Einstein-Cartan-Dirac equations in the Newman-Penrose formalism

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(Received 29 June 2018; published 24 September 2018)

We formulate the Einstein-Cartan-Dirac equations in the Newman-Penrose (NP) formalism, thereby presenting a more accurate and explicit analysis of previous such studies. The equations show in a transparent way how the Einstein-Dirac equations are modified by the inclusion of torsion. In particular, the Hehl-Datta equation is presented in NP notation. We then describe a few solutions of the Hehl-Datta equation on Minkowski space-time, and in particular report a solitonic solution which removes the unphysical behavior of the corresponding Dirac solution. The present work serves as a prelude to similar studies for nondegenerate Poincaré gauge gravity.

DOI: 10.1103/PhysRevD.98.064046

I. INTRODUCTION

Einstein's general theory of relativity (GR)—published in 1915—has been described as the most beautiful of all the existing physical theories [1]. The background space-time on which classical GR is formulated is a Riemannian manifold (denoted by V_4) which is torsionless. In this case, the affine connection coincides uniquely with the Levi-Civita connection and geodesics coincide with the path of shortest distance. This is, however, not generally true for other, *torsional* manifolds, such as the manifold on which the Einstein-Cartan-Sciama-Kibble (ECSK)—or simply, Einstein-Cartan (EC)—theory is formulated. In such a theory, the geometrical structure of the manifold is modified such that the affine connection is no longer required to be symmetric, and no longer coincides uniquely with the Levi-Civita connection [2–7].

Torsion, as the antisymmetric part of the affine connection, was introduced by Cartan [4]. Also termed U_4 theories of gravitation, Einstein-Cartan theories work with an underlying manifold that is non-Riemannian (unlike classical GR which is formulated on V_4). The non-Riemannian part of the manifold is associated with the spin density of matter, which plays the role of a source analogous to the role of mass in Riemannian curvature. Here, mass and spin *both* play a dynamical role. While mass "adds up" on classical length scales due to its monopole character, spin, being of dipole character, usually averages out in the absence of external forces.

For this reason, matter, in the macrophysical regime, can be dynamically characterized entirely by the energy-momentum tensor. In the microregime, heuristic arguments suggest that a spin density tensor plays an analogous role for spin, and related, as with mass and curvature, to some other geometrical property of space-time. It is this requirement that EC/ECSK theory satisfies (the reader is referred to [2] for a detailed treatment). When we minimally couple the Dirac field on U_4, we term this *Einstein-Cartan-Dirac (ECD) theory*. There are two independent geometric fields—the metric and torsion—and one matter field ψ in this theory. Varying the corresponding Lagrangian, we get three equations of motion, corresponding to the modified Einstein field equations, modified Dirac equation, and a torsional coupling. On U_4, the Dirac equation becomes nonlinear; and is known as the *Hehl-Datta (HD) equation* after the seminal work in [3].

The usual method in approaching solutions to problems in GR is to use a *local coordinate basis* $\hat{e}^\mu$ such that $\hat{e}^\mu = \partial_\mu$. This coordinate basis field is covariant under general coordinate transformations. However, it has been found useful to employ noncoordinate basis techniques in problems involving spinors. Moreover, choosing the tetrad basis vectors as *null vectors* is extremely useful in certain situations. This formalism—where a given theory is expressed in a basis of null tetrads—is the celebrated Newman-Penrose (NP) formalism. In this formalism, we replace tensors by their null tetrad components and represent these components with certain distinctive symbols. Most of the important and physically relevant geometrical

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objects and identities (e.g., the Riemann curvature tensor, Weyl tensor, Bianchi identities, Ricci identities etc.) on U_4 have been formulated in the NP formalism (such as in [8]).

It can be shown that there is a natural connection between spin dyads (a detailed account of spin dyads can be found in [9]) and null tetrads [9,10]. Physical systems involving spinor fields can be fully expressed in the NP formalism (e.g., the Dirac equation on V_4 has been studied extensively, Ref. Chap. 12 in [9]). In addition, many systems in gravitational physics are also studied in the NP formalism [9]. It appears that the NP formalism is the shared vocabulary between the physics of relativistic quantum mechanical systems (with spinor fields) and classical gravitational systems (having a metric and/or torsion). Apart from NP formalism, there are other approaches to the problem of a Dirac field in a Riemann-Cartan manifold. One such literature uses Foldy-Wouthyusen transformation for studying the relativistic Dirac fermion interacting with general electromagnetic fields in Riemann-Cartan spacetimes [11,12].

In the present paper, we aim to formulate the full ECD equations in the NP formalism. We know that the contortion (which is also spelled as contorsion) tensor is completely expressible in terms of the Dirac state [2]. We wish to then find expressions for the contortion spin coefficients—which are the standard NP variables that account for spin—explicitly in terms of the Dirac state. Using this, we can write the complete set of HD equations in the NP formalism. In a sense, this work is to be read as a sequel to the work of Chandrasekhar in [9] (see Chap. 12), where the Dirac equation on V_4 has been given a full treatment in the NP formalism. Some recent works [13–15] attempt to do that but have not provided explicit corrections to the standard NP variables due to torsion. Further, there are notational and sign inconsistencies in many such examples of existing literature in the field, and we aim to provide a comprehensive and self-contained treatment.

Finally, we attempt to find solutions to the HD equations in a Minkowski space with torsion. This, apart from being the simplest case to consider, is also motivated by certain physical intuitions which can be considered as supporting, but nonessential, corollaries to this work. A recent essay [16–18] suggests the incorporation of a new length scale in quantum gravity, thereby providing a symmetry between large and small masses; a conjecture has been proposed therein to establish a duality between these two limits. This conjecture is predicated on the necessary existence of solutions to the Hehl-Datta equations on Minkowski space, representing the balance between the Riemannian and torsional effects which reduce to small and large masses in the respective limits. However, notwithstanding the duality conjecture and the new length scale proposed, our results hold for the standard theory as well. All equations are expressed in terms of two relevant generic length scales, $l_1 = L_{\rm Pl}$ and $l_2 = \frac{1}{2}\lambda_C$, the first being Planck length, and the second being one half the Compton wavelength. In case of the modified ECD theory with a new length scale $L_{\rm CS}$ (as defined below), we will instead have $l_1 = l_2 = L_{\rm CS}$: the Planck length and Compton wavelength no longer appear in the ECD equations, and are both replaced by $L_{\rm CS}$.

A. Notation and conventions

The following conventions are in use for the remainder of this paper:

(i) The Lorentz signature used is $(+ - - -)$ throughout.

(ii) V_4 is a nontorsional space-time, while a space-time endowed with torsion is specified by U_4.

(iii) Greek indices, e.g., α, ζ, δ refer to world components, which transform according to *general coordinate transformations* and are raised or lowered using the metric $g_{\mu\nu}$.

(iv) Latin indices within parenthesis e.g., (a) or (i) are tetrad indices, which transform according to *local Lorentz transformations* in the flat tangent space, and are raised or lowered using $\eta_{(i)(k)}$.

(v) Latin indices (without parenthesis) e.g., i, j, b, c indicate objects in Minkowski space, which transform according to *global Lorentz transformations*.

(vi) In general 0,1,2,3 refer to world indices while (0), (1),(2),(3) refer to tetrad indices.

(vii) The total covariant derivative is denoted by ∇, while $\{\}$ denotes the Christoffel connection. Correspondingly, $\nabla^{\{\}}$ represents a covariant derivative with respect to the Christofel connections.

(viii) Commas $(,)$ indicate partial derivatives while semicolons $(;)$ indicate the Riemannian covariant derivative. Thus, for tensors, ; and $\nabla^{\{\}}$ are same, while for spinors, $(;)$ involves both partial derivatives and the Riemannian part of the spin connection, γ, as defined in the following.

(ix) The four component Dirac spinor is written as

$$\psi = \begin{bmatrix} P^A \\ \bar{Q}_{B'} \end{bmatrix} \tag{1}$$

where P^A and $\bar{Q}_{B'}$ are two dimensional complex vectors in $\mathbb{C}^2$ space. We redefine the spinors as $P^0 = F_1$, $P^1 = F_2$, $\bar{Q}^{1'} = G_1$, and $\bar{Q}^{0'} = -G_2$. This is in accordance with our primary source [9]; the notations, conventions, and representations wherein are generally adhered to in this paper.

II. EINSTEIN-CARTAN THEORY AND ITS COUPLING TO THE DIRACH FIELD

A. Einstein-Cartan theory

In the Einstein-Cartan theory, the Riemannian manifold of ordinary GR (V_4) is promoted to the corresponding non-Riemannian manifold U_4. As discussed, this latter manifold admits, in addition to the structure of ordinary GR, a nonvanishing torsion. Torsion is a (rank 3) tensorial object defined as the antisymmetric part of the affine connection:

$$Q_{\alpha\beta}{}^{\mu} = \Gamma_{[\alpha\beta]}{}^{\mu} = \frac{1}{2}(\Gamma_{\alpha\beta}{}^{\mu} - \Gamma_{\beta\alpha}{}^{\mu}). \tag{2}$$

Similarly, the *contortion* tensor $K_{\alpha\beta}{}^{\mu}$ is given by $K_{\alpha\beta}{}^{\mu} = -Q_{\alpha\beta}{}^{\mu} - Q^{\mu}{}_{\alpha\beta} + Q_{\beta}{}^{\mu}{}_{\alpha}$. This allows us to write—in terms of the usual Christoffel symbols—the following relation:

$$\Gamma_{\alpha\beta}{}^{\mu} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} - K{}_{\alpha\beta}{}^{\mu}. \tag{3}$$

When a matter field ψ is minimally coupled with gravity and torsion, its action is given as follows [2]:

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_m(\psi, \nabla\psi, g) - \frac{1}{2k}R(g, \partial g)\right]. \tag{4}$$

Here $k = 8\pi G/c^4$, $\mathcal{L}_m$ is the matter Lagrangian density, and the second term represents the Lagrangian density for the gravitational field. There are three fields in this Lagrangian: ψ, $g_{\mu\nu}$, and $K_{\alpha\beta\mu}$, representing the matter field, the metric, and the contortion, respectively. Varying the action with respect to these, one arrives at the following three field equations:

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi} = 0, \tag{5}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \tag{6}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta K_{\alpha\beta\mu}}. \tag{7}$$

Here, (5) leads us to the matter field equations on a curved space-time with torsion. The right-hand side of (6) is associated with $\sqrt{-g}kT_{\mu\nu}$ via the definition of the metric energy-momentum tensor $T_{\mu\nu}$. Similarly, the right-hand side of (7) is associated with $2\sqrt{-g}kS^{\mu\beta\alpha}$ where $S^{\mu\beta\alpha}$ is the spin density tensor. Together, these two yield the Einstein-Cartan field equations:

$$G^{\mu\nu} = k\Sigma^{\mu\nu}, \tag{8}$$

$$T^{\mu\beta\alpha} = kS^{\mu\beta\alpha}. \tag{9}$$

In (8), the $G^{\mu\nu}$ on the left-hand side is the asymmetric Einstein tensor built from the asymmetric connection, while $\Sigma^{\mu\nu}$ is the asymmetric canonical (total) energy momentum tensor, constructed out of the symmetric (metric) energy-momentum tensor and the spin density tensor. In (9), the so-called "modified" torsion $T^{\mu\beta\alpha}$ is the traceless part of the torsion tensor, and is algebraically related to $S^{\mu\beta\alpha}$ on the right. In the limit torsion $\rightarrow 0$, we recover classical GR—(9) vanishes, and (8) reduces to the Einstein field equations which couple the (symmetric) Einstein tensor to the (symmetric) metric energy-momentum tensor. It should also be noted that in the recent works [19], it was shown rigorously that in the structure of the Poincaré gauge (PG) theory of gravity (of which the EC theory is an example of), torsion couples only to the elementary particle spin and not to the orbital angular momentum under any circumstances.

B. EC coupling to the Dirac field

The theory generated from the minimal coupling of the Dirac field on U_4 is what we term *Einstein-Cartan-Dirac (ECD) theory*. In this theory, the matter field is the spinorial Dirac field ψ, for which the Lagrangian density is given by [20] (note the noncommuting covariant derivatives)

$$\mathcal{L}_m = \frac{i\hbar c}{2}(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi) - mc^2\bar{\psi}\psi. \tag{10}$$

In ECD theory, the addition of spin degrees of freedom necessitates a more careful treatment of anholonomic objects. As we define the affine connection, Γ, to facilitate parallel transport of geometrical objects with world (Greek) indices, so do we define the spin connection γ for anholonomic objects (with Latin indices). The affine connection can be decomposed into a Riemannian ($\{\}$) and a torsional part (made up of the contortion tensor, K) and similarly, the spin connection γ can also be decomposed into a Riemannian (γ^o) and torsional part (once again, formed of the contortion tensor). These components are related via the following equation (following the notation in [8]):

$$\gamma_{\mu}{}^{(i)(k)} = \gamma_{\mu}^{o(i)(k)} - K_{\mu}{}^{(k)(i)} \tag{11}$$

where $\gamma_{\mu}^{o(i)(k)}$ is Riemannian part and $K_{\mu}{}^{(k)(i)}$ is the torsional part. Using these, we define the covariant derivative for spinors, on V_4 and U_4:

$$\psi_{;\mu} = \partial_{\mu}\psi + \frac{1}{4}\gamma^{o}_{\mu(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi \quad (\text{on } V_4), \tag{12}$$

$$\nabla_{\mu}\psi = \partial_{\mu}\psi + \frac{1}{4}\gamma^{0}_{\mu(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{1}{4}K_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi \quad (\text{on } U_4). \tag{13}$$

Substituting this into (10) we obtain the explicit form of Lagrangian density; varying with respect to $\bar{\psi}$ as in (5) yields the Dirac equation on V_4 and U_4:

$$i\gamma^{\mu}\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 \quad (\text{on } V_4), \tag{14}$$

$$i\gamma^{\mu}\psi_{;\mu} + \frac{i}{4}K_{(a)(b)(c)}\gamma^{[(a)}\gamma^{(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 \quad (\text{on } U_4). \tag{15}$$

Next, we use (6) and Lagrangian density defined in (10) to obtain the gravitational field equations on V_4 and U_4:

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (\text{on } V_4), \tag{16}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \quad (\text{on } U_4). \tag{17}$$

Here, $T_{\mu\nu}$ is the metric EM tensor which is symmetric and defined as

$$\begin{aligned} T_{\mu\nu} &= \Sigma_{(\mu\nu)}(\{\}) \\ &= \frac{i\hbar c}{4}[\bar{\psi}\gamma_{\mu}\psi_{;\nu} + \bar{\psi}\gamma_{\nu}\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_{\nu}\psi - \bar{\psi}_{;\nu}\gamma_{\mu}\psi]. \end{aligned} \tag{18}$$

Equations (14) and (16) together form the system of equations of Einstein-Dirac theory. We now move to the full Einstein-Cartan-Dirac theory. Using the Lagrangian density defined in (10), we can define the spin density tensor:

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi. \tag{19}$$

Using (19) and (7), (15) can be simplified to give the Hehl-Datta equation [2,3]. This, along with (17) and the relation between the modified torsion tensor and spin density tensor, define the field equations of the Einstein-Cartan-Dirac theory:

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}, \tag{20}$$

$$T_{\mu\nu\alpha} = -K_{\mu\nu\alpha} = \frac{8\pi G}{c^4}S_{\mu\nu\alpha}, \tag{21}$$

$$i\gamma^{\mu}\psi_{;\mu} = +\frac{3}{8}L_{\mathrm{Pl}}^2\bar{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{mc}{\hbar}\psi, \tag{22}$$

where L_{Pl} is the Planck length.

III. INTRODUCING A UNIFIED LENGTH SCALE L_{CS} IN QUANTUM GRAVITY

Recent work [16,17] has provided motivation for unifying the Compton wavelength ($\frac{\lambda}{\hbar c}$) and Schwarzschild radius ($R_s = \frac{2GM}{c^2}$) of a point particle with mass m into one single length scale, the Compton-Schwarzschild length (L_{CS}). Such a treatment compels us to introduce torsion, and relate the Dirac field to the torsion field. An action principle has been proposed with this new length scale which permits the Dirac equation and the Einstein field equations as mutually dual limiting cases. The modified action proposed is as follows:

$$\frac{L_{\mathrm{Pl}}^2}{\hbar}S = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}L_{\mathrm{CS}}\bar{\psi}\psi + L_{\mathrm{CS}}^2\bar{\psi}i\gamma^{\mu}\nabla_{\mu}\psi\right]. \tag{23}$$

Using this new length scale, L_{CS}, we can rewrite the Einstein-Cartan-Dirac equations as [17]

$$G_{\mu\nu}(\{\}) = \frac{8\pi L_{\mathrm{CS}}^2}{\hbar c}T_{\mu\nu} + \left(\frac{8\pi L_{\mathrm{CS}}^2}{\hbar c}\right)^2\tau_{\mu\nu}, \tag{24}$$

$$T_{\mu\nu\gamma} = \frac{8\pi L_{\mathrm{CS}}^2}{\hbar c}S_{\mu\nu\gamma}, \tag{25}$$

$$i\gamma^a\psi_{;a} = +\frac{3}{8}L_{\mathrm{CS}}^2\bar{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi + \frac{1}{2L_{\mathrm{CS}}}\psi = 0. \tag{26}$$

A note on length scales: We use l to denote a length scale in the theory. For standard ECD theory, the two scales that appear are the Planck length $l_1 = L_{\mathrm{Pl}} = \sqrt{\frac{G\hbar}{c^3}}$, and half the Compton wavelength $l_2 = \frac{\lambda_C}{2} = \frac{\hbar}{2mc}$. For the modified ECD theory, we have $l_1 = l_2 = L_{\mathrm{CS}}$, in terms of the new unified length scale.

IV. THE NEWMAN-PENROSE FORMALISM AND ECD IN NP

A. Tetrads

It is common in the literature [9] to use tetrads (or *vierbeins*) to define spinors on a curved space-time (in V_4 as well as U_4).[1] In this formalism, the transformation properties of spinors are defined in a flat (Minkowski) space, locally tangent to U_4. At each point in space-time, we can define a coordinate basis vector field $\hat{e}^{\mu} = g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}$ [9] which is covariant under general coordinate transformations, with $g^{\mu\nu}$ being the metric. The basis vectors associated with spinors, however, are covariant under *local* Lorentz transformations. Hence, we define, at each point of our manifold, a set of four orthonormal basis vectors (forming the tetrad field) given by $\hat{e}^{(i)}(x)$. These comprise

[1]While this is often the case, there are other formalisms that can be used [21].

four vectors (one for each μ) at each point, and the tetrad field is governed by the relation $\hat{e}^{(i)}(x) = e_\mu^{(i)}(x)\hat{e}^\mu$ where $e_\mu^{(i)}$ is the transformation matrix.

The following convolution relation follows:

$$g_{\mu\nu} = e_\mu^{(i)} e_\nu^{(k)} \eta_{(i)(k).} \tag{27}$$

The inverse of transformation matrix viz. $e_{(i)}^\mu$ follows:

$$g^{\mu\nu} = e_{(i)}^\mu e_{(k)}^\nu \eta^{(i)(k)} \quad \text{and} \quad e_{(i)}^\mu = g^{\mu\nu}\eta_{(i)(k)} e_\nu^{(k)}. \tag{28}$$

The transformation matrix $e_\mu^{(i)}$ allows us to convert the components of any world tensor (a tensor which transforms according to general coordinate transformation) to the corresponding components in local Minkowskian space (these latter components being covariant under local Lorentz transformation). Greek indices are raised or lowered using the metric $g_{\mu\nu}$, while the Latin indices are raised or lowered using $\eta_{(i)(k)}$. Parentheses around indices is a matter of convention (see "Notations and conventions" in the Introduction). In general, given a world tensor $W_{\mu\nu}$, its corresponding components $W_{(i)(j)}$ in the flat tangent manifold can be obtained using a tetrad transformation matrix such that

$$W_{(i)(j)} = e_{(i)}^\mu e_{(j)}^\nu W_{\mu\nu}. \tag{29}$$

B. Introduction to the NP formalism

The Newman-Penrose (NP) formalism was formulated by Newman and Penrose in their work [22]. It is a special case of the tetrad formalism, in which we choose our tetrad as a set of four null vectors:

$$e_{(0)}^\mu = l^\mu, \qquad e_{(1)}^\mu = n^\mu, \qquad e_{(2)}^\mu = m^\mu, \qquad e_{(3)}^\mu = \bar{m}^\mu \tag{30}$$

where l^μ, n^μ are real and m^μ, $\bar{m}^\mu$ are complex. The null tetrad indices are raised and lowered using the flat space-time metric

$$\eta_{(i)(j)} = \eta^{(i)(j)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{31}$$

and the tetrad vectors satisfy the equation $g_{\mu\nu} = e_\mu^{(i)} e_\nu^{(j)} \eta_{(i)(j)}$. In this formalism, we replace tensors by their tetrad components and represent these components with a collection of distinctive symbols which are now standard in the literature.

C. Spinor analysis

We define four null tetrads (and their corresponding covectors) on Minkowski space (raised and lowered using $\eta_{\mu\nu}$):

$$l^a = \frac{1}{\sqrt{2}}(1,0,0,1), \qquad m^a = \frac{1}{\sqrt{2}}(0,1,-i,0),$$
$$\bar{m}^a = \frac{1}{\sqrt{2}}(0,1,i,0), \qquad n^a = \frac{1}{\sqrt{2}}(1,0,0,-1). \tag{32}$$

We also define the following Van der Waarden symbols:

$$\sigma^a = \sqrt{2}\begin{bmatrix} l^a & m^a \\ \bar{m}^a & n^a \end{bmatrix}, \qquad \tilde{\sigma}^a = \sqrt{2}\begin{bmatrix} n^a & -m^a \\ -\bar{m}^a & l^a \end{bmatrix}. \tag{33}$$

For the Dirac gamma matrices, we use the complex version of the Weyl (chiral) representation:

$$\gamma^a = \begin{bmatrix} 0 & (\tilde{\sigma}^a)^* \\ (\sigma^a)^* & 0 \end{bmatrix} \quad \text{where} \quad \gamma^0 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix},$$
$$\gamma^i = \begin{bmatrix} 0 & (-\sigma^i)^* \\ (\sigma^i)^* & 0 \end{bmatrix} \tag{34}$$

where $a = (0,1,2,3)$.

The complex Weyl representation is used so that the Dirac bispinor and gamma matrices defined in (1) and (34) remain consistent with Eqs. (97) and (98) of Sec. 103 in [9] [comparing with our standard reference, [9], we recover Eq. (99) in complex form].

In order to represent spinorial objects (objects comprising spinors and gamma matrices) on a curved space-time, we use the following prescription on the tetrad formalism [10], viz. let $\mathcal{M}$ be a curved manifold with all conditions necessary for the existence of spin structure, and let U be a chart on $\mathcal{M}$ with coordinate functions (x^α). Then, for representing spinorial objects, we (i) choose an orthonormal tetrad field $e_{(a)}^\mu(x^\alpha)$ on U, (ii) define the Van der Waarden symbols $\sigma^{(a)}$ and $\tilde{\sigma}^{(a)}$ in this tetrad basis exactly as defined on Minkowski space in (33) and choose a γ representation (34), (iii) then, the σ's in a local coordinate frame are obtained via

$$\sigma^\mu(x^\alpha) = e_{(a)}^\mu(x^\alpha)\sigma^{(a)} = \sqrt{2}\begin{bmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{bmatrix},$$
$$\tilde{\sigma}^\mu = e_{(a)}^\mu(x^\alpha)\tilde{\sigma}^{(a)} = \sqrt{2}\begin{bmatrix} n^\mu & -m^\mu \\ -\bar{m}^\mu & l^\mu \end{bmatrix} \tag{35}$$

with the γ matrices obeying a similar transformation.

Thus, objects with world indices (containing world-indexed γ matrices or spinors) are now functions of chosen orthonormal tetrads. These are defined *a priori* in a local tetrad basis (with components identical to those defined on a flat Minkowski space-time) and *then* carried into a curved space via the tetrads. This is unlike other geometrical world objects which are first defined naturally at a point in a manifold and subsequently carried to a local tangent space via tetrads. We now aim to carry the Dirac equation (in NP) on V_4 into the U_4 space, building upon Sec. 102(d) of [9]. In order to calculate the covariant derivative of a spinor in U_4, we require the spinor affine connection coefficients. They are defined via the requirement that ϵ_{AB} and σ's are covariantly constant. The analysis in [9]—until Eq. (91) in the book—still stands; however, the covariant derivatives are promoted to those acting on U_4. They are defined as follows:

$$\nabla_\mu P^A = \partial_\mu P^A + \Gamma^A_{\mu B} P^B, \tag{36}$$

$$\nabla_\mu \bar{Q}^{A'} = \partial_\mu \bar{Q}^{A'} + \bar{\Gamma}^{A'}_{\mu B'} \bar{Q}^{B'}. \tag{37}$$

The Γ terms here are added to the partial derivative when working with objects in U_4. Their values can completely be determined in terms of the spin coefficients, and we can readily evaluate its tetrad components using the following formulas and the spin dyads [10]:

$$\Gamma^A_{\mu B} = \frac{1}{2}\sigma^{AY'}_\nu (\nabla_\mu \sigma^\nu_{BY'}), \qquad \bar{\Gamma}^{A'}_{\mu B'} = \frac{1}{2}\bar{\sigma}^{A'Y}_\nu (\bar{\nabla}_\mu \bar{\sigma}^\nu_{B'Y}). \tag{38}$$

Using *Friedman's lemma* (see p. 542 of [9] for a full proof), we can express the various spin coefficients $\Gamma_{(a)(b)(c)(d')}$ in terms of covariant derivatives of the basis null vectors l, n, m and $\bar{m}$. The covariant derivative here is exactly as defined in Eq. (3.3) [and explicitly written in Eq. (3.5)] of [8].

Using this covariant derivative, it is readily seen how Eqs. (95) and (96) in [9] get modified; viz, $\Gamma_{0000'} = \kappa^o + \kappa_1$ and $\Gamma_{1101'} = \mu^o + \mu_1$ (noughts in the superscript are used to indicate the original spin coefficients defined on V_4). The 12 independent spin coefficients are calculated in terms of covariant derivatives of null vectors and defined in the following table[2] (39):

$\Gamma_{(a)(b)(c)(d')} =$

(c)(d') \ (a)(b)	00	01 or 10	11
00'	$\kappa^o + \kappa_1$	$\epsilon^o + \epsilon_1$	$\pi^o + \pi_1$
10'	$\rho^o + \rho_1$	$\alpha^o + \alpha_1$	$\lambda^o + \lambda_1$
01'	$\sigma^o + \sigma_1$	$\beta^o + \beta_1$	$\mu^o + \mu_1$
11'	$\tau^o + \tau_1$	$\gamma^o + \gamma_1$	$\nu^o + \nu_1$

(39)

D. Contortion spin coefficients in terms of Dirac spinor components

The spin density tensor of matter ($S^{\mu\nu\lambda}$) can be written as a world tensor in U_4 made up of the Dirac spinor, its adjoint, and gamma matrices:

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}\psi. \tag{40}$$

The ECD field equations show that $T^{\mu\nu\alpha} = kS^{\mu\nu\alpha}$, where $T^{\mu\nu\alpha}$ is the modified torsion tensor defined in Eq. (2.3) of [2]. It can be shown that, for the Dirac field, $T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = kS^{\mu\nu\alpha}$ as in Eq. (5.6) of [3]. Here, k is a gravitational coupling constant containing the length scale l_1, i.e., $k = \frac{8\pi l_1^2}{\hbar c}$. For the standard theory, $l_1 = L_{\rm Pl}$. Substituting (40) in the field equations, we obtain the following:

$$K^{\mu\nu\alpha} = -kS^{\mu\nu\alpha} = 2i\pi l_1^2 \bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}\psi \tag{41}$$

where the γ^μ's are those defined in (34), generalized with world indices using orthonormal tetrads. We subsequently rewrite $K^{\mu\nu\alpha}$ (of which only four independent components are excited by the Dirac field) in the NP formalism; i.e., in the null tetrad basis, as follows:

$$K_{(i)(j)(k)} = e_{(i)\mu} e_{(j)\nu} e_{(k)\alpha} K^{\mu\nu\alpha} \tag{42}$$

where $e_{(i)\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ for $i = 0, 1, 2, 3$ To calculate the contortion spin coefficients, we need to evaluate the contortion tensor with world indices as defined in (A1). Consider the product $\gamma^\alpha\gamma^\beta\gamma^\mu$, which is defined as

$$\gamma^\alpha\gamma^\beta\gamma^\mu = \begin{pmatrix} 0 & (\tilde{\sigma}^\alpha)^*(\sigma^\beta)^*(\tilde{\sigma}^\mu)^* \\ (\sigma^\alpha)^*(\tilde{\sigma}^\beta)^*(\sigma^\mu)^* & 0 \end{pmatrix}. \tag{43}$$

The explicit form of this matrix is fairly expansive, and a full treatment is given in Appendix A. Eventually, we substitute in for the Dirac bispinor (as defined in [9]), and obtain the expressions for the contortion spin coefficients in terms of the spinor components. We have, e.g., for ρ

$$\rho = -K_{(0)(2)(3)} = -2\sqrt{2}i\pi l_1^2[F_2\bar{F}_2 - G_1\bar{G}_1]. \tag{44}$$

All the contortion spin coefficients can be found in a similar fashion. After evaluating those, the eight nonzero spin coefficients excited by the Dirac spinor given in (1)—of which four are independent—are as follows:

$$\tau_1 = -2\beta_1 = K_{012} = 2\sqrt{2}i\pi l_1^2(F_2\bar{F}_1 + G_2\bar{G}_1), \tag{45}$$

$$\pi_1 = -2\alpha_1 = K_{013} = 2\sqrt{2}i\pi l_1^2(-F_1\bar{F}_2 - G_1\bar{G}_2), \tag{46}$$

$$\mu_1 = -2\gamma_1 = -K_{123} = 2\sqrt{2}i\pi l_1^2(F_1\bar{F}_1 - G_2\bar{G}_2), \tag{47}$$

$$\rho_1 = -2\epsilon_1 = -K_{023} = 2\sqrt{2}i\pi l_1^2(G_1\bar{G}_1 - F_2\bar{F}_2). \tag{48}$$

From the above relations, we have

$$\mu_1 = -\mu_1^*, \tag{49}$$

[2] In the generic case, all 12 have contortion spin coefficients.

$$\rho_1 = -\rho_1^*, \tag{50}$$

$$\pi_1 = +\tau_1^*. \tag{51}$$

The table (39) is modified as follows:

$\Gamma_{(a)(b)(c)(d')} =$

(c)(d') \ (a)(b)	00	01 or 10	11
00'	κ_0	$\epsilon_0 - \rho_1/2$	$\pi_0 + \pi_1$
10'	$\rho_0 + \rho_1$	$\alpha_0 - \pi_1/2$	λ_0
01'	σ_0	$\beta_0 - \tau_1/2$	$\mu_0 + \mu_1$
11'	$\tau_0 + \tau_1$	$\gamma_0 - \mu_1/2$	ν_0

(52)

Next, we formulate ECD theory in the NP formalism. There are three equations in this theory—the Dirac equation on U_4 (known as the Hehl-Datta equation), the gravitation field equation on U_4, and an algebraic equation relating torsion and spin. The algebraic equation is given in Eq. (A1). In the next two sections, we formulate the Dirac equation and the gravitation field equations explicitly on U_4 respectively.

E. The Dirac equation with torsion in the NP formalism

The Dirac equation on U_4 (also known as the Hehl-Datta equation) is

$$i\gamma^\mu \nabla_\mu \psi = \frac{mc}{\hbar}\psi = \frac{\psi}{2l_2} \tag{53}$$

where ∇ here denotes covariant derivative on U_4 and $l_2 = \frac{\lambda_c}{2}$ for standard theory. It can be written in the following matrix form:

$$i\begin{pmatrix} 0 & (\tilde{\sigma}^\mu)^* \\ (\sigma^\mu)^* & 0 \end{pmatrix}\nabla_\mu \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} = \frac{1}{2\sqrt{2}l_2}\begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix}. \tag{54}$$

This can be written as a pair of matrix equations:

$$\begin{pmatrix} \sigma^\mu_{00'} & \sigma^\mu_{10'} \\ \sigma^\mu_{01'} & \sigma^\mu_{11'} \end{pmatrix}\nabla_\mu \begin{pmatrix} P^0 \\ P^1 \end{pmatrix} + \frac{i}{2\sqrt{2}l_2}\begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} = 0, \tag{55}$$

$$\begin{pmatrix} \sigma^\mu_{11'} & -\sigma^\mu_{10'} \\ -\sigma^\mu_{01'} & \sigma^\mu_{00'} \end{pmatrix}\nabla_\mu \begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} + \frac{i}{2\sqrt{2}l_2}\begin{pmatrix} P^0 \\ P^1 \end{pmatrix} = 0. \tag{56}$$

Working out explicitly, the first equation is

$$\begin{aligned} \frac{i}{2\sqrt{2}l_2}\bar{Q}^{1'} &= \sigma^\mu_{00'}\nabla_\mu P^0 + \sigma^\mu_{10'}\nabla_\mu P^1 = (\partial_{00'}P^0 + \Gamma^0{}_{i00'}P^i) + (\partial_{10'}P^1 + \Gamma^1{}_{i10'}P^i) \\ &= (D + \Gamma^0{}_{000'}P^0 + \Gamma^0{}_{100'}P^1) + (\delta^* + \Gamma^1{}_{010'}P^0 + \Gamma^1{}_{110'}P^1) \\ \Rightarrow \frac{i}{2\sqrt{2}l_2}G_1 &= (D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1), \end{aligned} \tag{57}$$

where we have used the gamma matrices as defined in (34), computed the covariant derivatives using (36), (37), and the spin connections in terms of contortion spin coefficients as given in (52). Using this procedure (a full treatment given in Appendix B), the four Dirac equations are obtained as

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1) = ib(l_2)G_1, \tag{58}$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + \frac{3}{2}(\mu_1 F_2 - \tau_1 F_1) = ib(l_2)G_2, \tag{59}$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 - \frac{3}{2}(\tau_1 G_1 - \rho_1 G_2) = ib(l_2)F_2, \tag{60}$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 - \frac{3}{2}(\mu_1 G_1 - \pi_1 G_2) = ib(l_2)F_1. \tag{61}$$

Substituting in the spinorial form of the contortion spin coefficients in (45)–(48), we obtain

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + ia(l_1)[(-F_1\bar{F}_2 - G_1\bar{G}_2)F_2 + (F_2\bar{F}_2 - G_1\bar{G}_1)F_1] = ib(l_2)G_1, \tag{62}$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + ia(l_1)[(F_1\bar{F}_1 - G_2\bar{G}_2)F_2 - (F_2\bar{F}_1 + G_2\bar{G}_1)F_1] = ib(l_2)G_2, \tag{63}$$

$$(D+\epsilon_0^*-\rho_0^*)G_2-(\delta+\pi_0^*-\alpha_0^*)G_1-ia(l_1)[(F_2\bar{F}_2-G_1\bar{G}_1)G_2+(F_2\bar{F}_1+G_2\bar{G}_1)G_1]=ib(l_2)F_2, \tag{64}$$

$$(\Delta+\mu_0^*-\gamma_0^*)G_1-(\delta^*+\beta_0^*-\tau_0^*)G_2-ia(l_1)[(F_1\bar{F}_1-G_2\bar{G}_2)G_1-(-F_1\bar{F}_2-G_1\bar{G}_2)G_2]=ib(l_2)F_1, \tag{65}$$

where $a(l_1)=3\sqrt{2}\pi l_1^2$ and $b(l_2)=\frac{1}{2\sqrt{2}l_2}$.

These equations can be condensed into the following form:

$$(D+\epsilon_0-\rho_0)F_1+(\delta^*+\pi_0-\alpha_0)F_2=i[b(l_2)+a(l_1)\xi]G_1, \tag{66}$$

$$(\Delta+\mu_0-\gamma_0)F_2+(\delta+\beta_0-\tau_0)F_1=i[b(l_2)+a(l_1)\xi]G_2, \tag{67}$$

$$(D+\epsilon_0^*-\rho_0^*)G_2-(\delta+\pi_0^*-\alpha_0^*)G_1=i[b(l_2)+a(l_1)\xi^*]F_2, \tag{68}$$

$$(\Delta+\mu_0^*-\gamma_0^*)G_1-(\delta^*+\beta_0^*-\tau_0^*)G_2=i[b(l_2)+a(l_1)\xi^*]F_1, \tag{69}$$

where $\xi=F_1\bar{G}_1+F_2\bar{G}_2$ and $\xi^*=\bar{F}_1G_1+\bar{F}_2G_2$. These equations should be compared and contrasted with the torsionless Dirac equations in [9], and then we see that the impact of torsion is to include the term $a\xi$ on the right-hand side of the first two equations, and $a\xi^*$ in the last two equations.

F. The gravitation equations on U_4 in NP formalsim

The equation of interest here is (17), reproduced here:

$$G_{\mu\nu}(\{\})=\frac{8\pi l_1^2}{\hbar c}T_{\mu\nu}-\frac{1}{2}\left(\frac{8\pi l_1^2}{\hbar c}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}. \tag{70}$$

On the left-hand side, we have $G_{\mu\nu}(\{\})$, which has been completely evaluated in the NP formalism in [9]. There are two terms on right-hand side—the first of these is the metric energy-momentum tensor ($T_{\mu\nu}$) formulated on U_4 and is given by Eq. (18). In what follows, we will give a prescription to compute the various components of $T_{\mu\nu}$, under the definition

$$T_{\mu\nu}=\frac{i\hbar c}{4}[\bar{\psi}\gamma_\mu\nabla_\nu^{\{\}}\psi+\bar{\psi}\gamma_\nu\nabla_\mu^{\{\}}\psi-\nabla_\mu^{\{\}}\bar{\psi}\gamma_\nu\psi-\nabla_\nu^{\{\}}\bar{\psi}\gamma_\mu\psi]. \tag{71}$$

First, we choose a tetrad basis and construct Van der Waarden symbols as defined in (35). Using these, we construct Dirac gamma matrices in the complex Weyl representation as defined in (34). Now, the expression for the covariant derivatives of spinors—see (36)–(38)—can be expressed in terms of the gamma matrices, yielding

$$\begin{aligned}T_{\mu\nu}=\frac{i\hbar c}{4}\Big[&\bar{\psi}\gamma_\mu\partial_\nu\psi+\frac{1}{4}\bar{\psi}(\gamma_\mu\gamma^\alpha\nabla_\nu^{\{\}}\gamma_\alpha)\psi+\bar{\psi}\gamma_\nu\partial_\mu\psi+\frac{1}{4}\bar{\psi}(\gamma_\nu\gamma^\alpha\nabla_\mu^{\{\}}\gamma_\alpha)\psi\\&-\partial_\mu\bar{\psi}\gamma_\nu\psi-\frac{1}{4}(\bar{\gamma}^\alpha\bar{\nabla}_\mu^{\{\}}\bar{\gamma}_\alpha)\bar{\psi}\gamma_\nu\psi-\partial_\nu\bar{\psi}\gamma_\mu\psi-\frac{1}{4}(\bar{\gamma}^\alpha\bar{\nabla}_\nu^{\{\}}\bar{\gamma}_\alpha)\bar{\psi}\gamma_\mu\psi\Big].\end{aligned} \tag{72}$$

Here, the gamma matrices and other variables are expressed in the basis of null vectors l, n, m and $\bar{m}$. For the generic metric energy-momentum tensor $T_{\mu\nu}$, no further simplification is possible. The expression for $T_{\mu\nu}$ in the NP formalism will however simplify under certain symmetries or specific conditions that the system in question is subjected to. For example, if the background metric is $\eta_{\mu\nu}$, then (for illustration purposes) the T_{12} component of metric EM tensor is given by

$$\begin{aligned}T_{12}^{(\mathrm{NP})}=\frac{i\hbar c}{4\sqrt{2}}&(i\bar{F}_2(\delta+\delta^*)F_1-i\bar{F}_1(\delta+\delta^*)F_2-i\bar{G}_2(\delta+\delta^*)G_1+i\bar{G}_1(\delta+\delta^*)G_2\\&-i\bar{F}_2(\delta-\delta^*)F_1-i\bar{F}_1(\delta-\delta^*)F_2+i\bar{G}_2(\delta-\delta^*)G_1+i\bar{G}_1(\delta-\delta^*)G_2)\\&-i(\delta+\delta^*)\bar{F}_2F_1+(\delta+\delta^*)i\bar{F}_1F_2+(\delta+\delta^*)i\bar{G}_2G_1-(\delta+\delta^*)i\bar{G}_1G_2\\&+(\delta-\delta^*)i\bar{F}_2F_1+(\delta-\delta^*)i\bar{F}_1F_2-(\delta-\delta^*)i\bar{G}_2G_1-(\delta-\delta^*)i\bar{G}_1G_2).\end{aligned} \tag{73}$$

With this prescription, we are able to evaluate all the components of $T_{\mu\nu}$, achieving a particularly simple form in the case of a Minkowskian background metric.

In (17), we also have an additional term in terms of the spin density tensor, given as $\frac{4\pi l_1^2}{\hbar c}g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}$. Using our expression for the spin density, we can evaluate this term:

$$\frac{4\pi l_1^2}{\hbar c} g_{\mu\nu} S^{\alpha\beta\lambda} S_{\alpha\beta\lambda}$$

$$= \frac{-\pi l_1^2 \hbar c}{4} (\bar{\psi}\gamma^{[\alpha}\gamma^{\beta}\gamma^{\lambda]}\psi)(\bar{\psi}\gamma_{[\alpha}\gamma_{\beta}\gamma_{\lambda]}\psi) \tag{74}$$

$$= \frac{-\pi l_1^2 \hbar c}{4} (\bar{\psi}\gamma^{[(i)}\gamma^{(j)}\gamma^{(k)]}\psi)(\bar{\psi}\gamma_{[(i)}\gamma_{(j)}\gamma_{(k)]}\psi) \tag{75}$$

$$= 6\pi\hbar c l_1^2 g_{\mu\nu}(F_1\bar{G}_1 + F_2\bar{G}_2)(\bar{F}_1 G_1 + \bar{F}_2 G_2) \tag{76}$$

$$= 6\pi\hbar c l_1^2 g_{\mu\nu}\xi\xi^* \tag{77}$$

$$= 12\pi\hbar c l_1^2 (l_{(\mu}n_{\nu)} - m_{(\mu}\bar{m}_{\nu)})\xi\xi^* \tag{78}$$

i.e., we find that it turns out to be proportional to the ξ parameter introduced.

This completes the formulation of the Einstein-Cartan-Dirac equations in the NP formalism. The formalism can be used to examine how torsion modifies the properties of the Einstein-Dirac system. Next, we investigate some solutions of the Hehl-Datta equations. In future work we hope to extend these studies to Poincaré gauge gravity with propagating torsion.

V. SOLUTIONS TO HD EQUATIONS IN MINKOWSKI SPACE

A. Motivation

In the previous section, we formulated the ECD equations in the NP formalism. In this section, we aim to solve them. The simplest space-time with torsion is the Minkowski ($\eta_{\mu\nu}$) space-time with a manifold that has nonzero torsion. In this space-time, the Dirac equation on U_4 looks very similar to the linear Dirac equation with modified mass (the torsion-related term which modifies it is bilinear in the Dirac states). In this spirit, we will consider modifications (due to torsion) to well-studied solutions to the linear Dirac equation (e.g., plane wave solutions).

In addition, there are good (physical) reasons to work within Minkowski space-time, to find solution(s) of the HD equations incorporating torsion. In a recent work [16–18], a duality between large and small masses (correspondingly, between Riemannian curvature and torsion) was proposed, explicitly constructed in the "curvature-torsion duality conjecture" therein. For this conjecture to hold true, a solution to Dirac equation on Minkowski space with torsion must exist—along with certain other conditions. One such additional condition is the vanishing of the $(T-S)_{\mu\nu}$ tensor, as defined in Appendix C.

While we proceed in the following section to find solutions to the HD equations on Minkowski space for their own sake, the reader may find, in [18], useful extensions to this work. To this end, in the Appendices (reference Appendix C) we have also computed the $(T-S)_{\mu\nu}$ tensor in certain cases, for completeness.

B. The Hehl-Datta equations on Minkowski space with torsion

The HD equations on Minkowski space with torsion (in the NP formalism) are as follows:

$$DF_1 + \delta^* F_2 = i[b(l_2) + a(l_1)\xi]G_1, \tag{79}$$

$$\Delta F_2 + \delta F_1 = i[b(l_2) + a(l_1)\xi]G_2, \tag{80}$$

$$DG_2 - \delta G_1 = i[b(l_2) + a(l_1)\xi^*]F_2, \tag{81}$$

$$\Delta G_1 - \delta^* G_2 = i[b(l_2) + a(l_1)\xi^*]F_1. \tag{82}$$

In a Cartesian coordinate system (ct, x, y, z)[3] we have

$$(\partial_0 + \partial_3)F_1 + (\partial_1 + i\partial_2)F_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_1, \tag{83}$$

$$(\partial_0 - \partial_3)F_2 + (\partial_1 - i\partial_2)F_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_2, \tag{84}$$

$$(\partial_0 + \partial_3)G_2 - (\partial_1 - i\partial_2)G_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2, \tag{85}$$

$$(\partial_0 - \partial_3)G_1 - (\partial_1 + i\partial_2)G_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1. \tag{86}$$

In cylindrical polar coordinates (ct, r, ϕ, z), we have

$$r\partial_t F_1 + e^{i\phi} r\partial_r F_2 + ie^{i\phi}\partial_\phi F_2 + r\partial_z F_1 = ir\sqrt{2}[b(l_2) + a(l_1)\xi]G_1, \tag{87}$$

$$r\partial_t F_2 + e^{-i\phi} r\partial_r F_1 - ie^{-i\phi}\partial_\phi F_1 - r\partial_z F_2 = ir\sqrt{2}[b(l_2) + a(l_1)\xi]G_2, \tag{88}$$

$$r\partial_t G_2 - e^{-i\phi} r\partial_r G_1 + ie^{-i\phi}\partial_\phi G_1 + cr\partial_z G_2 = ir\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2, \tag{89}$$

$$r\partial_t G_1 - e^{i\phi} r\partial_r G_2 - ie^{i\phi}\partial_\phi G_2 - r\partial_z G_1 = ir\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1. \tag{90}$$

Likewise, in spherical polar coordinates (ct, r, θ, ϕ)

$$\partial_t F_1 + \cos\theta\partial_r F_1 - \frac{\sin\theta}{r}\partial_\theta F_1 + \frac{ie^{i\phi}}{r\sin\theta}\partial_\phi F_2 + e^{i\phi}\sin\theta\partial_r F_2 + \frac{e^{i\phi}\cos\theta}{r}\partial_\theta F_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_1, \tag{91}$$

[3] Setting $c = 1$ by convention.

$$\partial_t F_2 - \cos\theta \partial_r F_2 - \frac{\sin\theta}{r}\partial_\theta F_2 + \frac{ie^{-i\phi}}{r\sin\theta}\partial_\phi F_1 + e^{-i\phi}\sin\theta\partial_r F_1 - \frac{e^{-i\phi}\cos\theta}{r}\partial_\theta F_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi]G_2, \tag{92}$$

$$\partial_t G_2 + \cos\theta \partial_r G_2 - \frac{\sin\theta}{r}\partial_\theta G_2 - \frac{ie^{-i\phi}}{r\sin\theta}\partial_\phi G_1 - e^{-i\phi}\sin\theta\partial_r G_1 + \frac{e^{-i\phi}\cos\theta}{r}\partial_\theta G_1 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2, \tag{93}$$

$$\partial_t G_1 - \cos\theta \partial_r G_1 - \frac{\sin\theta}{r}\partial_\theta G_1 - \frac{ie^{i\phi}}{r\sin\theta}\partial_\phi G_2 - e^{i\phi}\sin\theta\partial_r G_2 - \frac{e^{i\phi}\cos\theta}{r}\partial_\theta G_2 = i\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1. \tag{94}$$

C. A nonstatic solution in 1+1 dimensions

In the following analysis, we will assume an ansatz of the form $F_1 = G_2$ and $F_2 = G_1$, and further assume that the Dirac states are a function of only t and z. The four equations—in Cartesian (83)–(86) as well as cylindrical polar coordinates (87)–(90)—reduce to the following two independent equations[4]:

$$\partial_t\psi_1 + \partial_z\psi_2 - i\sqrt{2}b\psi_1 + \frac{ia}{\sqrt{2}}(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0,$$
$$\partial_t\psi_2 + \partial_z\psi_1 + i\sqrt{2}b\psi_2 + \frac{ia}{\sqrt{2}}(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0, \tag{95}$$

where $\psi_1 = F_1 + F_2$ and $\psi_2 = F_1 - F_2$. If we were to define $\sqrt{2}b \equiv -m$ and $a = 2\sqrt{2}\lambda$, we would get

$$\partial_t\psi_1 + \partial_z\psi_2 + im\psi_1 + 2i\lambda(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0,$$
$$\partial_t\psi_2 + \partial_z\psi_1 - im\psi_2 + 2i\lambda(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0. \tag{96}$$

These equations are identical to those studied in [23], which investigates the convergence and stability of the difference scheme for the nonlinear Dirac equation in $1+1$ dimensions. Proceeding as in [23], we use the following solitary wave ansatz:

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}A(z)\\ iB(z)\end{pmatrix}e^{-i\Lambda t} \tag{97}$$

where $A(z)$ and $B(z)$ are real functions. Substituting in (96), we have

$$B' - (\sqrt{2}b + \Lambda)A - \frac{a}{\sqrt{2}}(A^2 - B^2)A = 0,$$
$$A' - (\sqrt{2}b - \Lambda)B - \frac{a}{\sqrt{2}}(A^2 - B^2)B = 0, \tag{98}$$

which admits the following solutions:

$$A(z) = \frac{-i2^{3/4}(\sqrt{2}b - \Lambda)}{\sqrt{a}}\frac{\sqrt{(\sqrt{2}b + \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}, \tag{99}$$

$$B(z) = \frac{-i2^{3/4}(\sqrt{2}b + \Lambda)}{\sqrt{a}}\frac{\sqrt{(\sqrt{2}b - \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}. \tag{100}$$

It can be seen upon the substitutions $\lambda = 0.5$ (equivalently $a = \sqrt{2}$) and $m = 1$ (equivalently $m_0 = -1$), that this is a generalization of the equations for $A(z)$ and $B(z)$ in [23] (see Sec. III). A similar solution is found in [24], with $a(l_1) = a(L_{pl})$ and $b(l_2) = b(\lambda_c)$. In terms of the spinor components:

$$F_1 = G_2 = \frac{\sqrt{(2b^2 - \Lambda^2)}}{2}\left[\frac{-i2^{3/4}}{\sqrt{a}}\frac{\sqrt{(\sqrt{2}b - \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} + \frac{2^{3/4}}{\sqrt{a}}\frac{\sqrt{(\sqrt{2}b + \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}\right]e^{-i\Lambda t}, \tag{101}$$

$$F_2 = G_1 = \frac{\sqrt{(2b^2 - \Lambda^2)}}{2}\left[\frac{-i2^{3/4}}{\sqrt{a}}\frac{\sqrt{(\sqrt{2}b - \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} - \frac{2^{3/4}}{\sqrt{a}}\frac{\sqrt{(\sqrt{2}b + \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}\right]e^{-i\Lambda t} \tag{102}$$

and the parameter ξ characterizing torsion takes the form

$$\xi = \frac{-2\sqrt{2}(2b^2 - \Lambda^2)(\sqrt{2}b - \Lambda)\cosh(2z\sqrt{2b^2 - \Lambda^2})}{a[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]^2}. \tag{103}$$

The probability density is given by the zeroth component of the four-vector fermion current $j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = 2(|F_1|^2 + |F_2|^2) = (|A|^2 + |B|^2)$ For the subsequent analysis, we define the following dimensionless variables:

$$p = \sqrt{2}bz, \qquad w = -\frac{\Lambda}{\sqrt{2}b},$$
$$\tilde{A}(p) = \frac{\sqrt{a}}{2\sqrt{b}}A(z), \qquad \tilde{B}(p) = \frac{\sqrt{a}}{2\sqrt{b}}B(z),$$
$$\tilde{j}^0 = \frac{a}{4b}j^0. \tag{104}$$

With these definitions, we have $[p] = [w] = [\tilde{A}(p)] = [\tilde{B}(p)] = [\tilde{j}^0] = 0$; i.e., all these quantities are now dimensionless. Scaled thus, $A(p)$ and $B(p)$ take the form

$$A(p) = \frac{2i(1+w)}{\sqrt{a}}\frac{\sqrt{b(1-w)}\cosh(p\sqrt{1-w^2})}{(w\cosh(2p\sqrt{1-w^2})+1)}, \tag{105}$$

$$B(p) = \frac{2i(1-w)}{\sqrt{a}}\frac{\sqrt{b(1+w)}\sinh(p\sqrt{1-w^2})}{(w\cosh(2p\sqrt{1-w^2})+1)}. \tag{106}$$

There are six unique cases (corresponding to values of w) which give different solutions. In each case, we will consider torsion-less limit (the linear Dirac equation) in order to compare and contrast the behavior. The equations and plots for the linear case can be found in Appendix D.

Case I: $w \in (-\infty, -1)$: The equations reduce to

$$\tilde{A}(p) = i(1+w)\frac{\sqrt{(|w|+1)}\cos(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))}, \tag{107}$$

$$\tilde{B}(p) = i(w-1)\frac{\sqrt{(|w|-1)}\sin(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))}, \tag{108}$$

$$\tilde{j}^0 = \left[\frac{(w+1)^2(|w|+1)\cos^2(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))^2} + \frac{(w-1)^2(|w|-1)\sin^2(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))^2}\right]. \tag{109}$$

Comments: This solution has an infinite number of singularities placed periodically at nonzero values of p, and is clearly unphysical. An example of this case (with $w = -2$) can be seen in the left column of Fig. 1.

Comparison with torsionless case: For $w \in (-\infty, -1)$, the linear Dirac equation gives plane waves solutions, which are physically meaningful, and the probability density fluctuates sinusoidally. It is the addition of torsion that makes this case unphysical. A plot has been made (for $w = -2$) in Fig. 2.

Case II: $w = \pm 1$ (trivial case): The equations reduce to

$$\tilde{A}(p) = 0, \qquad \tilde{B}(p) = 0, \qquad \tilde{j}^0 = 0. \tag{110}$$

Case III: $w \in (-1, 0)$: The equations reduce to

$$\tilde{A}(p) = i(1+w)\frac{\sqrt{(1+|w|)}\cosh(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))}, \tag{111}$$

$$\tilde{B}(p) = i(1-w)\frac{\sqrt{(1-|w|)}\sinh(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))}, \tag{112}$$

$$\tilde{j}^0 = \left[\frac{(w+1)^2(|w|+1)\cosh^2(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))^2} + \frac{(1-w)^2(1-|w|)\sinh^2(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2}))^2}\right]. \tag{113}$$

Comments: This solution has two singularities placed symmetrically around the origin at two finite (nonzero) values of p. In the infinite limit, it decays to zero. However, owing to the presence of singularities, we may still consider it an unphysical solution. An example (with $w = -0.5$) can be seen in the left column of Fig. 3.

Comparison with torsionless case: For $w \in (-1, 0)$ the linear Dirac equation has unphysical solutions. The solutions grow exponentially to infinity as $p \to \pm\infty$. For $w = -0.5$, this solution is plotted in Fig. 2. As can be seen, for this case, both the linear (torsionless) and nonlinear (with torsion) Dirac equations give unphysical solutions.

Case IV: $w = 0$: The equations reduce to

$$\tilde{A}(p) = i\cosh(p), \tag{114}$$

$$\tilde{B}(p) = i\sinh(p), \tag{115}$$

$$\tilde{j}^0 = [\cosh^2(p) + \sinh^2(p)]. \tag{116}$$

Comments: This solution blows up exponentially as $p \to \pm\infty$. Thus, it is clearly unphysical. This case (with $w = 0$) has been plotted in the right column of Fig. 3.

Comparison with torsionless case: For $w = 0$, the linear Dirac equation is unphysical. The solutions exponentially increase to infinity as $p \to +\infty$. A plot of the solutions (for $w = 0$) is available in Fig. 2. Thus, for this case, both the linear and nonlinear Dirac equations give unphysical solutions.

Case V: $w \in (0, 1)$: The equations reduce to

$$\tilde{A}(p) = i(1+w)\frac{\sqrt{(1-w)}\cosh(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))}, \tag{117}$$

$$\tilde{B}(p) = i(1-w)\frac{\sqrt{(1+w)}\sinh(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))}, \tag{118}$$

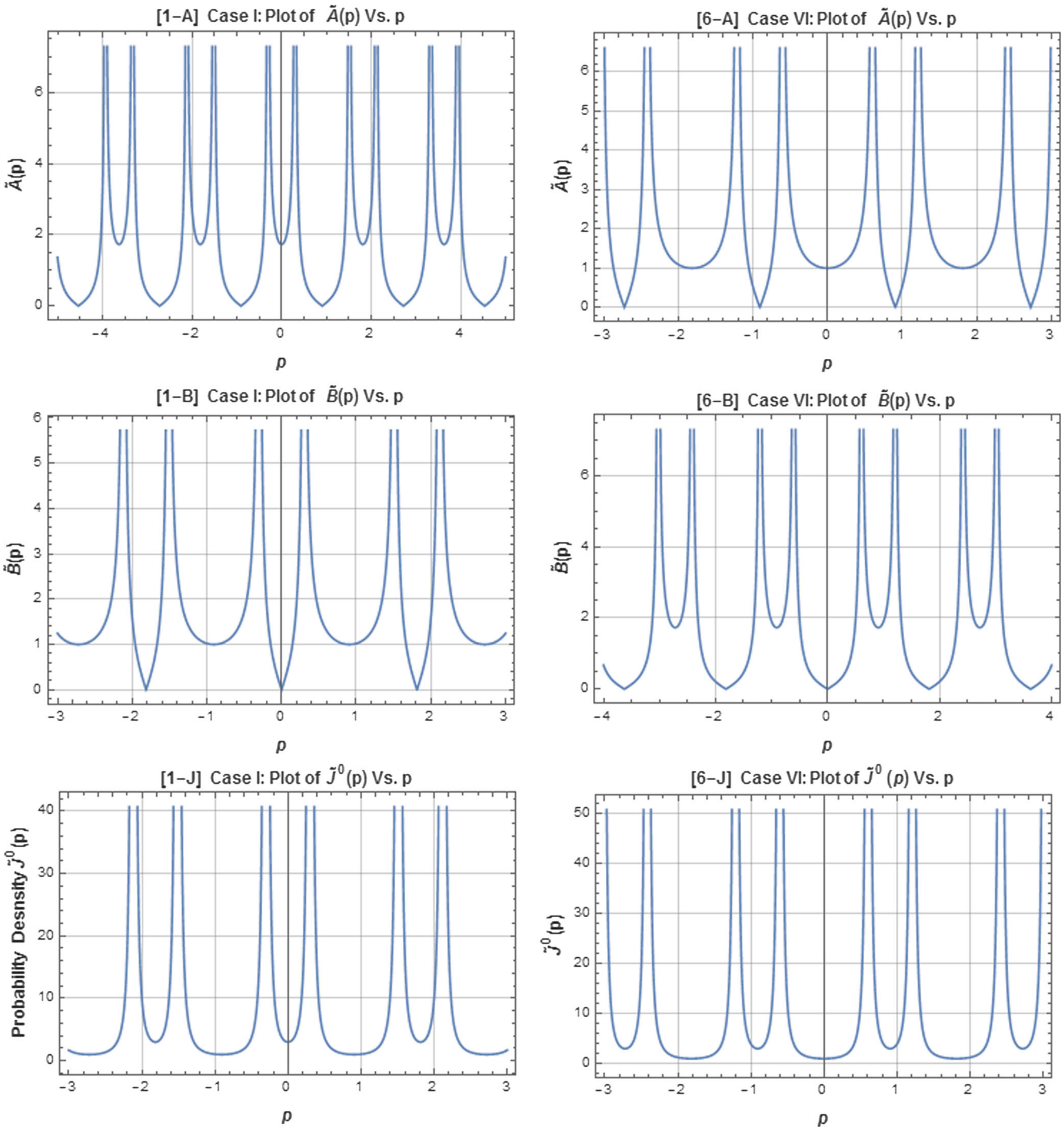

FIG. 1. Case I and case VI. The left column shows plots for case I with $w = -2$. The right column shows plots for case 6 with $w = +2$. Both the cases have unphysical solutions.

$$\tilde{j}^0 = \left[\frac{(1+w)^2(1-w)\cosh^2(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))^2} + \frac{(1-w)^2(1+w)\sinh^2(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))^2}\right]. \tag{119}$$

Comments: In this case, we have no singularities anywhere. All the functions (including the probability density) asymptotically vanish. Therefore, this case represents a physically viable solution. Depending on the exact nature of solution, we can consider two subcases: (a) with $w \in (0, \frac{1}{2})$ and (b) with $w \in [\frac{1}{2}, 1)$. We see that (a) has a local minimum at the origin and two global maxima symmetric around the origin at nonzero p. A plot is provided in Fig. 4 (in blue). On the other hand, (b) has global maximum at the origin and monotonically decays to zero at infinity. Two examples of this can be seen in

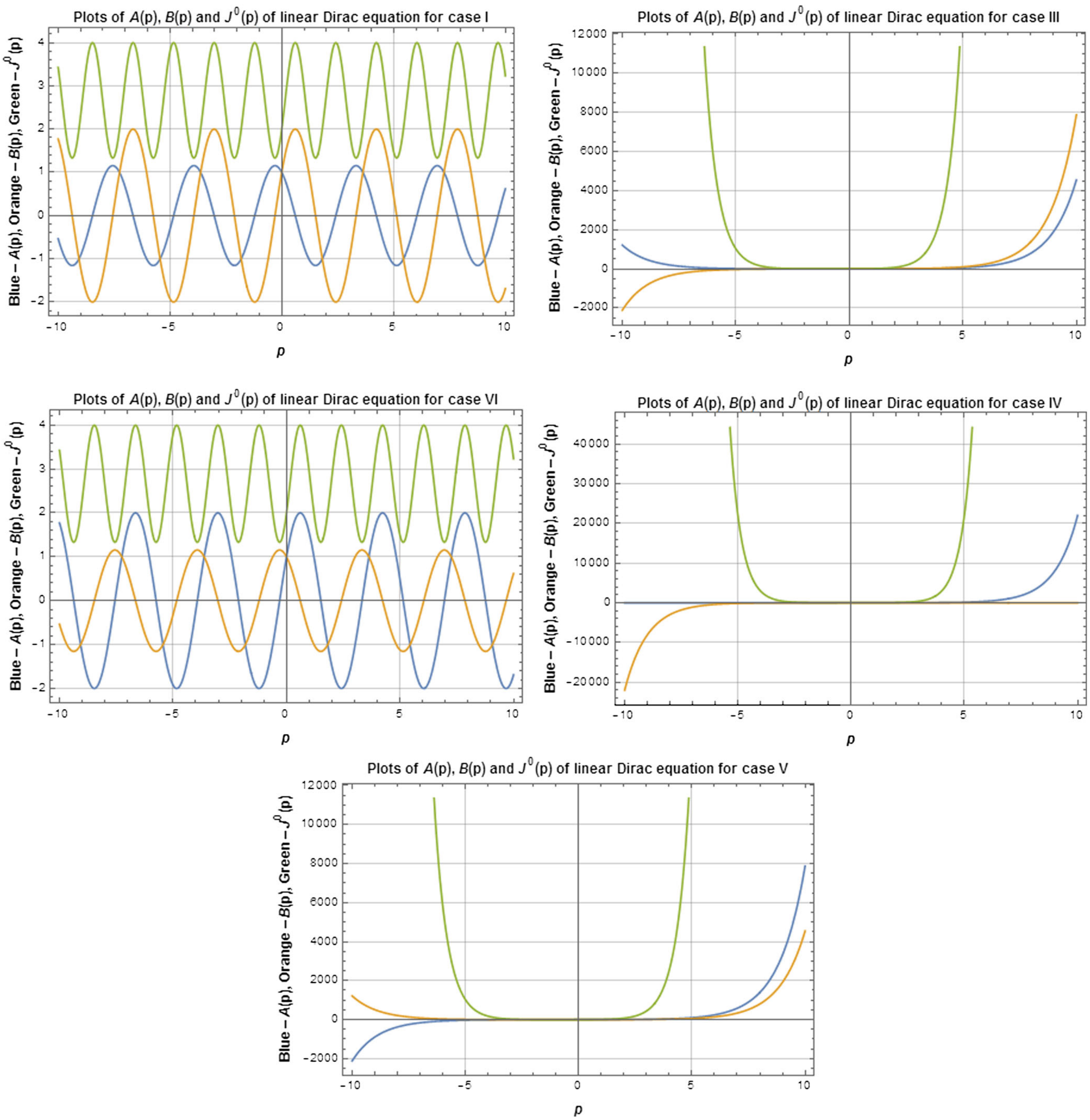

FIG. 2. Solutions to the linear (torsionless) Dirac equations. Only the plane-wave solutions (Cases I, VI) are physical.

Fig. 4 (in orange and green). This classification of case V into 2 subcases has been done by analyzing the behavior of probability density. The solution for case (b) is solitonlike; further analysis of this can be found in the discussion.

Comparison with torsionless case: For $w \in (0, 1)$ the linear Dirac equation gives unphysical solutions. The solutions increase exponentially to infinity as $p \to \pm\infty$. A plot of this solution (with $w = 0.5$) can be seen in Fig. 2. The addition of torsion, as seen, makes the solutions physically meaningful.

Case VI: $w \in (1, \infty)$: The equations reduce to

$$\tilde{A}(p) = -(1+w)\frac{\sqrt{(w-1)}\cos(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1})}, \qquad (120)$$

$$\tilde{B}(p) = -(1-w)\frac{\sqrt{(w+1)}\sin(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1})}, \qquad (121)$$

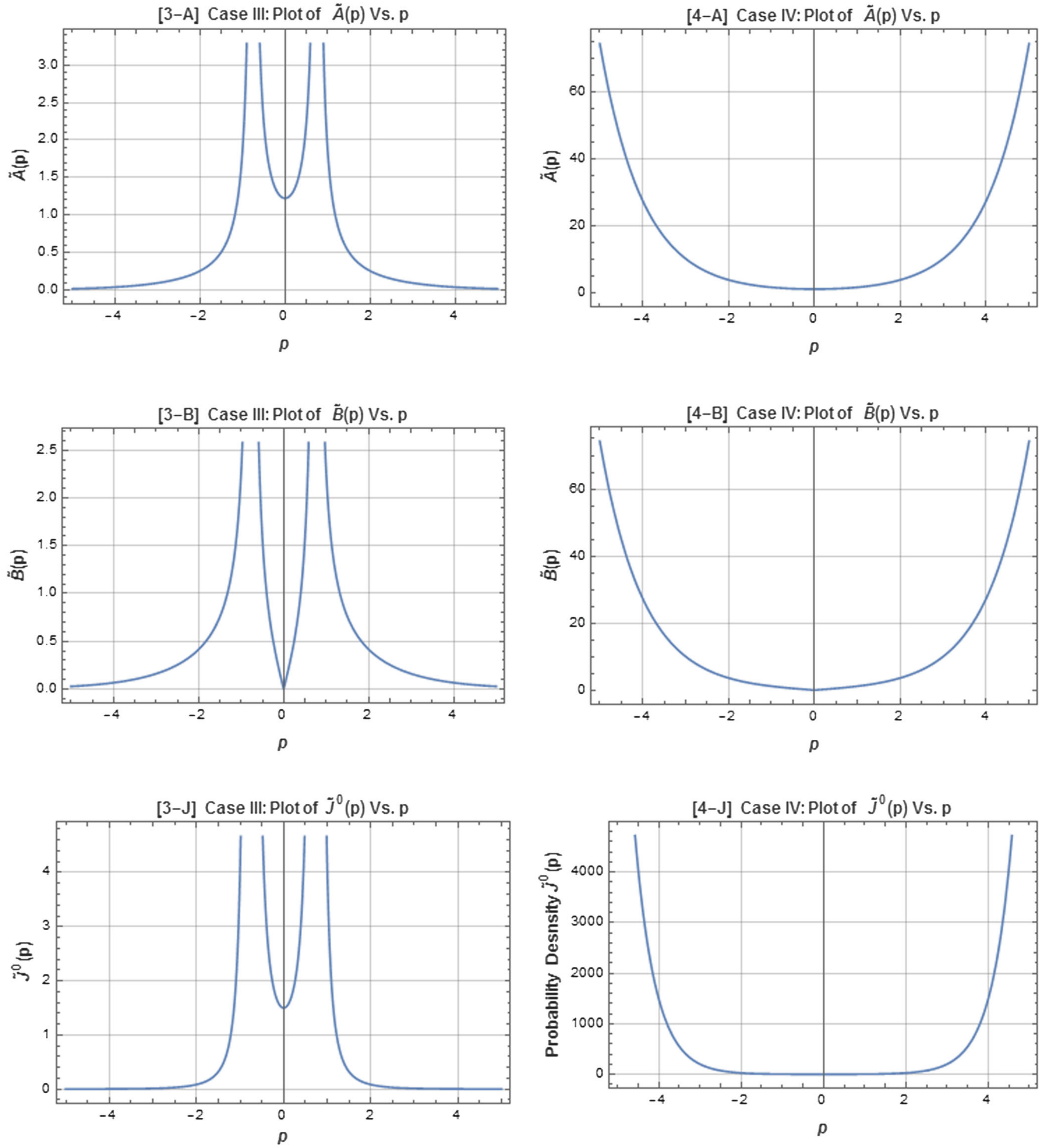

FIG. 3. Case III and case IV. Case III on the left, with $w = -0.5$. Case IV on the right, with $w = 0$. Both the cases have unphysical solutions.

$$\tilde{j}^0 = \left[\frac{(1+w)^2(w-1)\cos^2(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1}))^2} + \frac{(1-w)^2(w+1)\sin^2(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1}))^2}\right]. \tag{122}$$

Comments: This solution has an infinite number of singularities placed periodically over nonzero values of p, and is thus clearly unphysical. A plot (with $w = 2$) is given in the left column of Fig. 1.

Comparison with torsionless case: For $w \in (1, \infty)$ the linear Dirac equation gives (physically meaningful) plane

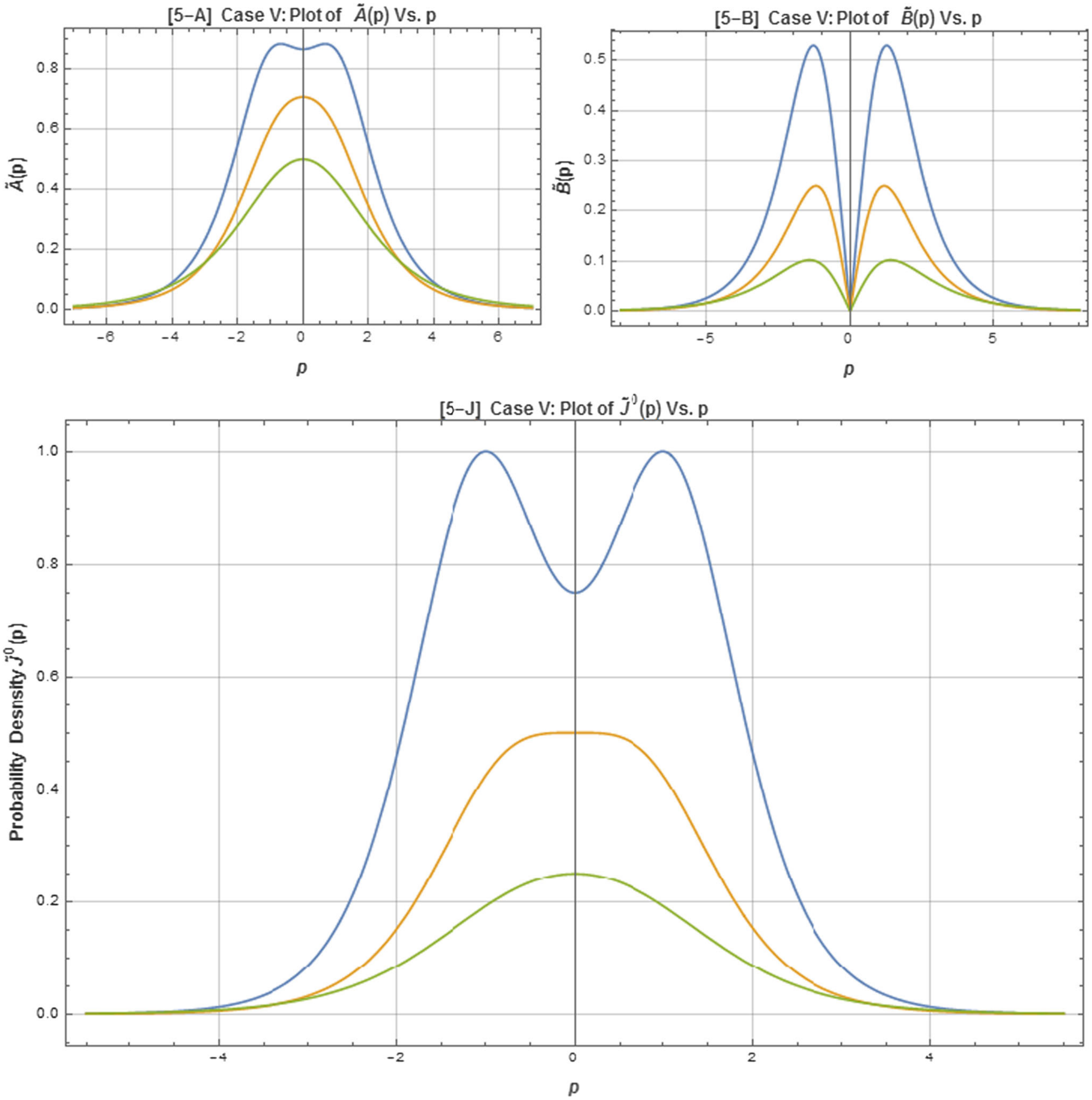

FIG. 4. Case V. In all plots: green: $w = 0.75$; orange: $w = 0.5$; blue: $w = 0.25$. For probability density plot, we have two subcases. Case V(b) has global maxima at origin (2 candidates of this case are shown in orange and green). Case V(a) has local minima at origin and two maximas at the two symmetrically opposite sides of origin at *nonzero* p (blue graph represents this case). Both cases V(a) and V(b) are asymptotically vanishing.

waves solutions. The probability density fluctuates sinusoidally. The addition of torsion makes this solution ultimately unphysical. A plot (with $w = 2$) is available in Fig. 2.

Table I summarises the various cases.

D. Attempting plane wave solutions

For previous work on plane wave solutions of the nonlinear Dirac equation see [24,25]. Our work in this section provides a more detailed analysis. We begin by considering the following plane wave ansatz:

$$\begin{bmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{bmatrix} e^{ik.x}. \tag{123}$$

TABLE I. While the physical cases for the Dirac solution (plane waves) are case I and case VI, case V is the only feasible solution in the case with torsion—while the corresponding Dirac solution blows up at infinity.

Cases	Solution(s) of the linear Dirac equation	Solution(s) of the Dirac equation with torsion
Case I	Physical (plane wave)	Unphysical (infinite singularities)
Case II	Trivial solution	Trivial solution
Case III	Unphysical (blows up exponentially at infinity)	Unphysical (two singularities)
Case IV	Unphysical (blows up exponentially at infinity)	Unphysical (blows up exponentially at infinity)
Case V	Unphysical (blows up exponentially at infinity)	Physical (No singularity)
Case VI	Physical (plane wave)	Unphysical (infinite singularities)

With this ansatz, ξ and ξ^* are as follows:

$$\xi = u^A \bar{v}_{A'}, \tag{124}$$

$$\xi^* = \bar{u}^{A'} v_A. \tag{125}$$

Substituting the above ansatz in (83)–(86), we obtain the following equations:

$$(k_0 + k_3)u^0 + (k_1 + ik_2)u^1 - \mu(\xi)\bar{v}_{0'} = 0, \tag{126}$$

$$(k_0 - k_3)u^1 + (k_1 - ik_2)u^0 - \mu(\xi)\bar{v}_{1'} = 0, \tag{127}$$

$$(k_0 + k_3)\bar{v}_{1'} - (k_1 - ik_2)\bar{v}_{0'} - \mu(\xi)u^1 = 0, \tag{128}$$

$$(k_0 - k_3)\bar{v}_{0'} - (k_1 + ik_2)\bar{v}_{1'} - \mu(\xi)u^0 = 0. \tag{129}$$

Here $\mu(\xi) = \sqrt{2}[b(l_2) + a(l_1)\xi]$ remains an undetermined quantity until a complete solution is obtained since ξ is a function of the spinor. However, if we assume that ξ is a real constant, we essentially end up with the usual Dirac equation with a "modified mass" $\mu(\xi)$. The equations can then be cast in matrix form:

$$\begin{pmatrix} (k_0 + k_3) & (k_1 + ik_2) & -\mu(\xi) & 0 \\ (k_1 - ik_2) & (k_0 - k_3) & 0 & -\mu(\xi) \\ 0 & -\mu(\xi) & -(k_1 - ik_2) & (k_0 + k_3) \\ -\mu(\xi) & 0 & (k_0 - k_3) & -(k_1 + ik_2) \end{pmatrix} \times \begin{pmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{130}$$

We work in the rest frame, and set $k_1 = k_2 = k_3 = 0$. The matrix equation then reduces to

$$\begin{pmatrix} k_0 & 0 & -\mu(\xi) & 0 \\ 0 & k_0 & 0 & -\mu(\xi) \\ 0 & -\mu(\xi) & 0 & k_0 \\ -\mu(\xi) & 0 & k_0 & 0 \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{131}$$

For a solution to exist, we require a null determinant. In other words,

$$(k_0^2 - \mu(\xi)^2)^2 = 0 \Rightarrow k_0 = \pm\mu(\xi).$$

Case I: $k_0 = \mu(\xi)$

The general solution is of the form

$$\begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix} = \frac{\alpha_1}{\sqrt{V}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{i\mu(\xi)x_0} + \frac{\beta_1}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{i\mu(\xi)x_0} \tag{132}$$

where $|\alpha_1|^2 + |\beta_1|^2 = 1$, and $V = 6\pi l_0^3$ is the volume of the box in which the theory lives.

Here, ξ and μ are as follows:

$$\xi = \frac{|\alpha_2|^2 + |\beta_2|^2}{V} = \frac{1}{V}, \tag{133}$$

$$\mu = \sqrt{2}\left(b + \frac{a}{V}\right) = \left(\frac{1}{2l_2} + \frac{l_1^2}{l_0^3}\right) \text{ where } l_0^3 > 2l_1^2 l_2. \tag{134}$$

ξ is indeed a real constant, verifying our approach. Further we recall that,

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} = \begin{pmatrix} P^0 \\ P^1 \\ \bar{Q}_{0'} \\ \bar{Q}_{1'} \end{pmatrix} = \begin{pmatrix} P^0 \\ P^1 \\ -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ -G_1 \\ -G_2 \end{pmatrix}. \tag{135}$$

Therefore, the actual spinor is given by

$$\Psi = \frac{\alpha_1}{\sqrt{V}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} e^{i(\mu_-)x_0} + \frac{\beta_1}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{i(\mu_-)x_0}. \tag{136}$$

Here, we have redefined $\mu(\xi) = \mu_-$ since the solution looks like the negative frequency solutions to the Dirac equation with a mass μ_-. This "modified mass" μ_- is always positive. Hence $k_0 = \mu_-$ is always positive in this case.

Case II: $k_0 = -\mu(\xi)$

In this case, the general solution is of the form

$$\begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix} = \frac{\alpha_2}{\sqrt{V}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} e^{-i\mu(\xi)x_0} + \frac{\beta_2}{\sqrt{V}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-i\mu(\xi)x_0} \tag{137}$$

where $|\alpha_2|^2 + |\beta_2|^2 = 1$ is the normalization condition.

The quantities ξ, μ and Ψ are given by

$$\xi = \frac{-|\alpha_2|^2 - |\beta_2|^2}{V} = \frac{-1}{V}, \tag{138}$$

$$\mu = \sqrt{2}\left(b - \frac{a}{V}\right) = \left(\frac{1}{2l_2} - \frac{l_1^2}{l_0^3}\right) \quad \text{where } l_0^3 > 2l_1^2 l_2, \tag{139}$$

$$\Psi = \frac{\alpha_2}{\sqrt{V}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{-i\mu_+ x_0} + \frac{\beta_2}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-i\mu_+ x_0}. \tag{140}$$

Once again we define $\mu(\xi) = \mu_+$ since this spinor looks like the positive frequency solution to the Dirac equation with a mass μ_+. This modified mass "μ_+" is always positive. Hence $k_0 = -\mu_+$ is always negative in this case.

By substituting the expressions for the suitable length scales in various theories ($l_1 = 0$, $l_2 = \lambda_c/2$ for a torsionless theory, $l_1 = L_{pl}$, $l_2 = \lambda_c/2$ for standard ECD, $l_1 = l_2 = L_{\rm CS}$ for modified ECD), and setting the value of fundamental constants to unity, we obtain the following table for μ_+ and μ_- in the various cases:

	No torsion	Standard ECD	Modified ECD
μ_+	$m_{1,2}$	$m_{1,2} - \frac{L_{pl}^2}{l_0^3}$	$\frac{1}{2L_{\rm CS}} - \frac{L_{\rm CS}^2}{l_0^3}$
μ_-	$m_{1,2}$	$m_{1,2} + \frac{L_{pl}^2}{l_0^3}$	$\frac{1}{2L_{\rm CS}} + \frac{L_{\rm CS}^2}{l_0^3}$

Corresponding to each value of $L_{\rm CS}$, there are two values of mass m_1 and m_2. For the theory with no torsion $\mu_+(l_1, l_2) = \mu_-(l_1, l_2)$, this equality breaks down when torsion is introduced, but is restored as l_0 tends to infinity. Note also that while $|m_{1,2} - \mu_+| = |m_{1,2} - \mu_-|$ is independent of $m_{1,2}$ for standard ECD, this is not the case for modified ECD.

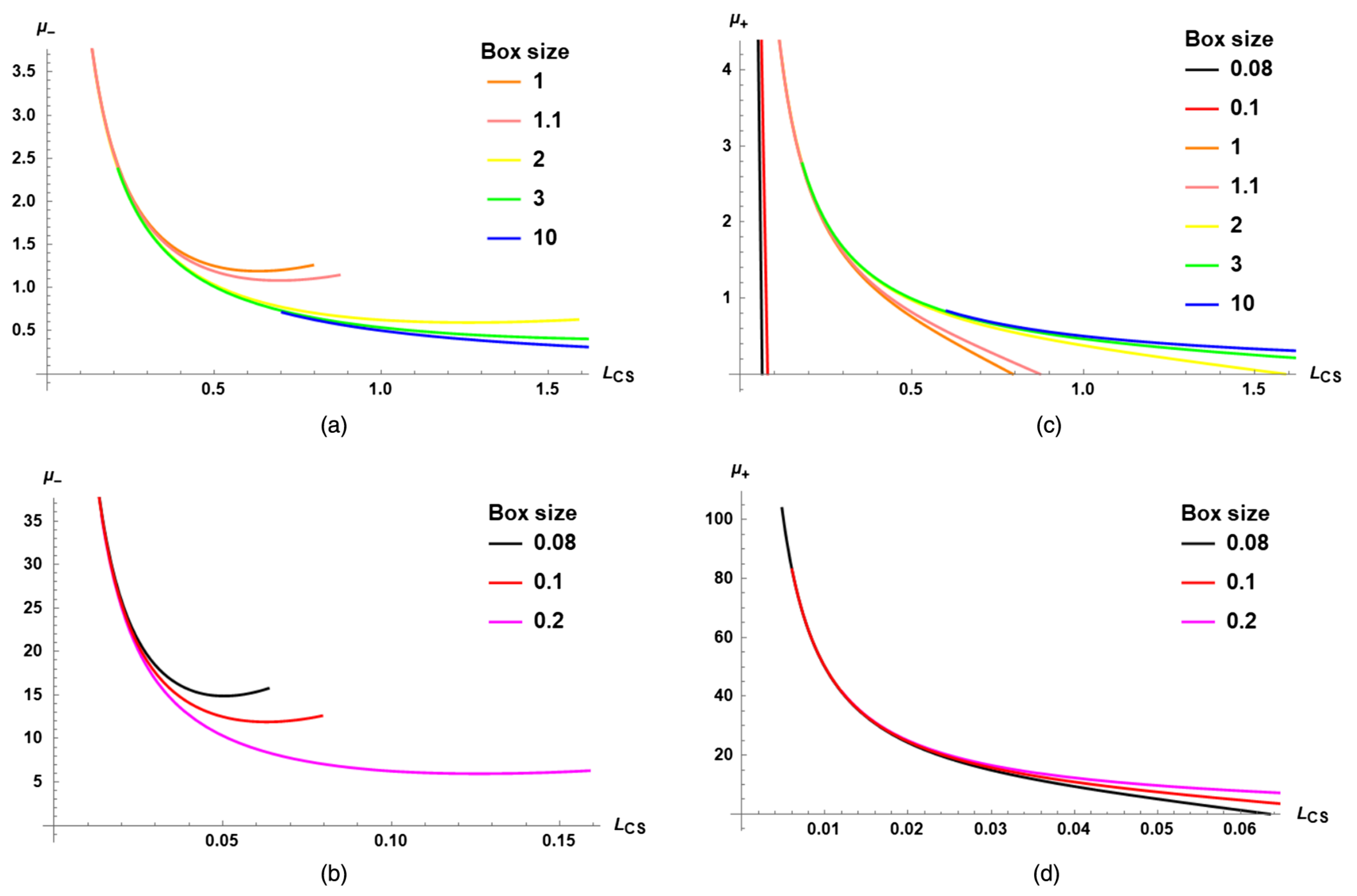

FIG. 5. Plots for μ_- and μ_+ as a function of $L_{\rm CS}$ for various values of l_0.

Figure 5 shows plots of μ_+ and μ_- as a function of L_{CS} (in the range L_{pl} to l_0) for various values of l_0. Lengths are measured in units of $10^{23}\ L_{pl}$. For a sense of scale, the L_{CS} for an electron (and for its dual mass) is $\sim 10^{22} L_{pl} = 0.1$ in these units.

The symmetry between positive and negative frequency solutions is broken by torsion in a peculiar way. Further, the introduction of L_{CS} introduces an interesting dependence of μ_+ and μ_- on L_{CS}. In the standard ECD theory, μ_+ (μ_-) acquires a very small subtractive (additive) "correction term" which is proportional to $\frac{1}{l_0^3}$ and independent of the mass $m_{1,2}$. This term becomes insignificant as the box size becomes larger. But this situation changes dramatically for the modified L_{CS} theory. μ_+ decreases monotonically with L_{CS} and increases monotonically with l_0. While μ_- decreases for $L_{CS} \leq l_0/4^{\frac{1}{3}}$, acquires a minimum at $L_{CS} = l_0/4^{\frac{1}{3}}$ and increases thereafter, it increases monotonically with l_0. The significance of the "modified mass" μ in this case is still being investigated.

E. Solution by reduction to (2 + 1) dim in cylindrical coordinates $(\mathbf{t},\mathbf{r},\boldsymbol{\phi},\mathbf{z})$

After assuming $\partial_z = 0$, the HD equations in cylindrical coordinates [(87)–(90)] are as follows:

$$r\partial_t F_1 + cr\partial_r F_2 e^{i\phi} + ic\partial_\phi F_2 e^{i\phi} F_1 = icr\sqrt{2}(b + a\xi)G_1, \tag{141}$$

$$r\partial_t F_2 + cr\partial_r F_1 e^{-i\phi} - ic\partial_\phi F_1 e^{-i\phi} = icr\sqrt{2}(b + a\xi)G_2, \tag{142}$$

$$r\partial_t G_2 - cr\partial_r G_1 e^{-i\phi} + ic\partial_\phi G_1 e^{-i\phi} = icr\sqrt{2}(b + a\xi^*)F_2, \tag{143}$$

$$r\partial_t G_1 - cr\partial_r G_2 e^{i\phi} - ic\partial_\phi G_2 e^{i\phi} = icr\sqrt{2}(b + a\xi^*)F_1. \tag{144}$$

We now take the ansatz, $F_2 = G_2$ and $F_1 = -G_1$

$$r\partial_t F_1 + r\partial_r F_2 e^{i\phi} + i\partial_\phi F_2 e^{i\phi} = -ir\sqrt{2}(b + a\xi)F_1, \tag{145}$$

$$r\partial_t F_2 + r\partial_r F_1 e^{-i\phi} - i\partial_\phi F_1 e^{-i\phi} = ir\sqrt{2}(b + a\xi)F_2. \tag{146}$$

We choose the following ansatz in the above equation:

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} iA(r)e^{\frac{i\phi}{2}} \\ B(r)e^{\frac{-i\phi}{2}} \end{bmatrix} e^{-i\omega t}. \tag{147}$$

Putting this ansatz in above equations, we obtain the two differential equations as follows:

$$-rB\omega + r\partial_r A + \frac{A}{2} = r\sqrt{2}[b + a(B^2 - A^2)]B, \tag{148}$$

$$rA\omega + r\partial_r B + \frac{B}{2} = r\sqrt{2}[b + a(B^2 - A^2)]A. \tag{149}$$

We add and subtract the two equations above and make the following substitution:

$$\psi_1 = B(r) + A(r), \tag{150}$$

$$\psi_2 = B(r) - A(r), \tag{151}$$

in order to obtain

$$-r\omega\psi_2 + r\psi_1' + \frac{\psi_1}{2} - r\sqrt{2}(b + a\psi_1\psi_2)\psi_1 = 0, \tag{152}$$

$$r\omega\psi_1 + r\psi_2' + \frac{\psi_2}{2} + r\sqrt{2}(b + a\psi_1\psi_2)\psi_2 = 0. \tag{153}$$

With $\omega = 0$, we have the solutions

$$\psi_1 = \left[\frac{c_2 e^{\sqrt{2}br}}{r^{(\frac{1-2\sqrt{2}ac_1}{2})}}\right], \qquad \psi_2 = \left[\frac{c_1 e^{-\sqrt{2}br} r^{(\frac{-1-2\sqrt{2}ac_1}{2})}}{c_2}\right]. \tag{154}$$

This is clearly unphysical because ψ_1 blows up $\forall$ nonzero c_2, and setting $c_2 = 0$ results in ψ_2 diverging. Thus, we conclude that a static solution to the above system of equation is unphysical, and ω cannot be zero. Further work to solve these equations numerically is in progress.

F. Solution by reduction to (3 + 1) dim in spherical coordinates $(\mathbf{t},\mathbf{r},\boldsymbol{\theta},\boldsymbol{\phi})$

We begin by putting following ansatz in HD equations with spherical coordinates:

$$\begin{bmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)e^{+i\phi/2} \\ R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)e^{-i\phi/2} \\ R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)e^{+i\phi/2} \\ R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)e^{-i\phi/2} \end{bmatrix} e^{-i\omega t}. \tag{155}$$

With this ansatz, (91)–(94) become

$$\left(-i\omega R_{-\frac{1}{2}}S_{-\frac{1}{2}}+\cos\theta R'_{-\frac{1}{2}}S_{-\frac{1}{2}}-\frac{\sin\theta}{r}R_{-\frac{1}{2}}S'_{-\frac{1}{2}}+\frac{1}{2r\sin\theta}R_{+\frac{1}{2}}S_{+\frac{1}{2}}+\sin\theta R'_{+\frac{1}{2}}S_{+\frac{1}{2}}+\frac{\cos\theta}{r}R_{+\frac{1}{2}}S'_{+\frac{1}{2}}\right)=i\sqrt{2}(b+a\xi)R_{+\frac{1}{2}}S_{-\frac{1}{2}} \tag{156}$$

$$\left(-i\omega R_{+\frac{1}{2}}S_{+\frac{1}{2}}-\cos\theta R'_{+\frac{1}{2}}S_{+\frac{1}{2}}+\frac{\sin\theta}{r}R_{+\frac{1}{2}}S'_{+\frac{1}{2}}-\frac{1}{2r\sin\theta}R_{-\frac{1}{2}}S_{-\frac{1}{2}}+\sin\theta R'_{-\frac{1}{2}}S_{-\frac{1}{2}}+\frac{\cos\theta}{r}R_{-\frac{1}{2}}S_{-\frac{1}{2}}\right)'=i\sqrt{2}(b+a\xi)R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta) \tag{157}$$

$$\left(-i\omega R_{-\frac{1}{2}}S_{+\frac{1}{2}}+\cos\theta R'_{-\frac{1}{2}}S_{+\frac{1}{2}}-\frac{\sin\theta}{r}R_{-\frac{1}{2}}S'_{+\frac{1}{2}}+\frac{1}{2r\sin\theta}R_{+\frac{1}{2}}S_{-\frac{1}{2}}-\sin\theta R'_{+\frac{1}{2}}S_{-\frac{1}{2}}-\frac{\cos\theta}{r}R_{+\frac{1}{2}}S'_{-\frac{1}{2}}\right)=i\sqrt{2}(b+a\xi^*)R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta) \tag{158}$$

$$\left(-i\omega R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)-\cos\theta R'_{+\frac{1}{2}}S_{-\frac{1}{2}}+\frac{\sin\theta}{r}R_{+\frac{1}{2}}S'_{-\frac{1}{2}}-\frac{1}{2r\sin\theta}R_{-\frac{1}{2}}S_{+\frac{1}{2}}-\sin\theta R'_{-\frac{1}{2}}S_{+\frac{1}{2}}-\frac{\cos\theta}{r}R_{-\frac{1}{2}}S'_{+\frac{1}{2}}\right)=i\sqrt{2}(b+a\xi^*)R_{-\frac{1}{2}}S_{-\frac{1}{2}} \tag{159}$$

where

$$\xi=R_{-\frac{1}{2}}S_{-\frac{1}{2}}\bar{R}_{+\frac{1}{2}}\bar{S}_{-\frac{1}{2}}+R_{+\frac{1}{2}}S_{+\frac{1}{2}}\bar{R}_{-\frac{1}{2}}\bar{S}_{-\frac{1}{2}}, \tag{160}$$

$$\xi^*=\bar{R}_{-\frac{1}{2}}\bar{S}_{-\frac{1}{2}}R_{+\frac{1}{2}}S_{-\frac{1}{2}}+\bar{R}_{+\frac{1}{2}}\bar{S}_{+\frac{1}{2}}R_{-\frac{1}{2}}S_{-\frac{1}{2}}. \tag{161}$$

Further work is in progress to investigate if this system of equations admits solitonic solutions.

VI. SUMMARY

The Einstein-Cartan-Dirac equations provide the most elegant classical system for describing the coupling of matter to space-time geometry. Torsion arises naturally because of the presence of spin; mass couples to gravity whereas torsion couples to spin. It is also expected that spin dominates mass in the small mass limit, whereas mass dominates spin for large masses. Correspondingly, it is expected that torsion dominates gravity in the microscopic limit, whereas gravity dominates torsion in the macroscopic limit. Furthermore, if one were to consider the fields of a point mass m, we expect it to behave like a black hole when say $m \gg m_{\mathrm{Pl}}$, and like a Dirac fermion when $m \ll m_{\mathrm{Pl}}$. This intriguingly suggests that the ECD equations ought to admit an exact solution which interpolates between Dirac fermions and black holes. There is an interplay between Compton wavelength and Schwarzschild radius of the particle, which will decide the nature of the solution (fermion or black hole) and one can expect some novel properties in the transition region. In the small mass limit, since torsion is present, the Dirac equation gets modified to the Hehl-Datta equation, and it is important to investigate the role that torsion might play in particle physics, and to put experimentally motivated bounds on torsion in the modified Dirac equation. It is with these motivations, that the present study has been initiated. The Newman-Penrose formalism is an elegant way to display the symmetry between torsion and gravity, especially in the context of the Dirac equation.

In this paper, we formulated ECD theory in the NP formalism. To this intent, we first described the standard field equations of the ECD theory. We also described how these equations are modified by the introduction of a new length scale, so that the two length scales in the problem are Planck length and Compton wavelength, or modifications thereof. We then introduced tetrads and the NP formalism. The contortion tensor is expressed in terms of Dirac spinors. The Dirac equation is carried to U_4 and presented (in NP) in (66)–(69). We have also provided a prescription for finding the covariant derivative on U_4 in NP formalism, thereby allowing one to calculate objects like the generic EM tensor on U_4 etc. We have calculated the spin density term which acts as a correction to the metric EM tensor; the two of which contribute together to the Einstein tensor (made up of Christoffel connections). In addition, the NP variables for the contortion spin coefficients are also expressed in terms of the Dirac state. Written in this formalism, the Dirac equation clearly shows an elegant symmetry between torsion and curvature.

Solutions to the linear Dirac equation on Minkowski space have been studied extensively. In this work, we attempted finding solutions to HD equations on Minkowski space with torsion. To begin with, we wrote these equations in Cartesian, cylindrical polar, and spherical polar coordinates. We explored whether presence of torsion induces any nontrivial (and physically relevant) modifications to the solutions for linear (nontorsional) case. Solutions after reducing the problem to $(1+1)$ dimension in the variables (t, z) were found. We found a finite parameter range

$w \in (0,1)$ (corresponding to the range $0 < \Lambda < m$), where this solution vanishes at infinity in the nonstatic case and has finite maxima (or finite local minima) at origin. For $w \in [\frac{1}{2}, 1)$, the solution (and the probability density) decreases monotonically from a finite value at center and asymptotically reaches zero at infinity. This is the sought after finite, solitonlike solution. This gives us hope that a $3 + 1$ solitonic solution exists, which interpolates between a black hole and a Dirac fermion.

Plane wave solutions were found in Sec. (V D). In so doing we have provided a more detailed analysis of earlier work on plane wave solutions of the nonlinear Dirac equation. The presence of torsion gives rise to a modified mass. We showed how the modified mass for the positive and negative frequency case depends on the bare mass and on the two length scales in the problem.

Next, we attempted finding solutions by reducing the problem to $(2 + 1)$ dimensions in cylindrical coordinates with variables (t, r, ϕ). Static solutions to this were also found to be unphysical. However, finding nonstatic solutions to $(2 + 1)$ case (given in Sec. V E) and the $(3 + 1)$ case (given in Sec. V F) is work under progress.

In future work we also hope to extend this investigation to Poincaré gauge gravity with propagating torsion. One of the principal goals of these studies is to look for torsion-induced nonsingular solitonic solutions of the nonlinear Dirac equation.

ACKNOWLEDGMENTS

S. K. and N. G. thank the Department of Science and Technology (DST), Govt. of India for the (INSPIRE)-SHE and KVPY fellowships, respectively. N. M. thank IIT-Bombay for letting him do a project at TIFR. S. K., A. V., N. G., and N. M. thank the Department of Astronomy and Astrophysics of TIFR for its hospitality.

APPENDIX A: CONTORTION TENSOR ($K^{\mu\nu\alpha}$) COMPONENTS

Our aim is to write the contortion tensor ($K^{\mu\nu\alpha}$) in the NP formalism eventually in terms of spinor components, with the contortion tensor given by

$$K^{\mu\nu\alpha} = -kS^{\mu\nu\alpha} = 2i\pi l^2 \bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{A1}$$

Note, only four independent components of this tensor is excited by the Dirac field. Writing explicitly in the NP formalism, i.e., null tetrad basis, we have

$$K_{(i)(j)(k)} = e_{(i)\mu}e_{(j)\nu}e_{(k)\alpha}K^{\mu\nu\alpha} \tag{A2}$$

where $e_{(i)\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ for $i = 0$, 1, 2, 3 First, we consider the product $\gamma^\alpha\gamma^\beta\gamma^\mu$, defined as follows:

$$\gamma^\alpha\gamma^\beta\gamma^\mu = \begin{pmatrix} 0 & (\tilde{\sigma}^\alpha)^*(\sigma^\beta)^*(\tilde{\sigma}^\mu)^* \\ (\sigma^\alpha)^*(\tilde{\sigma}^\beta)^*(\sigma^\mu)^* & 0 \end{pmatrix}$$
$$= 2\sqrt{2}\begin{pmatrix} 0_{2\times2} & K_{01} \\ K_{10} & 0_{2\times2} \end{pmatrix} \tag{A3}$$

where, explicitly, expanding out the Van der Waarden symbols, we have

$$K_{01} = \begin{bmatrix} +nln - n\bar{m}m - \bar{m}mn + \bar{m}nm & -nl\bar{m} + n\bar{m}l + \bar{m}m\bar{m} - \bar{m}nl \\ -mln + m\bar{m}m + lmn - lnm & +ml\bar{m} - m\bar{m}l - lm\bar{m} + lnl \end{bmatrix}^{\alpha\beta\mu}, \tag{A4}$$

$$K_{10} = \begin{bmatrix} +lnl - l\bar{m}m - \bar{m}ml + \bar{m}lm & +ln\bar{m} - l\bar{m}n - \bar{m}m\bar{m} + \bar{m}ln \\ +mnl - m\bar{m}m - nml + nlm & +mn\bar{m} - m\bar{m}n - nm\bar{m} + nln \end{bmatrix}^{\alpha\beta\mu}. \tag{A5}$$

With the expression for $\gamma^\alpha\gamma^\beta\gamma^\mu$, we can now define the world components of K. Next, we use (A2) to calculate the contortion spin coefficients [8] in the NP (null tetrad) basis. An an example, the solution for ρ_1 is given as

$$\rho_1 = -K_{(0)(2)(3)} = -l_\mu m_\nu \bar{m}_\alpha K^{\mu\nu\alpha} = -2i\pi l^2[l_\mu m_\nu \bar{m}_\alpha]\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi. \tag{A6}$$

The only quantity giving a nonzero scalar product when contracted with $l_\mu m_\nu \bar{m}_\alpha$ is $n^\mu \bar{m}^\nu m^\alpha$ and corresponding permutations (given the definition of $\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}$), giving $l_\mu m_\nu \bar{m}_\alpha n^\mu \bar{m}^\nu m^\alpha = 1$. Thus,

$$[l_\mu m_\nu \bar{m}_\alpha]\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}\psi = \frac{\sqrt{2}}{3}\bar{\psi}\left(\begin{bmatrix}0&0&-1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&-1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{bmatrix}\right.$$
$$\left. + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-1&0&0\end{bmatrix}\right)$$
$$= \frac{\sqrt{2}}{3}\begin{pmatrix}Q_0 & Q_1 & \bar{P}^{0'} & \bar{P}^{1'}\end{pmatrix}\begin{pmatrix}0&0&-3&0\\0&0&0&0\\0&0&0&0\\0&3&0&0\end{pmatrix}\begin{pmatrix}P^0\\P^1\\\bar{Q}_{0'}\\\bar{Q}_{1'}\end{pmatrix}$$
$$= \sqrt{2}(\bar{P}^{1'}P^1 - Q^1\bar{Q}^{1'}).$$

This gives the full expression for ρ (redefining the spinor components as prescribed):

$$\rho = -K_{(0)(2)(3)} = -2\sqrt{2}i\pi l^2[F_2\bar{F}_2 - G_1\bar{G}_1] \tag{A7}$$

and similarly for the other spin coefficients.

APPENDIX B: THE DIRAC EQUATION IN U_4

The Dirac equation on U_4 (the *Hehl-Datta* equation) is given, in matrix form, as

$$i\begin{pmatrix}0 & (\tilde{\sigma}^\mu)^*\\(\sigma^\mu)^* & 0\end{pmatrix}\nabla_\mu\begin{pmatrix}P^A\\\bar{Q}_{B'}\end{pmatrix} = \frac{1}{2\sqrt{2}l}\begin{pmatrix}P^A\\\bar{Q}_{B'}\end{pmatrix}. \tag{B1}$$

Rewriting as a pair of matrix equations, we have

$$\begin{pmatrix}\sigma^\mu_{00'} & \sigma^\mu_{10'}\\\sigma^\mu_{01'} & \sigma^\mu_{11'}\end{pmatrix}\nabla_\mu\begin{pmatrix}P^0\\P^1\end{pmatrix} + \frac{i}{2\sqrt{2}l}\begin{pmatrix}-\bar{Q}^{1'}\\\bar{Q}^{0'}\end{pmatrix} = 0, \tag{B2}$$

$$\begin{pmatrix}\sigma^\mu_{11'} & -\sigma^\mu_{10'}\\-\sigma^\mu_{01'} & \sigma^\mu_{00'}\end{pmatrix}\nabla_\mu\begin{pmatrix}-\bar{Q}^{1'}\\\bar{Q}^{0'}\end{pmatrix} + \frac{i}{2\sqrt{2}l}\begin{pmatrix}P^0\\P^1\end{pmatrix} = 0. \tag{B3}$$

We will proceed to work through a solution for the first and third equation generated by this pair; the second and fourth follow along similar lines.

Equation 1:

$$\begin{aligned}\frac{i}{2\sqrt{2}l}\bar{Q}^{1'} &= \sigma^\mu_{00'}\nabla_\mu P^0 + \sigma^\mu_{10'}\nabla_\mu P^1\\
&= (\partial_{00'}P^0 + \Gamma^0{}_{i00'}P^i) + (\partial_{10'}P^1 + \Gamma^1{}_{i10'}P^i)\\
&= (D + \Gamma^0{}_{000'}P^0 + \Gamma^0{}_{100'}P^1) + (\delta^* + \Gamma^1{}_{010'}P^0 + \Gamma^1{}_{110'}P^1)\\
&= (D + \Gamma_{1000'} - \Gamma_{0010'})P^0 + (\delta^* + \Gamma_{1100'} - \Gamma_{0110'})P^1\\
&= (D + \epsilon^o + \epsilon_1 - \rho^o - \rho_1)P^0 + (\delta^* + \pi^o + \pi_1 - \alpha^o - \alpha_1)P^1\\
&= (D + \epsilon_0 - \rho_0)P^0 + (\delta^* + \pi_0 - \alpha_0)P^1 + \frac{3}{2}(\pi_1 P^1 - \rho_1 P^0).\end{aligned} \tag{B4}$$

Equation 3:

$$\begin{aligned}
\frac{i}{2\sqrt{2}l}P^0 &= -\sigma^{\mu}_{11'}\nabla_\mu \bar{Q}^{1'} - \sigma^{\mu}_{10'}\nabla_\mu \bar{Q}^{0'} + \frac{i}{2\sqrt{2}l}P^0 \\
&= -\bar{\sigma}^{\mu}_{11'}\nabla_\mu \bar{Q}^{1'} - \bar{\sigma}^{\mu}_{0'1}\nabla_\mu \bar{Q}^{0'} + \frac{i}{2\sqrt{2}l}P^0 \\
&= (\partial_{11'}\bar{Q}^{1'} + \bar{\Gamma}^{1'}{}_{i'1'1}\bar{Q}^{i'}) + (\partial_{10'}\bar{Q}^{0'} + \bar{\Gamma}^{0'}{}_{i'0'1}\bar{Q}^{i'}) \\
&= (\Delta\bar{Q}^{1'} + \bar{\Gamma}^{1'}{}_{0'1'1}\bar{Q}^{0'} + \bar{\Gamma}^{1'}{}_{1'1'1}\bar{Q}^{1'}) + (\delta^*\bar{Q}^{0'} + \bar{\Gamma}^{0'}{}_{0'0'1}\bar{Q}^{0'} + \bar{\Gamma}^{0'}{}_{1'0'1}\bar{Q}^{1'}) \\
&= (\Delta + \bar{\Gamma}_{1'1'0'1} - \bar{\Gamma}_{0'1'1'1})\bar{Q}^{1'} + (\delta^* + \bar{\Gamma}_{1'0'0'1} - \bar{\Gamma}_{0'0'1'1})\bar{Q}^{0'} \\
&= (\Delta + \mu^o + \mu_1 - \gamma^o - \gamma_1)\bar{Q}^{1'} + (\delta^* + \beta^o + \beta_1 - \tau^o - \tau_1)\bar{Q}^{0'} \\
&= (\Delta + \mu_0^* - \gamma_0^*)\bar{Q}^{1'} - (\delta^* + \beta_0^* - \tau_0^*)\bar{Q}^{0'} - \frac{3}{2}(\mu_1\bar{Q}^{1'} - \pi_1\bar{Q}^{0'})
\end{aligned} \tag{B5}$$

where we have used the gamma matrices as defined in (34), computed the covariant derivatives using (36), (37) and the spin connections in terms of contortion spin coefficients as given in (52). Using this procedure, the four Dirac equations in U_4 are obtained as

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1) = ib(l)G_1, \tag{B6}$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + \frac{3}{2}(\mu_1 F_2 - \tau_1 F_1) = ib(l)G_2, \tag{B7}$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 - \frac{3}{2}(\tau_1 G_1 - \rho_1 G_2) = ib(l)F_2, \tag{B8}$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 - \frac{3}{2}(\mu_1 G_1 - \pi_1 G_2) = ib(l)F_1, \tag{B9}$$

where we have also redefined $\{P, Q\} \to \{F, G\}$, as per the substitution in (1) and to obtain a form that can be consistently compared with the primary source material in [9] [Eq. (108)].

APPENDIX C: CALCULATING $(T-S)_{\mu\nu}$

In theories which consider a balance between the Riemannian and torsional curvatures (such as in [18]), the tensor $(T-S)_{\mu\nu}$ is of paramount importance. Vanishing $(T-S)_{\mu\nu}$ would take the form of a "balance condition," and represent a space with nonzero Riemannian curvature and torsion, but where the two exactly cancel each other out. The $(T-S)_{\mu\nu}$ tensor is defined as

$$(T-S)_{\mu\nu} = T_{\mu\nu} - \frac{4\pi l^2}{\hbar c}\eta_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}. \tag{C1}$$

This tensor has 10 components. The 6 off-diagonal components are as follows:

$$\begin{aligned}
(T-S)_{10} = \frac{i\hbar c}{4}(&\bar{F}_1\partial_1 F_1 + \bar{F}_2\partial_1 F_2 + \bar{G}_1\partial_1 G_1 + \bar{G}_2\partial_1 G_2 - \bar{F}_2\partial_0 F_1 - \bar{F}_1\partial_0 F_2 + \bar{G}_2\partial_0 G_1 + \bar{G}_1\partial_0 G_2 \\
&- \partial_1\bar{F}_1 F_1 - \partial_1\bar{F}_2 F_2 - \partial_1\bar{G}_1 G_1 - \partial_1\bar{G}_2 G_2 + \partial_0\bar{F}_2 F_1 + \partial_0\bar{F}_1 F_2 - \partial_0\bar{G}_2 G_1 - \partial_0\bar{G}_1 G_2),
\end{aligned} \tag{C2}$$

$$\begin{aligned}
(T-S)_{20} = \frac{i\hbar c}{4}(&\bar{F}_1\partial_2 F_1 + \bar{F}_2\partial_2 F_2 + \bar{G}_1\partial_2 G_1 + \bar{G}_2\partial_2 G_2 + i\bar{F}_2\partial_0 F_1 - i\bar{F}_1\partial_0 F_2 - i\bar{G}_2\partial_0 G_1 + i\bar{G}_1\partial_0 G_2 \\
&- \partial_2\bar{F}_1 F_1 - \partial_2\bar{F}_2 F_2 - G_1\partial_2\bar{G}_1 - \partial_2\bar{G}_2 G_2 - i\partial_0\bar{F}_2 F_1 + i\partial_0\bar{F}_1 F_2 + i\partial_0\bar{G}_2 G_1 - i\partial_0\bar{G}_1 G_2),
\end{aligned} \tag{C3}$$

$$\begin{aligned}
(T-S)_{30} = \frac{i\hbar c}{4}(&\bar{F}_1\partial_3 F_1 + \bar{F}_2\partial_3 F_2 + \bar{G}_1\partial_3 G_1 + \bar{G}_2\partial_3 G_2 - \bar{F}_1\partial_0 F_1 + \bar{F}_2\partial_0 F_2 + \bar{G}_1\partial_0 G_1 - \bar{G}_2\partial_0 G_2 \\
&- \partial_3\bar{F}_1 F_1 - \partial_3\bar{F}_2 F_2 - \partial_3\bar{G}_1 G_1 - \partial_3\bar{G}_2 G_2 + \partial_0\bar{F}_1 F_1 - \partial_0\bar{F}_2 F_2 - \partial_0\bar{G}_1 G_1 + \partial_0\bar{G}_2 G_2),
\end{aligned} \tag{C4}$$

$$(T-S)_{21} = \frac{i\hbar c}{4}(i\bar{F}_2\partial_1 F_1 - i\bar{F}_1\partial_1 F_2 - i\bar{G}_2\partial_1 G_1 + i\bar{G}_1\partial_1 G_2 - \bar{F}_2\partial_2 F_1 - \bar{F}_1\partial_2 F_2 + \bar{G}_2\partial_2 G_1 + \bar{G}_1\partial_2 G_2 - i\partial_1\bar{F}_2 F_1 + i\partial_1\bar{F}_1 F_2 + i\partial_1\bar{G}_2 G_1 - i\partial_1\bar{G}_1 G_2 + \partial_2\bar{F}_2 F_1 + \partial_2\bar{F}_1 F_2 - \partial_2\bar{G}_2 G_1 - \partial_2\bar{G}_1 G_2), \tag{C5}$$

$$(T-S)_{31} = \frac{i\hbar c}{4}(-\bar{F}_1\partial_1 F_1 + \bar{F}_2\partial_1 F_2 + \bar{G}_1\partial_1 G_1 - \bar{G}_2\partial_1 G_2 - \bar{F}_2\partial_3 F_1 - \bar{F}_1\partial_3 F_2 + \bar{G}_2\partial_3 G_1 + \bar{G}_1\partial_3 G_2 + \partial_1\bar{F}_1 F_1 - \partial_1\bar{F}_2 F_2 - \partial_1\bar{G}_1 G_1 + \partial_1\bar{G}_2 G_2 + \partial_3\bar{F}_2 F_1 + \partial_3\bar{F}_1 F_2 - \partial_3\bar{G}_2 G_1 - \partial_3\bar{G}_1 G_2), \tag{C6}$$

$$(T-S)_{32} = \frac{i\hbar c}{4}(-\bar{F}_1\partial_2 F_1 + \bar{F}_2\partial_2 F_2 + \bar{G}_1\partial_2 G_1 - \bar{G}_2\partial_2 G_2 + i\bar{F}_2\partial_3 F_1 - i\bar{F}_1\partial_3 F_2 - i\bar{G}_2\partial_3 G_1 + i\bar{G}_1\partial_3 G_2 + \partial_2\bar{F}_1 F_1 - \partial_2\bar{F}_2 F_2 - \partial_2\bar{G}_1 G_1 + \partial_2\bar{G}_2 G_2 - i\partial_3\bar{F}_2 F_1 + i\partial_3\bar{F}_1 F_2 + i\partial_3\bar{G}_2 G_1 - i\partial_3\bar{G}_1 G_2). \tag{C7}$$

The diagonal components are as follows:

$$(T-S)_{00} = \frac{i\hbar c}{2}(\bar{G}_1\partial_0 G_1 + \bar{G}_2\partial_0 G_2 - \partial_0\bar{G}_1 G_1 - \partial_0\bar{G}_2 G_2 + \bar{F}_1\partial_0 F_1 + \bar{F}_2\partial_0 F_2 - \partial_0\bar{F}_1 F_1 - \partial_0\bar{F}_2 F_2) - 6\pi\hbar c l^2\xi\xi^*, \tag{C8}$$

$$(T-S)_{11} = \frac{i\hbar c}{2}(-\bar{F}_2\partial_1 F_1 - \bar{F}_1\partial_1 F_2 + \bar{G}_2\partial_1 G_1 + \bar{G}_1\partial_1 G_2 + \partial_1\bar{F}_2 F_1 + \partial_1\bar{F}_1 F_2 - \partial_1\bar{G}_2 G_1 - \partial_1\bar{G}_1 G_2) + 6\pi\hbar c l^2\xi\xi^*, \tag{C9}$$

$$(T-S)_{22} = \frac{i\hbar c}{2}(i\bar{F}_2\partial_2 F_1 - i\bar{F}_1\partial_2 F_2 - i\bar{G}_2\partial_2 G_1 + i\bar{G}_1\partial_2 G_2 - i\partial_2\bar{F}_2 F_1 + i\partial_2\bar{F}_1 F_2 + i\partial_2\bar{G}_2 G_1 - i\partial_2\bar{G}_1 G_2) + 6\pi\hbar c l^2\xi\xi^*, \tag{C10}$$

$$(T-S)_{33} = \frac{i\hbar c}{2}(-\bar{F}_1\partial_3 F_1 + \bar{F}_2\partial_3 F_2 + \bar{G}_1\partial_3 G_1 - \bar{G}_2\partial_3 G_2 + \partial_3\bar{F}_1 F_1 - \partial_3\bar{F}_2 F_2 - \partial_3\bar{G}_1 G_1 + \partial_3\bar{G}_2 G_2) + 6\pi\hbar c l^2\xi\xi^*. \tag{C11}$$

We can now calculate this tensor for the various solutions to the HD equations on Minkowski space with torsion, to probe the feasibility of a "balance condition." For example, $(T-S)_{\mu\nu}$ for nonstatic solutions in $1+1$ dim (t,z),

$$(T-S)_{\mu\nu} = \hbar c\begin{bmatrix} \left(\Lambda[A^2+B^2] - \frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 & -\Lambda AB & 0 \\ 0 & \left(\frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 & 0 \\ -\Lambda AB & 0 & \left(\frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 \\ 0 & 0 & 0 & \left([AB'-BA'] + \frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) \end{bmatrix} \tag{C12}$$

Λ is a free parameter in the solution; we can analyze the tensor $T-S$ for various types of values of Λ.

APPENDIX D: THE LINEAR (TORSIONLESS) DIRAC EQUATION IN 1+1 DIMENSIONS

The vanishing of torsion is characterized by the limit $a(l_2) = 3\sqrt{2}\pi L_{\mathrm{Pl}}^2 \to 0$. So in a torsionless case, the differential equations become (with dimensionless constants):

$$B' = (1-w)A, \tag{D1}$$

$$A' = (1+w)B. \tag{D2}$$

Their solutions in various special cases are plotted in Fig. 2.

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