Consistent relativistic chiral kinetic theory: A derivation from on-shell effective field theory

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We formulate the on-shell effective field theory (OSEFT) in an arbitrary frame and study its reparametrization invariance, which ensures that it respects Lorentz symmetry. In this formulation, the OSEFT Lagrangian looks formally equivalent to the sum over lightlike velocities of soft collinear effective field theory in the Abelian limit, but differences remain in the scale of the gauge fields involved in the two effective theories. We then use the OSEFT Lagrangian expanded in inverse powers of the on-shell energy to derive how the classical transport equations for charged massless fermions are corrected by quantum effects, as derived from quantum field theory. We provide a formulation in a full covariant way and explain how the consistent form of the chiral anomaly equation can be recovered from our results. We also show how the side-jump transformation of the distribution function associated with massless charged fermions can be derived from the reparametrization invariance transformation rules of the OSEFT quantum fields. Finally, we discuss the differences in our results with respect to others found in the literature.

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I. INTRODUCTION

In this paper, we use the so-called on-shell effective field theory (OSEFT) [1–3] to provide a derivation of the transport equations obeyed by charged chiral fermions beyond the classical limit approximation.

A formulation of transport theory for chiral fermions has been developed in Refs. [4–7], starting with the action of a point particle modified by the Berry curvature, together with a modified Poisson bracket structure. Other alternative approaches to derive the same transport equation can be found in the literature [1,8–17].

The first derivation of chiral kinetic theory (CKT) from quantum field theory was made in Ref. [6] for systems at finite density and zero temperature, using the so-called high density effective field theory (HDET) [18]. OSEFT was actually proposed to provide a similar derivation that could be valid also in a thermal background, where antifermions are also relevant degrees of freedom. Regardless of the background, transport equations describe the propagation of on-shell quasiparticles, and therefore it seems natural to use for their derivation an effective field theory approach that describes only the propagation of on-shell degrees of freedom, as OSEFT, while off-shell modes are integrated out. Let us stress that the notion of an on-shell quasiparticle depends on the energy scales one is looking at in the system under consideration. It is well known that for plasmas at finite temperature T only the high energy modes of order T can be considered as quasiparticles and their evolution studied with classical transport equations [19–21], while the same picture does not apply to lower energy modes. To get corrections to the classical point-particle picture described above from quantum field theory, one simply has to study how the off-shell modes modify the evolution of the highly energetic modes. These corrections are taken into account in the OSEFT Lagrangian and expressed as operators of increasing dimension over powers of the on-shell energy scale so that these modifications can be described with the accuracy one desires. The OSEFT Lagrangian can then be used to derive how the classical

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transport picture is modified, by using for their derivation an increasing number of terms in the high energy expansion.

One of the advantages of our formulation is that it may allow us to derive transport equations in full covariant form and derive their properties under Lorentz transformations. While the initial proposals of CKT were not given in a covariant form, it was soon realized that it would have peculiar properties under Lorentz transformations [22,23], especially seen when formulating two-body collisions but also expressed in the so-called side-jump behavior of the distribution function of CKT, that expresses that it is frame dependent.

We present in this paper a derivation of CKT in a covariant way, as derived from OSEFT, and explain how the side-jump effects can be deduced from the same symmetries of that effective field theory. While previous formulations of OSEFT were given in the preferred frame of the thermal bath, we generalize it to an arbitrary frame, introducing the frame vector u^μ. The resulting OSEFT Lagrangian then looks formally equivalent to that corresponding to a sum over velocities of the so-called soft collinear effective field theory (SCET) [24–27], although there are some differences, as will be discussed in the following. We further study the reparametrization invariance of OSEFT, that ensures that our formalism is respectful of Lorentz invariance.

We compute both the vector current and axial current in the OSEFT, by taking functional derivatives to the action, and take these expressions to deduce the corresponding values in the transport framework, which requires a Wigner transformation of a two-point function, together with a gradient expansion. As very clearly explained in the review [28], such a definition can only lead to the consistent form of the chiral anomaly, rather than the covariant form. We check from our expressions that this is indeed the case.

Our final form of the relativistic chiral transport equation mainly differs from that introduced in Refs. [9,10,13], in pieces that may be subleading when considering effects close to thermal equilibrium, but that might be relevant for studies off equilibrium, and also in the gradient terms of the gauge fields. It also differs, when fixing the frame, with the chiral transport equation obtained from the modified form of the one-point particle action.

Our paper is organized as follows. In Sec. II, we formulate OSEFT in an arbitrary frame, introducing a frame vector and showing its formal equivalence with soft collinear effective field theory. In Sec. III, we study the reparametrization invariance of this effective field theory, a basic ingredient to show that it is respectful of Lorentz symmetry. In Sec. IV, we introduce the basic two-point function in the OSEFT that will be used to derive the basic set of transport equations. The main content of the paper is in Sec. V, with the derivation of the collisionless transport equation, first using the OSEFT variables in Sec. V A and then expressed in terms of the QED original variables in Sec. V B. In Sec. VI, we derive both the vector and axial current obtained in the OSEFT approach and check that they obey the consistent form of the quantum anomalies. In Sec. VII, we derive the side-jump transformation of the distribution function from the reparametrization invariance transformations of the OSEFT quantum fields. We conclude in Sec. VIII, where we summarize our main findings and give a possible interpretation of the origin of the discrepancy of our results with alternative approaches. In Appendix A, we give some details of our computations, while in Appendix B, we show how to obtain the chiral magnetic effect from our formulation.

We use natural units $\hbar = c = k_B = 1$ and metric conventions $g^{\mu\nu} = (1, -1, -1, -1)$. We also use boldface letters to denote 3-vectors.

II. OSEFT IN AN ARBITRARY FRAME AND SCET

Let us review the OSEFT as originally formulated [1,2], introducing the basic fields and notation. Let us recall that the propagation of an on-shell massless fermion is described by its energy p, with $p > 0$, and the lightlike 4-velocity $v^\mu = (1, \mathbf{v})$, where $\mathbf{v}$ is three-dimensional unit vector, and thus its 4-momentum is $p^\mu = p v^\mu$. However, for a fermion close to being on shell, its 4-momentum can be expressed as

$$q^\mu = p v^\mu + k^\mu, \tag{1}$$

where k^μ is the residual momentum ($k_\mu \ll p$), i.e., the part of the momentum which makes q^μ off shell. A similar decomposition of the momentum for almost on-shell antifermions can be done as follows,

$$q^\mu = -p \tilde{v}^\mu + k^\mu, \tag{2}$$

where $\tilde{v}^\mu = (1, -\mathbf{v})$.

The Dirac field can be written as

$$\begin{aligned}\psi_{v,\tilde{v}} = {}& e^{-ipv\cdot x}(P_v \chi_v(x) + P_{\tilde{v}} H^{(1)}_{\tilde{v}}(x)) \\ &+ e^{ip\tilde{v}\cdot x}(P_{\tilde{v}} \xi_{\tilde{v}}(x) + P_v H^{(2)}_v(x)),\end{aligned} \tag{3}$$

where the basic OSEFT quantum fields obey

$$P_v \chi_v = \chi_v, \qquad P_{\tilde{v}} \chi_v = 0, \tag{4}$$

$$P_{\tilde{v}} \xi_{\tilde{v}} = \xi_{\tilde{v}}, \qquad P_v \xi_{\tilde{v}} = 0 \tag{5}$$

and the particle/antiparticle projectors are expressed as

$$P_v = \frac{1}{2}\gamma \cdot v\,\gamma_0, \qquad P_{\tilde{v}} = \frac{1}{2}\gamma \cdot \tilde{v}\,\gamma_0. \tag{6}$$

It is possible to integrate out the $H^{(1,2)}$ fields of the QED Lagrangian [1], to have an effective theory for the fields χ_v and $\xi_{\tilde{v}}$ only.

If we assume that the physical phenomena we aim to describe are dominated by the contribution of on-shell particles, then the corresponding OSEFT Lagrangian can be written as a sum over the different values of the on-shell momenta as

$$\mathcal{L} = \sum_{p,\mathbf{v}} \mathcal{L}_{p,\mathbf{v}}, \tag{7}$$

where the precise meaning of the sum displayed in Eq. (7) is not needed at this stage (we will come back to this point later on; see also Ref. [2]), and

$$\begin{aligned} \mathcal{L}_{p,\mathbf{v}} &= \mathcal{L}_{p,v} + \tilde{\mathcal{L}}_{p,\tilde{v}} \\ &= \bar{\chi}_v(x)\left(iv\cdot D + i\not{D}_\perp \frac{1}{2p + i\tilde{v}\cdot D} i\not{D}_\perp\right)\gamma^0 \chi_v(x) \\ &\quad + \bar{\xi}_{\tilde{v}}(x)\left(i\tilde{v}\cdot D + i\not{D}_\perp \frac{1}{-2p + iv\cdot D} i\not{D}_\perp\right)\gamma^0 \xi_{\tilde{v}}(x), \end{aligned} \tag{8}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, $\not{D}_\perp = P_\perp^{\mu\nu}\gamma_\mu D_\nu$, and

$$P_\perp^{\mu\nu} = g^{\mu\nu} - \frac{1}{2}(v^\mu \tilde{v}^\nu + v^\nu \tilde{v}^\mu) \tag{9}$$

is minus the transverse projector to $\mathbf{v}$, written in covariant form. Note that with our conventions $k_\perp^2 = P_\perp^{\mu\nu} k_\mu k_\nu = -\mathbf{k}_\perp^2$. From now on, and as done in Ref. [2], whenever we write a tensor with the symbol $\perp$, it means that a transverse projector applies to all its Lorentz indices. If only the transverse projector is applied to one of the indices, we will write $\perp$ only affecting that index. Thus, $\sigma_\perp^{\mu\nu} = P_\perp^{\mu\alpha} P_\perp^{\nu\beta}\sigma_{\alpha\beta}$, while $\sigma^{\mu_\perp \nu} = P_\perp^{\mu\alpha} g^{\nu\beta}\sigma_{\alpha\beta}$.

In the original formulation of the OSEFT, a choice of frame was made [1,2]. The energies of the on-shell particles in Eq. (1) are measured in the same frame where, e.g., the thermal bath is defined. If we want to express the same OSEFT Lagrangian in an arbitrary frame, we will then have to introduce a timelike vector u^μ which defines that frame. Then, one could write all the above different equations simply by replacing

$$p \to u^\mu p_\mu \equiv E, \qquad \gamma_0 \to \gamma_\mu u^\mu. \tag{10}$$

With our specific choice of variables v^μ and $\tilde{v}^\mu$, then it is not difficult to see that

$$u^\mu = \frac{v^\mu + \tilde{v}^\mu}{2}. \tag{11}$$

Note that in OSEFT u^μ is not an independent vector, once v^μ and $\tilde{v}^\mu$ have been defined. While in the static frame we chose a particular definition of the vectors v^μ and $\tilde{v}^\mu$, which implicitly assumed that $u^\mu = (1,0,0,0)$, in an arbitrary frame, we will only ask that these lightlike vectors obey

$$v^2 = \tilde{v}^2 = 0, \qquad v\cdot\tilde{v} = 2. \tag{12}$$

Thus, $u\cdot v = 1$ and $u^2 = 1$ are automatically fulfilled.

In our formulation of the OSEFT in an arbitrary frame, we will sometimes use $\tilde{v}^\mu$, and sometimes we will use u^μ. The last option is convenient, as in kinetic theory it may appear also in the thermal equilibrium distribution associated with the massless particles.

As for the particle/antiparticle projectors in an arbitrary frame, we will write them as

$$P_v = \frac{1}{2}\not{v}\not{u} = \frac{1}{4}\not{v}\not{\tilde{v}} \tag{13}$$

$$P_{\tilde{v}} = \frac{1}{2}\not{\tilde{v}}\not{u} = \frac{1}{4}\not{\tilde{v}}\not{v}, \tag{14}$$

where we used that $\not{v}\not{v} = \not{\tilde{v}}\not{\tilde{v}} = 0$.

The OSEFT Lagrangian in a general frame is then written down as

$$\mathcal{L} = \sum_{E,v}(\mathcal{L}_{E,v} + \mathcal{L}_{-E,\tilde{v}}), \tag{15}$$

where

$$\begin{aligned} &\mathcal{L}_{E,v} + \mathcal{L}_{-E,\tilde{v}} \\ &= \bar{\chi}_v(x)\left(iv\cdot D + i\not{D}_\perp \frac{1}{2E + i\tilde{v}\cdot D} i\not{D}_\perp\right)\frac{\not{\tilde{v}}}{2}\chi_v(x) \\ &\quad + \bar{\xi}_{\tilde{v}}(x)\left(i\tilde{v}\cdot D + i\not{D}_\perp \frac{1}{-2E + iv\cdot D} i\not{D}_\perp\right)\frac{\not{v}}{2}\xi_{\tilde{v}}(x), \end{aligned} \tag{16}$$

where we have used that

$$\not{v}\chi_v = 0, \qquad \not{\tilde{v}}\xi_{\tilde{v}} = 0.$$

It is noteworthy that Eq. (16) formally looks similar to the Lagrangian of soft-collinear effective field theory [24–27]. The corresponding projectors Eqs. (13) and (14) are also those used in SCET. We note that the explicit forms of the OSEFT and SCET Lagrangians differ because of our different convention in defining the quantum fluctuating fields: in SCET, the exponential terms of Eq. (3) have been included in the quantum fields of the effective theory. We also explicitly separate the contribution of particles and antiparticles. Further, we recall that we are considering an effective field theory for QED, while SCET is an effective field theory for QCD.

After noticing the above formal similarities of SCET and OSEFT when the latter is formulated in an arbitrary frame, it has to be stressed that they are still different effective field theories. SCET was originally formulated to describe the physics associated with highly energetic jets in vacuum, and there are only two lightlike vectors in the theory, v^μ and $\tilde{v}^\mu$, fixed by the direction of the jet. In SCET, the covariant derivatives are associated with collinear and ultrasoft gauge fields. OSEFT was in principle developed to describe many body particle systems, close to thermal equilibrium, where one can consider having many on-shell particles and their propagation in the background of soft gauge fields. Thus, for a fixed value of the energy, there might be particles moving in all arbitrary (lightlike) directions, and a sum over v^μ is displayed in the final Lagrangian, which is absent in SCET. In OSEFT, the covariant derivatives we use mainly contain soft gauge fields.

OSEFT also uses a different notation, which makes clear that its main goal is to make an analytical expansion in powers of the inverse of the on-shell energy $1/E$. At finite temperature and/or density, we will obtain different expressions multiplied by a particle distribution function. After integration over momenta, this expansion on the inverse of the on-shell energy will turn out to give an expansion in powers of the inverse of the temperature and/or chemical potential [2,3].

After mentioning the explicit similarities and differences of these two effective field theories, it is possible to use some of the results obtained in SCET to learn about some properties of OSEFT, such as that of reparametrization invariance, which will be discussed in the following section.

III. REPARAMETRIZATION INVARIANCE OF OSEFT

Reparametrization invariance (RI) is the symmetry associated with the ambiguity of the decomposition of the momentum q^μ performed in Eq. (1). If $M^{\mu\nu}$ defines the six Lorentz generators of $SO(3,1)$, the decomposition of Eq. (1) suggests an apparent breaking of five Lorentz generators, namely, $\{v_\mu M^{\mu\nu}, u_\mu M^{\mu\nu}\}$ or, equivalently, $\{v_\mu M^{\mu\nu}, \tilde{v}_\mu M^{\mu\nu}\}$. However, it is possible to show that the OSEFT Lagrangian is RI invariant, which is equivalent to saying that is Lorentz invariant. Let us stress that this reduces to the study of the RI of SCET for every sector of the theory defined by the vectors v_μ and $\tilde{v}_\mu$, something which has been extensively investigated [29]. The fact that the covariant derivatives displayed in SCET and OSEFT contain gauge fields of different scales does not, however, affect the proof of RI, which turns out to be formally equivalent in the two effective field theories.

Let us review how this effectively works. The Dirac field defined in Eq. (3) should be the same independent of the choice of the parameters used to define the effective field theory; thus,

$$\psi_{v,\tilde{v}}(x) = \psi'_{v',\tilde{v}'}(x). \tag{17}$$

As in SCET, we will see that the effective field theory action remains invariant under infinitesimal changes of the vectors v^μ and $\tilde{v}^\mu$ that preserve their basic properties expressed in Eq. (12). It is possible to show that the OSEFT Lagrangian is invariant under the following symmetries,

$$(\mathrm{I})\begin{cases} v^\mu \to v^\mu + \lambda_\perp^\mu \\ \tilde{v}^\mu \to \tilde{v}^\mu \end{cases} \qquad (\mathrm{II})\begin{cases} v^\mu \to v^\mu \\ \tilde{v}^\mu \to \tilde{v}^\mu + \epsilon_\perp^\mu \end{cases} \qquad (\mathrm{III})\begin{cases} v^\mu \to (1+\alpha)v^\mu \\ \tilde{v}^\mu \to (1-\alpha)\tilde{v}^\mu, \end{cases} \tag{18}$$

where $\{\lambda_\perp^\mu, \epsilon_\perp^\mu, \alpha\}$ are five infinitesimal parameters and $v\cdot\lambda_\perp = v\cdot\epsilon_\perp = \tilde{v}\cdot\lambda_\perp = \tilde{v}\cdot\epsilon_\perp = 0$. Please note that the transformation rule of the vector u^μ can be deduced from Eq. (11).

Just to have a flavor of the meaning of the above symmetries, let us imagine one fixes the values of the two lightlike vectors as $v^\mu = (1,0,0,1)$ and $\tilde{v}^\mu = (1,0,0,-1)$. Then, apparently, there are five broken generators in the OSEFT, which are $Q_1^\pm = J_1 \pm K_2$, $Q_2^\pm = J_2 \pm K_1$, and K_3, where J_i and K_i are the generators of rotations and boosts, respectively. Then, type I refers to the combined action of an infinitesimal boost in the $x(y)$ direction and a rotation around the $y(x)$ axis, such that $\tilde{v}^\mu$ is left invariant, with generators (Q_1^-, Q_2^+). Type II transformations are similar, but (Q_1^+, Q_2^-) leave v^μ invariant, while type III is a boost along the direction 3, K_3.

It is also worth it to note that the generators (Q_1^+, Q_2^-, J_3) obey the $SE(2)$ Lie algebra, that is the symmetry group of the two-dimensional Euclidean plane. They correspond to what is known as the Wigner little group associated with the vector $p^\mu = p v^\mu$ [30]; see also Refs. [31–33]. Similarly, the generators (Q_1^-, Q_2^+, J_3) correspond to Wigner's little group associated with $p^\mu = -p\tilde{v}^\mu$ (antiparticles). As discussed in Ref. [30], these Wigner translations are associated with shifts of the trajectory of finite wave packets of massless particles proportional to the particle's helicity.

It is possible to check easily that our Lagrangian is invariant under the above three RI transformations [29], which formally is equivalent to saying that it is Lorentz invariant. Let us discuss these briefly, as they are the same RI symmetries of SCET. We will mainly focus now on what our different notation implies. We will concentrate in the following in the particle sector, as for antiparticles things work analogously after trivial changes (namely, $u\cdot p \to -u\cdot p$ and $v^\mu \leftrightarrow \tilde{v}^\mu$). We will also see that the type II symmetry will allow us to generate the side jumps that were discussed in the framework of chiral kinetic theory in Ref. [22]. This point will be discussed in Sec. VII.

Let us first start with type I symmetry. The change in the vector v^μ implies a relabeling of what is called on-shell and residual parts of the momentum defined in Eq. (1). After a type I symmetry, the on-shell part and residual momenta change as

$$(u\cdot p)v^\mu \to (u\cdot p)v^\mu + \frac{1}{2}(\lambda_\perp \cdot p)v^\mu + (u\cdot p)\lambda_\perp^\mu, \tag{19}$$

$$k^\mu \to k^\mu - \frac{1}{2}(\lambda_\perp \cdot p)v^\mu - (u\cdot p)\lambda_\perp^\mu, \tag{20}$$

respectively. This implies that under a type I transformation the covariant derivatives acting on the fluctuating fields also transform.

Type II symmetry implies that the new on-shell and residual momenta change as

$$(u\cdot p)v^\mu \to (u\cdot p)v^\mu + \frac{1}{2}(p\cdot \epsilon_\perp)v^\mu, \tag{21}$$

$$k^\mu \to k^\mu - \frac{1}{2}(p\cdot\epsilon_\perp)v^\mu, \tag{22}$$

while the type III transformation leads to the changes

$$(u\cdot p)v^\mu \to (u\cdot p)v^\mu(1+2\alpha) - \alpha(\tilde{v}\cdot p)v^\mu, \tag{23}$$

$$k^\mu \to k^\mu - 2\alpha E v^\mu + \alpha(\tilde{v}\cdot p)v^\mu \tag{24}$$

in the on-shell and residual momenta, respectively.

In Table I, we summarize the transformation rules under all three types of transformations.

The OSEFT Lagrangian is invariant under these three RI transformations [29]:

$$\delta_{(\mathrm{I})}\mathcal{L}_{E,v} = \delta_{(\mathrm{II})}\mathcal{L}_{E,v} = \delta_{(\mathrm{III})}\mathcal{L}_{E,v} = 0. \tag{25}$$

In explicit computations of Feynman diagrams, or derivations of transport equations, we will expand the Lagrangian in power series of $1/E$. While Eq. (25) is exact to all orders in a $1/E$ expansion, in a perturbative analysis in $1/E$, it is important to note that RI invariance implies that different terms in the expansion are connected by symmetry. This comes from the fact that the covariant derivatives, or the fields, transform with terms proportional to E.

For completeness, we will also mention other discrete symmetries of the OSEFT. Under parity, charge conjugation, and time reversal, the basic OSEFT fields transform as

$$\chi_v(x) \to \gamma_0 \chi_{\tilde{v}}(\tilde{x}_P), \qquad \xi_{\tilde{v}}(x) \to \gamma_0 \xi_v(\tilde{x}_P) \tag{26}$$

$$\chi_v(x) \to -i\gamma^2 \xi_v^*(x), \qquad \xi_{\tilde{v}}(x) \to -i\gamma^2 \chi_{\tilde{v}}^*(x) \tag{27}$$

$$\chi_v(x) \to -\gamma^1\gamma^3 \chi_{\tilde{v}}(-\tilde{x}_T), \quad \xi_{\tilde{v}}(x) \to -\gamma^1\gamma^3 \xi_v(-\tilde{x}_T), \tag{28}$$

respectively, where if $x^\mu = (x_0, \mathbf{x})$, then $\tilde{x}_P^\mu = (x_0, -\mathbf{x})$, and $\tilde{x}_T^\mu = (-x_0, \mathbf{x})$.

There is also a spin symmetry, which is not a $SU(2)$ symmetry but a $U(1)$ symmetry, which corresponds to helicity [33].

IV. WIGNER FUNCTION IN THE OSEFT

We focus our attention here on the basic Wigner function used in the following part of the paper for the derivation of the transport equations from OSEFT. We will use the Keldysh-Schwinger formulation, allowing the time variables to take complex values, and define the two-point Green's functions of the OSEFT on the closed time-path contour. These are represented by a 2×2 matrix,

TABLE I. Transformation rules in OSEFT under RI transformations of types I, II, and III.

	Type I	Type II	Type III
v^μ	$v^\mu + \lambda_\perp^\mu$	v^μ	$v^\mu(1+\alpha)$
$\tilde{v}^\mu$	$\tilde{v}^\mu$	$\tilde{v}^\mu + \epsilon_\perp^\mu$	$\tilde{v}^\mu(1-\alpha)$
u^μ	$u^\mu + \frac{\lambda_\perp^\mu}{2}$	$u^\mu + \frac{\epsilon_\perp^\mu}{2}$	$u^\mu(1-\alpha) + \alpha v^\mu$
E	$E + \frac{1}{2}\lambda^\perp \cdot p$	$E + \frac{1}{2}(\epsilon_\perp \cdot p)$	$E(1+\alpha) - \alpha(\tilde{v}\cdot p)$
D_μ	$D_\mu + iE\lambda_\mu^\perp + \frac{i}{2}(\lambda_\perp\cdot p)v_\mu$	$D_\mu + \frac{i}{2}(\epsilon_\perp \cdot p)v_\mu$	$D_\mu + 2i\alpha E v_\mu - i\alpha(\tilde{v}\cdot p)v_\mu$
$(v\cdot D)$	$(v\cdot D) + \lambda^\perp \cdot D^\perp$	$(v\cdot D)$	$(v\cdot D)(1+\alpha)$
$(\tilde{v}\cdot D)$	$(\tilde{v}\cdot D) + i\lambda^\perp \cdot p$	$(\tilde{v}\cdot D) + i\epsilon_\perp \cdot p + \epsilon_\perp \cdot D_\perp$	$(\tilde{v}\cdot D)(1-\alpha) + 4iE\alpha - 2i\alpha(\tilde{v}\cdot p)$
$D_\mu^\perp$	$D_\mu^\perp - \frac{\lambda_\mu^\perp}{2}(\tilde{v}\cdot D) - \frac{\tilde{v}_\mu}{2}\lambda^\perp \cdot D^\perp + iE\lambda_\mu^\perp$	$D_\mu^\perp - \frac{\epsilon_\mu^\perp}{2}(v\cdot D) - \frac{v_\mu}{2}\epsilon^\perp \cdot D^\perp$	$D_\mu^\perp$
P_v	$P_v + \frac{1}{4}\not{\lambda}_\perp \not{\tilde{v}}$	$P_v - \frac{1}{4}\not{\epsilon}_\perp \not{v}$	P_v
$\chi_v(x)$	$(1 + \frac{1}{4}\not{\lambda}_\perp \not{\tilde{v}})\chi_v(x)$	$(1 + \frac{1}{2}\not{\epsilon}_\perp \frac{1}{2E + i\tilde{v}\cdot D} i\not{D}_\perp)\chi_v(x)$	$\chi_v(x)$

$$S_{E,v}(x,y) = \begin{pmatrix} S^c_{E,v}(x,y) & S^<_{E,v}(x,y) \\ S^>_{E,v}(x,y) & S^a_{E,v}(x,y) \end{pmatrix} = \begin{pmatrix} \langle T\chi_v(x)\bar{\chi}_v(y)\rangle & -\langle\bar{\chi}_v(y)\chi_v(x)\rangle \\ \langle\chi_v(x)\bar{\chi}_v(y)\rangle & \langle\tilde{T}\chi_v(x)\bar{\chi}_v(y)\rangle \end{pmatrix}, \quad (29)$$

where T denotes time ordering and $\tilde{T}$ denotes anti–time ordering.

We will focus on one of the entries only, namely, $S^<_{E,v}$, as this two-point function depends only on medium effects, while the diagonal entries of Eq. (29) do also contain vacuum contributions. We will drop the superindex $^<$ in what follows to make the notation lighter.

A similar two-point function can be introduced for the antiparticle quantum fluctuations. From now on, we will focus on the particle's sector, as the antiparticle's transport equations may be derived similarly, and only involve some few changes to the particle's derivation ($E \to -E$, and $v^\mu \leftrightarrow \tilde{v}^\mu$). However, we will have to take into account both degrees of freedom when computing physical observables.

In order to make contact with transport theory, one defines the (gauge-covariantly modified) Wigner transform of the above two-point functions. If $X = \frac{1}{2}(x+y)$ and $s = x-y$ define the center of mass and relative coordinates, respectively, then

$$S_{E,v}(X,k) = \int d^4s e^{ik\cdot s} U\left(X, X+\frac{s}{2}\right) \times S_{E,v}\left(X+\frac{s}{2}, X-\frac{s}{2}\right) U\left(X-\frac{s}{2}, X\right), \quad (30)$$

where U is the Wilson line,

$$U(x,y) = P\exp\left[-ie\int_\gamma dx^\mu A_\mu(x)\right], \quad (31)$$

and P denotes path-ordering along the path γ from x to y. Using that

$$U\left(X, X+\frac{s}{2}\right) U\left(X-\frac{s}{2}, X\right) \approx e^{ies\cdot A(X)}, \quad (32)$$

then one can show that the introduction of the Wilson line allows us to define the Wigner function in terms of the kinetic momentum $\bar{k}^\mu = k^\mu - eA^\mu(X)$. From now on, we will denote the kinetic momentum without the bar to keep the notation light.

We will focus on the construction of the transport equation associated with the vector and axial vector components of the above two-point function and define

$$\mathrm{Tr}(\gamma^\mu S_{E,v}(X,k)) = \sum_{\chi=\pm}\mathrm{Tr}(\gamma^\mu P_\chi \gamma_\nu J^{\nu,\chi}_{E,v}(X,k)) = 2\sum_{\chi=\pm} J^{\mu,\chi}_{E,v}(X,k), \quad (33)$$

where χ is an index that indicates the helicity/chirality of the particle and

$$P_\chi = \frac{(1+\chi\gamma_5)}{2} \quad (34)$$

is a chirality projector.

Now, simply by using that

$$g^{\mu\nu} = P^{\mu\nu}_\perp + \frac{1}{2}(v^\mu\tilde{v}^\nu + v^\nu\tilde{v}^\mu), \quad (35)$$

one can decompose

$$J^{\mu,\chi}_{E,v}(X,k) = v^\mu G^\chi_{E,v}(X,k) + \tilde{v}^\mu H^\chi_{E,v}(X,k) + J^{\mu,\chi}_{(E,v),\perp}(X,k). \quad (36)$$

Further, for the constraint $\not{\tilde{v}}\chi_v = 0$ for particles, one can deduce that $H^\chi_{E,v} = 0$. One can also show that $\langle\bar{\chi}_v(x)\gamma^\perp_\mu\chi_v(x)\rangle = 0$, and thus, $J^{\mu,\chi}_{(E,v),\perp}(X,k) = 0$.

We will thus write our transport equations in terms of the two-point function

$$G_{E,v}(x,y) = \left\langle\bar{\chi}_v(y)\frac{\not{\tilde{v}}}{2}\chi_v(x)\right\rangle \quad (37)$$

and its (gauge-covariantly modified) Wigner transform.

A basic ingredient to derive classical or semiclassical transport equations is to perform the gradient expansion, which assumes

$$\partial_X \ll \partial_s. \quad (38)$$

By doing this, we will consistently neglect gradients of the gauge fields. This does not mean that we are considering only situations of constant background fields but rather that their variation is consistently neglected, as we will not take into account second order derivatives on X of the two-point Green function.

V. DERIVATION OF THE COLLISIONLESS TRANSPORT EQUATION

A. Computation using the OSEFT variables

For our derivation, we substantially follow the approach of Ref. [6], where a chiral transport equation valid for Fermi systems at $T = 0$ was derived from HDET [18]. Actually, one of the motivations to develop OSEFT in Ref. [1] was to extend the validity of the same derivation at finite temperature, where also antiparticles have to be taken into account. While in a system at finite density and vanishing temperature the Fermi sea provides a natural privileged frame, our derivation will be valid for an arbitrary frame. With some minor technical differences (the use of Dirac rather than Weyl fermions, use of local field redefinitions, and consideration of nonhomogeneous distribution functions), we will find the final form of the chiral transport equation in an arbitrary frame, respectful of reparametrization invariance, and therefore Lorentz invariance. We will point out an important difference from Ref. [6] in our final results.

We start by considering the equations obeyed by the two-point Green's functions, as follows from the OSEFT Lagrangian. To derive the collisionless transport equation, it is enough to consider the tree level equations. These can be expressed as

$$\sum_{n=0}(\mathcal{O}_x^{(n)})S_{E,v}(x,y)=0 \tag{39}$$

and

$$\sum_{n=0}S_{E,v}(x,y)(\mathcal{O}_y^{(n)})^\dagger=0, \tag{40}$$

where from the OSEFT Lagrangian we can extract [34]

$$\mathcal{O}_x^{(0)}=iv\cdot D\frac{\not{\tilde{v}}}{2}, \tag{41}$$

$$\mathcal{O}_x^{(1)}=-\frac{1}{2E}\left(D_\perp^2+\frac{e}{2}\sigma_\perp^{\mu\nu}F_{\mu\nu}\right)\frac{\not{\tilde{v}}}{2}, \tag{42}$$

$$\begin{aligned}\mathcal{O}_x^{(2)}&=-\frac{1}{4E^2}i\not{D}_\perp(i\tilde{v}\cdot D)i\not{D}_\perp\frac{\not{\tilde{v}}}{2}\\&=\frac{1}{8E^2}([\not{D}_\perp,[i\tilde{v}\cdot D,\not{D}_\perp]]+\{(\not{D}_\perp)^2,i\tilde{v}\cdot D\})\frac{\not{\tilde{v}}}{2},\end{aligned} \tag{43}$$

and we limit our study to operators up to $1/E^2$ in the energy expansion.

It is convenient to introduce local field redefinitions to eliminate the temporal derivative in Eq. (43), as in Ref. [2], as these simplify quite a lot the computations at higher orders [35]. Local field redefinitions might not be respectful of RI if one considers off-shell quantities, but they will not affect the result of on-shell quantities. Thus, after the field redefinition

$$\chi_v\to\chi_v'=\left(1+\frac{\not{D}_\perp^2}{8E^2}\right)\chi_v, \tag{44}$$

the second order differential operator becomes

$$\begin{aligned}\mathcal{O}_{x,\mathrm{LFR}}^{(2)}=\frac{1}{8E^2}\Bigg(&[\not{D}_\perp,[i\tilde{v}\cdot D,\not{D}_\perp]]\\&-\left\{D_\perp^2+\frac{e}{2}\sigma_\perp^{\mu\nu}F_{\mu\nu},(iv\cdot D-i\tilde{v}\cdot D)\right\}\Bigg)\frac{\not{\tilde{v}}}{2}\end{aligned} \tag{45}$$

We have checked that these two forms of the second-order Lagrangian lead to an equivalent form of the (on-shell) transport equation.

We now combine the sum and difference of Eqs. (39) and (40) and compute their Wigner transform. For every order in the energy expansion, we define

$$\begin{aligned}I_\pm^{(n)}=\int d^4s e^{ik\cdot s}(&\mathcal{O}_x^{(n)}U(x,y)S_{E,v}(x,y)\\&\pm S_{E,v}(x,y)U(x,y)\mathcal{O}_y^{(n)\dagger});\end{aligned} \tag{46}$$

however, note that these are matrix equations in the Dirac subspace of the particles. In order to recover the transport equation, we trace the above equations,

$$\mathrm{Tr}(I_\pm^{(n)})=\sum_{\chi=\pm}I_{\chi,\pm}^{(n)}. \tag{47}$$

We can also derive separate equations for each helicity by multiplying by the appropriate chiral projector.

Furthermore, from Eqs. (33) and (36), one can write

$$G_{E,v}^\chi(X,k)=\frac{1}{2}(\tilde{v}\cdot J_{E,v}^\chi)(X,k). \tag{48}$$

We leave for the Appendix A some details of the computations and present here our final results. For $n=0$,

$$I_{\chi,+}^{(0)}=4k\cdot vG_{E,v}^\chi(X,k), \tag{49}$$

$$I_{\chi,-}^{(0)}=2iv_\mu[\partial_X^\mu-eF^{\mu\nu}(X)\partial_{k,\nu}]G_{E,v}^\chi(X,k); \tag{50}$$

for $n=1$,

$$I_{\chi,+}^{(1)}=\frac{2}{E}\left(k_\perp^2-\frac{e\chi}{4}\epsilon^{\alpha\beta\mu\nu}\tilde{v}_\beta v_\alpha F_{\mu\nu}^\perp\right)G_{E,v}^\chi(X,k), \tag{51}$$

$$I_{\chi,-}^{(1)}=2\frac{i}{E}k_\perp^\mu[\partial_{X,\mu}-eF_{\mu\nu}\partial_k^\nu]G_{E,v}^\chi(X,k); \tag{52}$$

while for $n=2$, one gets

$$\begin{aligned}I_{\chi,+}^{(2)}=-\frac{2}{E^2}\Bigg(&\left[k_\perp^2-\frac{e\chi}{4}\epsilon^{\alpha\beta\mu_\perp\nu_\perp}\tilde{v}_\beta v_\alpha F_{\mu\nu}\right]\frac{\tilde{v}\cdot k-v\cdot k}{2}\\&+\frac{e\chi}{4}\epsilon^{\alpha\beta\mu_\perp\nu_\perp}\tilde{v}_\beta v_\alpha F_{\nu\rho}\tilde{v}^\rho k_\mu\Bigg)G_{E,v}^\chi(X,k),\end{aligned} \tag{53}$$

and

$$\begin{aligned}I_{\chi,-}^{(2)}=\frac{2}{E^2}\Bigg(&-k_\perp^\mu\frac{\tilde{v}\cdot k-v\cdot k}{2}+\frac{1}{4}\left[k_\perp^2-\frac{e\chi}{4}\epsilon^{\alpha\beta\delta\gamma}\tilde{v}_\beta v_\alpha F_{\delta\gamma}^\perp\right]\\&\times(v^\mu-\tilde{v}^\mu)-\frac{e\chi}{8}\epsilon^{\alpha\beta\mu_\perp\nu_\perp}\tilde{v}_\beta v_\alpha F_{\nu\rho}\tilde{v}^\rho\Bigg)\\&\times i[\partial_{X,\mu}-eF_\mu^\sigma(X)\partial_{k,\sigma}]G_{E,v}^\chi(X,k).\end{aligned} \tag{54}$$

We can check that, when computed in the static frame defined by fixing the frame vector as $u^\mu = (1, 0, 0, 0)$, and using Eq. (11), our results agree with those computed from HDET in Ref. [6] if we replace the chemical potential μ by the energy E, except in what follows. With the local field redefinition, the factor multiplying the time derivative in the transport equation is 1, while without it, one gets a nontrivial factor. We have checked that the same equation is obtained if we normalize the transport equation of Ref. [6] so as to obtain the same normalisation of the time derivative term. We, however, disagree in the numerical factor of the piece proportional to $F_{\nu\rho}\tilde{v}^\rho$ in Eqs. (53) and (54), in what it is apparently an algebraic mistake. The numerical factors found above turn out to be essential to deriving both the proper form of the dispersion relation and the consistent form of the anomaly equation.

B. Going backward to the original variables

Having derived the relevant equations in terms of the OSEFT variables, let us now go back and express them in terms of the original momenta of the full theory.

1. Dispersion relation

The dispersion relation is fixed after imposing

$$I^{(0)}_{\chi,+} + I^{(1)}_{\chi,+} + I^{(2)}_{\chi,+} = 0, \tag{55}$$

which suggests that the Wigner function can be written as

$$G^\chi_{E,v}(X,k) = 2\pi\delta(K^\chi) f^\chi_{E,v}(X,k), \tag{56}$$

where $f^\chi_{E,v}(X,k)$ is the particle distribution function, and we have introduced a (2π) factor in order to reproduce, to leading order, the expected density in a QED plasma. We keep the labels E and v in the distribution function, as this function will depend on the on-shell variables; see e.g., Ref. [2], where it was explicitly seen that close to equilibrium the on-shell energy acts as a sort of chemical potential for the residual momentum. The function K^χ fixes then the dispersion relation, to the order considered, and can be read from the $I_{\chi,+}$ functions. In particular, up to order $n = 2$,

$$\begin{aligned} K^\chi = 2k\cdot v &+ \frac{1}{E}\left(k_\perp^2 - \frac{e\chi}{4}\epsilon^{\alpha\beta\mu\nu}\tilde{v}_\beta v_\alpha F^\perp_{\mu\nu}\right) \\ &- \frac{1}{E^2}\left(\left[k_\perp^2 - \frac{e\chi}{4}\epsilon^{\alpha\beta\mu_\perp\nu_\perp}\tilde{v}_\beta v_\alpha F_{\mu\nu}\right]\frac{\tilde{v}\cdot k - v\cdot k}{2}\right. \\ &\left. + \frac{e\chi}{4}\epsilon^{\alpha\beta\mu_\perp\nu_\perp}\tilde{v}_\beta v_\alpha F_{\nu\rho}\tilde{v}^\rho k_\mu\right). \end{aligned} \tag{57}$$

Note that we could replace $\epsilon^{\alpha\beta\mu\nu}\tilde{v}_\beta v_\alpha = 2\epsilon^{\alpha\beta\mu\nu}u_\beta v_\alpha$ in the above expression. The on-shell constraint can be solved to different orders in the energy expansion. To leading order it is simply

$$2k\cdot v = 0, \tag{58}$$

while at the following order,

$$2k\cdot v + \frac{1}{E}\left(k_\perp^2 - \frac{e\chi}{4}\epsilon^{\alpha\beta\mu\nu}\tilde{v}_\beta v_\alpha F^\perp_{\mu\nu}\right) = 0, \tag{59}$$

showing that $(v\cdot k)$ turns out to be subleading in the $1/E$ expansion when taken on shell.

It turns out convenient to express the on-shell constraint in terms of the original momentum q^μ. Then, one can check that it leads to the constraint

$$q^2 - eS^{\mu\nu}_\chi F_{\mu\nu} = 0, \tag{60}$$

where $S^{\mu\nu}_\chi$ is the spin tensor defined as

$$S^{\mu\nu}_\chi = \chi\frac{\epsilon^{\alpha\beta\mu\nu}u_\beta q_\alpha}{2(q\cdot u)}, \tag{61}$$

if solved up to order $1/E^2$ in the OSEFT variables. To see this, we can express Eq. (60) in terms of on-shell and residual momenta. Using

$$E_q \equiv q\cdot u = E + k\cdot u, \tag{62}$$

and also that we can write for the residual momentum

$$\begin{aligned} k^\mu &= k^\mu_\perp + \frac{1}{2}(v\cdot k)\tilde{v}^\mu + \frac{1}{2}(\tilde{v}\cdot k)v^\mu, \\ k^2 &= k_\perp^2 + (v\cdot k)(\tilde{v}\cdot k), \end{aligned} \tag{63}$$

then the spin tensor can be written as

$$S^{\mu\nu}_\chi = \frac{\chi}{2}\epsilon^{\alpha\beta\mu\nu}u_\beta\left(v_\alpha + \frac{k^\perp_\alpha}{E}\right) + \mathcal{O}\left(\frac{1}{E^2}\right). \tag{64}$$

We can then easily obtain

$$\begin{aligned} q^2 - eS^{\mu\nu}_\chi F_{\mu\nu} = 2E&\left[v\cdot k + \frac{1}{2E}(k_\perp^2 - eS^{\mu\nu}_\chi F_{\mu\nu})\left(1 - \frac{(\tilde{v}\cdot k)}{2E}\right)\right] \\ &+ \mathcal{O}\left(\frac{1}{E^2}\right), \end{aligned} \tag{65}$$

where in the last expression we used Eq. (59) and the fact that we are considering expansions in powers of $1/E$. Furthermore, employing once again the decomposition in Eq. (35) both for k_α and $F_{\mu\nu}$, we can express $S^{\mu\nu}_\chi F_{\mu\nu}$ in terms of the OSEFT variables

$$\begin{aligned} S^{\mu\nu}_\chi F_{\mu\nu} = \frac{\chi}{2}\epsilon^{\alpha\beta\mu\nu}u_\beta\left(v_\alpha + \frac{k^\perp_\alpha}{E}\right)&(F^\perp_{\mu\nu} + F_{\mu_\perp\rho}\tilde{v}^\rho v_\nu \\ &+ F_{\mu_\perp\rho}v^\rho\tilde{v}_\nu) + \mathcal{O}\left(\frac{1}{E^2}\right). \end{aligned} \tag{66}$$

Finally, we can replace the above vector u_β by $\tilde{v}_\beta/2$, the difference being a higher $1/E$ effect. This can be checked by noticing that $v^\mu A_\mu \ll \tilde{v}^\mu A_\mu$. Note that the condition Eq. (58) involves the kinetic, rather than canonical, momentum, which implies that not all the vector gauge field components are equally relevant in the $1/E$ expansion.

Under these conditions, one can then check that Eq. (65) becomes exactly EK^χ. Equation (55) thus enforces the on-shell condition Eq. (60), as anticipated.

Thus, in returning to the original variables, we will identify, to order $n = 2$ accuracy in the $1/E$ expansion,

$$\begin{aligned} G^\chi_{E,v}(X,k) &= (2\pi)\delta(K^\chi) f^\chi_{E,v}(X,k) \\ &= (2\pi)E\delta(EK^\chi) f^\chi_{E,v}(X,k) \\ &= \pi E \delta_+(Q^\chi) f^\chi(X,q), \end{aligned} \tag{67}$$

where we have defined

$$\delta_+(Q^\chi) = \delta(q^2 - eS^{\mu\nu}_\chi F_{\mu\nu}) 2\theta(E_q). \tag{68}$$

When the Wigner function is expressed in terms of the original variables, there is still an E dependence. In explicit computations of physical parameters, such as the vector current (see Sec. VI), this E dependence disappears when one finally expresses the whole current in terms of the original variables.

2. Transport equation

The transport equation is obtained from

$$I^{(0)}_{\chi,-} + I^{(1)}_{\chi,-} + I^{(2)}_{\chi,-} = 0. \tag{69}$$

We will express the transport equation in terms of the original momentum q^μ. Let us define the vector

$$v^q_\mu \equiv \frac{q^\mu}{E_q} = \frac{E}{E_q} v^\mu + \frac{k^\mu}{E_q}, \tag{70}$$

which satisfies $u \cdot v_q = 1$. In the absence of gauge fields, this vector can be written as

$$v^q_\mu = v^\mu + \frac{k^\mu - v^\mu(k\cdot u)}{E} - (k\cdot u)\frac{k^\mu - v^\mu(k\cdot u)}{E^2} + \cdots \tag{71}$$

If we further consider the on-shell condition at lowest order $v \cdot k = 0$, then

$$k^\mu - v^\mu(k\cdot u)|_{\text{o.s.}} = k^\mu_\perp, \tag{72}$$

and it is not difficult to realize that

$$v^q_\mu|_{\text{o.s.}} = v^\mu + \frac{k^\mu_\perp}{E} - (k\cdot\tilde{v})\frac{k^\mu_\perp}{2E^2} + \frac{v^\mu - \tilde{v}^\mu}{4E^2}k^2_\perp + \mathcal{O}\left(\frac{1}{E^3}\right). \tag{73}$$

If we now we include the gauge fields, after using Eq. (59), we then get

$$\begin{aligned} v^q_\mu|_{\text{o.s.}} = v^\mu &+ \frac{k^\mu_\perp}{E} - (k\cdot\tilde{v})\frac{k^\mu_\perp}{2E^2} \\ &+ \frac{v^\mu - \tilde{v}^\mu}{4E^2}\left(k^2_\perp - \frac{e\chi}{4}\epsilon^{\alpha\beta\mu\nu}\tilde{v}_\beta v_\alpha F^\perp_{\mu\nu}\right) + \mathcal{O}\left(\frac{1}{E^3}\right), \end{aligned} \tag{74}$$

which is the combination that appears in the $I_{\chi,-}$ functions.

If we define

$$\Delta^\mu \equiv \partial^\mu_X - eF^{\mu\nu}(X)\partial_{q,\nu}, \tag{75}$$

one can write the transport equation in terms of the original variables as

$$\left(v^q_\mu - \frac{e}{2E_q^2}S^{\mu\nu}_\chi F_{\nu\rho}(2u^\rho - v^\rho_q)\right)\Delta_\mu f(X,q)\delta_+(Q) = 0, \tag{76}$$

where we have used that $\tilde{v}^\rho = 2u^\rho - v^\rho_q$ in the last term only. In the absence of the $1/E_q$ corrections, Eq. (76) corresponds to a classical transport equation of a charged fermion in the collisionless limit [37].

After taking into account the on-shell condition, Eq. (76) is similar, but not identical, to the one proposed in Ref. [10], see also Refs. [9,13], if we identify their frame vector n^μ with our u^μ. For homogeneous backgrounds, Eq. (76) contains a term, the piece proportional to $S^{\mu\nu}_\chi F_{\nu\rho} v^\rho_q$, which is absent in Eq. (11) of Ref. [10]. It could be eliminated by introducing a new term in the OSEFT Lagrangian at order $1/E^2$, namely, the same that appears in Eq. (43), but changing the $(\tilde{v}\cdot D)$ by $(v \cdot D)$. However, this could only be done at the expense of breaking reparametrization invariance and, ultimately, Lorentz invariance.

For nonhomogeneous backgrounds, Eq. (11) of Ref. [10] kept some gradient terms of the gauge fields and frame vector. The gradient expansion used to reach to the above transport equation was made by neglecting gradients of the electromagnetic fields (see Appendix A), which would otherwise naturally emerge in the computations of the functions $I_{\chi,-}$; thus, not all the gradient terms were kept in Refs. [9,13], and in a close to thermal equilibrium situation, it might be nonconsistent to keep those gradient terms while neglecting $\partial^2_X G$.

Let us consider now our covariant relativistic equation and write it in the frame $u^\mu = (1,0,0,0)$. In this frame, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ijk}B^k$, and also

$$S_\chi^{\mu\nu} \to S_\chi^{ij} = \chi \frac{\epsilon^{ijk} q^k}{2q_0}, \qquad S_\chi^{\mu\nu} F_{\mu\nu} = -\chi \mathbf{B} \cdot \frac{\mathbf{q}}{q_0}. \tag{77}$$

After considering the on-shell condition, it is not difficult to arrive at

$$\left(\Delta_0 + \hat{\mathbf{q}}^i \left(1 + e\chi \frac{\mathbf{B}\cdot\hat{\mathbf{q}}}{2q^2}\right)\Delta_i + e\chi \frac{\epsilon^{ijk} E^j \hat{q}^k - B^i_{\perp,\mathbf{q}}}{4q^2} \Delta_i\right) \times f^\chi(X, \mathbf{q}) = 0, \tag{78}$$

where we have defined $B^i_{\perp,\mathbf{q}} \equiv B^i - \hat{\mathbf{q}}^i (\mathbf{B}\cdot\hat{\mathbf{q}})$. This equation differs from Eq. (13) of Ref. [9], which for homogeneous backgrounds reads

$$\left(\Delta_0 + \hat{\mathbf{q}}^i \left(1 + e\chi \frac{\mathbf{B}\cdot\hat{\mathbf{q}}}{2q^2}\right)\Delta_i + e\chi \frac{\epsilon^{ijk} E^j \hat{q}^k}{2q^2} \Delta_i\right) f^\chi(X, \mathbf{q}) = 0. \tag{79}$$

Eq. (78) also differs from the transport equation described in Sec. IIB of Ref. [6], although that equation leads to the covariant chiral anomaly equation, while ours leads to the consistent form of the chiral anomaly equation, as we discuss in the following section.

VI. CONSISTENT CURRENT AND CHIRAL ANOMALY EQUATION

In this section, we compute both the consistent electromagnetic and chiral currents. For the computation of the latter, the best option is to introduce an artificial chiral gauge field A^5_μ and an artificial gauge field tensor $F^5_{\mu\nu}$, which are finally sent to zero, as advocated in Ref. [28] and in Ref. [14], for example. Thus, we assume that the original QED Lagrangian reads

$$\mathcal{L} = \sum_{E,v} (\bar{\psi}_{v,\tilde{v}}(x) i\gamma^\mu (\partial_\mu + ieA_\mu + ie\gamma_5 A^5_\mu)\psi_{v,\tilde{v}}(x)). \tag{80}$$

One can proceed with the same derivation of the OSEFT Lagrangian in the presence of the chiral field. After introducing the chiral projectors, it is not difficult to realize that all our equations remain valid if we replace

$$A_\mu \to A_\mu + \chi A^5_\mu, \qquad F_{\mu\nu} \to F_{\mu\nu} + \chi F^5_{\mu\nu}, \tag{81}$$

in all our final formulas, in agreement with the prescription of Ref. [14].

The electromagnetic and chiral currents are obtained from the OSEFT action, simply by performing the functional derivatives

$$j^\mu(x) = -\frac{\delta \mathcal{S}}{\delta A_\mu(x)}, \qquad j^5_\mu(x) = -\frac{\delta \mathcal{S}}{\delta A^5_\mu(x)}, \tag{82}$$

respectively. Alternatively, one could start with the QED currents and plug the explicit expression of the Dirac fields in Eq. (3) to finally write the current in terms of the OSEFT fields. For example, considering only the contribution of the particles,

$$\bar{\psi}_{v,\tilde{v}}(x)\gamma^\mu \psi_{v,\tilde{v}}(x) \to (\bar{\chi}_v(x) + \bar{H}^{(1)}_{\tilde{v}}(x))\gamma^\mu (\chi_v(x) + H^{(1)}_{\tilde{v}}(x)) \equiv j^\mu(x). \tag{83}$$

Using the expression of the $H^{(1)}_{\tilde{v}}$ of Ref. [1] generalized to an arbitrary frame, we find

$$\begin{aligned} j^\mu(x) = {} & v^\mu \bar{\chi}_v \frac{\tilde{\not{v}}}{2}\chi_v + \frac{1}{2E}\left(\bar{\chi}_v \gamma^\mu_\perp i\not{D}_\perp \frac{\tilde{\not{v}}}{2}\chi_v + \bar{\chi}_v (i\overleftarrow{\not{D}})_\perp \gamma^\mu_\perp \frac{\tilde{\not{v}}}{2}\chi_v\right) \\ & - \frac{\tilde{v}^\mu}{4E^2}\left(\bar{\chi}_v (i\overleftarrow{\not{D}})_\perp (i\not{D})_\perp \frac{\tilde{\not{v}}}{2}\chi_v\right) + \frac{v^\mu}{8E^2}\left(\bar{\chi}_v (\overleftarrow{\not{D}})^2_\perp \frac{\tilde{\not{v}}}{2}\chi_v + \bar{\chi}_v (\not{D})^2_\perp \frac{\tilde{\not{v}}}{2}\chi_v\right) \\ & - \frac{1}{4E^2}\left(\bar{\chi}_v (i\tilde{v}\cdot D)\gamma^\mu_\perp (i\not{D})_\perp \frac{\tilde{\not{v}}}{2}\chi_v + \bar{\chi}_v (i\overleftarrow{\not{D}})_\perp (i\tilde{v}\cdot \overleftarrow{D})\gamma^\mu_\perp \frac{\tilde{\not{v}}}{2}\chi_v\right) + \mathcal{O}\left(\frac{1}{E^3}\right), \end{aligned} \tag{84}$$

where we have to take into account the local field redefinition, Eq. (44), so as to compute the current in the same way as the corrections to the transport equations. A completely analogous computation can be carried out for the chiral current.

At leading order in the energy expansion, one can immediately express the current in terms of the two-point function. After a Wigner transform, one finds

$$j^\mu_{(0)}(X) = e\sum_{E,v,\chi} \int \frac{d^4k}{(2\pi)^4} v^\mu 2G^\chi_{E,v}(X,k). \tag{85}$$

We can use now the explicit form of the Wigner function at order $n = 0$; see Eq. (56). If we further make the identification [2,38]

$$\sum_{E,v} \int \frac{d^4k}{(2\pi)^4} \equiv \int \frac{d^4q}{(2\pi)^4}, \tag{86}$$

then, at leading order, the current is expressed as

$$j^\mu_{(0)}(X) = e\sum_{\chi=\pm} \int \frac{d^4q}{(2\pi)^3} 2\theta(E_q)\delta(q^2) q^\mu f^\chi(X,q), \tag{87}$$

where we have approximated $Ev^\mu \approx q^\mu$ at leading order, and it is understood that the on-shell condition is taken to leading order, thus, without the gauge field contribution. Similarly, the axial current at leading order reads

$$j^\mu_{5,(0)}(X) = e\sum_{\chi=\pm}\chi \int \frac{d^4q}{(2\pi)^3} 2\theta(E_q)\delta(q^2)q^\mu f^\chi(X,q). \qquad (88)$$

At the following orders in the energy expansion, and due to the presence of derivative terms in the explicit expression of the current, a point-splitting regularization is needed. This means that we take the field $\bar{\chi}_v$ at the value y. We then perform the (gauge-covariantly modified) Wigner transform, together with the derivative expansion, to finally take the limit $y \to x$. Note that this point-splitting regularization is only needed to properly define the Wigner transform (see, e.g., the scalar QED example explained in Ref. [39] for the proper definition of the current) and not to regulate ultraviolet problems, which are absent in the two-point function we are studying.

If one considers corrections up to order $n = 2$, then the vector current reads

$$\begin{aligned} j^\mu_{(2)}(X) = e\sum_{E,v,\chi}\int \frac{d^4k}{(2\pi)^4}\Bigg\{&\left(v^\mu + \frac{k^\mu_\perp}{E} - (k\cdot\tilde{v})\frac{k^\mu_\perp}{2E^2} + \frac{v^\mu - \tilde{v}^\mu}{4E^2}\left(k^2_\perp - \frac{e\chi}{4}\epsilon^{\alpha\beta\mu\nu}\tilde{v}_\beta v_\alpha F^\perp_{\mu\nu}\right)\right).\\ &- \frac{\chi}{4E}\left(\epsilon^{\mu\nu\alpha\beta}\tilde{v}_\alpha v_\beta - \frac{(k\cdot\tilde{v})}{2E}\epsilon^{\mu\nu\alpha\beta}\tilde{v}_\alpha v_\beta\right)[\partial^X_\nu - eF_{\nu\sigma}\partial^\sigma_k] + \frac{\chi}{8E^2}\epsilon^{\mu\nu\alpha\beta}\tilde{v}_\alpha v_\beta k_\nu \tilde{v}^\rho[\partial^X_\rho - eF_{\rho\sigma}\partial^\sigma_k]\\ &+ \frac{e\chi}{8E^2}\epsilon^{\mu\rho\alpha\beta}\tilde{v}_\alpha v_\beta F_{\rho\sigma}\tilde{v}^\sigma\Bigg\}2G^\chi_{E,v}(X,k), \end{aligned} \qquad (89)$$

which, if converted to the original momentum, reads

$$j^\mu_{(2)}(X) = e\sum_{\chi=\pm}\int \frac{d^4q}{(2\pi)^3}\left\{q^\mu + S^{\mu\nu}_\chi \Delta_\nu - \frac{e}{2E_q}S^{\mu\nu}_\chi F_{\nu\rho}(2u^\rho - v^\rho_q)\right\} f^\chi(X,q)\delta_+(Q^\chi). \qquad (90)$$

For the axial current, we get the same expression, but the whole integral is multiplied by χ.

In order to get the complete current, the antiparticle contribution has to be added. As mentioned in Sec. IV, this can be recovered from the OSEFT particle contribution, Eq. (89), by simply replacing $v^\mu \leftrightarrow \tilde{v}^\mu$ and $E \to -E$.

Let us consider the current associated with one single value of the chirality. Using the transport equation (76) and the antisymmetry of the spin tensor, it is not difficult to deduce

$$\partial_\mu j^\mu_\chi(X) = e^2\int \frac{d^4q}{(2\pi)^3}\left\{q^\mu + S^{\mu\nu}_\chi\Delta_\nu - \frac{e}{2E_q}S^{\mu\nu}_\chi F_{\nu\rho}(2u^\rho - v^\rho_q)\right\} \times F_{\mu\lambda}\frac{\partial}{\partial q^\lambda}(f^\chi\delta_+(Q^\chi)). \qquad (91)$$

To deduce the form of the chiral anomaly, we will now consider the frame $u^\mu = (1,0,0,0)$, as then the analysis simplifies quite a lot. We will also consider the situation where, to leading order, the distribution function corresponds to a thermal distribution function, with a chemical potential that depends on the chirality: that is, there is a fermion chiral imbalance in the system. The proof, however, can also be extended to distribution functions which, when the on-shell condition to leading order is considered, are parity invariant. One can express the integral on the rhs of Eq. (91), after taking into account the on-shell condition, as a surface integral. As the distribution function vanishes for $|\mathbf{q}| \to \infty$, the only nonvanishing contribution arises for low values of the momenta, where the quasiparticle picture breaks down. We proceed as in Ref. [5] and Refs. [1,8], and define a sphere centered in $|\mathbf{q}| = 0$ of radius R and then compute the only nonvanishing surface integral

$$\begin{aligned} \partial_\mu j^\mu_\chi(X) &= -e^2\chi\lim_{R\to 0}\left(\int\frac{d\mathbf{S}_R}{(2\pi)^3}\cdot\mathbf{E}\frac{\hat{\mathbf{q}}\cdot\mathbf{B}}{4R^2}f^\chi(|\mathbf{q}|=R) - \int\frac{d\mathbf{S}_R}{(2\pi)^3}\cdot\frac{\hat{\mathbf{q}}}{4R^2}\mathbf{E}\cdot\mathbf{B}f^\chi(|\mathbf{q}|=R)\right)\\ &= e^2\chi\frac{\mathbf{E}\cdot\mathbf{B}}{2\pi^2}\frac{1}{6}f^\chi(|\mathbf{q}|=0). \end{aligned} \qquad (92)$$

At this point, we should consider the contribution of all the chiralities, of both fermions and antifermions so as to obtain the full complete contribution to the axial and vector currents. We thus assume the following fermion and antifermion distribution functions,

$$f^\chi(|\mathbf{q}|) = \frac{1}{e^{(|\mathbf{q}|-\mu_\chi)/T}+1}, \qquad \tilde{f}^\chi(|\mathbf{q}|) = \frac{1}{e^{(|\mathbf{q}|+\mu_\chi)/T}+1}, \qquad (93)$$

respectively, to obtain the nonconservation of the chiral current

$$\partial_\mu \mathcal{J}_5^\mu(X) = \frac{1}{3}\frac{e^2}{2\pi^2}(\mathbf{E}\cdot\mathbf{B} + \mathbf{E}_5\cdot\mathbf{B}_5). \tag{94}$$

The vector current also has a quantum anomaly also in the presence of chiral gauge fields

$$\partial_\mu \mathcal{J}^\mu(X) = \frac{1}{3}\frac{e^2}{2\pi^2}(\mathbf{E_5}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{B}_5). \tag{95}$$

Eq. (94) gives account of the consistent form of the chiral anomaly equation, rather than its covariant form. We refer the reader to the excellent review [28] that gives very clear explanations about these two different forms of the quantum anomaly. After defining our currents as functional derivatives of the action, one cannot get anything else than the consistent currents. Unfortunately, the vector current is also nonconserved. It is possible to add the so-called Bardeen counterterms [40] to the quantum action

$$e^2\int d^4x\epsilon^{\mu\nu\rho\lambda}A_\mu A_\nu^5(c_1F_{\rho\lambda} + c_2F^5_{\rho\lambda}), \tag{96}$$

with the choice $c_1 = \frac{1}{12\pi^2}$ and $c_2 = 0$, and then one can get a vector conserved current [28].

Previous approaches to CKT have shown to provide both the covariant currents and also the covariant form of the chiral anomaly [1–3]; see also Ref. [14]. One can relate the consistent and covariant currents by adding Chern-Simons currents [28].

VII. SIDE JUMPS DERIVED FROM REPARAMETRIZATION INVARIANCE OF THE OSEFT

Once we know how the fields of the OSEFT behave under the three types of RI transformations, we can deduce how the different two-point functions behave under the same transformations. Then, after performing the (gauge-covariantly modified) Wigner transform and a gradient expansion, we can deduce how the distribution function behaves under the same sort of transformations.

It is actually easy to show that under the type I and type III symmetries of RI the distribution function in the OSEFT remains invariant. For example, under type I symmetry, the basic two-point function transforms as (see Table I)

$$\begin{aligned}&\left\langle\bar\chi_v(y)\frac{\tilde{v}\!\!\!/}{2}\chi_v(x)\right\rangle' \\ &\to \left\langle\bar\chi_v(y)\left(1+\frac{1}{4}\tilde{v}\!\!\!/\lambda\!\!\!/_\perp\right)\frac{\tilde{v}\!\!\!/}{2}\left(1+\frac{1}{4}\lambda\!\!\!/_\perp\tilde{v}\!\!\!/\right)\chi_v(x)\right\rangle \\ &= \left\langle\bar\chi_v(y)\frac{\tilde{v}\!\!\!/}{2}\chi_v(x)\right\rangle,\end{aligned} \tag{97}$$

where we have used that $\lambda\!\!\!/_\perp\tilde{v}\!\!\!/ = -\tilde{v}\!\!\!/\lambda\!\!\!/_\perp$, and $\tilde{v}\!\!\!/\tilde{v}\!\!\!/ = 0$. It then follows that

$$(f^\chi_{E,v}(X,k))' = f^\chi_{E,v}(X,k) \tag{98}$$

under a type I transformation. Similarly, it is possible to show that the distribution function does not change under a type III transformation.

The Green function (37) used in our derivation of the transport equation has, however, a nontrivial transformation under type II symmetry. Using the transformation rules of Table I, we obtain

$$\begin{aligned}\left\langle\bar\chi_v(y)\frac{\tilde{v}\!\!\!/}{2}\chi_v(x)\right\rangle' &\to \left\langle\bar\chi_v(y)\frac{\tilde{v}\!\!\!/+\epsilon\!\!/_\perp}{2}\chi_v(x)\right\rangle \\ &\quad+\left\langle\bar\chi_v(y)\left(\frac{(i\overleftarrow{D\!\!\!/}_{\perp,y})^\dagger\epsilon\!\!/_\perp^\dagger}{2E}\right)\frac{\tilde{v}\!\!\!/}{2}\chi_v(x)\right\rangle \\ &\quad+\left\langle\bar\chi_v(y)\frac{\tilde{v}\!\!\!/}{2}\left(\frac{1}{2}\frac{\epsilon\!\!/_\perp iD\!\!\!/_{\perp,x}}{2E}\right)\chi_v(x)\right\rangle \\ &\quad+\mathcal{O}\left(\frac{1}{E^2}\right).\end{aligned} \tag{99}$$

In OSEFT, $\langle\bar\chi_v(y)\gamma_\perp^\mu\chi_v(x)\rangle = 0$. After the Wigner transform, together with the gradient expansion, we end up with

$$\begin{aligned}(G^\chi_{E,v}(X,k))' &\to G^\chi_{E,v}(X,k) - \frac{1}{2E}k_\perp\cdot\epsilon_\perp G^\chi_{E,v}(X,k) \\ &\quad-\frac{\chi}{E}\epsilon^{\mu_\perp\nu_\perp\alpha\beta}v_\alpha\tilde{v}_\beta\epsilon_\nu^\perp(\partial_\mu^X - eF_{\mu\lambda}\partial_k^\lambda)G^\chi_{E,v}(X,k).\end{aligned} \tag{100}$$

Taking into account the definition of the two-point function at order $1/E$ involves the current density that might be computed [see the integrand of Eq. (89) at order $1/E$] as

$$G^\chi_{E,v}(X,k) = \frac{1}{2}\tilde{v}_\mu\cdot\left(v^\mu + \frac{k_\perp^\mu}{E} + \cdots\right)(2\pi)f^\chi_{E,v}(X,k)\delta_+(K^\chi); \tag{101}$$

this implies that the distribution function should change as

$$\begin{aligned}(f^\chi_{E,v}(X,k))' &\to f^\chi_{E,v}(X,k) - \frac{\chi}{E}\epsilon^{\mu_\perp\nu_\perp\alpha\beta}v_\alpha\tilde{v}_\beta\epsilon_\nu^\perp(\partial_\mu^X - eF_{\mu\lambda}\partial_k^\lambda) \\ &\quad\times f^\chi_{E,v}(X,k),\end{aligned} \tag{102}$$

under a type II transformation.

In terms of the original variables, one then gets

$$(f^\chi(X,q))' \to f^\chi(X,q) - \frac{1}{E_q}S_\chi^{\mu\nu}\epsilon_\nu^\perp\Delta_\mu f^\chi(X,q) + \mathcal{O}\left(\epsilon_\perp^2,\frac{1}{E_q^2}\right). \tag{103}$$

Taking into account that $\epsilon_\perp^\mu/2 = u'^\mu - u^\mu$, we see that Eq. (103) agrees with the infinitesimal form of the side-jump transformation first discussed in Ref. [23] in the absence of gauge fields, later generalized in the presence of the gauge fields in Ref. [9].

VIII. DISCUSSION

We have derived from OSEFT the corrections to the classical transport equations associated with on-shell massless charged fermions and antifermions. We have seen how from the proposed equations one can derive the consistent form of the chiral anomaly equation when considering a chiral imbalance system in thermal equilibrium. Our formulation turns out to be the proper generalization of the HDET approach to chiral transport theory of Ref. [6], but valid also for finite temperature systems and formulated in an arbitrary frame. The study of reparametrization invariance of the theory allows us to claim that the results are consistent with Lorentz symmetry, even if the kinetic equation depends on a frame vector. We have also deduced the side jumps of the distribution function of the theory from the transformation rule under RI of the OSEFT quantum fields.

Let us insist that when we consider the frame vector as $u^\mu = (1, \mathbf{0})$ our equations almost agree with those of Ref. [6], except in a couple of factors, in what apparently was an algebraic mistake. It is, however, important to stress that the transport equation obtained either in Ref. [6] or in this paper do not match exactly with the transport equation in Sec. IIB of Ref. [6], which were obtained starting with a corrected form of the classical point-particle action, with modified Poisson brackets. This starting point can be justified by performing a Foldy-Wouthuysen diagonalization of the quantum Dirac Hamiltonian, as seen in Ref. [1]. However, the same exact form of the transport equation is not obtained if the starting point is a quantum field theory. Let us stress that in such a formulation one obtains the covariant form of the chiral anomaly, as the chiral current is not defined by performing a functional derivative of an action, but from the equation obeyed by the current in the transport approach.

The question remains whether there can be more than one possible transport equation describing the same system equally well. The Foldy-Wouthuysen diagonalization used in Ref. [1] suggests that the starting quantum fields used there or those used in our OSEFT approach are not the same beyond the classical limit approximation. Thus, probably it is not so surprising that one does not end up with the same exact form of the corresponding kinetic equations, while the two approaches give an equivalent description of the system.

Probably more surprising are the discrepancies we obtained from the results of Refs. [9,10,13], obtained from massless QED, assuming homogenous gauge field backgrounds. OSEFT only helps in organizing the quantum field theory computation at large energies, as it has already been checked in the computation of Feynman diagrams at high T [2,41]. We cannot comment on the possible origin of these discrepancies, although it seems that the approach should also lead to the consistent form of the chiral anomaly, rather than its covariant form, as claimed in Ref. [10].

Let us, however, stress that discrepancies of our results with others published in the literature only appear at order $n = 2$ in the energy expansion both in the transport equation and the current. Let us mention that, since the chiral magnetic effect as well as other chiral transport effects appear already at order $n = 1$, our formulation gives the same description as that of other formulations (see Appendix B for the computation of the chiral magnetic effect).

While in this manuscript we have focused our attention to the collisionless form of the transport equation, a much more challenging task is to derive the collision terms from OSEFT, such that the Lorentz symmetry is respected and the side jumps are properly described. This will be the subject of a different project.

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APPENDIX A: DERIVATION OF THE $I_{\chi,\pm}$ FUNCTIONS

We provide in this Appendix some details of the computation of the $I_{\chi,\pm}$ functions. We take here $e = 1$ for simplicity.

We start from the equation of motion for quantum fields χ_v,

$$(\mathcal{O}_x^{(0)} + \mathcal{O}_x^{(1)} + \mathcal{O}_x^{(2)})\chi_v(x) = 0, \tag{A1}$$

and similarly its Hermitian conjugate for y. By adding and subtracting them, we can build equations for the two-point function. For each piece, we isolate the different possible Dirac structures, so we write

$$\mathcal{O}_x^{(n)} = (\alpha_x^{(n)} + \beta_{x,\mu\nu}^{(n)}\sigma_\perp^{\mu\nu})\frac{\tilde{\not{v}}}{2}; \tag{A2}$$

then, taking the trace of Eq. (46), one gets

$$\mathrm{Tr}(I_{\pm}^{(n)}) = \int d^4 s e^{ik\cdot s}\left\{(\alpha_x^{(n)} \pm \alpha_y^{(n)*})\mathrm{Tr}\left[\frac{\not{v}}{2}S_{E,v}(x,y)\right] + (\beta_{x,\mu\nu}^{(n)} \pm \beta_{y,\mu\nu}^{(n)*})\mathrm{Tr}\left[\sigma^{\mu\nu}\frac{\not{v}}{2}S_{E,v}(x,y)\right]\right\}. \quad (A3)$$

For the α and β coefficients, we find (after neglecting terms of higher order in the gradient expansion like $\partial_\alpha^X F_{\mu\nu}$)

$$\alpha^{(0)} = iv\cdot D, \qquad \beta_{\mu\nu}^{(0)} = 0, \quad (A4)$$

$$\alpha^{(1)} = -\frac{1}{2E}D_\perp^2, \qquad \beta_{\mu\nu}^{(1)} = -\frac{1}{4E}F_{\mu\nu}, \quad (A5)$$

$$\alpha^{(2)} = \frac{1}{4E^2}(v^\alpha - \tilde{v}^\alpha)(F_{\mu\alpha}D^\mu - iD_\alpha D_\perp^2), \qquad \beta_{\mu\nu}^{(2)} = \frac{i}{4E^2}\left(F_{\mu\alpha}\tilde{v}^\alpha D_\nu - \frac{1}{2}F_{\mu\nu}(v\cdot D - \tilde{v}\cdot D)\right). \quad (A6)$$

We now perform the change of variables to the center of mass and relative coordinates X, s. The recurring combinations will be

$$D_\alpha^x - (D_\alpha^y)^* = 2(\partial_\alpha^s + iA_\alpha(X)), \qquad D_\alpha^x + (D_\alpha^y)^* = \partial_\alpha^X + is_\beta\partial^\beta A_\alpha(X), \quad (A7)$$

together with

$$(D_\perp^x)^2 + ((D_\perp^y)^*)^2 = 2(\partial_X\cdot\partial_s + i(\partial_X\cdot A(X) + A(X)\cdot\partial_X) + is^\beta\partial_\beta^X A^\alpha(X)(\partial_\alpha^s + iA_\alpha(X))), \quad (A8)$$

$$(D_\perp^x)^2 + ((D_\perp^y)^*)^2 = 2(\partial_s^2 + 2iA(X)\cdot\partial_s - A(X)^2). \quad (A9)$$

We also use that

$$\mathrm{Tr}\left[\frac{\not{v}}{2}S_{E,v}\right] = 2\sum_{\chi=\pm}G_{E,v}^\chi, \qquad \mathrm{Tr}\left[\sigma^{\mu\nu}\frac{\not{v}}{2}S_{E,v}\right] = -\sum_{\chi=\pm}\chi\epsilon^{\mu\nu\alpha\rho}\tilde{v}_\alpha J_{(E,v),\rho}^\chi, \quad (A10)$$

where G and J are defined in Eqs. (36) and (37), respectively.

For an example, we can work out the lowest order function. If here k^μ denotes the canonical momentum, then

$$\begin{aligned} I_+^{(0)} &= \int d^4 s e^{ik\cdot s} iv\cdot(D_x - D_y^*)\sum_{\chi=\pm}2G_{E,v}^\chi(X,s)e^{-iAs} \\ &= \int d^4 s e^{ik\cdot s} iv\cdot 2(-ik + iA(X))\sum_{\chi=\pm}2G_{E,v}^\chi(X,s)e^{-iAs} \\ &= 4(v\cdot\bar{k})\int d^4 s e^{i\bar{k}\cdot s}\sum_{\chi=\pm}G_{E,v}^\chi(X,s) = 4(v\cdot\bar{k})\sum_{\chi=\pm}G_{E,v}^\chi(X,\bar{k}), \end{aligned} \quad (A11)$$

where now $\bar{k}^\mu = k^\mu - A^\mu$ is the canonical momentum.

APPENDIX B: CHIRAL MAGNETIC EFFECT

In this Appendix, we briefly show how from our formulation one can reproduce the chiral magnetic effect. We start from the current Eq. (90) and focus on its spatial components in the local rest frame $u^\mu = (1,0,0,0)$. After performing the q_0 integration, we get

$$j^i(X) = e\sum_{\chi=\pm}\int\frac{d^3q}{(2\pi)^3}\left(\frac{q^i}{E_q} + \frac{S_\chi^{ij}\Delta_j}{E_q} - \frac{e}{2E_q^2}S_\chi^{ij}F_{j\sigma}\tilde{v}^\sigma\right)f^\chi(X,q)\Big|_{q_0=E_q}, \quad (B1)$$

with the dispersion relation in this frame given by

$$q_0 = E_q = |\mathbf{q}|\left(1 - e\chi \frac{\mathbf{B}\cdot\hat{\mathbf{q}}}{2|\mathbf{q}|^2}\right). \tag{B2}$$

We now expand the distribution function using the dispersion relation and assume we are in equilibrium so we can use the standard Fermi-Dirac expressions:

$$f^\chi(X,q)|_{q_0=E_q} = f^\chi(|\mathbf{q}|) - e\chi \frac{\mathbf{B}\cdot\hat{\mathbf{q}}}{2|\mathbf{q}|}\frac{df^\chi(|\mathbf{q}|)}{d|\mathbf{q}|},$$
$$f^\chi(|\mathbf{q}|) = \frac{1}{1+e^{(|\mathbf{q}|-\mu_\chi)/T}}; \tag{B3}$$

this in turn eliminates all terms containing spatial derivatives, and keeping only the leading terms in $1/|\mathbf{q}|$, we are left with

$$j^i(X) = e\sum_{\chi=\pm}\int\frac{d^3q}{(2\pi)^3}\left[\left(\frac{q^i}{|\mathbf{q}|} - e\epsilon^{jkl}\frac{S_\chi^{ij}}{|\mathbf{q}|}B^l\frac{\partial}{\partial q^k}\right)f^\chi(|\mathbf{q}|) - e\frac{q^i}{|\mathbf{q}|}\chi\frac{\mathbf{B}\cdot\hat{\mathbf{q}}}{2|\mathbf{q}|}\frac{df^\chi(|\mathbf{q}|)}{d|\mathbf{q}|}\right]. \tag{B4}$$

After an integration by parts and performing angular integration, we finally arrive at

$$j^i(X) = -\frac{e^2}{4\pi^2}B^i\sum_{\chi=\pm}\chi\int d|\mathbf{q}||\mathbf{q}|\frac{df^\chi(|\mathbf{q}|)}{d|\mathbf{q}|} = e^2\frac{\mu_5}{4\pi^2}B^i, \tag{B5}$$

where $\mu_5 = \mu_1 - \mu_{-1}$, which is exactly the expected result for the chiral magnetic effect [42–44]; see also Ref. [8].

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