameters are given by [41,101]

$$S_0^2 = 9.7 \times 10^{34}\ \mathrm{kg\,s^{-1}}, \tag{63}$$

$$\alpha_g = 3.3 \times 10^{75}\ \mathrm{kg\,m^2\,s^{-1}}, \tag{64}$$

$$\gamma = 5.4 \times 10^{-39}\ \mathrm{m^{-1}}, \tag{65}$$

$$2\beta = 3 \times 10^3\ \mathrm{m}, \tag{66}$$

$$k = 9.5 \times 10^{-50}\ \mathrm{m^{-2}}. \tag{67}$$

For the graviton mass this yields

$$m_{g,\mathrm{CG}} = 1.9 \times 10^{-58}\ \mathrm{kg} = 1.1 \times 10^{-22}\ \mathrm{eV}. \tag{68}$$

Note that using the line element (62) has been criticized in the literature; see [56,72,73,102–104]. Nevertheless, in the following we show that it does not matter for our analysis of the gravitational radiation whether these additional terms are there or not. For the parameter values that are needed to fit galaxy rotation curves (corresponding to a small graviton mass) the additional terms do not affect the gravitational radiation of the system under study.

To be consistent with solar system tests we have to choose $\tilde{G} = G = \beta/M_\odot$ and $\gamma_\odot = \gamma/M_\odot$. The gravitational potential energy and its time derivative for CG is given by [24,28,37,101]

$$E_{\mathrm{pot}} = -\frac{G\mu M}{R} + \frac{\gamma_\odot \mu M}{2} R, \tag{69}$$

$$\dot{E}_{\mathrm{pot}} = G\mu M \frac{\dot{R}}{R^2}\left(1 + \frac{\gamma_\odot R^2}{2G}\right), \tag{70}$$

where $M = m_1 + m_2$ is the total mass of the system. Inserting this into (61), we find

$$\frac{\dot{P}}{P} \approx -\frac{|E_{\mathrm{GR}}|^{\cdot}}{|E_{\mathrm{GR}}|}\left(\frac{3}{2} - \frac{\gamma_\odot}{2G}R^2\right), \tag{71}$$

where $|E_{\mathrm{GR}}| = GM\mu/(2R)$ and $\gamma_\odot R^2/G \ll 1$. To verify that this combination is indeed small, we insert the distance between the sources of the binary system under study; cf. Table II and assume that for a binary system in circular motion, we have $R \approx a \approx 8.9 \times 10^{14}\ \mathrm{eV^{-1}}$, where a is the semimajor axis of PSR J1012 + 5307. Also we use the parameters determined by the analysis of galaxy rotation curves in (63)–(67), which shows that the second term in Eq. (71) is indeed negligible since $\gamma_\odot a^2/G \approx 10^{-26}$. This demonstrates that in CG with a small graviton mass the orbital energy, which is lost by the system, is, up to small modifications, the same as in GR, because on solar system distance scales the second term in (69) can be neglected with respect to the first one. Therefore, we can treat the binary system in the Newtonian limit.

2. Massive conformal gravity

Now, let us apply the same analysis for MCG ($\epsilon = +1$) in the case of a large graviton mass ($m_{g,>}^2 > \omega^2$), where $m_{g,>}$ denotes the graviton mass for this case.

In [103,104] or in Appendix D it is pointed out that this model cannot fit galaxy rotation curves without dark matter, but it is still interesting because of its GR limit.

In this case the massive part of the graviton becomes damped, and we are left with a theory that is just GR modified by exponentially suppressed contributions. Nevertheless, there is a profound difference to GR, since it is claimed that this theory is power-counting renormalizable [95,105].

In Appendix D it is shown that the gravitational potential in the Newtonian limit is given by

$$\Phi(r) = -\frac{GM}{r}\left(1 - \frac{4}{3}e^{-m_{g,>}r}\right), \tag{72}$$

where $\tilde{G} = G$ has been chosen.

We have to constrain the graviton mass with data from short range tests of the inverse square law. From [106], we get

$$m_{g,>} > 10^{-38}\ \mathrm{kg} \approx 10^{-2}\ \mathrm{eV}. \tag{73}$$

This means that the Yukawa term in (72) becomes important only on submillimeter distance scales. For binary systems in the inspiral phase the distance between the objects is always macroscopic ($m_{g,>}a \approx 3.5 \times 10^{12}$), and hence we can completely neglect this term for the analysis of gravitational radiation. The result for the decay of the orbital period is

$$\frac{\dot{P}}{P} \approx -\frac{3|E_{\mathrm{GR}}|^{\cdot}}{2|E_{\mathrm{GR}}|}. \tag{74}$$

Further, we can look at the case of a small graviton mass ($m_{g,<}^2 < \omega^2$) in MCG. Let us first assume the same potential as for the case of a large graviton mass in Eq. (72).

From the constraint $m_{g,<}^2 < \omega^2$ it is clear that one cannot make the graviton mass large enough to push the Yukawa contribution to the submillimeter scale. Rather we get an upper bound on the graviton mass from solar system tests on the inverse square law of the gravitational force [106]

$$m_{g,<} < 10^{-58}\ \mathrm{kg} \approx 10^{-22}\ \mathrm{eV}. \tag{75}$$

With this bound even on galactic distance scales the Yukawa contribution is too small and has the wrong sign to compensate for dark matter.

Here we get for the decay of the orbital period

$$\frac{\dot{P}}{P} \approx -\frac{|E_{\mathrm{GR}}|^{\cdot}}{|E_{\mathrm{GR}}|}\left(\frac{3}{2} - \frac{2}{3} m_{g,<}^2 R^2 e^{-m_{g,<}R}\right). \tag{76}$$

The second term in the bracket is negligible, since $(m_{g,<}a)^2 \lesssim 10^{-36}$ for $R \approx a$, where a is the semimajor axis of the system.

We have shown that in all cases of interest the choice $\tilde{G} = G$ leads to the Newtonian limit and hence this relation should hold from now on. Note, however, that $\tilde{G}$ is not a free parameter of the theory. The theory is independent of $\tilde{G}$, because of the Weyl invariance. The only free parameter is α_g (or equivalently m_g). Hence, the choice of $\tilde{G} = G$ is just convenient to recover expressions that look familiar and to compare to GR.

B. Gravitational waves from binary systems

We discuss the GW solutions for an explicit binary system in circular motion and in the Newtonian limit.

But before doing so, we show that for a small graviton mass monopole and dipole radiation can be neglected. For the massless part, since it is the same as in GR, there is no monopole and dipole radiation and the leading contribution comes from the quadrupole term. The reason for this is that the metric perturbation is a massless spin-2 field and that the matter energy-momentum tensor is conserved far away from the source. But for a small graviton mass there are nonvanishing contributions from the monopole and dipole radiation. Nevertheless, in the following we will show that in the quadrupole approximation these do not contribute to the radiated energy and that only 2 of the 5 additional d.o.f. of the massive mode are excited by a conserved matter energy-momentum tensor.

The quadrupole approximation requires that the typical velocities of the source are much smaller than the velocity of the gravitational waves such that $k_\omega d \ll 1$ is fulfilled. In GR this holds true for nonrelativistic sources, since the gravitational waves travel with the speed of light.

For a small graviton mass we can apply the quadrupole approximation in (58) because $k_{\omega,\epsilon} \approx \omega(1 - \epsilon m_g^2/(2\omega^2))$ for $m_g^2/\omega^2 \ll 1$ and, as we will verify later in this section, $\omega = 2\omega_s$. Thus the speed of the massive mode of the gravitational waves is nearly the speed of light and hence much higher than the orbital speed of the source.

However, in the case of MCG with a large graviton mass we have $k_{\omega,>} \approx m_g(1 - \omega^2/(2m_g^2))$ for $\omega^2/m_g^2 \ll 1$, which leads to $k_{\omega,>} d \gg 1$. This shows that the quadrupole approximation cannot be used in (59). Nevertheless, the term $\exp(-m_g r)$ in (59) leads to an exponential suppression of the massive mode anyway. Hence, we do not need the quadrupole approximation and keep only the leading order term of the far field approximation. For more details to the multipole expansion, see, e.g., [107].

Before we apply the quadrupole approximation let us define the mass-energy moments

$$M(t) = \int d^3x T^{00}(t, \mathbf{x}), \tag{77}$$

$$D^i(t) = \int d^3x x^i T^{00}(t, \mathbf{x}), \tag{78}$$

$$M^{ij}(t) = \int d^3x x^i x^j T^{00}(t, \mathbf{x}). \tag{79}$$

These quantities are called monopole, dipole, and quadrupole moments, and we denote their time Fourier transformations as $\tilde{M}(\omega)$, $\tilde{D}^i(\omega)$, and $\tilde{M}^{ij}(\omega)$. We further introduce relations between the energy-momentum tensor and the mass-energy moments using energy-momentum conservation in flat spacetime,

$$\int d^3x \tilde{T}^{ij}(\omega, \mathbf{x}) = -\frac{\omega^2}{2}\int d^3x x^i x^j \tilde{T}^{00}(\omega, \mathbf{x}) = -\frac{\omega^2}{2}\tilde{M}^{ij}(\omega), \tag{80}$$

$$\int d^3x \tilde{T}^{0i}(\omega, \mathbf{x}) = -i\omega \int d^3x x^i \tilde{T}^{00}(\omega, \mathbf{x}) = -i\omega \tilde{D}^i(\omega), \tag{81}$$

$$\int d^3x \tilde{T}^{ij}(\omega, \mathbf{x}) = -i\omega \int d^3x x^i \tilde{T}^{j0}(\omega, \mathbf{x}) = -\frac{\omega^2}{2}\tilde{M}^{ij}(\omega). \tag{82}$$

Now, we transform (58) back to real space, insert it into (54), expand in $k_\omega |\mathbf{x}' \cdot \mathbf{n}| \ll 1$, and keep terms up to the quadrupole contribution. This yields

$$\hat{\Psi}_{\mu\nu}(t, \mathbf{x}) = -\frac{4G}{r}\int d^3x' \left[\int_{m_g}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} e^{ik_\omega r}\left(1 - ik_{\omega,\epsilon}\mathbf{x}' \cdot \mathbf{n} - \frac{k_{\omega,\epsilon}^2}{2}(\mathbf{x}' \cdot \mathbf{n})^2\right)\tilde{T}_{\mu\nu}(\omega, \mathbf{x}') + \int_{-\infty}^{-m_g} \ldots\right], \tag{83}$$

where $k_{\omega,\epsilon} = \sqrt{\omega^2 - \epsilon m_g^2}$. The integral $\int_{-\infty}^{-m_g}$ represents the contribution from the second term in (58), which we suppress in the steps below because its analysis is analogous to the first integral. This expression is exact up to the quadrupole contribution. For the components we find

$$\hat{\Psi}^{00} = -\frac{4G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\left(\tilde{M}(\omega) - ik_{\omega,\epsilon}n_k\tilde{D}^k(\omega) - \frac{k_{\omega,\epsilon}^2}{2}n_kn_l\tilde{M}^{kl}(\omega)\right), \tag{84}$$

$$\hat{\Psi}^{0i} = -\frac{4G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\left(-i\omega\tilde{D}^i(\omega) - \frac{\omega}{2}k_{\omega,\epsilon}n_k\tilde{M}^{ki}(\omega)\right), \tag{85}$$

$$\hat{\Psi}^{ij} = \frac{2G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}(\omega^2\tilde{M}^{ij}(\omega)). \tag{86}$$

We will later see that for the radiated energy we only need time derivatives of these components, because all spatial derivatives can be translated into time derivatives. They are given by

$$\dot{\hat{\Psi}}^{00} = -\frac{4G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\left(-i\omega\tilde{M}(\omega) - \omega k_{\omega,\epsilon}n_k\tilde{D}^k(\omega) + i\omega\frac{k_{\omega,\epsilon}^2}{2}n_kn_l\tilde{M}^{kl}(\omega)\right), \tag{87}$$

$$\dot{\hat{\Psi}}^{0i} = -\frac{4G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\left(-\omega^2\tilde{D}^i(\omega) + i\frac{\omega^2}{2}k_{\omega,\epsilon}n_k\tilde{M}^{ki}(\omega)\right), \tag{88}$$

$$\dot{\hat{\Psi}}^{ij} = -i\frac{2\tilde{G}}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\omega^3\tilde{M}^{ij}(\omega). \tag{89}$$

Note that we can expand $k_{\omega,\epsilon} \approx \omega(1 - \epsilon m_g^2/(2\omega^2))$ for $m_g^2/\omega^2 \ll 1$ and $\omega > 0$. The validity of this expansion will be shown in Sec. VI. Using this expansion Eqs. (87)–(89) simplify to

$$\dot{\hat{\Psi}}^{00} \approx -\frac{4G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\left(-i\omega\tilde{M}(\omega) + \omega^2n_k\tilde{D}^k + i\frac{\omega^3}{2}n_kn_l\tilde{M}^{kl}(\omega)\right), \tag{90}$$

$$\dot{\hat{\Psi}}^{0i} \approx -\frac{4G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\left(\omega^2\tilde{D}^i - i\frac{\omega^3}{2}n_k\tilde{M}^{ki}(\omega)\right), \tag{91}$$

$$\dot{\hat{\Psi}}^{ij} \approx -i\frac{2G}{r}\int_{m_g}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}e^{ik_{\omega,\epsilon}r}\omega^3\tilde{M}^{ij}(\omega). \tag{92}$$

In GR, one can go to the transverse-traceless (TT) gauge ($h_\mu^{\mathrm{TT}\mu} = 0, h_{\mathrm{TT}}^{0\mu} = 0, \partial^j h_{ij}^{\mathrm{TT}} = 0$) in vacuum, and one only needs to calculate the spatial components of the metric perturbation. In CG/MCG, the choice of the gauge is more subtle, since there are more d.o.f. than in GR. In principle, the massive graviton contributes 5 additional d.o.f. Hence, by using the additional coordinate freedom left over after choosing the Teyssandier gauge, we can find the analog to the TT gauge (see Appendix B for details),

$$H_\mu^{\mathrm{TT}\mu} = 0, \qquad \partial_\mu H_{\mathrm{TT}}^{\mu\nu} = 0, \qquad H_{\mu 0}^{\mathrm{TT}} = 0, \tag{93}$$

$$\Psi_i^{\mathrm{TT}i} = 0, \qquad \Psi_{0i}^{\mathrm{TT}} = 0, \qquad \partial_i\partial_j\Psi_{\mathrm{TT}}^{ij} = -\partial_i\partial^i\Psi_{00}^{\mathrm{TT}}. \tag{94}$$

Note that for the massive part only the spatial trace is zero and the 00 component does not vanish.

Nevertheless, we show that these additional modes are not excited by a conserved energy-momentum tensor. Contracting (54) with a partial derivative yields

$$\begin{aligned}\partial^\mu\hat{\Psi}_{\mu\nu} &= \int_V d^4x'\left(\frac{\partial}{\partial x_\mu}\mathcal{G}(x-x')\right)T_{\mu\nu}(x')\\ &= -\int_V d^4x'\left(\frac{\partial}{\partial x'_\mu}\mathcal{G}(x-x')\right)T_{\mu\nu}(x')\\ &= -\mathcal{G}(x-x')T_{\mu\nu}(x')|_{\partial V}\\ &\quad + \int_V d^4x'\mathcal{G}(x-x')\left(\frac{\partial}{\partial x'_\mu}T_{\mu\nu}(x')\right)\\ &= 0,\end{aligned} \tag{95}$$

where we have used $\frac{\partial}{\partial x^\mu}\mathcal{G}(x-x') = -\frac{\partial}{\partial x'^\mu}\mathcal{G}(x-x')$ for the second equal sign and integration by parts for the third equal sign. Furthermore, we have chosen an integration volume V that is larger than the source, such that $T_{\mu\nu}(x)$ vanishes on the boundary ∂V. The last expression vanishes due to matter energy-momentum conservation; see (31). Hence, although the Teyssandier gauge (see Appendix B) does not lead to the harmonic gauge for the massive mode of the metric perturbation, a conserved energy-momentum tensor only excites the transverse modes and we get the harmonic gauge condition for the massive part automatically. By applying a further coordinate transformation in an analogous way as in GR without spoiling the harmonic gauge, we can bring both parts of the wave to the standard GR-TT gauge

$$H^{\mathrm{TT}\mu}_{\mu} = 0, \qquad \partial_\mu H^{\mu\nu}_{\mathrm{TT}} = 0, \qquad H^{\mathrm{TT}}_{\mu 0} = 0, \tag{96}$$

$$\Psi^{\mathrm{TT}\mu}_{\mu} = 0, \qquad \partial_\mu \Psi^{\mu\nu}_{\mathrm{TT}} = 0, \qquad \Psi^{\mathrm{TT}}_{\mu 0} = 0. \tag{97}$$

Note that in the TT gauge $\bar{H}_{\mu\nu} = H_{\mu\nu}$ and $\hat{\Psi}_{\mu\nu} = \Psi_{\mu\nu}$, since the traces vanish.

Inserting (84)–(86) explicitly into (95) leads to

$$-i\int_{m_g}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} e^{ik_{\omega,\epsilon} r} \omega \tilde{M}(\omega) = 0, \tag{98}$$

$$-\int_{m_g}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} e^{ik_{\omega,\epsilon} r} \omega^2 \tilde{D}^i(\omega) = 0. \tag{99}$$

This shows that the monopole and dipole contributions in (90)–(92), which are the quantities that enter into the radiated energy, drop out and we are left with only the quadrupole contribution as for the massless mode. However, there is a phase difference between the massless and the massive mode varying with the distance to the source. This becomes obvious by the factor $\exp(-ik_\omega r)$ in (90)–(92).

Let us calculate the explicit solution for the gravitational wave that is generated by a simple binary system in circular motion in the Newtonian limit, which can be described in the center of mass frame as one particle with the reduced mass μ. We choose the orbit such that it lies in the xy plane and get for the relative coordinates

$$x_0^1(t) = -R\sin(\omega_s t), \tag{100}$$

$$x_0^2(t) = R\cos(\omega_s t), \tag{101}$$

$$x_0^3(t) = 0, \tag{102}$$

where R is the radius of the source. We do not need to calculate the 0μ components, because our aim is to calculate the radiated energy far away from the source, where we can use the TT gauge. Therefore, we restrict here to calculate only the spatial components in the harmonic gauge and project the solutions into the TT gauge when needed.

For a point particle of reduced mass μ in the nonrelativistic limit we get for the quadrupole moment

$$M^{ij} = \mu x_0^i x_0^j. \tag{103}$$

In components this reads

$$M_{11} = \mu R^2 \frac{1-\cos(2\omega_s t)}{2}, \tag{104}$$

$$M_{22} = \mu R^2 \frac{1+\cos(2\omega_s t)}{2}, \tag{105}$$

$$M_{12} = -\mu R^2 \frac{\sin(2\omega_s t)}{2}, \tag{106}$$

$$M_i^i = \mu R^2, \tag{107}$$

where M_i^i is the spatial trace of the mass moment and $\omega_s > 0$. The time Fourier transform of these expressions is given by

$$\tilde{M}_{11}(\omega) = \frac{\mu R^2 \pi}{2}[\delta(\omega) - \delta(\omega + 2\omega_s) - \delta(\omega - 2\omega_s)], \tag{108}$$

$$\tilde{M}_{22}(\omega) = \frac{\mu R^2 \pi}{2}[\delta(\omega) + \delta(\omega + 2\omega_s) + \delta(\omega - 2\omega_s)], \tag{109}$$

$$\tilde{M}_{12}(\omega) = \frac{\mu R^2 \pi}{2i}[\delta(\omega - 2\omega_s) - \delta(\omega + 2\omega_s)], \tag{110}$$

$$\tilde{M}_i^i(\omega) = \mu R^2 \pi \delta(\omega). \tag{111}$$

For CG and MCG with a small graviton mass, inserting (108)–(110) into (86) (note that we have to consider the $\int_{-\infty}^{-m_g} \ldots$ contribution here), we find the nonvanishing components for the massive mode,

$$\hat{\Psi}_{11}(t,r) = -\hat{\Psi}_{22}(t,r) = -\frac{4G\mu R^2\omega_s^2}{r}\cos(2\omega_s t_m), \tag{112}$$

$$\hat{\Psi}_{12}(t,r) = \hat{\Psi}_{21}(t,r) = -\frac{4G\mu R^2\omega_s^2}{r}\sin(2\omega_s t_m), \tag{113}$$

$$\hat{\Psi}_i^i(t,r) = 0, \tag{114}$$

where $t_m = t - v_{g,\epsilon} r$ is the travel time and $v_{g,\epsilon} = \sqrt{1 - \epsilon m_g^2/(4\omega_s^2)}$ is the speed of the massive gravitational wave.

We have calculated the massive part of (49). To get the full metric perturbation, we now add the massless mode $\bar{H}_{\mu\nu}$ of the metric perturbation to the massive mode in (112) and (113). The derivation of the solution for the massless mode can be found in nearly every standard textbook about GR and gravitational waves; see, e.g., [65,107]. We find

$$h_{11}(t,r) = -h_{22}(t,r) = \frac{4G\epsilon\mu R^2\omega_s^2}{r}[\cos(2\omega_s t_{\rm ret}) - \cos(2\omega_s t_m)], \quad (115)$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{4G\epsilon\mu R^2\omega_s^2}{r}[\sin(2\omega_s t_{\rm ret}) - \sin(2\omega_s t_m)], \quad (116)$$

where $t_{\rm ret} = t - r$ is the retarded time. This result is consistent with the statement made in Sec. III C that the metric perturbation vanishes in the limit $m_g \to 0$ (for $\bar{H} = 0$ and $\hat{\Psi} = 0$), which shows that this is not the GR limit.

For MCG with a large graviton mass, we find

$$\hat{\Psi}_{11}(t,r) = -\hat{\Psi}_{22}(t,r) = \frac{-4G\mu R^2\omega_s^2}{r} e^{-k_{\omega,>}r}\cos(2\omega_s t), \quad (117)$$

$$\hat{\Psi}_{12}(t,r) = \hat{\Psi}_{21}(t,r) = -\frac{4G\mu R^2\omega_s^2}{r} e^{-k_{\omega,>}r}\sin(2\omega_s t), \quad (118)$$

and $\hat{\Psi}^i_i(t,r) = 0$. We combine this with the massless mode and get the final result

$$h_{11}(t,r) = -h_{22}(t,r) = \frac{4G\mu R^2\omega_s^2}{r}[\cos(2\omega_s t_{\rm ret}) - e^{-k_{\omega,>}r}\cos(2\omega_s t)], \quad (119)$$

$$h_{12}(t,r) = h_{21}(t,r) = \frac{4G\mu R^2\omega_s^2}{r}[\sin(2\omega_s t_{\rm ret}) - e^{-k_{\omega,>}r}\sin(2\omega_s t)], \quad (120)$$

which is just the GR solution modified by an exponentially damped term.

V. ENERGY-MOMENTUM TENSOR OF GRAVITATIONAL WAVES

To analyze the radiation emitted by sources like binary systems, we need to calculate the explicit form of the gravitational energy-momentum tensor in CG and MCG.

We calculate the gravitational energy-momentum tensor via the corresponding Noether current. In order to do so, we have to expand the gravitational part of the total action

$$I_{\rm GRAV} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-m_g^{-2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) - \epsilon R\right] \quad (121)$$

to second order in $h_{\mu\nu}$ and apply the TT gauge. We find

$$I_{\rm GRAV}^{\rm TT(2)} = \frac{1}{64\pi G}\int d^4x(-m_g^{-2}\Box h^{\rm TT}_{\rho\sigma}\Box h^{\rho\sigma}_{\rm TT} + \epsilon\partial_\alpha h^{\rm TT}_{\rho\sigma}\partial^\alpha h^{\rho\sigma}_{\rm TT}). \quad (122)$$

The formula for an energy-momentum tensor of a fourth-order derivative theory is given by

$$(T^{\rm GRAV})^\lambda_\alpha = \frac{1}{\sqrt{-g}}\left\langle\left(\partial_\xi\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}} - \frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda}}\right)g_{\rho\sigma,\alpha} - \frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha} + \delta^\lambda_\alpha\mathcal{L}\right\rangle, \quad (123)$$

where the angle brackets denote the average over several wavelengths or periods of the wave. This leads to

$$(T^{(2)}_{\rm GRAV})^\lambda_\alpha = \frac{1}{32\pi G}\langle 2m_g^{-2}\Box h^{\rm TT}_{\rho\sigma}\partial_\alpha\partial^\lambda h^{\rho\sigma}_{\rm TT} + \epsilon\partial^\lambda h^{\rm TT}_{\rho\sigma}\partial_\alpha h^{\rho\sigma}_{\rm TT}\rangle. \quad (124)$$

Here, we have already discarded terms proportional to η^λ_α, since they do not contribute to the radiated energy; cf. (130). In vacuum with the help of (42), (50), and (51) it is possible to write this as

$$(T^{(2)}_{\rm GRAV})^\lambda_\alpha = \frac{1}{32\pi G}\langle 2\Psi^{\rm TT}_{\rho\sigma}\partial_\alpha\partial^\lambda h^{\rho\sigma}_{\rm TT} + \epsilon\partial_\alpha h^{\rm TT}_{\rho\sigma}\partial^\lambda h^{\rho\sigma}_{\rm TT}\rangle. \quad (125)$$

VI. ENERGY LOSS DUE TO GRAVITATIONAL WAVE EMISSION

A. Radiated energy

In this section we want to calculate the amount of energy that is radiated by binary systems. We use the conservation of the energy-momentum tensor in the far zone ($r \gg R$) and set $T_{\mu\nu} = 0$. Hence we can go to the TT gauge. We find

$$\partial_0 T^{0\nu}_{\rm GRAV} + \partial_s T^{s\nu}_{\rm GRAV} = 0. \quad (126)$$

The energy carried in volume V by gravitational waves is given by $E_V = \int d^3x T^{00}_{\rm GRAV}$. By combining with Eq. (126) we find

$$\dot{E}_V = \int_V d^3x\partial_0 T^{00}_{\rm GRAV} = -\int_V d^3x\partial_s T^{s0}_{\rm GRAV} = -r^2\int_{\partial V} d\Omega n_s T^{s0}_{\rm GRAV}, \quad (127)$$

where $d\Omega = \sin\theta d\theta d\phi$ is the differential solid angle, ∂V is the surface of the volume V, and $\mathbf{n}$ is the spatial unit vector pointing from the source to the observer. The minus sign means that gravitational waves carry away energy flux from the volume. Hence, the radiated energy of gravitational waves has the opposite sign, and we find

$$\dot{E} = r^2 \int_{\partial V} d\Omega n_s T^{s0}_{\mathrm{GRAV}}. \tag{128}$$

Therefore, the quantity of interest is

$$T^{s0}_{\mathrm{GRAV}} n_s = \frac{1}{32\pi G} n_s \langle 2\Psi^{\mathrm{TT}}_{\rho\sigma} \partial^s \partial^0 h^{\rho\sigma}_{\mathrm{TT}} + \epsilon \partial^s h^{\mathrm{TT}}_{\rho\sigma} \partial^0 h^{\rho\sigma}_{\mathrm{TT}} \rangle \tag{129}$$

$$= \frac{\epsilon}{32\pi G} n_s \langle -\partial^s \Psi^{\mathrm{TT}}_{ij} \partial^0 \Psi^{ij}_{\mathrm{TT}} + \partial^s H^{\mathrm{TT}}_{ij} \partial^0 H^{ij}_{\mathrm{TT}} \rangle, \tag{130}$$

where we used (42) and integration by parts in the second line. Only the spatial components contribute in the TT gauge.

1. Small graviton mass

For CG and MCG with a small graviton mass ($m_g^2 < 4\omega_s^2$, where $\omega_s = \omega/2$) we find

$$T^{s0}_{\mathrm{GRAV}} n_s = \frac{\epsilon}{32\pi G} \left\langle -\partial^0 \Psi^{\mathrm{TT}}_{ij} \partial^0 \Psi^{ij}_{\mathrm{TT}} + \partial^0 H^{\mathrm{TT}}_{ij} \partial^0 H^{ij}_{\mathrm{TT}} + \epsilon \frac{m_g^2}{8\omega_s^2} \partial^0 \Psi^{\mathrm{TT}}_{ij} \partial^0 \Psi^{ij}_{\mathrm{TT}} + \mathcal{O}\left(\frac{m_g^4}{\omega_s^4} \partial^0 \Psi^{\mathrm{TT}}_{ij} \partial^0 \Psi^{ij}_{\mathrm{TT}} \right) \right\rangle$$
$$\approx \frac{\epsilon}{32\pi G} \Lambda_{ijkl} \left\langle -\partial^0 \hat{\Psi}_{ij} \partial^0 \hat{\Psi}^{kl} + \partial^0 \bar{H}_{ij} \partial^0 \bar{H}^{kl} + \epsilon \frac{m_g^2}{8\omega_s^2} \partial^0 \hat{\Psi}_{ij} \partial^0 \hat{\Psi}^{kl} \right\rangle, \tag{131}$$

where we have used $\partial^s \bar{H}^{\rho\sigma} = \partial^0 \bar{H}^{\rho\sigma} n^s + \mathcal{O}(1/r^2)$, $\partial^s \hat{\Psi}^{\rho\sigma} = \partial^0 \hat{\Psi}^{\rho\sigma}[1 - \epsilon m_g^2/(8\omega_s^2) + \mathcal{O}(m_g^4/\omega_s^4)] n^s + \mathcal{O}(1/r^2)$, and $n_s n^s = 1$ to find the first line. In the second line we introduced the so-called Lambda tensor Λ_{ijkl}, which projects h^{ij} into the TT gauge (see Appendix A for details). Note that the second term is the same as in GR for $\epsilon = +1$. This shows that the contribution from the massless and the massive part of the metric perturbation have the same structure, but come with a relative sign.

We insert (131) in (127) and use

$$\int d\Omega \Lambda_{ijkl} = \frac{2\pi}{15} (11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}), \tag{132}$$

to find

$$\dot{E} \approx \frac{\epsilon r^2}{20G} \left\langle -\partial^0 \hat{\Psi}_{ij} \partial^0 \hat{\Psi}^{ij} + \partial^0 \bar{H}_{ij} \partial^0 \bar{H}^{ij} + \epsilon \frac{m_g^2}{8\omega_s^2} \partial^0 \hat{\Psi}_{ij} \partial^0 \hat{\Psi}^{ij} \right\rangle. \tag{133}$$

Inserting (115) and (116) yields

$$\dot{E} \approx \epsilon \dot{E}_{\mathrm{GR}} \langle -\sin^2(2\omega_s t_m) - \cos^2(2\omega_s t_m) + \sin^2(2\omega_s t_{\mathrm{ret}}) + \cos^2(2\omega_s t_{\mathrm{ret}}) \rangle + \frac{m_g^2}{8\omega_s^2} \dot{E}_{\mathrm{GR}} = \frac{m_g^2}{8\omega_s^2} \dot{E}_{\mathrm{GR}}, \tag{134}$$

where

$$\dot{E}_{\mathrm{GR}} = \frac{32 G \mu^2 R^4 \omega_s^6}{5}. \tag{135}$$

Note that (133) is independent of the signs of $\bar{H}_{ij}$ and $\hat{\Psi}_{ij}$ as they appear quadratically. Hence, depending on ϵ, the first or the second term comes with the wrong sign compared to GR. However, in (134) we see that the ϵ-dependent part vanishes and the remaining contribution has the same sign as in GR.

For CG we have $m_{g,\mathrm{CG}}^2/(8\omega_s^2) \approx 9 \times 10^{-6}$ and for MCG we have $m_{g,<}^2/(8\omega_s^2) < 10^{-5}$. Hence, the radiated energy is several orders of magnitude smaller than in GR.

Since we have shown in Sec. IV A that a too small radiated energy directly translates into a too small decay of the orbital period, it seems that gravitational radiation cannot explain the measured decrease of the orbital period of binary systems in these theories.

2. Large graviton mass

For MCG with a large graviton mass ($\epsilon = +1$, $m_g^2 > 4\omega_s^2$, where $\omega_s = \omega/2$) we use $\partial_r \hat{\Psi}_{\mu\nu} = -k_\omega \hat{\Psi}_{\mu\nu} + \mathcal{O}(1/r^2)$ in (130) to find

$$T^{s0}_{\mathrm{GRAV}} n_s \approx \frac{1}{32\pi G} \Lambda_{ij,kl} \langle k_\omega \hat{\Psi}_{ij} \partial^0 \hat{\Psi}^{kl} + \partial^0 \bar{H}_{ij} \partial^0 \bar{H}^{kl} \rangle, \tag{136}$$

with $k_\omega = \sqrt{m_g^2 - 4\omega_s^2}$. The second term in (136) gives the same contribution as in GR. To calculate the first term, we use Eqs. (117) and (118). We find $k_\omega \hat{\Psi}_{ij}\partial^0\hat{\Psi}^{kl} \propto k_\omega e^{-2k_\omega r}\sin(2\omega_s t)\cos(2\omega_s t)$, which vanishes in combination with the average over several periods of the wave. Hence, MCG with a large graviton mass reproduces the GR result exactly. We get

$$\dot{E} = \frac{r^2}{20G}\langle \partial^0 \bar{H}_{ij}\partial^0 \bar{H}^{ij}\rangle = \dot{E}_{\mathrm{GR}}. \tag{137}$$

Therefore, MCG with a large graviton mass represents a theory that still needs dark matter to explain galaxy rotation curves, but accounts for the decay of the orbital period due to gravitational waves. On macroscopic distance scales such as $r \gg m_g^{-1}$ it can be split into GR plus small contributions from the higher derivative terms. Therefore, it is reasonable to expect that also the other tests of gravity can be passed. Only on very small scales, where the higher derivative terms become important, is a significant deviation from GR expected. This is the reason why this theory is renormalizable [95].

VII. SUMMARY, CONCLUSION, AND OUTLOOK

In this work we have investigated gravitational radiation from the binary system PSR J1012+5307 in CG and MCG. Both theories belong to the class of models containing higher derivatives and are invariant under Weyl rescaling. The action is given by a C^2 term that contributes the higher-derivative part and a term that resembles the Einstein-Hilbert term in the Weyl gauge $S(x) = S_0$ and $\tilde{G} = G$. By introducing the parameter $\epsilon = \pm 1$, we distinguished between CG and MCG. The difference between these two theories is the sign in front of the Einstein-Hilbert term, and both signs are allowed by the Weyl symmetry. This choice of sign does not only change the results for gravitational radiation but also changes the properties of the gravitational wave. We have argued that in CG ($\epsilon = -1$) the choice of sign leads to metric perturbations that can be written as a massless ghost field and a massive tachyon. Whereas in MCG ($\epsilon = +1$), the massless mode is healthy and the massive mode is a ghost, but not a tachyon. A ghost field represents a severe problem for a theory, but as we discussed in Sec. III B, there seem to be solutions to the ghost problems in CG and MCG [30,105].

In Sec. III we derived the inhomogeneous linearized field equations in the Teyssandier gauge for the metric perturbation given in Eq. (35). These equations are higher-derivative partial differential equations for a partially massive field. It was shown that one can divide this equation into a massless and a massive mode; see (45) and (48). Since the solution to the massless part is known from GR, we only investigated the massive part. In principle, the massive part contributes 5 additional d.o.f., including monopole and dipole radiation. However, in Sec. IV B we have shown that these additional d.o.f. are not excited by a conserved energy-momentum tensor (for nonrelativistic binaries it is mass conservation), and hence monopole and dipole radiation vanish. This means that only the transverse modes contribute.

We found solutions for three different cases. For CG the massive mode has the same form as in GR, but travels faster than the speed of light. In the case of MCG with a small graviton mass the solution is the same, but the sign is different and the velocity is smaller than the speed of light. For MCG in the case of a large graviton mass, the massive terms are damped exponentially, such that in the limit of a large graviton mass GR is recovered.

To calculate the energy radiated by an idealized binary system, we derived the gravitational energy-momentum tensor in Sec. V. It has a contribution from the massless mode that is the same as in GR (the sign depends on ϵ) and additional contributions that depend on the massive mode of the metric perturbation. Most importantly, there is a relative sign between the various contributions that can lead to cancellations and that reduces the efficiency of gravitational wave emission in certain regions of the parameter space.

Finally, in Sec. VI we were able to calculate the radiated energy in the Newtonian limit for a binary system in circular motion. In Table III we summarize our results and compare them to the radiated energy in GR given in Eq. (135). In CG and MCG with a small graviton mass (small compared to the orbital frequency of the binary system), we find the radiated energy to be much smaller than in GR. For CG we fixed the graviton mass by the analysis of galaxy rotation curves without the introduction of dark matter, $m_{g,\mathrm{CG}} = 1.1 \times 10^{-22}$ eV, which turns out to fall into the small mass regime.

Hence, CG and MCG with a small graviton mass cannot explain the decay of the orbital period via gravitational radiation. Nevertheless, one could think of another mechanism to account for the shrinkage of the orbits of binary systems. A suggestion in this direction is given in [30]. Thus our result does not rule out CG, as we cannot exclude the existence of another such mechanism, but it makes CG a less attractive solution to the dark matter problem.

MCG cannot fit galaxy rotation curves without dark matter, but experiments on the inverse square law of the Newtonian potential constrain the graviton mass to the ranges $m_{g,<} < 10^{-22}$ eV for $m_g^2 < 4\omega_s^2$ and $m_{g,>} > 10^{-2}$ eV for $m_g^2 > 4\omega_s^2$. The orbital frequency of the binary system studied in this work is $\omega_s \approx 1.3 \times 10^{-20}$ eV. Most interestingly, MCG with a large graviton mass (i.e., $m_{g,>} > 10^{-2}$ eV) shows properties close to GR. As it contains GR as a limit (MCG with a small graviton mass does not contain GR as a limit, because $m_g^2 < 4\omega_s^2$ and $m_g \to \infty$ are inconsistent unless $\omega_s \to \infty$), MCG is expected to pass all tests of GR on length scales $r \gg m_g^{-1}$. And besides, due to its higher-derivative nature, it seems to be a renormalizable model for gravity [95]. Thus, this model seems to offer interesting opportunities for future work.

The application and extension of our findings to coalescing binaries, as observed by gravitational wave interferometers and for compact stars followed up by telescopes at various wavebands is most interesting, as it might result in further constraints on these theories, in the case in which another mechanism would account for the shrinkage of the orbits of binary systems far away from coalescence (in which case the above findings would not be relevant). Up to now, strong constraints on the graviton mass have been derived from the absence of differences in the arrival times of the gravitational wave and the electromagnetic signal, within the experimental error. This experimental constraint has not yet been applied to CG and MCG. Further studies are needed to figure out what one should expect for these theories, because there are two modes that travel with different speeds, and hence there will be interference between these modes. This will result in a beating pattern (over distance) for the total amplitude of the gravitational wave. On the other hand, for MCG with a large graviton mass the massive mode is not propagating, and hence one cannot expect a constraint on the graviton mass in this case. This analysis will be presented in a forthcoming paper.

ACKNOWLEDGMENTS

We wish to thank Felipe F. Faria and Carl Roberts for valuable comments and discussions. P. H. and D. J. S. acknowledge financial support from Deutsche Forschungsgemeinschaft (DFG) under Grant No. RTG 1620 "Models of Gravity." We also thank the COST Action CA15117 "Cosmology and Astrophysics Network for Theoretical Advances and Training Actions (CANTATA)," supported by COST (European Cooperation in Science and Technology).

APPENDIX A: CONVENTIONS

The signature of the metric is

$$g = \mathrm{diag}(-,+,+,+). \tag{A1}$$

The Christoffel symbols are defined by

$$\Gamma^{\lambda}_{\kappa\mu} = \frac{1}{2} g^{\lambda\rho}(\partial_\kappa g_{\rho\mu} + \partial_\mu g_{\rho\kappa} - \partial_\rho g_{\kappa\mu}), \tag{A2}$$

and the Riemann tensor is given by

$$R^{\lambda}_{\mu\nu\kappa} = -(\partial_\nu \Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\mu\kappa} - \Gamma^{\lambda}_{\kappa\alpha}\Gamma^{\alpha}_{\mu\nu}). \tag{A3}$$

From this we find the Ricci tensor $R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa}$ and the Ricci scalar $g^{\mu\kappa} R_{\mu\kappa}$. The Einstein equations in the convention used by Mannheim and Weinberg [65] read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{A4}$$

The Weyl tensor is given by the expression

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R[g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}] - \frac{1}{2}[g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}]. \tag{A5}$$

In the following we give a list of the curvature tensors, expanded around flat spacetime, at first order in $h_{\mu\nu}$:

$$R^{\mu(1)}_{\nu\rho\sigma} = \frac{1}{2}(-\partial_\nu \partial_\rho h^\mu_\sigma - \partial^\mu \partial_\sigma h_{\nu\rho} + \partial^\mu \partial_\rho h_{\nu\sigma} + \partial_\nu \partial_\sigma h^\mu_\rho), \tag{A6}$$

$$R^{(1)}_{\mu\nu} = \frac{1}{2}(\Box h_{\mu\nu} - \partial_\rho \partial_\mu h^\rho_\nu - \partial_\nu \partial_\rho h^\rho_\mu + \partial_\mu \partial_\nu h), \tag{A7}$$

$$R^{(1)} = \Box h - \partial_\mu \partial_\nu h^{\mu\nu}. \tag{A8}$$

At second order in $h_{\mu\nu}$ the Ricci tensor is given by

$$R^{(2)}_{\mu\nu} = -\frac{1}{2} h^{\rho\sigma}[\partial_\mu \partial_\nu h_{\rho\sigma} - \partial_\nu \partial_\rho h_{\mu\sigma} - \partial_\sigma \partial_\mu h_{\rho\nu} + \partial_\rho \partial_\sigma h_{\mu\nu}] + \frac{1}{4}[2\partial_\sigma h^\sigma_\rho - \partial_\rho h][\partial_\nu h^\rho_\mu + \partial_\mu h^\rho_\nu - \partial^\rho h_{\mu\nu}] - \frac{1}{4}[\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\sigma\rho} - \partial_\sigma h_{\rho\nu}][\partial^\rho h^\sigma_\mu + \partial_\mu h^{\sigma\rho} - \partial^\sigma h^\rho_\mu]. \tag{A9}$$

Since we use the Ricci tensor in the action integral, we can use integration by parts. Using the TT gauge we find

$$R^{(2)\mathrm{TT}}_{\mu\nu} = \frac{1}{4}\partial_\nu h^{\mathrm{TT}}_{\sigma\rho} \partial_\mu h^{\sigma\rho}_{\mathrm{TT}} - \frac{1}{2}\partial_\rho h^{\mathrm{TT}}_{\sigma\nu} \partial^\rho h^\sigma_{\mathrm{TT}\mu}. \tag{A10}$$

The Ricci scalar is given by

$$R^{(2)\mathrm{TT}} = \frac{1}{4} \Box h^{\rho\sigma}_{\mathrm{TT}} h^{\mathrm{TT}}_{\rho\sigma}. \tag{A11}$$

Let us also define the Lambda tensor, which is the projector into the TT gauge. It is given by

$$\Lambda_{ijkl} = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l, \tag{A12}$$

where n_i denotes the spatial unit vector pointing into the direction of wave propagation. The Lambda tensor has some useful properties:

$$\Lambda_{ijmn} = \Lambda_{ijkl}\Lambda^{kl}_{mn}, \tag{A13}$$

$$\Lambda^i_{ikl} = \Lambda^k_{ijk} = 0, \tag{A14}$$

$$n^i \Lambda_{ijkl} = 0, \tag{A15}$$

$$n^j \Lambda_{ijkl} = 0. \tag{A16}$$

APPENDIX B: GENERALIZED GAUGE CONDITION

In order to find the physical d.o.f. of CG and MCG we have to choose gauge fixing conditions. In higher-order derivative theories, it is convenient to choose a generalization of the harmonic gauge, the so-called Teyssandier gauge [81]. To show the usefulness of this gauge, let us start with gauging the theories in a naive way, similar to how it is usually done for GR.

To find the number of physical d.o.f., it is enough to study gravitational waves that propagate in a vacuum. In GR, the metric perturbation is a symmetric 4×4 matrix and has 10 independent components. We are free to perform a coordinate transformation

$$x_\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{B1}$$

where $|\partial_\mu \xi_\nu|$ is of order $|h_{\mu\nu}|$. The trace-reversed metric perturbation transforms as

$$\bar{h}_{\mu\nu}(x) \to \bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho), \tag{B2}$$

and hence

$$\partial^\nu \bar{h}_{\mu\nu} \to (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu. \tag{B3}$$

So to find the harmonic gauge condition one has to choose

$$\Box \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}. \tag{B4}$$

These four conditions reduce the d.o.f. to 6. Nevertheless, this does not fix the gauge freedom completely. One can do a residual coordinate transformation

$$x'^\mu \to x''^\mu = x'^\mu + \zeta^\mu, \tag{B5}$$

where $|\partial_\mu \zeta_\nu|$ is again of the order of $|h_{\mu\nu}|$. This leads to

$$\bar{h}'_{\mu\nu}(x) \to \bar{h}''_{\mu\nu}(x') = \bar{h}'_{\mu\nu}(x) - (\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu - \eta_{\mu\nu}\partial_\rho \zeta^\rho). \tag{B6}$$

Since we do not want to spoil the harmonic gauge condition, we have to demand

$$\Box \zeta_\mu = 0. \tag{B7}$$

For simplicity we only look at a single mode and find the plane wave solution to this equation

$$\zeta_\mu = c_\mu e^{ik_\rho x^\rho} + \text{c.c.} \tag{B8}$$

In the following, we will also suppress the complex conjugate (c.c.). Here c_μ represents four arbitrary constants for fixed wave number k_μ, which is lightlike $(k_\rho k^\rho = 0)$. Inserting this into (B6) one can explicitly use the four components of ζ_μ to set components of $\bar{h}'_{\mu\nu}$ to zero. In the TT gauge these functions are chosen in order to get

$$\bar{h}_{00} = 0, \tag{B9}$$

$$\partial^j \bar{h}_{ij} = 0, \tag{B10}$$

$$\bar{h}_{0i} = 0, \tag{B11}$$

$$\bar{h}^i_i = 0. \tag{B12}$$

For CG/MCG naively one could apply the same procedure. The difference is that there is now also a massive part of the metric perturbation

$$\bar{h}_{\mu\nu} = \epsilon(\bar{H}_{\mu\nu} + \bar{\Psi}_{\mu\nu}), \tag{B13}$$

where $\bar{H}_{\mu\nu}$ corresponds to the massless and $\bar{\Psi}_{\mu\nu}$ to the massive part of the spin-2 field. Hence, there are 20 independent components now. To analyze this, let us expand the metric perturbation in Fourier modes

$$\bar{h}_{\mu\nu} = \epsilon(\bar{a}_{\mu\nu} e^{ik_\rho x^\rho} + \bar{b}_{\mu\nu} e^{il_\rho x^\rho}), \tag{B14}$$

where $l^\rho l_\rho = -\epsilon m_g^2$. $\bar{a}_{\mu\nu}$ and $\bar{b}_{\mu\nu}$ are called the polarization tensors. Inserting Eq. (B14) into the harmonic gauge condition we get

$$k^\nu \bar{a}_{\mu\nu} = 0, \tag{B15}$$

$$l^\nu \bar{b}_{\mu\nu} = 0. \tag{B16}$$

These eight conditions reduce the d.o.f. to 12. Now, there appears a problem. Although it is possible to bring the massless part to the standard TT gauge, it is not possible to set terms of the massive part to zero, since ζ_μ is a lightlike vector field, which cannot cancel a massive wave. But since we still have eight independent components, there has to be one more condition, since a massless and a massive spin-2 field should have only 7 d.o.f.. This is the reason why it is more convenient to use the Teyssandier gauge. Let us briefly derive this gauge here.

The field equations and gauge conditions for the massless and the massive parts of the wave are shown in Sec. III.

In Eq. (34) we have chosen the Teyssandier gauge condition. But the gauge freedom is not fixed completely, and hence we can do another coordinate transformation. Under a coordinate transformation, $x^\mu \to x'^\mu = x^\mu + \zeta^\mu$, this quantity transforms as

$$Z'^\mu = Z^\mu - \epsilon m_g^{-2}(\Box - \epsilon m_g^2)\Box \zeta^\mu. \tag{B17}$$

Again, to not spoil the Teyssandier gauge condition $Z_\mu = 0$ we have to demand

$$(\Box - \epsilon m_g^2)\Box\zeta^\mu = 0. \tag{B18}$$

The solution to this equation is

$$\zeta^\mu = c^\mu e^{ik_\rho x^\rho} + d^\mu e^{il_\rho x^\rho}. \tag{B19}$$

We look only at one mode and discard the c.c. for simplicity. c^μ and d^μ are arbitrary constants for fixed wave numbers k_μ and l_μ ($k_\rho k^\rho = 0$ and $l_\rho l^\rho = -\epsilon m_g^2$). The second term describes a massive vector field, and hence it is possible to set components of the massive mode of the metric perturbation to zero. The massless and the massive parts of the metric perturbation expanded in Fourier modes transform as

$$a'_{\mu\nu} = a_{\mu\nu} - i(k_\mu c_\nu + k_\nu c_\mu), \tag{B20}$$

$$b'_{\mu\nu} = b_{\mu\nu} - i(l_\mu d_\nu + l_\nu d_\mu). \tag{B21}$$

We bring the massless part to the TT gauge as in GR. With no loss of generality we choose the wave propagating in the z direction, $k^\mu = (k, 0, 0, k)$. From the gauge (46) for the massless part we get

$$\bar{a}_{00} = -\bar{a}_{30}, \tag{B22}$$

$$\bar{a}_{01} = -\bar{a}_{31}, \tag{B23}$$

$$\bar{a}_{02} = -\bar{a}_{32}, \tag{B24}$$

$$\bar{a}_{03} = -\bar{a}_{33}, \tag{B25}$$

and hence $\bar{a}_{00} = \bar{a}_{33}$. Using this, one can show that $a = -a_{00} + a_{33}$ and $a_{11} + a_{22} = 0$. For the trace we find

$$a' = a - 2ik^\rho c_\rho. \tag{B26}$$

We can set $a' = 0$ if we choose

$$c_0 = \frac{-a_{00} + a_{33}}{2ik} - c_3. \tag{B27}$$

Using (B21) we also see that $a'_{11} + a'_{22} = 0$, because a_{11} and a_{22} do not transform under this coordinate transformation. To set $a'_{0i} = 0$, we have to choose

$$c_1 = -\frac{a_{01}}{ik}, \tag{B28}$$

$$c_2 = -\frac{a_{02}}{ik}, \tag{B29}$$

$$c_3 = \frac{a_{03} - a}{2ik}. \tag{B30}$$

Inserting this in the harmonic gauge condition yields

$$\bar{a}'_{00} = 0, \tag{B31}$$

$$\bar{a}'_{33} = 0, \tag{B32}$$

$$\bar{a}'_{31} = 0, \tag{B33}$$

$$\bar{a}'_{32} = 0. \tag{B34}$$

This brings the massless part to the convenient TT gauge

$$H_{00}^{\mathrm{TT}} = 0, \tag{B35}$$

$$\partial^j H_{ij}^{\mathrm{TT}} = 0, \tag{B36}$$

$$H_{0i}^{\mathrm{TT}} = 0, \tag{B37}$$

$$H_i^{\mathrm{TT}i} = 0. \tag{B38}$$

For the massive mode we can proceed analogously. The gauge condition $\partial_\rho\partial_\sigma\Psi^{\rho\sigma} = \Box\Psi$ yields

$$(l^0)^2 b_{00} + 2l^0 l_3 b_{03} + (l_3)^2 b_{33} = -\epsilon m_g^2 b, \tag{B39}$$

where we again have chosen the massive part to travel in the z direction with $l^\mu = \left(l, 0, 0, \sqrt{l^2 - \epsilon m_g^2}\right)$. We choose

$$d_1 = \frac{b_{01}}{il_0}, \tag{B40}$$

$$d_2 = \frac{b_{02}}{il_0}, \tag{B41}$$

$$d_3 = \frac{b_{03}}{il_0} - \frac{l_3}{2i(l_0)^2} b_{00}, \tag{B42}$$

to set $b'_{0i} = 0$. Inserting this back into the transformed (B39) we find the condition

$$(l^0)^2 b'_{00} + (l_3)^2 b'_{33} = -\epsilon m_g^2 b'^\mu_\mu. \tag{B43}$$

We have the freedom to choose d_0 to set to zero b'_{00}, b'_{33}, b'^i_i, or b'^μ_μ. Thus, one choice for the completely gauge-fixed massive mode is

$$\Psi_{0i}^{\mathrm{TT}} = 0, \tag{B44}$$

$$\Psi_i^{\mathrm{TT}i} = 0, \tag{B45}$$

$$\Psi_{00}^{\mathrm{TT}} = -\Psi_{33}^{\mathrm{TT}}, \tag{B46}$$

which reduce the d.o.f. of the massive mode to 5.

TABLE III. Summary of the radiated energy for CG and MCG with small and large graviton masses.

ϵ	-1	$+1$
$m_g^2 < 4\omega_s^2$	$\frac{m_g^2}{8\omega_s^2}\dot{E}_{\mathrm{GR}}$	$\frac{m_g^2}{8\omega_s^2}\dot{E}_{\mathrm{GR}}$
$m_g^2 > 4\omega_s^2$	No reasonable Newtonian limit	$\dot{E}_{\mathrm{GR}}$

APPENDIX C: GHOSTS AND TACHYONS FOR THE SCALAR FIELD

In this Appendix we want to study the properties of a free scalar field $S(x)$ in flat spacetime. We investigate

$$I_M = \int d^4x\sqrt{-g}\frac{\epsilon}{2}(\nabla_\mu S\nabla^\mu S - m_s^2 S^2), \tag{C1}$$

where $m_s^2 = R/6$ represents the mass of the scalar field $S(x)$. The equation of motion is given by

$$(\nabla_\mu\nabla^\mu - m_s^2)S = 0. \tag{C2}$$

By defining the conjugate momentum $\pi_S = \epsilon\sqrt{-g}\nabla_0 S$ we find the Hamiltonian density

$$\mathcal{H} = \pi_S\nabla_0 S - \mathcal{L} = \frac{\epsilon}{2}\sqrt{-g}(-\nabla_0 S\nabla_0 S - (\nabla S)^2 + m_s^2 S^2). \tag{C3}$$

From this we can derive the stability properties of the scalar field, which are summarized in Table IV. We see that the sign of ϵ and the Ricci scalar are crucial for the properties of the scalar field. Nevertheless, since in CG and MCG the scalar field represents no real d.o.f., it is not necessary that it is a healthy field. We can always choose a Weyl gauge, which sets the scalar field to a constant.

APPENDIX D: ANALYSIS OF THE NEWTONIAN LIMIT

Let us investigate the Newtonian limit of MCG. We use the wave equations (44) and (45) and make the following assumptions corresponding to the Newtonian limit:

$$\partial_t h_{\mu\nu} = 0, \tag{D1}$$

$$T_{00}^{\mathrm{Newt}} \approx M\eta_{\mu 0}\eta_{\nu 0}\delta^{(3)}(\mathbf{r}), \tag{D2}$$

TABLE IV. The stability properties of the scalar field minimizing the action (C1).

	$R<0$	$R>0$
$\epsilon = -1$	Healthy	Tachyon
$\epsilon = +1$	Tachyonic ghost	Ghost

$$T^{\mathrm{Newt}} \approx -M\delta^{(3)}(\mathbf{r}), \tag{D3}$$

$$\partial_t\rho = 0, \tag{D4}$$

where T_{00}^{Newt} is the time-time component of the matter energy-momentum tensor $T_{\mu\nu}^{\mathrm{Newt}}$ in the Newtonian limit and T^{Newt} is the trace. M is the mass of the point source at $r = 0$, and we have neglected the pressure p. Inserting this into (44) and (45) we find

$$\triangle H_{\mu\nu} = -16\pi\tilde{G}M\left(\eta_{\mu 0}\eta_{\nu 0} + \frac{1}{2}\eta_{\mu\nu}\right)\delta^{(3)}(\mathbf{r}), \tag{D5}$$

$$(\triangle - m_g^2)\Psi_{\mu\nu}(r) = 16\pi\tilde{G}M\left(\eta_{\mu 0}\eta_{\nu 0} + \frac{1}{3}\eta_{\mu\nu}\right)\delta^{(3)}(\mathbf{r}). \tag{D6}$$

First, let us find the vacuum solutions to these equations. In the Newtonian limit we can write the line element as

$$ds^2 = (-1 - 2\Phi(r))dt^2 + (1 - 2\Theta(r))dr^2 + r^2 d\Omega^2, \tag{D7}$$

where $h_{00} = -2\Phi$, $h_{rr} = -2\Theta$, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of a unit 2-sphere. The 00 component of (D5) and (D6) in the vacuum yields

$$\Phi(r) = c_0 + \frac{c_1}{r} + \frac{c_2 e^{-m_g r}}{r} + \frac{c_3 e^{m_g r}}{r}, \tag{D8}$$

where c_0, c_1, c_2, and c_3 are arbitrary constants. The constant term has no physical relevance, so we set $c_0 = 0$. The condition of asymptotic flatness demands $c_3 = 0$. For the other constants it is convenient to choose $c_1 = -GM$ and $c_2 = -GM\alpha$. This leads to

$$\Phi(r) = -\frac{GM}{r}(1 + \alpha e^{-m_g r}). \tag{D9}$$

The point source solution to (D5) and (D6) in spatial Fourier space reads

$$\tilde{h}_{\mu\nu}(k) = 16\pi\tilde{G}M\left[\frac{\eta_{\mu 0}\eta_{\nu 0} + \frac{1}{2}\eta_{\mu\nu}}{k^2} + \frac{\eta_{\mu 0}\eta_{\nu 0} + \frac{1}{3}\eta_{\mu\nu}}{k^2 + m_g^2}\right]. \tag{D10}$$

In real space this yields

$$h_{\mu\nu} = \frac{4\tilde{G}}{r}\left(\eta_{\mu 0}\eta_{\mu 0} + \frac{1}{2}\eta_{\mu\nu}\right) - \frac{4\tilde{G}}{r}e^{-m_g r}\left(\eta_{\mu 0}\eta_{\mu 0} + \frac{1}{3}\eta_{\mu\nu}\right), \tag{D11}$$

where we have chosen boundary conditions of asymptotic flatness. From the 00 component we find for the Newtonian potential

$$\Phi(r) = -\frac{\tilde{G}M}{r}\left(1 - \frac{4}{3}e^{-m_g r}\right). \tag{D12}$$

Choosing $\tilde{G} = G$ and comparing with (D9) we get $\alpha = -4/3$. The limit $m_g r \gg 1$ yields just the standard Newtonian potential. For $m_g r \ll 1$ one gets $\Phi(r) = GM/(3r)$, which leads to a repulsive gravitational force. This points out that the additional Yukawa potential cannot serve to fit galaxy rotation curves without dark matter in any parameter range, since it always comes with the wrong sign; see [108].

In literature there also exists another choice for the gravitational potential. This is the phenomenological approach by Sanders [109], which is also adopted by MCG [68,110]. In this case the gravitational potential exterior to a source, given by

$$\Phi = -\frac{GM}{r(1+\delta)}[1 + \delta e^{-m_{g,S} r}], \tag{D13}$$

with parameters $\delta = -0.92$ and $m_{g,S} \approx 1.6 \times 10^{-28}$ eV ($m_{g,S}$ is the graviton mass) [109], has been used to fit galaxy rotation curves without dark matter. The standard Newtonian potential is recovered in the limit $m_g r \ll 1$. Trying to match (D13) with (D12) seems to be impossible, unless the massive part of the metric perturbation couples differently to matter than the massless part.

Assuming that it is possible to derive such a potential in some way, let us calculate the decay of the orbital period of the binary system. We find

$$\frac{\dot{P}}{P} \approx -\frac{|E_{\rm GR}|^{\cdot}}{|E_{\rm GR}|}\left(\frac{3}{2} + \frac{\delta}{2} m_{g,S}^2 R^2 e^{-m_{g,s} R}\right), \tag{D14}$$

where we have assumed that $m_{g,S} R \ll 1$, which can be verified using Table II. We find $m_{g,S} R \approx 10^{-13}$. Hence, the contribution from the second term in the parentheses in Eq. (D14) is negligible.

Fourth-order theories, such as $\mathcal{L} = f(R, R_{\mu\nu})$, have been criticized for explaining galaxy rotation curves without dark matter; see [108]. In this reference the authors state that for a fourth-order theory, which includes squares of the Ricci tensor, the additional Yukawa potential term always comes with the wrong sign, such that it does not give additional but less attraction.

In the phenomenological approach of Sanders the Yukawa term also appears with the wrong sign, since δ is negative. Nevertheless, the reason why this approach is able to fit galaxy rotation curves without invoking dark matter is that also the gravitational constant G is changed to an effective gravitational constant $\tilde{G} = G/(1+\delta)$. This procedure has also been adopted in MCG [68] and in scalar-tensor-vector gravity [110]. However, in accordance with our findings, it is also criticized in [111] that it is not clear how such a modified gravitational potential as in (D13) can emerge from a standard matter source.

On top of that it also fails to explain the decrease of the orbital period of binary systems by gravitational radiation.

The ratio between the graviton mass and the orbital frequency of the binary system is $m_{g,S}/\omega_s \lesssim 10^{-8}$ and the radiated energy is to first order in m_g^2/ω_s^2

$$\dot{E} \approx \frac{m_g^2}{8\omega_s^2(1+\delta)}\dot{E}_{\rm GR}, \tag{D15}$$

which is much smaller than $\dot{E}_{\rm GR}$.

[1] C. M. Will, Living Rev. Relativity 17, 4 (2014).
[2] M. Kramer *et al.*, Science **314**, 97 (2006).
[3] J. M. Weisberg and J. H. Taylor, *Binary Radio Pulsars: Proc. Aspen Conference*, ASP Conf. Series, edited by F. A. Rasio and I. H. Stairs (ASP, San Francisco, 2005), Vol. 328, p. 25.
[4] B. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016).
[5] B. Abbott *et al.*, Phys. Rev. Lett. **116**, 241103 (2016).
[6] L. Scientific *et al.*, Phys. Rev. Lett. **118**, 221101 (2017).
[7] B. Abbott *et al.*, Astrophys. J. Lett. **851**, L35 (2017).
[8] B. P. Abbott *et al.*, Phys. Rev. Lett. **119**, 141101 (2017).
[9] B. P. Abbott *et al.*, Phys. Rev. Lett. **119**, 161101 (2017).
[10] B. Abbott *et al.*, Astrophys. J. Lett. **848**, L12 (2017).
[11] D. A. Coulter *et al.*, Science **358**, 1556 (2017).
[12] B. P. Abbott *et al.*, Astrophys. J. Lett. **848**, L13 (2017).
[13] L. Lombriser and N. A. Lima, Phys. Lett. B **765**, 382 (2017).
[14] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller, and I. Sawicki, Phys. Rev. Lett. **119**, 251301 (2017).
[15] P. Creminelli and F. Vernizzi, Phys. Rev. Lett. **119**, 251302 (2017).
[16] J. M. Ezquiaga and M. Zumalacárregui, Phys. Rev. Lett. **119**, 251304 (2017).
[17] J. Sakstein and B. Jain, Phys. Rev. Lett. **119**, 251303 (2017).
[18] H. Nersisyan, N. A. Lima, and L. Amendola, arXiv:1801.06683.
[19] Y. Akrami, P. Brax, A.-C. Davis, and V. Vardanyan, Phys. Rev. D **97**, 124010 (2018).
[20] M. Ostrogradski, Mem. Ac. St. Petersbourg **VI**, 385 (1850).
[21] R. Woodard, Avoiding dark energy with 1/r modifications of gravity, in *The Invisible Universe: Dark Matter and Dark Energy*, edited by L. Papantonopoulos (Springer,

Berlin, 2007), pp. 403–433, DOI: 10.1007/978-3-540-71013-4_14.
[22] H. Weyl, Ann. Phys. (Leipzig) **59**, 101 (1919).
[23] R. Bach, Math. Z. **9**, 110 (1921).
[24] P. D. Mannheim and D. Kazanas, Astrophys. J. **342**, 635 (1989).
[25] P. D. Mannheim, Gen. Relativ. Gravit. **22**, 289 (1990).
[26] D. Kazanas and P. D. Mannheim, Astrophys. J. Suppl. Ser. **76**, 431 (1991).
[27] P. D. Mannheim and D. Kazanas, Phys. Rev. D **44**, 417 (1991).
[28] P. D. Mannheim, Astrophys. J. **419**, 150 (1993).
[29] P. D. Mannheim, Prog. Part. Nucl. Phys. **56**, 340 (2006).
[30] P. D. Mannheim, Gen. Relativ. Gravit. **43**, 703 (2011).
[31] G. 't Hooft, arXiv:1011.0061.
[32] G. 't Hooft, arXiv:1009.0669.
[33] G. 't Hooft, Found. Phys. **41**, 1829 (2011).
[34] P. D. Mannheim, Found. Phys. **42**, 388 (2012).
[35] J. Maldacena, arXiv:1105.5632.
[36] P. D. Mannheim and D. Kazanas, Astrophys. J. **342**, 635 (1989).
[37] P. D. Mannheim and D. Kazanas, Gen. Relativ. Gravit. **26**, 337 (1994).
[38] C. Deliduman, O. Kasikci, and B. Yapiskan, arXiv:1511.07731.
[39] P. D. Mannheim, Astrophys. J. **479**, 659 (1997).
[40] P. D. Mannheim and J. G. O'Brien, Phys. Rev. D **85**, 124020 (2012).
[41] P. D. Mannheim and J. G. O'Brien, J. Phys. Conf. Ser. **437**, 012002 (2013).
[42] P. D. Mannheim, Astrophys. J. **391**, 429 (1992).
[43] P. D. Mannheim, arXiv:astro-ph/9901219.
[44] A. Diaferio, L. Ostorero, and V. Cardone, J. Cosmol. Astropart. Phys. 10 (2011) 008.
[45] P. D. Mannheim, Phys. Rev. D **85**, 124008 (2012).
[46] C. Roberts, K. Horne, A. O. Hodson, and A. D. Leggat, arXiv:1711.10369.
[47] L. Knox and A. Kosowsky, arXiv:astro-ph/9311006.
[48] D. Elizondo and G. Yepes, Astrophys. J. **428**, 17 (1994).
[49] A. Edery and M. B. Paranjape, Phys. Rev. D **58**, 024011 (1998).
[50] A. Edery, A. A. Methot, and M. B. Paranjape, Gen. Relativ. Gravit. **33**, 2075 (2001).
[51] J. Sultana and D. Kazanas, Phys. Rev. D **81**, 127502 (2010).
[52] J. Sultana, D. Kazanas, and J. L. Said, Phys. Rev. D **86**, 084008 (2012).
[53] C. Cattani, M. Scalia, E. Laserra, I. Bochicchio, and K. K. Nandi, Phys. Rev. D **87**, 047503 (2013).
[54] Y.-K. Lim and Q.-h. Wang, Phys. Rev. D **95**, 024004 (2017).
[55] J. Sultana and D. Kazanas, Mon. Not. R. Astron. Soc. **466**, 4847 (2017).
[56] M. Campigotto, A. Diaferio, and L. Fatibene, arXiv:1712.03969.
[57] V. Perlick and C. Xu, Astrophys. J. **449**, 47 (1995).
[58] J. Wood and W. Moreau, arXiv:gr-qc/0102056.
[59] Y. S. Myung and T. Moon, arXiv:1407.0441.
[60] L. Fabbri and M. B. Paranjape, Phys. Rev. D **83**, 104046 (2011).
[61] R.-J. Yang, Phys. Lett. B **784**, 212 (2018).
[62] A. Stabile and S. Capozziello, arXiv:1501.02187.
[63] W. Nelson, J. Ochoa, and M. Sakellariadou, Phys. Rev. D **82**, 085021 (2010).
[64] W. Nelson, J. Ochoa, and M. Sakellariadou, Phys. Rev. Lett. **105**, 101602 (2010).
[65] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of GR* (John Wiley & Sons, Inc., New York, 1972).
[66] C. Lanczos, Ann. Math. **39**, 842 (1938).
[67] We have used simplified notation. The kinetic part of the fermionic energy-momentum tensor is given by $i\bar{\psi}\gamma^{\mu}(x)[\partial_{\mu}+\Gamma_{\mu}(x)]\psi - i\bar{\psi}[\overleftarrow{\partial_{\mu}}+\Gamma_{\mu}(x)]\gamma^{\mu}(x)\psi$ in order to be Hermitian.
[68] F. F. Faria, Adv. High Energy Phys. **2014**, 520259 (2014).
[69] F. F. Faria, arXiv:1410.5104.
[70] In [68] there is only one scalar field coupled to the gravitational part of the theory, whereas in [69] a second scalar field coupled to the fermionic matter sector is introduced. Nevertheless, after choosing a Weyl gauge for the first scalar field and after a spontaneous breaking of Weyl symmetry for the second scalar field, they take constant values throughout spacetime. Effectively, this corresponds to $\epsilon = +1$ in this work.
[71] I. Oda, Adv. Stud. Theor. Phys. **9**, 595 (2015).
[72] K. Horne, Mon. Not. R. Astron. Soc. **458**, 4122 (2016).
[73] É. É. Flanagan, Phys. Rev. D **74**, 023002 (2006).
[74] F. Sbisa, Eur. J. Phys. **36**, 015009 (2015).
[75] H.-J. Schmidt, Astron. Nachr. **306**, 67 (1985).
[76] H.-J. Schmidt, Astron. Nachr. **306**, 231 (1985).
[77] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012).
[78] H. van Dam and M. Veltman, Nucl. Phys. **B22**, 397 (1970).
[79] V. I. Zakharov, JETP Lett. **12**, 312 (1970).
[80] F. F. Faria, arXiv:1703.01318.
[81] P. Teyssandier, Classical Quantum Gravity **6**, 219 (1989).
[82] T. Biswas, T. Koivisto, and A. Mazumdar, arXiv:1302.0532.
[83] In [82] it is unclear which sign conventions have been used, and hence it is not apparent if CG or MCG is studied. Therefore it is not clear whether the signs in their Eq. (33) are in agreement with our result or not. The difference in the signs for the massless and massive parts appears nevertheless.
[84] P. D. Mannheim and A. Davidson, arXiv:hep-th/0001115.
[85] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[86] C. M. Bender and P. D. Mannheim, Phys. Rev. D **78**, 025022 (2008).
[87] C. M. Bender and P. D. Mannheim, J. Phys. A **41**, 304018 (2008).
[88] C. M. Bender and P. D. Mannheim, Phys. Rev. Lett. **100**, 110402 (2008).
[89] P. D. Mannheim, Phil. Trans. R. Soc. A **371**, 20120060 (2013).
[90] P. D. Mannheim, J. Phys. A **51**, 315302 (2018).
[91] P. D. Mannheim, Phys. Lett. B **753**, 288 (2016).
[92] A. Pais and G. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
[93] In [94] the quantization procedure of Mannheim and Bender is criticized. It is claimed that the PT-symmetric quantum mechanics of CG is at odds with the classical-to-quantum correspondence.

[94] M. Raidal and H. Veermäe, Nucl. Phys. **B916**, 607 (2017).
[95] K. Stelle, Phys. Rev. D **16**, 953 (1977).
[96] P. D. Mannheim, J. Phys. G **44**, 115003 (2017).
[97] A. Salvio and A. Strumia, J. High Energy Phys. 14 (2014) 80.
[98] K. Lazaridis *et al.*, Mon. Not. R. Astron. Soc. **400**, 805 (2009).
[99] L. Nicastro, A. Lyne, D. Lorimer, P. Harrison, M. Bailes, and B. Skidmore, Mon. Not. R. Astron. Soc. **273**, L68 (1995).
[100] P. J. Callanan, P. M. Garnavich, and D. Koester, Mon. Not. R. Astron. Soc. **298**, 207 (1998).
[101] P. D. Mannheim, Phys. Rev. D **75**, 124006 (2007).
[102] M. Hohmann, C. Pfeifer, M. Raidal, and H. Veermäe, arXiv:1802.02184.
[103] O. V. Barabash and Y. V. Shtanov, Phys. Rev. D **60**, 064008 (1999).
[104] O. Barabash and H. Pyatkovska, Ukr. J. Phys. **53**, 737 (2008).
[105] F. Faria, Eur. Phys. J. C **76**, 188 (2016).
[106] E. Adelberger, J. Gundlach, B. Heckel, S. Hoedl, and S. Schlamminger, Prog. Part. Nucl. Phys. **62**, 102 (2009).
[107] M. Maggiore, *Gravitational Waves: Vol 1, Theory and Experiments* (Oxford University Press, Oxford, UK, 2008).
[108] A. Stabile and G. Scelza, Phys. Rev. D **84**, 124023 (2011).
[109] R. Sanders, Astron. Astrophys. Rev. **2**, 1 (1990).
[110] P. Mishra and T. P. Singh, Phys. Rev. D **88**, 104036 (2013).
[111] A. Stabile and S. Capozziello, Phys. Rev. D **87**, 064002 (2013).