Upper bound on the gravitational masses of stable spatially regular charged compact objects

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In a very interesting paper, Andréasson has recently proved that the gravitational mass of a spherically symmetric compact object of radius R and electric charge Q is bounded from above by the relation $\sqrt{M} \le \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}}$. In the present paper we prove that, in the dimensionless regime $Q/M < \sqrt{9/8}$, a stronger upper bound can be derived on the masses of physically realistic (*stable*) self-gravitating horizonless compact objects: $M < \frac{R}{3} + \frac{2Q^2}{3R}$.

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I. INTRODUCTION

The asymptotically measured gravitational mass M of a spherically symmetric asymptotically flat Schwarzschild black-hole spacetime is directly related to the horizon radius R by the simple relation $M = R/2$ [1,2]. It is well known that a stronger upper bound on the gravitational masses of spatially regular self-gravitating horizonless compact objects is provided by the physically important Buchdahl bound $M \le 4R/9$ [3].

Similar bounds are known to exist for charged self-gravitating compact objects. In particular, charged Reissner-Nordström black holes are characterized by the simple relation $M = R/2 + Q^2/2R$ [1]. In a physically interesting paper, Andréasson [4] has recently derived the stronger upper bound

$$\sqrt{M} \le \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}} \tag{1}$$

on the gravitational masses of spatially regular horizonless charged compact objects.

In the present paper we raise the following physically intriguing question: Can one improve the important upper bound (1) on the masses of self-gravitating charged compact objects by adding to the characteristic properties of the corresponding horizonless curved spacetimes the physically motivated requirement of dynamical *stability*?

As we shall explicitly show below, the above-stated question is directly related to the physically important theorem presented recently in [5] (see also [6,7]), according to which the innermost null circular geodesic of a horizonless compact object, if it exists, is stable [8]. In particular, combining this interesting physical property of the spatially regular self-gravitating compact objects that we consider in the present paper with the intriguing assertion made in [9] (see also [10,11]), according to which horizonless spacetimes which possess stable null circular geodesics (stable closed light rings) are expected to develop nonlinear instabilities in response to the presence of time-dependent massless perturbation fields [12], one concludes that spatially regular compact objects that possess light rings in their exterior spacetime regions are dynamically unstable.

Motivated by the physically important observations made in [5,9] regarding the (in)stability properties of horizonless compact objects, in the present paper we shall use *analytical* techniques in order to derive an improved upper bound [see Eq. (22) below] on the maximally allowed gravitational masses $M^{\max}(R, Q)$ of dynamically *stable* spatially regular charged compact objects.

II. DESCRIPTION OF THE SYSTEM

We consider self-gravitating horizonless charged compact objects whose spatially regular curved spacetimes are described by the spherically symmetric line element [1,10,11,13–15]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

The radially dependent metric functions $\mu = \mu(r)$ and $\delta = \delta(r)$ are related to the composed energy-momentum tensor $T^{\mu}_{\nu}(\text{total}) = T^{\mu}_{\nu}(\text{matter}) + T^{\mu}_{\nu}(\text{electromagnetic field})$ of the charged matter configurations by the Einstein field equations [1]

$$G^{\mu}_{\nu} = 8\pi[T^{\mu}_{\nu}(\text{matter}) + T^{\mu}_{\nu}(\text{electromagnetic field})], \tag{3}$$

which, using the curved line element (2) and the functional expressions [16]

$$T^{\mathrm{em}\,t}{}_{t} = T^{\mathrm{em}\,r}{}_{r} = -T^{\mathrm{em}\,\theta}{}_{\theta} = -T^{\mathrm{em}\,\phi}{}_{\phi} = -\frac{Q^2(r)}{8\pi r^4} \qquad (4)$$

for the components of the electromagnetic (em) energy-momentum tensor, can be expressed in the differential forms [11,16,17]

$$\mu' = -8\pi r\left[\rho + \frac{Q^2(r)}{8\pi r^4}\right] + \frac{1-\mu}{r} \qquad (5)$$

and

$$\delta' = -\frac{4\pi r(\rho + p)}{\mu}. \qquad (6)$$

Here $Q(r)$ is the electric charge contained within a sphere of areal radius r [16], $\rho \equiv -T^t_t(\text{matter})$, and $p \equiv T^r_r(\text{matter})$ [18].

The metric functions $\{\mu, \delta\}$ of the horizonless spatially regular asymptotically flat spacetime (2) are respectively characterized by the near-origin and far-region functional relations [11]

$$\mu(r \to 0) \to 1, \qquad \mu(r \to \infty) \to 1 \qquad (7)$$

and [11]

$$\delta(0) < \infty, \qquad \delta(r \to \infty) \to 0. \qquad (8)$$

In particular, the Einstein equation (5) implies that the radial metric function $\mu(r)$ can be expressed in the mathematically compact form [4]

$$\mu(r) = 1 - \frac{2m(r)}{r} + \frac{Q^2(r)}{r^2}, \qquad (9)$$

where $m(r)$ is the gravitational mass contained within a sphere of radius r [4,16]. For later purposes we note that, as explicitly proved in [11], regular self-gravitating matter configurations with asymptotically measured finite masses are characterized by the asymptotically decaying functional behavior

$$r^3 T^r_r(\text{total}) \to 0 \quad \text{for } r \to \infty. \qquad (10)$$

III. THE UPPER BOUND ON THE GRAVITATIONAL MASSES OF STABLE SPATIALLY REGULAR HORIZONLESS CHARGED COMPACT OBJECTS

In the present section we shall prove that, in the dimensionless regime [19,20]

$$\frac{Q}{M} \le \sqrt{\frac{9}{8}}, \qquad (11)$$

one can use the instability properties of spatially regular horizonless spacetimes which possess light rings [5–7,9] in order to derive an upper bound on the gravitational masses of physically realistic (*stable*) charged compact objects. In particular, below we shall explicitly show that the newly derived bound [see Eq. (22) below] is stronger than the important upper bound (1).

The functional equation which determines the radii of light rings in the curved spacetime (2) was derived in [1,10,11]. For completeness of the presentation, we shall first provide a brief sketch of the analytical derivation of the functional relation which characterizes the null circular geodesics of the charged spacetime. As explicitly shown in [1,10], the circular null trajectories which characterize the spherically symmetric spacetime (2) are determined by the two relations [21,22]

$$V_r = E^2 \quad \text{and} \quad V'_r = 0, \qquad (12)$$

where the effective potential V_r is given by the functional expression [1,10,11]

$$E^2 - V_r \equiv \dot{r}^2 = \mu\left(\frac{E^2}{e^{-2\delta}\mu} - \frac{L^2}{r^2}\right). \qquad (13)$$

Here the energy E and the angular momentum L are conserved quantities which reflect the fact that the metric components of the spherically symmetric static spacetime (2) are independent of the time and angular coordinates $\{t, \phi\}$ [1,10,11].

Substituting Eq. (13) into Eq. (12) and using the Einstein differential equations (5) and (6), one finds that the light rings of the spherically symmetric static curved spacetime (2) are determined by the compact functional relation

$$\mathcal{R}(r) \equiv 3\mu - 1 - 8\pi r^2\left[p - \frac{Q^2(r)}{8\pi r^4}\right] = 0 \quad \text{for } r = r_\gamma. \qquad (14)$$

In addition, taking cognizance of Eqs. (7), (10), and (14), one deduces that the dimensionless function $\mathcal{R}(r)$, whose zeroes determine the discrete radii of the null circular geodesics of the spherically symmetric spacetime (2), is characterized by the two boundary relations

$$\mathcal{R}(r = 0) = 2 \quad \text{and} \quad \mathcal{R}(r \to \infty) \to 2. \qquad (15)$$

These simple relations imply that spatially regular horizonless compact objects are generally characterized by an *even* number of null circular geodesics [5,6,23].

The stability properties of the null circular geodesics are generally determined by the second spatial derivative of the effective curvature potential (13) [1,10]. In particular, unstable light rings are characterized by locally concave radial potentials with $V''_r(r = r_\gamma) < 0$, whereas stable circular geodesics which, as discussed in [9], are associated with nonlinear instabilities of the corresponding curved spacetimes, are characterized by locally convex curvature potentials with the property $V_r''(r = r_\gamma) > 0$ [1,10]. Taking cognizance of Eqs. (5), (6), (12), and (13), and using the conservation equation $T^{\mu}_{r;\mu} = 0$ [16], one finds the simple functional relation [6,7]

$$V_r''(r = r_\gamma) = -\frac{E^2 e^{2\delta}}{\mu r_\gamma} \times \mathcal{R}'(r = r_\gamma). \tag{16}$$

From Eqs. (14) and (15) one deduces that the innermost null circular geodesic, $r = r_\gamma^{\text{innermost}}$, of a spatially regular horizonless compact object is generally [23] characterized by the properties

$$\mathcal{R}(r = r_\gamma^{\text{innermost}}) = 0 \quad \text{and} \quad \mathcal{R}'(r = r_\gamma^{\text{innermost}}) < 0. \tag{17}$$

In particular, the innermost light ring of a spatially regular compact object, if it exists, is generally *stable* with the property $V_r''(r = r_\gamma^{\text{innermost}}) > 0$ [see Eqs. (16) and (17)] [5–7].

The exterior spacetime regions ($r \geq R$) of the spherically symmetric charged compact objects that we consider in the present paper are characterized by the relations [4]

$$\rho = p = 0 \tag{18}$$

and [1]

$$\mu(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad \text{for } r \geq R, \tag{19}$$

where $\{M, Q\}$ are respectively the total gravitational mass and the total electric charge of the spherically symmetric spacetime as measured by asymptotic observers.

Let us assume that the spatially regular charged compact object possesses an external light ring with $r_\gamma > R$. Substituting Eqs. (18) and (19) into the functional relation (14), which characterizes the null circular geodesics of the spherically symmetric spacetime (2), one finds the remarkably simple expression

$$r_\gamma^{\text{outer}} = \frac{1}{2}(3M + \sqrt{9M^2 - 8Q^2}) \quad \text{for } \frac{Q}{M} \leq \sqrt{\frac{9}{8}} \tag{20}$$

for the radius of the outer light ring.

As discussed above, the presence of the light ring (20) outside the surface of a spatially regular horizonless compact object [24] implies the existence of a second (stable) light ring (with the property $r_\gamma^{\text{innermost}} < r_\gamma^{\text{outer}}$) in the charged curved spacetime. In particular, as suggested in [9], the presence of this inner *stable* null circular geodesic in the spherically symmetric curved spacetime (2) may *indicate* that the corresponding horizonless compact object is nonlinearly unstable to massless perturbation fields [12,25]. One therefore concludes that spatially regular horizonless spacetimes describing physically realistic (*stable*) compact objects must not possess light rings. This physical fact yields the lower bound [see Eq. (20)]

$$R > \frac{1}{2}\left(3M + \sqrt{9M^2 - 8Q^2}\right) \tag{21}$$

on the radii of stable horizonless charged compact objects.

IV. SUMMARY AND DISCUSSION

In a physically important paper, Andréasson [4] has recently derived the upper bound (1) on the gravitational masses of spatially regular horizonless charged compact objects. In the present paper we have raised the physically interesting related question: Can one derive a stronger upper bound on the gravitational masses of *stable* charged compact systems?

This physically intriguing question is motivated by the recent theorem presented in [5] which, when combined with the results presented in [9], asserts that horizonless compact objects whose curved spacetimes possess null circular geodesics (light rings) are nonlinearly unstable to massless perturbation fields.

Using analytical techniques, we have proved that the answer to the above-stated question is "yes." In particular, it has been explicitly proved that the masses of physically realistic (*stable*) self-gravitating horizonless compact objects are bounded from above by the compact functional relation [see Eq. (21)]

$$M < \frac{R}{3} + \frac{2Q^2}{3R} \quad \text{for } \frac{Q}{M} \leq \sqrt{\frac{9}{8}}. \tag{22}$$

Taking cognizance of (1) and (22) one finds that, in the dimensionless regime $Q/M \leq \sqrt{9/8}$, the analytically derived upper bound (22) on the gravitational masses of stable spatially regular charged compact objects is *stronger* than the physically important bound (1). In particular, one finds that the newly derived upper bound (22) is stronger than (1) in the $R \geq Q$ regime. In addition, we recall that in the present paper we consider compact objects which are characterized by the dimensionless inequality $Q/M \leq \sqrt{9/8}$ [see (11)] which, taking cognizance of Eq. (21), corresponds to $R > \sqrt{2}Q$. One therefore concludes that, in the $Q/M \leq \sqrt{9/8}$ regime, the bound (22) for *stable* charged compact systems is stronger than the bound (1).

Finally, it is worth mentioning that a universal upper bound on the entropies of charged compact systems has been presented in [2,26,27]

$$S \leq \pi(2MR - Q^2). \tag{23}$$

Interestingly, substituting the newly derived upper bound (22) on the masses of physically realistic charged compact objects into (23), one can express the entropy upper bound in terms of the surface area $A = 4\pi R^2$ of the corresponding charged stable physical system [28]:

$$S \leq \frac{A}{6} - \frac{\pi Q^2}{3}. \quad (24)$$

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[21] Note that the conditions (12), which characterize the null circular geodesics of the spherically symmetric spacetime (2), correspond to $\dot{r}^2 = (\dot{r}^2)' = 0$ [1,10].
[22] We use the dot symbol to denote a derivative with respect to an affine parameter.
[23] See [6] for the special case of horizonless compact objects which are characterized by degenerate light rings with $\mathcal{R} = \mathcal{R}' = 0$. As discussed in [6], these special matter configurations may possess an odd number of null circular geodesics.
[24] That is, the presence of a null circular geodesic in the exterior spacetime region with $\rho = p = 0$ [see Eq. (18)].
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