Acoustic analog of gravitational wave

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(Received 30 April 2018; revised manuscript received 18 August 2018; published 24 September 2018)

We explore nonlinear perturbations in different static fluid systems. We find that the equations, corresponding to the perturbation of the integrals of motion, i.e., Bernoulli's constant and the mass flow rate, satisfy the massless scalar field equation in a time dependent acoustic metric. When one is interested up to the second order behavior of the perturbations, the emergent time dependent acoustic metric of the system, derived from the massless scalar field equations of the perturbations of the integrals of motion, has some astounding similarities with the metric describing gravitational wave in Minkowski spacetime.

DOI: 10.1103/PhysRevD.98.064049

I. INTRODUCTION

Detection of gravitational waves [1] is not only one of the greatest achievement of this century in physics but it is also an another confirmation of the general theory of relativity [2]. Unruh's pioneering work [3] shows that the linear perturbation of velocity potential in an inviscid irrotational fluid medium behaves like a massless scalar field propagating in a curved spacetime. Several features of a classical black hole can be mimicked in different fluid systems [4]. There are also some works which show that instead of linear perturbation of velocity potential, one can work with linear perturbation of the integrals of motion of the fluid equations for irrotational inviscid medium, i.e., Bernoulli's constant and mass flow rate [5–10]. However it is evident that to mimic the Minkowski spacetime, the fluid medium has to be static, i.e., the background velocity of the fluid medium has to be zero everywhere and in such a medium, the propagation of linear perturbation of velocity potential would satisfy the massless scalar field equation in the acoustic analog of Minkowski spacetime.

References [11–13] study nonlinear perturbations in a moving fluid medium, more specifically, in an accreting medium, i.e., there is a position dependence of velocity. We consider apparently simpler fluid systems where there is no background velocity in the medium. The medium is static with respect to the indeterminable absolute space [14] or the medium has a uniform velocity with respect to the absolute space. Due to the Galilean relativity principle, all the inertial frames are equivalent in nature. Therefore for simplicity, we consider a reference frame where the medium is static and Newton's laws hold. Hence the emergent spacetime is analogous to Minkowski spacetime when one works with linear perturbations. Here we have shown that the emergent spacetime corresponding to the massless scalar field equation satisfied by the perturbation of the integrals of motion gives time dependent spacetime metric in general. In the case of the nonlinear wave equation, the acoustic metric is time dependent and when the expressions of the perturbations are expanded up to the second order, i.e., in the weak nonlinear limit, the emergent spacetime metric is very similar to the spacetime metric describing gravitational wave propagation in Minkowski spacetime. In our system, the background medium is static and fluid density and velocity have no time dependence. Therefore, if one now analyzes the flow around this solution by linearizing the density and velocity and assuming the flow to be barotropic and irrotational in nature just like Unruh did [3], the emergent spacetime will be an acoustic analog of Minkowski spacetime. There would be no time dependence in the metric. In Unruh's formulation [3], the fluid quantities are perturbed around a known solution of the fluid equations. If the known solution does not have explicit time dependence, the acoustic metric obtained through the linear perturbation in velocity potential is time independent. As the known solution of the fluid systems which are considered in this work are time independent, the acoustic metric obtained by following Unruh's formulation is time independent. There is only one way to get a time dependent metric, i.e., by introducing perturbations which are nonlinear in nature. The vindication of this statement is shown in the next section. This is the main difference between Unruh's result [3] and ours. Unlike Unruh's case, we extend the analysis to the second order in perturbation technique. Unlike Unruh's result, the wave equation we find is nonlinear in the perturbed quantities. Here the acoustic analog of a gravitational wave, propagating with the speed of sound in the medium, is not transverse in kind, rather it is of longitudinal type. Our work explores the longitudinal wave nature of the acoustic metric in different fluid systems.

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II. NONLINEAR ACOUSTICS

We first consider the simplest possible system, a medium of uniform density ρ_0 and of uniform pressure p_0, i.e., $\nabla\rho_0 \sim 0$, $\nabla p_0 \sim 0$. The effect of any external field is assumed to be negligible. This is how we are choosing the background flow, i.e., the known solutions (density and velocity) are not only independent of time but also they do not have any spatial dependence. The fluid equations for inviscid flow read as in general,

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} = -\frac{\nabla p}{\rho}. \tag{2}$$

We assume the medium to be barotropic, i.e., pressure is a function of density only. We introduce perturbations, not necessarily linear, as

$$p(\mathbf{x},t) = p_0 + p'(\mathbf{x},t),$$
$$\rho(\mathbf{x},t) = \rho_0 + \rho'(\mathbf{x},t),$$
$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}'(\mathbf{x},t).$$

Unlike Unruh's case, we are making no assumption about the linearity of the wave, we are assuming the wave to be nonlinear in general. The motion of the fluid is assumed to be irrotational, i.e., $\nabla\times\mathbf{v} = \nabla\times\mathbf{v}' = 0$. Hence

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{v}^2 + \int\frac{dp}{\rho}\right) = 0. \tag{3}$$

For the steady state problem, the conserved quantity derived from the momentum equation is Bernoulli's constant, ζ and $\zeta = (\frac{1}{2}\mathbf{v}^2 + \int\frac{dp}{\rho})$ in general. Equation (3) implies

$$\frac{\partial\mathbf{v}'}{\partial t} + \nabla\zeta' = 0. \tag{4}$$

One can write ζ as

$$\zeta(\mathbf{x},t) = \zeta_0 + \zeta'(\mathbf{x},t) \tag{5}$$

where ζ_0 corresponds to the background value of the Bernoulli's constant, which is a constant number and $\zeta'(\mathbf{x},t)$ is the nonlinear fluctuation around this value. From the expression (calculations are shown in detail in Appendix B)

$$\partial_t\zeta' = \mathbf{v}'.\partial_t\mathbf{v}' + \frac{c_s^2}{\rho}\partial_t\rho' \tag{6}$$

where $c_s^2 = \frac{dp}{d\rho}$ from definition. Using the continuity equation (1) and Euler momentum equation (2),

$$\partial_\mu(f^{\mu\nu}(\mathbf{x},t)\partial_\nu)\zeta'(\mathbf{x},t) = 0 \tag{7}$$

where

$$f^{\mu\nu}(\mathbf{x},t) \equiv \frac{\rho}{c_s^2}\begin{bmatrix} -1 & \vdots & -v'^j \\ \cdots & \cdots & \cdots\cdots \\ -v'^j & \vdots & c_s^2\delta^{ij} - v'^i v'^j \end{bmatrix} \tag{8}$$

where indices i, j run from 1 to 3 and the Greek indices run from 0 to 3. Unlike Unruh's case, the field equation (7) we find is nonlinear in perturbed quantities generally. Now one can find the time dependent acoustic metric by comparing the above equation with the massless scalar field equation as follows:

$$f^{\mu\nu} = \sqrt{-g}g^{\mu\nu} \tag{9}$$

where g, the determinant of the metric $g_{\mu\nu}$, is equal to $-\frac{\rho^4}{c_s^2}$. We find

$$g_{\mu\nu}(\mathbf{x},t) \equiv \frac{\rho}{c_s}\begin{bmatrix} -(c_s^2 - v'^2) & \vdots & -v'^i \\ \cdots & \cdots & \cdots\cdots \\ -v'^i & \vdots & \delta_{ij} \end{bmatrix}. \tag{10}$$

Hence the perturbation of Bernoulli's constant satisfies the massless scalar field equation in a time dependent spacetime.[1]

Now we make use of the perturbation method, writing every relevant quantity in the form of an infinite series, where each term becomes smaller and smaller in magnitude with succession. Therefore, we have

$$\rho(\mathbf{x},t) = \rho_0 + \rho'_{(1)}(\mathbf{x},t) + \cdots$$
$$p(\mathbf{x},t) = p_0 + p'_{(1)}(\mathbf{x},t) + \cdots$$
$$c_s(\mathbf{x},t) = c_{s0} + c_{s(1)}(\mathbf{x},t) + \cdots$$
$$c_s^2(\mathbf{x},t) = c_{s0}^2 + c_{s(1)}^2(\mathbf{x},t) + \cdots$$
$$\mathbf{v}'(\mathbf{x},t) = \mathbf{v}'_{(1)}(\mathbf{x},t) + \mathbf{v}'_{(2)}(\mathbf{x},t) + \cdots$$

The above expressions are not independent expressions. It is sufficient to introduce a pressure variation in the system and then write that in terms of a perturbation series to get the expression of other quantities in such a fashion (discussed in detail in Appendices B and C). $\zeta'(\mathbf{x},t), f^{\mu\nu}(\mathbf{x},t)$ can be written as

[1] Instead of working with ζ', one can work with ζ because ζ_0 is just a constant number.

$$\zeta'(\mathbf{x},t) = \zeta'_{(1)}(\mathbf{x},t) + \zeta'_{(2)}(\mathbf{x},t) + \cdots$$
$$f^{\mu\nu}(\mathbf{x},t) = f^{\mu\nu}_{(0)} + f^{\mu\nu}_{(1)}(\mathbf{x},t) + \cdots$$

Now we write down the governing equations for each order because the $(n+1)$th term is incomparably smaller than the nth term in series expansion of any quantity; one does this in the perturbation method.

As the pressure and density of the medium is assumed to be uniform within the region of our interest, the zeroth order momentum equation gives the conserved Bernoulli's constant in such a medium (see Appendix B).

In the first order, the continuity equation is given by

$$\partial_t \rho'_{(1)} + \rho_0 \nabla \mathbf{v}'_{(1)} = 0. \tag{11}$$

Similarly, separating out the first order terms from Eq. (4), we get

$$\partial_t \mathbf{v}'_{(1)} + \nabla \zeta'_{(1)} = 0 \tag{12}$$

where $\zeta'_{(1)} = \frac{c_{s0}^2}{\rho_0}\rho'_{(1)}$. Therefore, we find

$$\partial_t^2 \rho'_{(1)}(\mathbf{x},t) = c_{s0}^2 \nabla^2 \rho'_{(1)}(\mathbf{x},t) \tag{13}$$

and

$$\partial_t^2 v'^i_{(1)}(\mathbf{x},t) = c_{s0}^2 \partial_i(\partial_j v'^j_{(1)}(\mathbf{x},t)). \tag{14}$$

Separating out first order terms from Eq. (7), we get

$$\partial_\mu (f^{\mu\nu}_{(0)} \partial_\nu) \zeta'_{(1)}(\mathbf{x},t) = 0. \tag{15}$$

Therefore, the acoustic metric obtained from the terms in first order is given by

$$g_{\mu\nu} \equiv \frac{\rho_0}{c_{s0}} \begin{bmatrix} -c_{s0}^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

where $\frac{\rho_0}{c_{s0}}$ is a constant number. The above metric gives the acoustic analog of the Minkowski metric, $(\eta_A)_{\mu\nu} = \frac{c_{s0}}{\rho_0} g_{\mu\nu}$. Hence, when one is interested in the equations in the first order of smallness, the emergent spacetime metric is time independent and as the medium is static and uniformly dense, the acoustic metric is analogous to the Minkowski metric. It is evident that if one limits himself or herself up to the first order in smallness according to the strength of the disturbance in the medium the emergent spacetime will always be time independent.

In the weakly nonlinear limit, studying perturbation up to the first order is not going to give the correct result. In this limit, all the quantities are expanded up to the second order. Therefore, $\zeta' = \zeta'_{(1)}$ is not a good approximation to work with. In this limit $\zeta' = \zeta'_{(1)} + \zeta'_{(2)}$ (details in Appendix C). Therefore, we seek the wave equation for $\zeta' = \zeta'_{(1)} + \zeta'_{(2)}$.

$\zeta'_{(1)}$ already satisfies a wave equation [Eq. (15)]. Separating out the second order term from Eq. (7), we find

$$\partial_\mu (f^{\mu\nu}_{(0)} \partial_\nu) \zeta'_{(2)}(\mathbf{x},t) + \partial_\mu (f^{\mu\nu}_{(1)} \partial_\nu) \zeta'_{(1)}(\mathbf{x},t) = 0. \tag{17}$$

In the weak nonlinear limit, one neglects any term of third order in magnitude and higher order than the third order. At this point, one can first solve the wave equation of $\zeta'_{(1)}$ [Eq. (15)] and then using that solution of $\zeta'_{(1)}$ in Eq. (17), one can find the solution of ζ'_2 and thus ζ' can be found in the weakly nonlinear limit. In this paper, our motivation is not to find the solution of ζ'; here we are interested to find the acoustic metric through perturbation of Bernoulli's constant ζ'. Therefore, we add Eqs. (15) and (17) and we make use of the weak nonlinearity assumption by adding another term of cubic order, $\partial_\mu (f^{\mu\nu}_{(1)}) \zeta'_{(2)}$, and we get

$$\partial_\mu ((f^{\mu\nu}_{(0)} + f^{\mu\nu}_{(1)}(\mathbf{x},t)) \partial_\nu)(\zeta'_{(1)}(\mathbf{x},t) + \zeta'_{(2)}(\mathbf{x},t)) = O(3) \approx 0.$$

The right-hand side is approximately zero due to the assumption of weak nonlinearity. The above equation can also be viewed in a different manner. If one simply uses $\zeta' = \zeta'_{(1)} + \zeta'_{(2)}$ (the weak nonlinear limit) and then if one expands the terms in Eq. (7) up to the second order in smallness, one gets

$$\partial_\mu ((f^{\mu\nu}_{(0)} + f^{\mu\nu}_{(1)}(\mathbf{x},t)) \partial_\nu)(\zeta'_{(1)}(\mathbf{x},t) + \zeta'_{(2)}(\mathbf{x},t)) + O(3) \approx 0. \tag{18}$$

The terms higher than the third order can be equated to zero for each order due to the implementation of perturbation technique and the third order extra leftover terms are zero due to weak nonlinearity.

We do not proceed further by finding the equation for $\zeta' = \zeta'_{(1)} + \zeta'_{(2)} + \zeta'_{(3)}$ because of the weakly nonlinear assumption about the perturbation strength.

Therefore, to study second order behavior, $f^{\mu\nu}(\mathbf{x},t)$ is eventually expanded up to the first order in smallness; hence as a consequence, $g_{\mu\nu}(\mathbf{x},t)$ is expanded up to the linear order as

$$g_{\mu\nu}(\mathbf{x},t) = g_{(0)\mu\nu} + g_{(1)\mu\nu}(\mathbf{x},t). \tag{19}$$

Assuming isentropic perturbations in this isothermal medium,[2] we use barotropic equation for fluid,

[2]The medium is isothermal in the absence of any perturbation because the medium is taken to be uniform in density and pressure, hence from the equation of state of ideal gas, the medium has to be isothermal.

$$p = K\rho^{\gamma} \tag{20}$$

where K is a constant number and γ is the specific heat ratio. Therefore when one works with perturbations up to the first order in smallness, the speed of propagation of linear perturbation, i.e., the adiabatic sound speed c_{s0}, is given by [15]

$$c_{s0}^2 = \frac{p'_{(1)}(\mathbf{x},t)}{\rho'_{(1)}(\mathbf{x},t)} = \frac{\gamma p_0}{\rho_0}. \tag{21}$$

One can work with isothermal perturbation also, for sound wave propagating in the air medium, the adiabatic approximation works much better than the isothermal one [15].

Using Eq. (14), we find that

$$\frac{c_{s(1)}(\mathbf{x},t)}{c_{s0}} = \frac{(\gamma-1)}{2}\frac{\rho'_{(1)}(\mathbf{x},t)}{\rho_0},$$
$$\frac{c_{s(1)}^2(\mathbf{x},t)}{c_{s0}^2} = (\gamma-1)\frac{\rho'_{(1)}(\mathbf{x},t)}{\rho_0}.$$

Using the above expressions in the matrix of Eq. (10), we get

$$g_{\mu\nu}(\mathbf{x},t) = g_{(0)\mu\nu} + g_{(1)\mu\nu}(\mathbf{x},t) \equiv \frac{\rho_0}{c_{s0}}\begin{bmatrix} -c_{s0}^2\left(1+\frac{(\gamma+1)}{2}\frac{\rho'_{(1)}(\mathbf{x},t)}{\rho_0}\right) & \vdots & -v'^{i}_{(1)}(\mathbf{x},t) \\ \cdots & \cdots & \cdots\cdots \\ -v'^{i}_{(1)}(\mathbf{x},t) & \vdots & \delta_{ij}\left(1+\frac{(3-\gamma)}{2}\frac{\rho'_{(1)}(\mathbf{x},t)}{\rho_0}\right)\end{bmatrix}, \tag{22}$$

as $\frac{\rho_0}{c_{s0}}$ is just a constant number in front of the above matrix. We work with a better looking matrix, defined by

$$\tilde{g}_{\mu\nu}(\mathbf{x},t) = \frac{c_{s0}}{\rho_0}g_{\mu\nu}(\mathbf{x},t) = (\eta_A)_{\mu\nu} + h_{\mu\nu}(\mathbf{x},t) \tag{23}$$

where $(\eta_A)_{\mu\nu}$, the acoustic analog of the Minkowski metric, is $(\mathrm{diag}[-c_{s0}^2,+1,+1,+1])_{\mu\nu}$; the convention in [16] is used. $h_{\mu\nu}$ is the linear perturbation term of the acoustic metric.

Now we examine the behavior of the $h_{\mu\nu}(\mathbf{x},t)$.

We assume in our coordinate system that the z component of the linear perturbation of velocity is the only nonzero component, i.e., we are studying nonlinear sound wave propagating parallel to the z axis. Hence

$$v'^{1,2}_{(1)} = 0, \tag{24}$$

$$v'^{3}_{(1)} = v'^{3}_{(1)}(z,t), \tag{25}$$

$$\rho'_{(1)} = \rho'_{(1)}(z,t). \tag{26}$$

This assumption is compatible with the irrotationality condition. This is very similar to working in the harmonic coordinate system [17], i.e., choosing the Einstein gauge [18], in the case of studying real gravitational wave propagating parallel to the z axis. Therefore, in this coordinate system, we have from Eqs. (11) and (12),

$$\frac{\partial\rho'_{(1)}}{\partial t} + \rho_0\frac{\partial v'^{3}_{(1)}}{\partial z} = 0 \tag{27}$$

and

$$\frac{\partial v'^{3}_{(1)}}{\partial t} + \frac{c_{s0}^2}{\rho_0}\frac{\partial\rho'_{(1)}}{\partial z} = 0. \tag{28}$$

Similarly, from Eqs. (13) and (14), we get

$$\Box_A h_{\mu\nu}(z,t) = 0 \tag{29}$$

where $\Box_A$ is the acoustic analog of the d'Alembertian wave operator, given by

$$\Box_A = -\frac{1}{c_{s0}^2}\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2.$$

Hence, $h_{\mu\nu}(z,t)$ represents the acoustic analog of the gravitational wave propagating parallel to the z axis. In the case of the sound wave propagating uniformly in all directions, we would have chosen to work in the spherical polar coordinate system as the analogous harmonic coordinate system. We are doing two things simultaneously, one is that we are giving the wave vector of sound a certain direction (sound wave propagating uniformly in all directions or sound wave propagating along a particular direction, etc.) and we are considering a suitable coordinate system to describe it. As a result the d'Alembertian operator of Eq. (29) happens to be $1+1$ dimensional.

$h_{\mu\nu}(z,t)$ depends only on the linear terms of the perturbed quantities. Therefore, apparently it seems that analyzing equations up to the first order in smallness is enough to obtain Eq. (29) but to relate the linear perturbation term in density and linear perturbation term in velocity to $h_{\mu\nu}(z,t)$, i.e., the linear perturbation terms in the acoustic metric, one needs Eq. (7) followed by Eq. (18) which are nonlinear wave equations. However, after arriving at Eq. (29), one can argue that if one analyzes the equations up to the first order in smallness, one could predict what would happen to the acoustic metric if the equations are written down in the weakly nonlinear limit, but to predict such thing one should have the information about the nonlinear wave equation (18) in weak form.

One can also work in other coordinate systems to get Eq. (29), for example in a coordinate system such that $v'^{1,2}_{(1)}$ are constant numbers. This is similar to gauge freedom which we have in the case of a real gravitational wave.

Therefore, $h_{\mu\nu}(z,t)$ is given by

$$h_{\mu\nu}(z,t) \equiv \begin{bmatrix} -c_{s0}^2 \frac{(\gamma+1)}{2}\frac{\rho'_{(1)}(z,t)}{\rho_0} & 0 & 0 & -v'^3_{(1)}(z,t) \\ 0 & \frac{(3-\gamma)}{2}\frac{\rho'_{(1)}(z,t)}{\rho_0} & 0 & 0 \\ 0 & 0 & \frac{(3-\gamma)}{2}\frac{\rho'_{(1)}(z,t)}{\rho_0} & 0 \\ -v'^3_{(1)}(z,t) & 0 & 0 & \frac{(3-\gamma)}{2}\frac{\rho'_{(1)}(z,t)}{\rho_0} \end{bmatrix}. \tag{30}$$

Instead of Bernoulli's constant, one can start with studying perturbation of mass flow rate (Appendix A), the conserved quantity derived from the continuity equation when the motion of the fluid medium is assumed to be steady. In that case, in the very beginning, one has to assume the direction of the sound wave and a suitable coordinate system to describe it, and similarly, one would find the analog of the gravitational wave in the Minkowski spacetime.

Instead of adiabatic sound in the isothermal medium, one could start with isothermal sound in such a medium. Sound is approximately adiabatic in nature in air medium [15]. In the case of isothermal sound, the expressions in the previous equations would be the same except one has to put $\gamma = 1$.

Just like the gravitational wave propagating in the z direction, the acoustic analog of a gravitational wave has nontrivial components, $h_{00}(z,t)$ (proportional to the linear perturbation of density of the medium) and $h_{03}(z,t)$ (proportional to the linear perturbation of velocity along the z axis in the medium). Other nontrivial components are derivable from these two nonzero components as follows:

$$h_{11} = h_{22} = h_{33} = -\frac{(3-\gamma)}{c_{s0}^2(\gamma+1)} h_{00}, \tag{31}$$

$$h_{30} = h_{03}. \tag{32}$$

Equations (27) and (28) give that h_{00} and $h_{(03)}$ are related by the following expressions:

$$\frac{2}{c_{s0}^2(\gamma+1)}\frac{\partial h_{(00)}}{\partial t} + \frac{\partial h_{03}}{\partial z} = 0 \tag{33}$$

and

$$\frac{\partial h_{(03)}}{\partial t} + \frac{2}{(\gamma+1)}\frac{\partial h_{00}}{\partial z} = 0. \tag{34}$$

Therefore, the number of nontrivial independent components is one.

A real gravitational wave acquires the nontrivial components by gauge freedom and gauge fixing. In the case of a real gravitational wave, there is the Einstein equation in the linear order, i.e., in the weak field approximation, over which we have a gauge freedom. In the case of analog gravity models there is no analogous Einstein equation to begin with [4]. Therefore, the analog models of gravity give half of the picture of general relativity, only the kinematic picture not the dynamic picture. In the case of an acoustic gravitational wave, we have the fluid equations, continuity equation, and momentum equation in the Newtonian framework. The fluid systems we are talking about have a privileged coordinate system and each component of the acoustic metric has physical meaning, for example, density and velocity of the medium, appearing in the acoustic metric [Eqs. (16) and (22)]. This is not the case in the general relativistic context. Thus the diffeomorphism invariance is violated here ("diffeomorphism invariance" section in [19]). As we are beginning with fluid equations in a Newtonian framework, the diffeomorphism invariance is not reflected here, which is why the analysis can be thought of as a model of gravitational wave only describing the kinematic aspect.

Equation (22) is compatible with the symmetric property of the acoustic metric.

For a sound wave propagating along the x axis, $h_{\mu\nu} = h_{\mu\nu}(x,t)$, the nontrivial diagonal quantities will be similar looking as before, $h_{01} = h_{10} = v'^1_{(1)}(x,t)$ will be the nontrivial component instead of h_{03}. This same conclusion can be drawn similarly by applying a rotation operator over $h_{\mu\nu}$. As an acoustic metric is invariant under rotation, we have

$$h'_{\mu\nu} = R^{\rho}_{\mu} R^{\sigma}_{\nu} h_{\rho\sigma} \tag{35}$$

where R^{ρ}_{μ} denotes the rotation operator and $h'_{\mu\nu}$ denotes the linear perturbation term in the acoustic metric after rotation; writing the matrix corresponding to $h_{\mu\nu}$ as $\hat{h}$ and the rotation operator as $\hat{R}$, we get

$$\hat{h}' = \hat{R}^T \hat{h} \hat{R} \tag{36}$$

where $\hat{R}^T \hat{R} = \hat{R}\hat{R}^T = \mathbb{I}$, $\hat{R}^T$ is the transpose of $\hat{R}$ and $\mathbb{I}$ is the 4×4 identity matrix. The sound wave propagating along the z direction in one coordinate system has direction along the x axis in another coordinate system; the second coordinate system can be obtained from the first one by a rotation of $-90°$ about the y axis. For a rotation of angle θ about the y axis over the matrix of Eq. (28), we get

$$\hat{h}' = \hat{R}_y^T \hat{h} \hat{R}_y = \begin{bmatrix} -c_{s0}^2 \frac{(\gamma+1)}{2}\frac{\rho'_{(1)}}{\rho_0} & v'^3_{(1)}\sin\theta & 0 & -v'^3_{(1)}\cos\theta \\ v'^3_{(1)}\sin\theta & \frac{(3-\gamma)}{2}\frac{\rho'_{(1)}}{\rho_0} & 0 & 0 \\ 0 & 0 & \frac{(3-\gamma)}{2}\frac{\rho'_{(1)}}{\rho_0} & 0 \\ -v'^3_{(1)}\cos\theta & 0 & 0 & \frac{(3-\gamma)}{2}\frac{\rho'_{(1)}}{\rho_0} \end{bmatrix} \tag{37}$$

where $\hat{R}_y$ denotes the rotation operator corresponding to rotation about the y axis. $\theta = -90°$ in the above matrix equation gives the desired $\hat{h}'$. Therefore by rotation about the y axis, the diagonal elements in the $\hat{h}$ matrix remain same. Wherefore, the diagonal entries in the $\hat{h}$ matrix are proportional to the linear perturbation in density, the diagonal elements do not change under rotation of any kind; these quantities are scalar quantities under rotation. The nontrivial off-diagonal quantities are proportional to the linear perturbation in velocity, hence they transform under rotation.

However, rotation about the z axis on $\hat{h}$ do not have any effect. This is not the case for a real gravitational wave propagating along the z axis; the physically significant terms in the case of a real gravitational wave have helicity ± 2 [17].

The solution of Eq. (29), has the general form

$$h_{\mu\nu}(z,t) = h_{\mu\nu}(z \pm c_s t). \tag{38}$$

The speed of an analog gravitational wave is the sound speed; the "+" sign implies sound wave propagating along the negative z axis and the "−" sign implies the sound wave propagating along the positive z axis. For a plane wave propagating along the $+z$ axis, we write

$$h_{\mu\nu}(z,t) = e_{\mu\nu} \exp(i(kz - \omega t)), \tag{39}$$

where $e_{\mu\nu}$ is the amplitude of the wave, and ω and k are the angular frequency and the wave vector of the plane wave. Using Eq. (29), we find the dispersion relation as

$$\omega = c_s k. \tag{40}$$

This linear dispersion relation of the acoustic gravitational wave is similar to that for the real gravitational wave.

Hence when one extends the analysis of introducing perturbations to second order, the emergent metric gets to have some striking similarities with the real gravitational wave and there are some differences as well.

A. Implications in nonlinear acoustics

The above formalism of nonlinear perturbations can be used to understand some nonlinear phenomena. We have assumed the fluid to be inviscid even in the presence of perturbations and also there is no heat conduction or convection happening in the system. Hence we can describe nonlinear acoustics in lossless fluids [20]. Let us consider irrotational flow along the x axis. The velocity $v^1 = v'^1 = -\psi_x$, where ψ is the velocity potential and ψ_x is the partial derivative with respect to x. From Eq. (4)

$$\psi_t = \zeta'. \tag{41}$$

Using the expression of ζ, Eq. (6), the continuity equation and the above expression, we get

$$\psi_{tt} - 2\psi_{xt}\psi_x + (\psi_x^2)\psi_{xx} = c_s^2 \psi_{xx}. \tag{42}$$

This is the nonlinear acoustic wave equation in a lossless scenario in terms of velocity potential [20–23]. Using the barotropic equation (20) and the expression of ζ, we have

$$\begin{aligned} \zeta &= \frac{1}{2}\psi_x^2 + \frac{c_s^2}{(\gamma-1)} \\ &= \zeta_0 + \zeta'(x,t) \\ &= \frac{c_{s0}^2}{(\gamma-1)} + \psi_t. \end{aligned} \tag{43}$$

Therefore, using expression (43) in Eq. (42), we find

$$\psi_{tt} - c_{s0}^2 \psi_{xx} = 2\psi_{xt}\psi_x + (\gamma-1)\psi_{xx}\psi_t - \frac{(\gamma+1)}{2}\psi_x^2\psi_{xx}. \tag{44}$$

Furthermore, using expression (43), one can write density as a function of partial derivatives of velocity potential [22],

$$\rho = \rho_0\left(1+\frac{(\gamma-1)}{c_{s0}^2}\left(\psi_t-\frac{\psi_x^2}{2}\right)\right)^{\frac{1}{\gamma-1}}. \tag{45}$$

Neglecting smallness of cubic order, one can derive some interesting lossless wave equations in weakly nonlinear limit [24],

$$\psi_{tt} - c_{s0}^2\psi_{xx} = 2\psi_{xt}\psi_x + (\gamma-1)\psi_{xx}\psi_t. \tag{46}$$

Approximating $\psi_{xx} \sim \frac{1}{c_0^2}\psi_{tt}$, one can derive the lossless Kuznetsov equation [25],

$$\psi_{tt} - c_{s0}^2\psi_{xx} = 2\psi_{xt}\psi_x + \frac{(\gamma-1)}{c_{s0}^2}\psi_{tt}\psi_t. \tag{47}$$

Studying nonlinear acoustic phenomena is not the aim of this paper. Hence we are not going into more detail about it.

Thus the nonlinear wave in the lossless regime can also be described as acoustic gravitational wave propagating in the medium. Here the gravitational wavelike effect is the emergent phenomena in the system.

III. STRATIFIED MEDIUM

A. Isothermal stratified medium

Let us consider an isothermal medium of uniform temperature T_0, having a density variation along the z axis due to the external force along the z direction. For example, in a constant gravitational field g, acting along the $-z$ direction, the density $\rho(z) = \rho(0)\exp\left(-\frac{z}{(RT_0/gM_A)}\right)$ [15], where R is the ideal gas constant and M_A is the molar weight of the constituent particle of the medium. In the absence of any perturbation in such a system, we have

$$\frac{1}{\rho_0}\frac{dp_0}{dz} + F_{\rm ext}(z) = 0. \tag{48}$$

$F_{\rm ext}(z)$ is the external body force. Let us consider perturbations in the medium along the x direction,

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t), \tag{49}$$

$$v'^y = v'^z = 0, \tag{50}$$

$$v'^x = v'^x(x,t). \tag{51}$$

Equations (50) and (51) are compatible with the irrotationality condition. The system has a preferred direction, i.e., the z direction. As at the very outset, we are assuming perturbations propagating along the x direction, we expect to get a 2×2 acoustic metric instead of 4×4. Therefore, we have continuity equation and Euler momentum equation for the perturbed quantities as follows:

$$\partial_t\rho' + \partial_x(\rho v'^x) = 0, \tag{52}$$

$$\partial_t v'^x + \partial_x\zeta' = 0. \tag{53}$$

The nonlinear perturbation $\partial_t\zeta' = v'^x\partial_t v'^x + \frac{c_s^2}{\rho}\partial_t\rho'$. After some manipulations we find

$$\partial_\mu(f^{\mu\nu}\partial_\nu)\zeta' = 0 \tag{54}$$

where μ, ν run over t and x. $f^{\mu\nu}$ is given by

$$f^{\mu\nu}(x,z,t) \equiv \frac{\rho}{c_s^2}\begin{bmatrix} -1 & -v'^x \\ -v'^x & c_s^2-(v'^x)^2 \end{bmatrix}. \tag{55}$$

Defining $(g_{\mu\nu})_{\rm eff}$ [26], as the problem is not intrinsically $1+1$ dimensional rather it is $3+1$ dimensional [19], we can write[3]

$$(g_{\mu\nu})_{\rm eff} \equiv \frac{\rho}{c_s}\begin{bmatrix} -(c_s^2-v'^2) & -v'^x \\ -v'^x & 1 \end{bmatrix}. \tag{56}$$

After getting rid of the conformal factor, considering expansion of the terms up to second order and considering the disturbances to be of adiabatic type, we find in a similar fashion as before

$$\tilde{g}_{\mu\nu}(\mathbf{x},t) = (\eta_A)_{\mu\nu} + h_{\mu\nu}(\mathbf{x},t) \tag{57}$$

$$\equiv \begin{bmatrix} -c_{s0}^2\left(1+\frac{(\gamma+1)}{2}\frac{\rho'_{(1)}}{\rho_0}\right) & -v'^x_{(1)} \\ -v'^x_{(1)} & \left(1+\frac{(3-\gamma)}{2}\frac{\rho'_{(1)}}{\rho_0}\right) \end{bmatrix}. \tag{58}$$

As the medium has uniform constant temperature in the absence of any disturbances, the linear sound speed $c_{s0}(=\sqrt{\frac{\gamma RT_0}{M_A}})$ is the same everywhere.

Considering the continuity equation and Euler equation up to first order of smallness,

$$\partial_t\rho'_1 + \partial_x(\rho_0 v'^x_1) = 0, \tag{59}$$

$$\partial_t v'^x_1 + \frac{c_{s0}^2}{\rho_0}\partial_x\rho'_1 = 0, \tag{60}$$

[3]As the actual dimension (as in Sec. II) of the problem is $3+1$ because in this problem, the equation $\partial_\mu(f^{\mu\nu}\partial_\nu)\zeta' = 0$ still holds even for general perturbations, i.e., in the presence of nonzero v'^x, v'^y and v'^z; here in the very beginning choosing the symmetry and the direction of the wave reduces the dimension of the wave equation to $1+1$. Alternatively, for wave propagating in an arbitrary direction, after deriving $\partial_\mu(f^{\mu\nu}\partial_\nu)\zeta' = 0$, we could have chosen the symmetry and the direction of the wave as we did in Sec. II. Therefore we use the same conformal factor in front of the metric as before.

$$\partial_z\left(\frac{c_{s0}^2}{\rho_0}\rho_1'\right) = 0. \tag{61}$$

$\because v'^x_1 = v'^x_1(x,t)$ and $\rho_0 = \rho_0(z)$, $\rho'_1(x,y,z,t) = \frac{\rho_0(z)}{c_{s0}^2}\epsilon(x,t)$ from the above equations. $\epsilon(x,t)$ is a function within the first order of smallness. Therefore

$$\partial_t^2\rho_1' = c_{s0}^2\partial_x^2\rho_1', \tag{62}$$

$$\partial_t^2 v'^x_1 = c_{s0}^2\partial_x^2 v'^x_1, \tag{63}$$

$$\Rightarrow \left(-\frac{1}{c_{s0}^2}\partial_t^2 + \partial_x^2\right)h_{\mu\nu}(x,t) = 0. \tag{64}$$

The above equation has a solution of plane wave propagating along the $\pm x$ axis.

Here we have tacitly chosen the coordinate system first and we have assumed the perturbations across the perpendicular direction of stratification in the medium. We restrict ourselves by considering perturbations perpendicular to the direction of stratification. Nevertheless the form of Eq. (54) does not depend on the direction of perturbations but the form of Eqs. (62)–(64) depends on the relative orientation between the direction of the propagating wave and the direction of stratification. Unlike the previous case of uniform medium, in this case there is a preferred direction in the system, i.e., the direction of external body force, and the symmetry is lost. That is why the wave propagating along the direction of stratification is different from the wave propagating across it. Even disturbances linear in nature propagating parallel to the direction of external body force have attenuation and would be dispersive in nature [15].

B. Adiabatic stratified medium

Let us consider the direction of external body force is along the z axis as before. In the absence of any perturbation, pressure $p_0(z) \propto \rho_0(z)^\gamma$. Unlike the previous case, the sound speed, more precisely the speed of linear perturbation, is not a constant number but rather a function of z, for example, in the case of an adiabatic medium in a constant gravitational field $-g\hat{z}$, where $\hat{z}$ is the unit vector along the z axis. In this case, sound speed diminishes linearly with z as the temperature diminishes linearly with height z.

Introducing perturbations in the medium as below,

$$\rho(x,z,t) = \rho_0(z) + \rho'(x,z,t), \tag{65}$$

$$v^x(x,z,t) = v'^x(x,z,t), \tag{66}$$

$$v^z(x,z,t) = v'^z(x,z,t), \tag{67}$$

such that $\partial_x v'^z(x,z,t) = \partial_z v'^x(x,z,t)$, i.e., the irrotationality condition is satisfied.[4] Again after manipulation in a similar fashion, one gets

$$\partial_\mu(f^{\mu\nu}(x,z,t)\partial_\nu)\zeta'(x,z,t) = 0, \tag{68}$$

where

$$f^{\mu\nu}(x,z,t) \equiv \frac{\rho}{c_s^2}\begin{bmatrix} -1 & -v'^x & -v'^z \\ -v'^x & c_s^2 - (v'^x)^2 & -v'^x v'^z \\ -v'^z & -v'^z v'^x & c_s^2 - (v'^z)^2 \end{bmatrix}. \tag{69}$$

We get a three-dimensional matrix, because we choose the quantities having dependence on one time dimension and two spatial dimensions. The continuity equation and Euler equation in the first order of smallness,

$$\partial_t\rho_1' + \partial_x(\rho_0 v'^x_1) + \partial_z(\rho_0 v'^z_1) = 0, \tag{70}$$

$$\partial_t v'^x_1 + \frac{c_{s0}^2}{\rho_0}\partial_x(\rho_1') = 0, \tag{71}$$

$$\partial_t v'^z_1 + \partial_z\left(\frac{c_{s0}^2}{\rho_0}\rho_1'\right) = 0. \tag{72}$$

From the equation in the zeroth order of smallness, we have

$$\frac{1}{\rho_0(z)}\frac{dp_0(z)}{dz} = -F_{\mathrm{ext}}(z) = \frac{d\Phi(z)}{dz} \tag{73}$$

where $\Phi(z)$ corresponds to the potential corresponding to the conservative external body force. Hence at height $\frac{l}{2}$ from the height z_0, at

$$\rho_0\left(z_0 + \frac{l}{2}\right) = \rho_0(z_0)\left(1 + \frac{1}{c_{s0}^2(z_0)}\left(\Phi\left(z_0 + \frac{l}{2}\right) - \Phi(z_0)\right) + \cdots\right). \tag{74}$$

Free fall velocity from height $z_0 + \frac{l}{2}$ to z_0 is $\sqrt{2(\Phi(z_0 + \frac{l}{2}) - \Phi(z_0))}$. Hence $\rho_0(z_0 + \frac{l}{2}) \sim \rho_0(z_0)$ when the height difference is such that the free fall velocity is negligible compared to the sound speed within one's tolerance range of precision. Therefore, $\rho_1'(x, z_0 + \frac{l}{2}, t) \sim \rho_1'(x, z_0, t)$. We assume $v'^z(x,z,t)$ to be a slowly varying function of z which means $v'^z(x, z_0 + \frac{l}{2}, t) \sim v'^z(x, z_0, t)$.

[4] If one chooses simply as before, $v'^z = 0$ and $v'^x = v'^x(x,t)$, in the same manner, one could derive $\partial_t^2 v'^x_1 = c_{s0}^2\partial_x^2 v'^x_1$, which does not make any sense because $c_{s0} = c_{s0}(z)$, whereas $v'^x = v'^x(x,t)$. That is why we choose $v'^x = v'^x(x,z,t)$ and to satisfy the irrotationality condition we need $v'^z(x,z,t)$.

Basically, we are trying to generate wave propagating perpendicular to the direction of stratification which follows $v'^x(x,z_0,t) \sim v'^x(x,z_0+\frac{l}{2},t)$, otherwise the viscous effects due to shear force would come into play. From the irrotationality condition, $v'^z(x,z,t)$ is very small. As the sound speed is also more or less the same within the slice of space between heights, $z_0+\frac{l}{2}$ and $z_0-\frac{l}{2}$, we effectively reduce the problem to a problem of isothermal medium. Two things are done here simultaneously, one is that disturbance propagating perpendicular to the direction of stratification is chosen and secondly a slice of space having a thickness along the direction of external body force is chosen in such a way that all the variations along z become negligible.

Averaging out Eqs. (71)–(73) by integrating over z, over z in the limit $z_0-\frac{l}{2}$ to $z_0+\frac{l}{2}$, writing the average of $\rho'_1(x,z,t)$ as $\tilde{\rho}'_1(x,t) \sim \rho'_1(x,z_0,t)$ and the average of $v'^x_1(x,z,t)$ as $\tilde{v}'_x(x,t) \sim v'^x(x,z_0,t)$, we get

$$\partial_t\tilde{\rho}'_1 + \partial_x(\rho_0(z_0)\tilde{v}'^x_1) = 0,$$

$$\partial_t\tilde{v}'^x_1 + \frac{c^2_{s0}(z_0)}{\rho_0(z_0)}\partial_x\tilde{\rho}'_1 = 0,$$

$$\Rightarrow \partial^2_t\tilde{\rho}'_1 = c^2_{s0}(z_0)\partial^2_x\tilde{\rho}'_1, \tag{75}$$

$$\Rightarrow \partial^2_t\tilde{v}'^x_1 = c^2_{s0}(z_0)\partial^2_x\tilde{v}'^x_1. \tag{76}$$

Similarly averaging $f^{\mu\nu}$ over z and rewriting it as $\tilde{f}^{\mu\nu}$,

$$\tilde{f}^{\mu\nu}(x,t) \equiv \frac{\tilde{\rho}}{\tilde{c_s}^2}\begin{bmatrix} -1 & -\tilde{v}'^x & 0 \\ -\tilde{v}'^x & \tilde{c_s}^2-(\tilde{v}'^x)^2 & 0 \\ 0 & 0 & \tilde{c_s}^2-(\tilde{v}'^z)^2 \end{bmatrix}. \tag{77}$$

Now finding the acoustic metric after getting rid of the conformal factor and carrying the expressions up to second order of smallness, we find using Eqs. (76) and (77) in the same manner as done before

$$\left(-\frac{1}{c^2_{s0}(z_0)}\partial^2_t + \partial^2_x\right)h_{\mu\nu}(x,t) = 0. \tag{78}$$

IV. SHALLOW WATER WAVE

A shallow water wave or long gravity wave is a wave having wavelength longer than the depth of the incompressible liquid medium [27]. We assume constant gravity $-g\hat{z}$; the depth of the liquid is denoted by h. As we consider a shallow water wave not necessarily linear, the liquid flows through a channel (along the x axis) and the wave is longitudinal [27], i.e., the velocity along the z direction and the y direction is much smaller compared to the velocity v along the x axis; the continuity equation reads as [27]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0. \tag{79}$$

The Euler momentum equation has the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0. \tag{80}$$

Now we define a quantity ξ as $(\frac{1}{2}v^2 + gh)$ which is very similar to Bernoulli's constant ζ in the previous cases, and hence from the momentum equation is given by

$$\frac{\partial v}{\partial t} + \frac{\partial \xi}{\partial x} = 0. \tag{81}$$

The fluid velocity and height are given by

$$v(x,t) = v'(x,t), \tag{82}$$

$$h(x,t) = h_0 + h'(x,t), \tag{83}$$

where primed quantities are the perturbations in the system, and h_0 is the constant height of the liquid in the absence of any disturbances. The sound speed corresponding to linear perturbation c_{s0} is $\sqrt{gh_0}$ [27]. After manipulations with the perturbed quantities in the same manner as before, we find

$$\partial_\mu(f^{\mu\nu}\partial_\nu)\xi'(x,t) = 0 \tag{84}$$

where μ, ν run over t and x. $f^{\mu\nu}$ is given by

$$f^{\mu\nu}(x,t) \equiv \begin{bmatrix} -1 & -v' \\ -v' & gh - v'^2 \end{bmatrix}. \tag{85}$$

In the above matrix, gh can be denoted as c^2_s. There is no conformal factor in front of $f^{\mu\nu}$ because in this problem h is mathematically equivalent to ρ in the previous problems and c^2_s is proportional to h; that is why the conformal factor $\frac{\rho}{c^2_s}$ in the previous cases happens to be a constant number in this problem. Just like in the previous case of Sec. III A and due to symmetry, the problem becomes $1+1$ dimensional, therefore using the same argument, we use same conformal factor as before. Hence, effectively, the acoustic metric becomes

$$g_{\mu\nu}(x,t) \equiv \sqrt{h}\begin{bmatrix} -(c^2_s - v'^2) & -v'^x \\ -v'^x & 1 \end{bmatrix}. \tag{86}$$

The conformal factor is $\sqrt{h}$ because in the previous cases, the conformal factor $\frac{\rho}{c_s}$ in front of $g_{\mu\nu}$ is equivalent to $\frac{h}{\sqrt{gh}}$ here. After expanding the equations up to second order of smallness, i.e., in the weak nonlinear limit, we find after dropping the conformal factor $\sqrt{h_0}$,

$$\tilde{g}_{\mu\nu}(x,t) = (\eta_A)_{\mu\nu} + h_{\mu\nu}(x,t) \equiv \begin{bmatrix} -c_{s0}^2\left(1+\frac{3}{2}\frac{h_1'}{h_0}\right) & -v'^x_1 \\ -v'^x_1 & \left(1+\frac{1}{2}\frac{h_1'}{h_0}\right) \end{bmatrix}. \quad (87)$$

One can show in the same manner that

$$\left(-\frac{1}{c_{s0}^2}\partial_t^2 + \partial_x^2\right)h_{\mu\nu}(x,t) = 0. \quad (88)$$

A shallow water wave in a weakly nonlinear limit can also be experimentally realized [28].

V. BOSE EINSTEIN CONDENSATE IN A TIGHT RING TRAP

Experimentally, very cold temperature ($\sim 10^{-6}$ K) is achieved in alkali atoms, which are very dilute as well as weakly interacting [29]. A dilute, very cold (temperature $\sim 0+$ K), weakly interacting Bose Einstein condensate (BEC) can be described by a classical field $\Phi(\mathbf{x},t)$, having the meaning of the order parameter, satisfying the time dependent Gross-Pitaevskii equation [30],

$$i\hbar\frac{\partial\Phi(\mathbf{x},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + g|\Phi(\mathbf{x},t)|^2\right)\Phi(\mathbf{x},t) \quad (89)$$

where $V(\mathbf{x})$ is the external potential and g is the two body interaction coefficient related to s-wave scattering cross section,

$$g = \frac{4\pi\hbar^2 a}{m},$$

where a is the scattering length and g is positive for repulsive interaction and negative for attractive interaction. The stationary state $\Phi_s(\mathbf{x},t)$, satisfying the eigenvalue equation, i.e., the time independent Gross-Pitaevskii equation, is given by

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + g|\Phi_s(\mathbf{x},t)|^2\right)\Phi_s(\mathbf{x},t) = \mu\Phi_s(\mathbf{x},t), \quad (90)$$

where μ is the eigenvalue of the problem, which is also the chemical potential of the problem.

Hence from Eq. (91), $\Phi_s(\mathbf{x},t) = \Phi_s(\mathbf{x},0)e^{-i\frac{\mu t}{\hbar}} = \sqrt{n_0(\mathbf{x})}e^{iS(\mathbf{x})}e^{-i\frac{\mu t}{\hbar}}$ with $i=\sqrt{-1}$, $n_0(\mathbf{x})$ being the condensate number density and $S(\mathbf{x})$ being a phase factor. Superfluid speed (resistance less speed) of BEC is proportional to the gradient of $S(\mathbf{x})$ [30]. The energy functional $E[\Phi]$ is given by [30]

$$E[\Phi] = \int d^3\mathbf{x}\left(\frac{\hbar^2}{2m}|\nabla\Phi|^2 + V_{\text{ext}}(\mathbf{x})|\Phi|^2 + \frac{g}{2}|\Phi|^4\right). \quad (91)$$

The first, second, and third terms in the integral correspond to the kinetic energy (E_{kin}), the potential energy (E_V), and the interaction energy (E_{int}), respectively.

Ring traps as external potential are experimentally realized in many cases [31–35]. Here we discuss the toroidal ring trap, given by

$$V_{\text{ext}}(\mathbf{x}) = \frac{1}{2}m\omega^2((r-R)^2 + z^2). \quad (92)$$

For simplicity, we assume the trapping frequency along the cylindrical radial direction r is same as the trapping frequency along z, denoted by ω.

A. The energy scales and the length scales of the problem

In the ground state, i.e., the state with zero superfluid speed, $S(\mathbf{x})$ can be assumed to be zero. Hence the solution of the stationary Gross-Pitaevskii (GP) equation is effectively a function of density n_0 only, i.e., $\Phi_s(\mathbf{x},t) = \sqrt{n_0(\mathbf{x})}e^{-i\frac{\mu t}{\hbar}}$. Therefore the energy is a functional of number density only [30],

$$E[\sqrt{n_0}] = \int d^3\mathbf{x}\left(\frac{\hbar^2}{2m}|\nabla\sqrt{n_0}|^2 + V_{\text{ext}}(\mathbf{x})n_0 + \frac{g}{2}n_0^2\right). \quad (93)$$

The length scale along the z direction and the r direction around the radius R, i.e., around the circle of minima of the potential on $z=0$ plane, is $a_{ho} = (\frac{\hbar}{m\omega})^{1/2}$ [30]. The length scale along the azimuthal direction is R. Hence the volume scale of the problem is $a_{ho}^2 R$, $N \sim \bar{n}_0 a_{ho}^2 R$, where N is the total number of atoms in the trap. Therefore, we have

$$|\Phi_s| \sim \sqrt{\bar{n}_0}, \qquad \left|\frac{\partial\Phi_s}{\partial r}\right| \sim \left|\frac{\partial\Phi_s}{\partial z}\right| \sim \frac{\sqrt{\bar{n}_0}}{a_{ho}}, \qquad \left|\frac{\partial\Phi_s}{r\partial\varphi}\right| \sim \frac{\sqrt{\bar{n}_0}}{R},$$

where $\bar{n}_0$ is the spatially averaged number density of condensate atoms, r is the cylindrical radial coordinate, and φ is the azimuthal angle. Therefore,

$$E_{\text{int}} \sim g\bar{n}_0 N = \frac{4\pi\hbar^2 a}{m}\bar{n}_0 N, \qquad E_{\text{kin}}^{\varphi} \sim \left(\frac{\hbar^2}{2m}\right)\frac{\bar{n}_0 a_{ho}^2}{R},$$

$$E_{\text{kin}}^{r} \sim E_{\text{kin}}^{z} \sim \frac{\hbar^2}{2m}\bar{n}_0 R,$$

where E_{kin}^{φ}, E_{kin}^{r}, and E_{kin}^{z} are the kinetic energy components along φ, r, and z directions, respectively,

$$\Rightarrow \frac{E_{\rm kin}^{r,z}}{E_{\rm int}} \sim \frac{R}{Na}, \tag{94}$$

$$\Rightarrow \frac{E_{\rm kin}^{\varphi}}{E_{\rm int}} \sim \left(\frac{a_{ho}}{R}\right)\left(\frac{a_{ho}}{Na}\right). \tag{95}$$

For a tight toroidal trap, $a_{ho} \ll R$.

B. Ground state solution of the stationary GP equation

We seek solution for the ground state, as the external potential does not depend on the azimuthal angle φ, therefore $\Phi_s(\mathbf{x}, t) = \Phi_s(r, z, t)$. Under the Thomas-Fermi (T-F) approximation, the GP equation can be reduced to classical fluid equations [30]. We assume the bosonic atoms to be strongly repulsively interacting, i.e., the T-F approximation, $E_{\rm int} \gg E_{\rm kin}$. As there is no azimuthal angle dependence of the ground state function, from the above section, the T-F approximation in this case means $Na \gg R$ which automatically implies for a tight toroidal trap $Na \gg R \gg a_{ho}$. Therefore, dropping the kinetic term in Eq. (92), we find

$$n_0(r, z) = \frac{\mu - V_{\rm ext}(r, z)}{g}, \tag{96}$$

where $n_0(r, z) > 0$ for $\mu > V_{\rm ext}(r, z)$ and zero for $\mu \le V_{\rm ext}(r, z)$. Figure 1 describes the distribution of the condensate atoms.

We define a new coordinate system as

$$(r - R) = \chi \cos\alpha, \tag{97}$$

$$z = \chi \sin\alpha, \tag{98}$$

where χ is the distance from $r = R$ at a fixed ϕ; α is the angle of that distance with the $r - \varphi$ plane. Therefore $V_{\rm ext}(r, z) = V_{\rm ext}(\chi) = \frac{1}{2}m\omega^2\chi^2$. $n_0(\chi)(=\frac{\mu - \frac{1}{2}m\omega^2\chi^2}{g})$ is greater than zero for $\chi < \chi_0(=\frac{\sqrt{2\mu/m}}{\omega})$ and is zero for $\chi \ge \chi_0$. Therefore under the T-F approximation, the BEC atoms are confined within a torus of finite radius χ_0 surrounding the minima of the potential function at $r = R$ on the $z = 0$ plane. χ_0 is determined by the equation

$$N = \int_V n_0 d^3\mathbf{x} = 2\pi \int_0^{2\pi}\int_0^{\chi_0} d\chi d\alpha \chi (R + \chi\cos\alpha)\frac{1}{g}\left(\mu - \frac{1}{2}m\omega^2\chi^2\right). \tag{99}$$

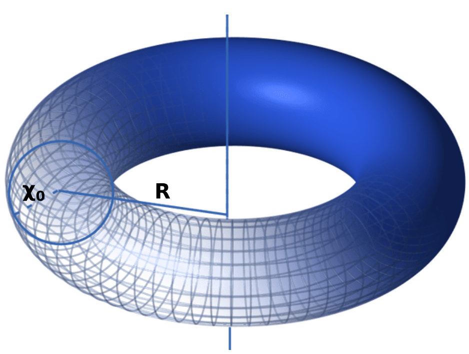
V is the volume of the torus with χ_0 being the radius of its circular section. $\mu(=\frac{1}{2}m\omega^2\chi_0^2)$ determines the maximum χ radius within which all the atoms stay. We find $\chi_0 = 2a_{ho}(\frac{aN}{2\pi R})^{\frac{1}{4}}$.

Therefore,

$$n_0(r, z) = n_0(\chi) = \frac{1}{8\pi a_{ho}^4 a}\left(4a_{ho}^2\left(\frac{aN}{2\pi R}\right)^{\frac{1}{2}} - \chi^2\right) = \frac{1}{8\pi a_{ho}^4 a}\left(4a_{ho}^2\left(\frac{aN}{2\pi R}\right)^{\frac{1}{2}} - (r-R)^2 - z^2\right). \tag{100}$$

Therefore, from Eqs. (91) and (92), the stationary ground state solution of the GP equation is

$$\Phi_s(r, z, t) = \sqrt{n_0(r, z)}e^{-i\frac{\mu t}{\hbar}}. \tag{101}$$



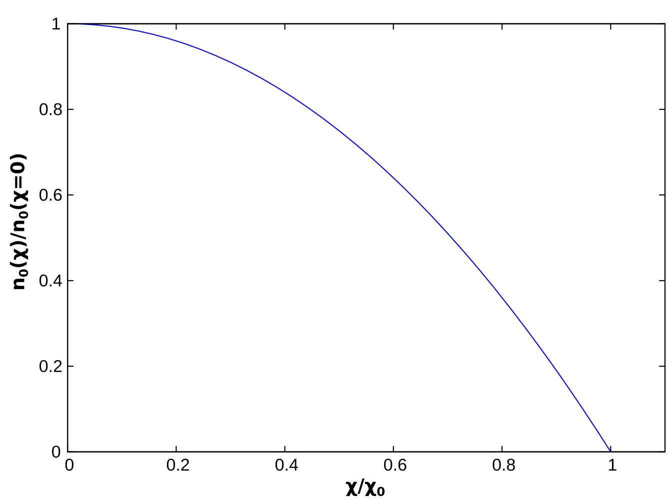


FIG. 1. Demonstrating the ring torus region of space (the blue region in the figure) within which all the BEC atoms are effectively trapped (left) and the Thomas-Fermi number density distribution (right).

C. Perturbative approach to the time dependent GP equation

With tight radial and axial components, the dynamics along the radial direction and the axial direction is "frozen" [36]. The problem becomes effectively one dimensional. Therefore, the wave function can be written as

$$\Phi = f(r,z)\psi(\varphi,t), \tag{102}$$

where $\int_B rdrdzf(r,z)^2 = 1$ and $\int_0^{2\pi} d\varphi|\psi|^2 = N$, where $B = \frac{V}{2\pi} = \pi\chi_0^2 R$,

$$\begin{aligned} i\hbar f(r,z)\frac{\partial\psi(\varphi,t)}{\partial t} = &-\frac{\hbar^2}{2m}\left(\frac{\psi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)f(r,z)\right. \\ &\left.+\frac{f}{r^2}\frac{\partial^2}{\partial\varphi^2}\psi(\varphi,t)+\psi\frac{\partial^2}{\partial z^2}f(r,z)\right) \\ &+\frac{1}{2}m\omega^2((r-R)^2+z^2)f\psi \\ &+g|f|^2|\psi|^2 f\psi. \end{aligned} \tag{103}$$

Now we insert the expression of the stationary ground state solution in the above equation as below

$$f(r,z) = \sqrt{\frac{2\pi}{N}}\sqrt{n_0(r,z)}. \tag{104}$$

Therefore, due to the T-F approximation, the first and third terms in the right-hand side of Eq. (105) vanishes. Hence

$$\begin{aligned} i\hbar f\frac{\partial\psi(\varphi,t)}{\partial t} = &-\frac{\hbar^2}{2m}\frac{f}{r^2}\frac{\partial^2}{\partial\varphi^2}\psi(\varphi,t) \\ &+\left(\frac{1}{2}m\omega^2((r-R)^2+z^2)+g|f|^2|\psi|^2\right)f\psi. \end{aligned} \tag{105}$$

The dependence on r and z can be projected out by multiplying the above equation by f^* and integrating the above equation over r and z within the volume B. We find

$$i\hbar\frac{\partial\psi}{\partial t} = -\left(\frac{\hbar^2}{2mR^2}\right)\frac{\partial^2\psi}{\partial\varphi^2}+\left(\frac{\mathfrak{n}}{12\bar{n}_0}\right)m\omega^2\chi_0^2\psi+\tilde{g}|\psi|^2\psi, \tag{106}$$

where $\tilde{g} = g\frac{2\pi\mathfrak{n}^2}{3N\bar{n}_0}$, $\mathfrak{n} = n_0(\chi = 0)$, and the average number density $\bar{n}_0 = \frac{N}{2\pi R\pi\chi_0^2}$. The factor $\frac{1}{R^2}$ in the first term of the right-hand side is appearing because we are considering the wave function to be concentrated around the minima circle of potential due to tightness of the trap [36]. Thus the problem becomes effectively one dimensional. The second term in the right-hand side of the equation is a constant shift in potential; we make it zero by translation in the potential. Therefore, finally we have

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2\psi}{\partial\varphi^2}+\tilde{g}|\psi|^2\psi. \tag{107}$$

We decompose ψ as

$$\psi = \psi_S + \psi'(\varphi,t), \tag{108}$$

where ψ_S corresponds to ψ in the ground state which is proportional to $e^{-i\frac{\mu t}{\hbar}}$. The number density $n(\varphi,t) = |\psi|^2 = \text{const} + n'(\varphi,t)$ and $\psi = \sqrt{n}e^{i\gamma(\varphi,t)}$, where $n'(\varphi,t)$ is the perturbation in number density which is not necessarily linear. The velocity along φ, v_φ is proportional to $\frac{\partial\gamma}{\partial\varphi}$. Putting this value of ψ and using the T-F approximation, we get classical inviscid irrotational fluid equations [30]

$$\partial_t\rho + \frac{1}{R}\frac{\partial}{\partial\varphi}(\rho v) = 0, \tag{109}$$

$$\boldsymbol{\nabla}\times\mathbf{v} = 0, \tag{110}$$

$$\frac{\partial v_\varphi}{\partial t}+\frac{v_\varphi}{R}\frac{\partial v_\varphi}{\partial\varphi} = -\frac{1}{\rho R}\frac{\partial p}{\partial\varphi}, \tag{111}$$

where $p = \frac{1}{2}\tilde{g}n^2$, and sound speed $c_{s0} = \sqrt{\frac{\tilde{g}n}{m}}$. We have

$$\rho = \text{const} + \rho' \tag{112}$$

and

$$v_\varphi = v'_\varphi. \tag{113}$$

After manipulations in same manner as before, we find

$$\partial_\mu(f^{\mu\nu}(\varphi,t)\partial_\nu)\zeta'(\varphi,t) = 0. \tag{114}$$

The greek indices in the above equation run over time t and the compact dimension, $\mathcal{R} = R\varphi$.

$$f^{\mu\nu}(\varphi,t) \equiv \frac{\rho}{c_s^2}\begin{bmatrix} -1 & -v'_\varphi \\ -v'_\varphi & c_s^2-(v'_\varphi)^2 \end{bmatrix} \tag{115}$$

and

$$(g_{\mu\nu})_{\text{eff}} \equiv \frac{\rho}{c_s}\begin{bmatrix} -(c_s^2-(v'_\varphi)^2) & -v'_\varphi \\ -v'_\varphi & 1 \end{bmatrix}. \tag{116}$$

Proceeding in the same fashion, we find

$$\tilde{g}_{\mu\nu}(\mathcal{R},t) = (\eta_A)_{\mu\nu} + h_{\mu\nu}(\mathcal{R},t) \equiv \begin{bmatrix} -c_{s0}^2\left(1+\frac{3}{2}\frac{\rho_1'}{\rho_0}\right) & -(v_\varphi')_1 \\ -(v_\varphi')_1 & \left(1+\frac{1}{2}\frac{\rho_1'}{\rho_0}\right) \end{bmatrix}. \tag{117}$$

This is very similar to Eq. (89). One can show in the same manner that

$$\left(-\frac{1}{c_{s0}^2}\partial_t^2 + \partial_{\mathcal{R}}^2\right)h_{\mu\nu}(\mathcal{R},t) = 0. \tag{118}$$

Here the difference from the previous cases is that the spatial dimension is a compact dimension. Any perturbation produced in the toroidal ring will propagate in clockwise and anticlockwise senses and eventually will superimpose with each other and thus a standing wave will be produced. Here we can view the scenario as the standing wave of the acoustic analog gravitational wave.

One can do the same kind of analysis in other tight traps of different geometries. The methods would be very similar to the method discussed in this section.

VI. SUMMARY AND CONCLUSIONS

We find that if one extends the perturbative method of analysis to study sound in the weakly nonlinear limit for irrotational inviscid barotropic flow in a static medium, one discovers that acoustic metric components satisfy the wave equation and in that sense, the acoustic geometry happens to get some similarities with the gravitational wave propagation in Minkowski spacetime, where the metric components also have wave nature. The transverse nature of the gravitational wave is missing in the acoustic metric; rather it describes the longitudinal wave in the acoustic analog of Minkowski spacetime. Our analysis also makes a connection between two seemingly different subjects; one is nonlinear acoustics and the other one is the study of emergent spacetime. In the weakly nonlinear limit, the acoustic analog of the gravitational wave is the emergent phenomena.

As Unruh's work shows, the acoustic metric components for linear perturbation depend on the known solution, i.e., the background solution over which perturbations in the fluid quantities are introduced. Therefore, in principle, one can design a time dependent system (the known solutions) in such a manner that, after linearizing the fluid quantities, the linear wave equation satisfied by the velocity potential or integrals of motion gives an acoustic metric similar to the metric describing propagation of the gravitational wave. That can be an another potential way to get an analogous gravitational wave, but to do that first one has to find out such realistic or experimentally possible systems. If one can find such a system, that can be thought of as another alternative analog model of the gravitational wave. The novelty of our work lies in two things. One is that, as completely different from Unruh's work [3], we have shown a way of getting a time dependent acoustic metric from a time independent known solution (or background solution) by introducing perturbations which are not linear. Second, we have made a connection between nonlinear acoustics in weak form with an analog model of a gravitational wave.

ACKNOWLEDGMENTS

The author is thankful to Mr. Subhroneel Chakrabarti, senior research fellow of Harish-Chandra Research Institute, India, for useful discussions and suggestions.

APPENDIX A: PERTURBATION IN MASS FLOW RATE

Mass flow rate is defined as the amount of mass of fluid passing through a unit area perpendicularly per unit time and that is why we first specify the direction of the disturbance and we consider the medium to be uniform in the absence of any perturbations,

$$p(z,t) = p_0 + p'(z,t) \qquad \rho(z,t) = \rho_0 + \rho'(z,t)$$
$$v^z = v'^z(z,t) \qquad v^x = 0 \qquad v^y = 0.$$

Therefore, mass flow rate $f = \rho v^z = f'(z,t) = \rho v'^z$. From the continuity equation and the Euler equation, the fluid equations can be written as

$$\frac{\partial\rho'}{\partial t} + \partial_z(f') = 0. \tag{A1}$$

$$\partial_{tt}v'^z + \partial_z(v'^z\partial_t v'^z) = \partial_z\left(\frac{c_s^2}{\rho}\partial_t f'\right). \tag{A2}$$

From the definition of the mass accretion rate

$$\partial_t v'^z = \frac{1}{\rho}(\partial_t f' + v'^z\partial_z f'). \tag{A3}$$

Therefore, using Eqs. (A1)–(A3), we get

$$\partial_\mu(f^{\mu\nu}(z,t)\partial_\nu)f'(z,t) = 0 \tag{A4}$$

where

$$f^{\mu\nu}(z,t) \equiv \frac{1}{\rho}\begin{bmatrix} -1 & -v'^z \\ -v'^z & c_s^2\delta^{ij} - (v'^z)^2 \end{bmatrix}. \tag{A5}$$

The above matrix is a 2×2 matrix because we have chosen the direction of the perturbation first. One can not derive $g^{\mu\nu}$ from $f^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ for 2×2 matrices. As the problem is intrinsically $3+1$ dimensional, we conventionally use the same $g^{\mu\nu}$ obtained from the wave equation of Bernoulli's constant. As a result, the rest of the analysis becomes the same.

APPENDIX B: PROPAGATION OF SOUND IN A MEDIUM

We assume that the pressure in the medium is a function of density only even in the presence of perturbation in the medium. Hence, the barotropic condition is given by

$$p = F(\rho), \tag{B1}$$

where $p \propto \rho$ for the isothermal relation and $p \propto \rho^\gamma$ ($\gamma \neq 1$) for the adiabatic relation between pressure and density

$$p(\mathbf{x}, t) = p_0 + p'(\mathbf{x}, t) \tag{B2}$$

$$\rho(\mathbf{x}, t) = \rho_0 + \rho'(\mathbf{x}, t). \tag{B3}$$

p_0 and ρ_0 are the background value of pressure and density, which can be treated as constant numbers, i.e., these quantities neither depend on time nor on position.

Due to the barotropic condition, perturbation of pressure induces the density to change. Therefore, the change in density does not happen independently. The medium is static. From either of the continuity equation (1) or momentum equation (2), there has to be a change in velocity as well due to this change in pressure. Therefore, it is sufficient to introduce change in pressure in the medium, and that change in pressure will cause a change in density which will cause a change in the velocity field.

Therefore, the velocity of the medium can be written as

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}'(\mathbf{x}, t). \tag{B4}$$

Velocity change is the rate of change of displacement $\boldsymbol{\delta}$ in unit time of the fluid elements from their position of equilibrium (the position in the absence of disturbance in the fluid system) (Chap. 6 of Ref. [15]),

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\delta}(\mathbf{x}, t)}{\partial t} = \frac{d\delta}{dt}. \tag{B5}$$

The second equality in Eq. (B5) is due to the zero unperturbed velocity. Therefore, the event can be described as follows. First, there is a mechanical external agent (for example, the moving boundary in the experimental setup depicted in Ref. [28]), which undulates the medium by transfer of mechanical energy, i.e., the fluid elements are displaced from their positions of equilibrium, causing pressure and density in the medium to change. This mechanical energy source is outside our region of interest where we are considering the momentum equation and the continuity equation of fluid. This is exactly like considering the source free Einstein equation in weak field approximation while studying a gravitational wave. There is of course a source of energy but that thing is not situated in the region where we are writing down the equations.

The Bernoulli's constant, defined for irrotational inviscid barotropic flow, is written as

$$\zeta = \left(\frac{1}{2}\mathbf{v}^2 + \int \frac{dp}{\rho}\right). \tag{B6}$$

The first term in the above expression is the kinetic energy per unit mass of a fluid element and the second term can be thought of as the pressure energy, the enthalpy per unit mass, arising from the random motion of the constituent fluid molecules. Equation (4) implies that the energy from the external mechanical agent is expelled in two ways: one is that the bulk velocity of each fluid element is changed causing the kinetic energy of each fluid element to increase, and also the kinetic energy due to random motion of the constituent particles in a fluid element is changed, causing a change in the enthalpy.

In the absence of disturbance in such a static medium, Bernoulli's constant is a conserved quantity, given by (Refs. [15,27])

$$\zeta_0 = \frac{c_{s0}^2}{\gamma - 1} \tag{B7}$$

for the adiabatic relation between pressure and density,

$$\zeta_0 = c_{s0}^2 \ln(\rho_0), \tag{B8}$$

for the isothermal relation between pressure and density. Therefore, Bernoulli's constant and mass flow rate (conserved quantity derived from the continuity equation) govern the steady state flow. That is why studying the change in these two quantities due to disturbances in the medium is important.

APPENDIX C: PERTURBATION METHOD AND ITS LIMITATION

In this Appendix, we have shown some calculations explicitly. Let us consider a continuous and differentiable function of $\rho(\mathbf{x}, t)$, $\mathcal{F}(\rho)$. ρ is given by Eq. (B1). Therefore, using rules of partial derivatives

$$\partial_t \mathcal{F} = \frac{d\mathcal{F}}{d\rho}\partial_t \rho = \frac{d\mathcal{F}}{d\rho}\partial_t \rho', \tag{C1}$$

because ρ_0 in expression (B3) is not a function of time.The second equality in the above expression can also be shown by expanding $\mathcal{F}$ in Taylor series of ρ' around ρ_0 and then by taking a partial derivative in time. We have defined c_s^2 as

$$c_s^2 = \frac{dp}{d\rho} = \frac{dF}{d\rho}. \tag{C2}$$

Therefore, using Eqs. (B1) and (C1), we find

$$\partial_t p = c_s^2 \partial_t \rho \Rightarrow \partial_t p' = c_s^2 \partial_t \rho'. \tag{C3}$$

The enthalpy, defined for barotropic flow, is given by

$$h(\rho) = \int \frac{dp}{\rho} = \int \frac{c_s^2 d\rho}{\rho}. \tag{C4}$$

Therefore, we have

$$\partial_t h = \frac{c_s^2}{\rho} \partial_t \rho \Rightarrow \partial_t \rho = \frac{c_s^2}{\rho} \partial_t \rho', \tag{C5}$$

$$\Rightarrow \partial_t \zeta' = \mathbf{v}'.\partial_t \mathbf{v}' + \frac{c_s^2}{\rho} \partial_t \rho'. \tag{C6}$$

This is how the quantities like enthalpy and Bernoulli's constant are automatically changed when there is a change in density or pressure due to a disturbance in the medium.

Let us evaluate h in the presence of perturbation in the system, in powers of ρ' by using Taylor series,

$$\begin{aligned} h(\rho) &= h(\rho_0 + \rho') \\ &= h(\rho_0) + \frac{dh}{d\rho}\bigg|_{\rho=\rho_0} \rho' + \frac{1}{2!} \frac{d^2 h}{d\rho^2}\bigg|_{\rho=\rho_0} \rho'^2 \\ &\quad + \frac{1}{3!} \frac{d^3 h}{d\rho^3}\bigg|_{\rho=\rho_0} \rho'^3 + \cdots \end{aligned} \tag{C7}$$

$\frac{dh}{d\rho}|_{\rho=\rho_0} = \frac{c_{s0}^2}{\rho_0}$ from Eq. (C4). For the adiabatic case, $p = K\rho^\gamma$, K being proportionality constant. Therefore,

$$\frac{d^n}{d\rho^n}\left(\frac{c_s^2}{\rho}\right) = c_s^2 \rho^{-(n+1)} \prod_{i=1}^{i=n} (\gamma - 1 - i). \tag{C8}$$

From Eq. (C7), we have

$$\begin{aligned} h(\rho) &= h_0 + h' \\ &= \frac{c_{s0}^2}{\gamma - 1} \\ &\quad + c_{s0}^2 \left[\frac{\rho'}{\rho_0} + \frac{\gamma - 2}{2!}\left(\frac{\rho'}{\rho_0}\right)^2 + \frac{(\gamma-2)(\gamma-3)}{3!}\left(\frac{\rho'}{\rho_0}\right)^3 + \cdots\right]. \end{aligned} \tag{C9}$$

Therefore, we have

$$\begin{aligned} \zeta &= \frac{1}{2}\mathbf{v}'.\mathbf{v}' + \frac{c_{s0}^2}{\gamma - 1} \\ &\quad + c_{s0}^2 \left[\frac{\rho'}{\rho_0} + \frac{\gamma - 2}{2!}\left(\frac{\rho'}{\rho_0}\right)^2 + \frac{(\gamma-2)(\gamma-3)}{3!}\left(\frac{\rho'}{\rho_0}\right)^3 + \cdots\right]. \end{aligned} \tag{C10}$$

For the isothermal case, $p \propto \rho$, we have $c_s^2 = \text{constant} = c_{s0}^2$. Therefore, we have in the similar way,

$$\begin{aligned} \zeta &= \frac{1}{2}\mathbf{v}'.\mathbf{v}' + c_{s0}^2 \ln(\rho_0) \\ &\quad + c_{s0}^2 \left[\frac{\rho'}{\rho_0} - \frac{1}{2}\left(\frac{\rho'}{\rho_0}\right)^2 + \frac{1}{3}\left(\frac{\rho'}{\rho_0}\right)^3 - \cdots\right]. \end{aligned} \tag{C11}$$

Therefore, the order of magnitude of the terms in ζ depends on the magnitude of the perturbations.

In the perturbation method, one writes

$$\rho(\mathbf{x}, t) = \rho_0 + \rho' = \rho_0 + \rho'_{(1)}(\mathbf{x}, t) + \rho'_{(2)}(\mathbf{x}, t)\ldots \tag{C12}$$

$$\mathbf{v}'(\mathbf{x}, t) = \mathbf{v}'_{(1)}(\mathbf{x}, t) + \mathbf{v}'_{(2)}(\mathbf{x}, t) + \cdots \tag{C13}$$

such that $\rho_0 \gg |\rho'_{(1)}| \gg |\rho'_{(2)}| \gg \ldots$ and $|\mathbf{v}'_{(1)}| \gg |\mathbf{v}'_{(2)}| \gg \ldots$, $|\rho'_{(1)}|$, $|\mathbf{v}'_{(1)}|$ are the terms having order of magnitude of ϵ, ϵ being a small dimensionless number ($\epsilon < 1$); $|\rho'_2|$ and $|\mathbf{v}'_{(2)}|$ are the terms having magnitude of the order of ϵ^2, and so on. As a result, we can expand the other quantities, depending on the density perturbation and velocity perturbation, as below

$$p(\mathbf{x}, t) = p_0 + p'_{(1)}(\mathbf{x}, t) + \cdots \tag{C14}$$

$$c_s(\mathbf{x}, t) = c_{s0} + c_{s(1)}(\mathbf{x}, t) + \cdots \tag{C15}$$

$$c_s^2(\mathbf{x}, t) = c_{s0}^2 + c_{s(1)}^2(\mathbf{x}, t) + \cdots \tag{C16}$$

$$\zeta'(\mathbf{x}, t) = \zeta'_{(1)}(\mathbf{x}, t) + \zeta'_{(2)}(\mathbf{x}, t) + \cdots \tag{C17}$$

Therefore, from Eq. (C10), we find for the adiabatic case,

$$\zeta'_{(1)} = c_{s0}^2 \frac{\rho'_{(1)}}{\rho_0}, \tag{C18}$$

$$\zeta'_{(2)} = \frac{1}{2}\mathbf{v}'_{(1)}.\mathbf{v}'_{(1)} + c_{s0}^2 \frac{\gamma - 2}{2!}\left(\frac{\rho'_{(1)}}{\rho_0}\right)^2 + c_{s0}^2 \frac{\rho'_{(2)}}{\rho_0}, \tag{C19}$$

$$\begin{aligned} \zeta'_{(3)} &= \mathbf{v}'_{(1)}.\mathbf{v}'_{(2)} + c_{s0}^2 \frac{(\gamma-2)(\gamma-3)}{3!}\left(\frac{\rho'_{(1)}}{\rho_0}\right)^3 \\ &\quad + c_{s0}^2 \frac{\gamma - 2}{2!}\left(\frac{2\rho'_{(1)}\rho'_{(2)}}{\rho_0^2}\right) + c_{s0}^2 \frac{\rho'_{(3)}}{\rho_0}, \end{aligned} \tag{C20}$$

and so on. Similarly, for the isothermal case,

$$\zeta'_{(1)} = \frac{c_{s0}^2}{\rho_0}\rho'_{(1)}, \tag{C21}$$

$$\zeta'_{(2)} = \frac{1}{2}\mathbf{v}'_{(1)}.\mathbf{v}'_{(1)} - \frac{c_{s0}^2}{2}\left(\frac{\rho'_{(1)}}{\rho_0}\right)^2 + c_{s0}^2 \frac{\rho'_{(2)}}{\rho_0}, \tag{C22}$$

$$\zeta'_{(3)} = \mathbf{v}'_{(1)}.\mathbf{v}'_{(2)} + c_{s0}^2 \frac{1}{3}\left(\frac{\rho'_{(1)}}{\rho_0}\right)^3 - c_{s0}^2 \frac{1}{2}\left(\frac{2\rho'_{(1)}\rho'_{(2)}}{\rho_0^2}\right) + c_{s0}^2 \frac{\rho'_{(3)}}{\rho_0}, \quad \text{(C23)}$$

and so on. In the same way, the other quantities like c_s^2, c_s, p, etc. can be expanded in terms of density perturbation and velocity perturbation.

Depending on the magnitude of the perturbation, one has to consider the higher order terms. For example, in the case of blast waves and shock waves, one has to consider the higher order terms and possibly one has to solve it numerically order by order in perturbation theory to match the phenomena (finding the solution of such a wave is not taken into account in this paper). It might happen that the higher order terms may appear to be bigger in magnitude or having magnitude of same order as the lower order terms. In that case, the perturbation method will fail, one has to consider different techniques. For example, the similarity solution technique is commonly used to describe dynamics of a spherical blast wave (Refs. [15,27]). As we are working in the weakly nonlinear limit (slight improvement over linear limit), perturbation theory is useful here. Weakly nonlinear waves are also studied in the literature (Refs. [24,25]). Weakly nonlinear waves have experimental significance too [28].

In the weakly nonlinear limit, one neglects terms having magnitude higher than ϵ^2. Therefore, in the weakly nonlinear limit, for the adiabatic case, ζ' can be written as

$$\zeta' = \zeta'_{(1)} + \zeta'_{(2)} = \frac{c_{s0}^2}{\rho_0}\rho'_{(1)} + \frac{1}{2}\mathbf{v}'_{(1)}.\mathbf{v}'_{(1)} + c_{s0}^2 \frac{\gamma - 2}{2!}\left(\frac{\rho'_{(1)}}{\rho_0}\right)^2 + c_{s0}^2 \frac{\rho'_{(2)}}{\rho_0}. \quad \text{(C24)}$$

For the isothermal case, in the weakly nonlinear limit,

$$\zeta' = \zeta'_{(1)} + \zeta'_{(2)} = \frac{c_{s0}^2}{\rho_0}\rho'_{(1)} + \frac{1}{2}\mathbf{v}'_{(1)}.\mathbf{v}'_{(1)} - \frac{c_{s0}^2}{2}\left(\frac{\rho'_{(1)}}{\rho_0}\right)^2 + c_{s0}^2 \frac{\rho'_{(2)}}{\rho_0}. \quad \text{(C25)}$$

We have shown in the paper that, in the weakly linear limit, the variation of ζ' can be described in a way as if ζ' is a massless scalar field in a spacetime background, and that spacetime, i.e., the emergent spacetime, has some similarities with the real spacetime corresponding to the gravitational wave in Minkowski spacetime.

We restrict ourselves to weakly nonlinear perturbation over a steady state flow and that steady state flow does not have any explicit time dependency, rather the flow is taken to be independent of position in some cases. In the case of linear perturbation, $\zeta' = \zeta'_{(1)}$. Using the perturbation method, we get from Eq. (18), expanding terms up to first order in smallness,

$$\partial_\mu (f_0^{\mu\nu} \partial_\nu) \zeta'_{(1)} = 0. \quad \text{(C26)}$$

One gets an emergent metric, analogous to the Minkowski spacetime metric, $(\eta_A)_{\mu\nu}$ of Eq. (23), from $f_0^{\mu\nu}$. As we are considering stationary background medium, within our scope, it is possible to get a time dependent acoustic metric if and only if nonlinear perturbation is introduced in the flow, wherefore to get an analog gravitational wave, one has to work with nonlinearity in weak form.

APPENDIX D: PERTURBATION IN VELOCITY POTENTIAL

From the irrotationality condition,

$$\mathbf{v} = -\nabla \psi. \quad \text{(D1)}$$

Therefore, from Eq. (3), we find

$$-\frac{\partial \psi}{\partial t} + \zeta = -\frac{\partial \psi}{\partial t} + \left(\frac{1}{2}\mathbf{v}^2 + \int \frac{dp}{\rho}\right) = 0$$

and by subtracting the equation for the unperturbed quantities,

$$\Rightarrow -\frac{\partial \psi'}{\partial t} + \zeta' = -\frac{\partial \psi'}{\partial t} + \frac{1}{2}\mathbf{v}'^2 + h' = 0, \quad \text{(D2)}$$

where h and h' are defined as enthalpy and its variation by Eqs. (C4) and (C9), respectively. From Eq. (1),

$$\frac{\partial \rho}{\partial t} = \nabla.(\rho \nabla \psi)$$
$$\Rightarrow \frac{\partial \rho'}{\partial t} = \nabla.(\rho \nabla \psi') \quad \text{(D3)}$$

After some manipulations, one finds that

$$-\frac{\rho}{c_s^2}\partial_{tt}\psi' + \nabla.(\rho \nabla \psi') + \frac{\rho}{c_s^2}\mathbf{v}'.\partial_t \nabla \psi' = 0. \quad \text{(D4)}$$

This above equation can never be put into the form of $\partial_\mu (f^{\mu\nu}\partial_\nu)\psi' = 0$ (where, from $f^{\mu\nu}$, one derives the time dependent acoustic metric), as opposed to the equations satisfied by the perturbation of Bernoulli's constant ζ' [Eq. (7)] and mass accretion rate f' [Eq. (A4)].

This is the reason, unlike Unruh's case [3], we choose to analyze the nonlinear wave equation for the variation of the integrals of motion instead of velocity potential in the system to find out the acoustic metric.

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