Spherical symmetric dust collapse in vector-tensor gravity

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There is a viable vector-tensor gravity (VTG) theory, the vector field of which produces repulsive forces leading to important effects. In the background universe, the effect of these forces is an accelerated expansion identical to that produced by vacuum energy (cosmological constant). Here, we prove that another of these effects arises for great enough collapsing masses which lead to Schwarzschild black holes and singularities in general relativity. For these masses, pressure becomes negligible against gravitational attraction, and the complete collapse cannot be stopped in the context of general relativity; however, in VTG, a strong gravitational repulsion could stop the falling of the shells toward the symmetry center. A certain study of a collapsing dust cloud is then developed, and in order to undertake this task, the VTG equations in comoving coordinates are written. In this sense, as it happens in general relativity for a pressureless dust ball, three different solutions are found. These three situations are analyzed, and the problem of the shell crossings is approached. The apparent horizons and trapped surfaces, the analysis of which will lead to diverse situations, depending on a certain theory characteristic parameter value, are also examined.

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I. INTRODUCTION

Any vector-tensor theory of gravitation involves the metric tensor $g^{\mu\nu}$ and a vector field A^{μ}. These fields are coupled to build up an appropriate action leading to the basic equations via variational calculations. There are many actions and vector-tensor theories [1,2], but one of them has been extensively studied to conclude that (i) it has neither classical nor quantum instabilities and (ii) it explains—as well as general relativity (GR)—both cosmological and solar system observations [3–7]; hence, new applications of this viable and promising theory are worthwhile. Since there are opposite gravitational forces, this theory will be hereafter called attractive-repulsive vector-tensor gravity (AR-VTG).

As was shown in Ref. [6], for appropriate values of the AR-VTG parameters (see below), there are black hole event horizons with admissible radii that are a little smaller than those of GR; nevertheless, for other values of these parameters, there are no horizons of this kind.

Our signature is $(-,+,+,+)$. Greek indexes run from 0 to 3. The symbol ∇ (∂) stands for a covariant (partial) derivative. The antisymmetric tensor $F_{\mu\nu}$ is defined by the relation $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. It has nothing to do with the electromagnetic field. Quantities $R_{\mu\nu}$, R, and g are the covariant components of the Ricci tensor, the scalar curvature, and the determinant of the matrix $g_{\mu\nu}$ formed by the covariant components of the metric, respectively. Units are chosen in such a way that the gravitational constant, G, and the speed of light, c, take the values $c = G = 1$; namely, we use geometrized units.

This paper is structured as follows. The AR-VTG theory is described in Sec. II. The vacuum stationary spherically symmetric solutions of the field equations are presented in Sec. III. The collapsing systems to be considered are described in Sec. IV. The behavior of the collapse of a spatially bounded spherical dust cloud, modeled into shells, is considered in Sec. V. Finally, Sec. VI contains conclusions, a certain discussion, and prospects.

II. AR-VTG FOUNDATIONS

Let us now briefly summarize the AR-VTG basic equations, which were derived in Refs. [3,4] from an appropriated action, which is a particularization of the

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general vector-tensor action given in Ref. [1]. The resulting field equations are

$$G^{\mu\nu} = 8\pi(T^{\mu\nu}_{\rm GR} + T^{\mu\nu}_{\rm VT}), \tag{1}$$

$$2(2\epsilon - \gamma)\nabla^{\nu} F_{\mu\nu} = J^{A}_{\mu}, \tag{2}$$

where $G^{\mu\nu}$ is the Einstein tensor, $T^{\mu\nu}_{\rm GR}$ is the GR energy-momentum tensor, $J^{A}_{\mu} \equiv -2\gamma\nabla_{\mu}(\nabla\cdot A)$ with $\nabla\cdot A = \nabla_{\mu}A^{\mu}$, and

$$\begin{aligned} T^{\mu\nu}_{VT} = {} & 2(2\epsilon-\gamma)\left[F^{\mu}_{\alpha}F^{\nu\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \\ & - 2\gamma\left[\left\{A^{\alpha}\nabla_{\alpha}(\nabla\cdot A) + \frac{1}{2}(\nabla\cdot A)^{2}\right\}g^{\mu\nu} \right. \\ & \left. - A^{\mu}\nabla^{\nu}(\nabla\cdot A) - A^{\nu}\nabla^{\mu}(\nabla\cdot A)\right]. \end{aligned} \tag{3}$$

Equation (2) leads to the conservation law

$$\nabla^{\mu} J^{A}_{\mu} = 0 \tag{4}$$

for the fictitious current J^{A}_{μ}. Moreover, the conservation laws $\nabla_{\mu}T^{\mu\nu}_{GR} = 0$ and $\nabla_{\mu}T^{\mu\nu}_{VT} = 0$ are satisfied by any solution of Eqs. (1) and (2) (see Ref. [1]).

The pair of parameters (ϵ, γ) must satisfy the inequality $2\epsilon - \gamma > 0$ to prevent the existence of quantum ghosts and unstable modes in AR-VTG (see Ref. [5] and references cited therein). The condition $\gamma > 0$ must be required to have a positive A^{μ} energy density in the background universe, which will play the role of vacuum energy; hence, the inequalities $\epsilon > \frac{\gamma}{2} > 0$ must be satisfied.

III. VACUUM STATIONARY SPHERICALLY SYMMETRIC METRICS IN AR-VTG

In the stationary spherically symmetric case, in Schwarzschild coordinates, the line element may be written as (see, e.g., Ref. [8])

$$\begin{aligned} ds^{2} = {} & -e^{\nu}dt^{2} + e^{\lambda}dr^{2} \\ & + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}), \end{aligned} \tag{5}$$

where ν and λ are functions of r. Moreover, the covariant components A_{μ} have the form

$$A_{\mu} \equiv (A_0, A_1, 0, 0). \tag{6}$$

In the absence of any matter content, the form of functions $\nu(r)$, $\lambda(r)$, $A_0(r)$, $A_1(r)$, and $\nabla\cdot A$ may be found in Ref. [6]. The resulting functions involve integration constants, and some of them have not yet been fixed; nevertheless, the involved constants in $A_0(r)$ and $A_1(r)$ are not necessary to perform this research since, whatever their values may be, the AR-VTG line element is

$$\begin{aligned} ds^{2} = {} & -h(r)dt^{2} + h^{-1}(r)dr^{2} \\ & + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}), \end{aligned} \tag{7}$$

where

$$h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^{2} + \frac{\alpha^{2}M^{2}}{r^{2}}. \tag{8}$$

The dimensionless quantity $\alpha^2 > 0$ is proportional to the positive number $2\epsilon - \gamma$. It is obvious that the metric (8) is formally identical to the Reissner-Nordström-de Sitter metric of Einstein-Maxwell theory for a stationary spherically symmetric charged system, with charge Q such that $Q^2 = \alpha^2 M^2$. For $Q = 0$, this metric reduces to the Kottler-Schwarzschild-de Sitter metric [9] of GR.

Finally, as was shown in Ref. [6], the scalar $\nabla\cdot A$ is constant. Its value may be easily fixed by taking into account Eq. (3). In fact, according to this equation, as r tends to infinity, the energy-momentum tensor $T^{\mu\nu}_{VT}$ created by the total mass M tends to $-\gamma(\nabla\cdot A)^2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the Minkowski metric. Since this energy-momentum tensor must asymptotically vanish, one concludes that the relation $\nabla\cdot A = 0$ must be satisfied for the vacuum solution under consideration. This relation and the fact that M is a constant will be taken into account to fit the inner and outer solutions on the collapsing star boundary.

IV. GENERAL CONSIDERATIONS

In terms of the function $h(r) = -g_{00}(r)$, the event horizons are the hypersurfaces $r = r_h$ defined by the condition $h(r_h) = 0$. A complete discussion about horizons—for the line element defined by Eqs. (7) and (8)—may be found in Ref. [10]. The number of horizons and their radius depend on the values of M, Λ, and α^2.

In the standard ΛCDM cosmological model of GR, most current observations are explained for a vacuum energy density parameter $\Omega_{\Lambda} \simeq 0.7$, namely, for $\Lambda \simeq 1.2 \times 10^{-46}$ km^{-2}. The same value also explains current observations in the framework of AR-VTG (see Ref. [5]); hence, this value of the cosmological constant is hereafter fixed. In AR-VTG cosmology, quantity $\nabla\cdot A$ takes on a constant value, and the cosmological constant is proportional to the square of this value. Nevertheless, this constant must have another unknown origin in GR; anyway, in the stationary spherically symmetric case, it is denoted Λ, and its treatment is the same in both theories. The line element (5) does not depend on time, and consequently, it does not describe a cosmological spacetime but the spacetime in a region located well inside the so-called cosmological horizon and outside the black hole event horizon (if it exists). In this region, one has $h(r) > 0$ as it should be. Sometimes, the black hole horizon does not exist, and the metric is well defined everywhere—inside the cosmological horizon—excepting the singular point $r = 0$.

For realistic masses, M, ranging from those of the smallest black holes (star collapses) to the masses of the greatest supermassive black holes located in galactic centers, the product ΛM^2 is many orders of magnitude smaller than unity, and as a result of this fact, the general discussion about horizons presented in Ref. [10] may be simplified. First of all, it may be easily proved that, for arbitrary realistic values of M and α^2 (see below), there is a cosmological horizon with radius $r_c \simeq 1.73 \times 10^{23}$ km; this is the outermost horizon, the radius of which essentially depends on the cosmological constant, which has been fixed. Furthermore, for r of the order of M ($r \ll r_c$), other inner horizons may exist. The radius of these horizons may be easily estimated, with very high precision, by neglecting the term $\Lambda r^2/3$ -order ΛM^2—in the $h(r)$ formula; the error due to this approximation is fully negligible. By solving then the equation $h(r) = 0$, one easily finds the following:

(i) For $\alpha^2 > 1$, there are no solutions, and consequently, only the cosmological horizon exists.
(ii) For $\alpha^2 = 1$, there is a unique double solution $r = M$ corresponding to an inner event horizon.
(iii) For $\alpha^2 < 1$, there are two solutions r_{H_-} and r_{H_+} with $r_{H_-} < r_{H_+}$. In such a situation, there are two inner horizons plus the cosmological one.

In the context of GR, after a supernova explosion, three cases may be distinguished:

(i) If the mass M of the supernova core is smaller than the Chandrasechar limit ($\sim 1.4\ M_\odot$), a white dwarf is formed. As an example, let us mention Sirius B [11], with a mass $M \sim M_\odot$ and a radius $R \sim 8 \times 10^{-3} R_\odot$; this white dwarf has a ratio $M/r \sim 3 \times 10^{-4}$ and a mean density $\rho \sim 3 \times 10^6$ gr/cm^3. In this situation, the pressure of degenerate electrons prevents collapse, and the value of M/r is so small that the term $\alpha^2 M^2/r^2$ in Eq. (8) is negligible against $2M/r$ for $\alpha^2 \leq 2$, which means that AR-VTG, with $\alpha^2 \leq 2$ (see below), and GR ($\alpha^2 = 0$) lead to the same description of Sirius B. The same occurs for any white dwarf,
(ii) For a mass $M \gtrsim 1.4\ M_\odot$, the pressure due to degenerate electrons cannot balance gravity, and star contraction continues to reach densities much greater than $\rho \sim 10^6$ gr/cm^3. For masses $1.4 \lesssim M \lesssim 3\ M_\odot$ [12], the pressure due to degenerate neutrons and their strong interactions may prevent collapse to form a dense neutron star with a small radius of the order of 10 km [13]; e.g., a great neutron star, with mass $M = 2\ M_\odot$ and radius $R \sim 10$ km, has a ratio $M/r \sim 0.15$ and a mean density $\rho \sim 2 \times 10^{15}$ gr/cm^3 (see Ref. [14]). For this value of M/r and $\alpha^2 \leq 2$, the term $\alpha^2 M^2/r^2$ is less than 15% of $2M/r$; hence, small but non-negligible deviations between the neutron star structures in GR and AR-VTG should exist. Since these deviations could be too great for $\alpha^2 \gg 2$, in this paper, the condition $\alpha^2 \leq 2$ has been tentatively assumed. The mass-radius relation must be estimated in the context of AR-VTG, namely, taking into account the A^μ repulsion for different α^2 values. This will be done elsewhere,
(iii) For masses $M \gtrsim 3\ M_\odot$, the pressure of degenerate neutrons cannot prevent collapse. All the particles would reach the singularity $r = 0$ in a finite proper time, although this fall toward $r = 0$ cannot be observed from points with $r > 2M$ due to the existence of an event horizon at $r = 2M$. As the system approaches the singularity, its density grows, reaching greater values than the so-called Planck density. For these huge densities, classical gravity does not apply, and quantum gravity might prevent the singularity; however, as we will show in next sections, the shells of a collapsing star cannot reach the singularity ($r = 0$) in AR-VTG due to the action of a strong A^μ repulsion. Nothing similar occurs in GR (just gravitation, that is, no external fields) in which any neutral (not charged) shell collapses.

The question is what happens with star and black hole formation in the context of AR-VTG for $\alpha^2 \leq 2$. A qualitative but accurate answer follows from points i–iii. For $M \lesssim 3\ M_\odot$, GR and AR-VTG predict similar qualitative evolutions, even for the most massive observed neutron stars. The differences between both theories grow as M increases. For $M \gtrsim 3\ M_\odot$, the nuclear density $\rho_N \sim 2 \times 10^{14}$ gr/cm^3 will be reached at a certain moment, t_N, as in neutron stars, but the pressure of degenerated nucleons cannot stop contraction when this density is reached, and consequently, collapse continues. Let us study this last phase starting at time t_N. Close to this time, RG and AR-VTG evolutions would be comparable, but these evolutions dramatically deviate later. In GR, contraction cannot be stopped, and particles converge toward $r = 0$; however, in the context of AR-VTG, it may be proved that the repulsive gravitational forces quickly grow as r decreases, in such a way that the fall of the fluid shells toward $r = 0$ is stopped when a certain minimum bounce radius is reached. Nevertheless, specifically because of this increasing repulsive force, the innermost shells are gravitationally less bounded and hence collapse more slowly than the outer ones, and as it happens with an electromagnetic repulsive force for a collapsing charged dust, shell crossings are a possibility that have to be reviewed; this is complemented by an analysis about the formation of horizons and trapped surfaces. The determination of the conditions under which trapped surfaces are formed (if they exist) is an interesting, or furthermore necessary, task because it may reveal the presence of singularities in a gravitational collapse [15]. Analytical models for the evolution of the dust collapsing cloud is considered in next section, in which the aforementioned issues are also examined.

V. SINGULARITIES IN GR AND AR-VTG

In the framework of GR, any shell—including the outermost one (see Ref. [16])—of the pressureless collapsing dust reaches point $r = 0$ in a proper finite time $\Delta\tau$; thus, the collapse of the whole dust cloud is unavoidable. However, in AR-VTG, the outermost shell has a very different behavior. One could expect that the dust sphere's surface moves as a test particle in the AR-VTG stationary spherically symmetric spacetime defined by the line element given by Eqs. (7) and (8). Neglecting the term $\Lambda r^2/3$, the radial motion is ruled by the equation

$$\frac{d\tau}{dx} = \pm M\left[(E^2 - 1) + \frac{2}{x} - \frac{\alpha^2}{x^2}\right]^{-1/2}, \tag{9}$$

where E is the energy per unit rest mass, τ is the proper time, and x is defined by the ratio $x \equiv r/M$. The causal structure of this spacetime is analogous to the Reissner-Nordström one (see Fig. 1 in Ref. [17]) for $\alpha^2 < 1$, and the apparent horizons are defined by the expression

$$r_{H_\pm} = M\left(1 \pm \sqrt{1 - \alpha^2}\right). \tag{10}$$

A possible solution of Eq. (9) would lead to damped oscillations with a small period driving the test particle to a state of minimum gravitational energy with a finite radius. Point $r = 0$ would be not reached by the aforementioned test particle; this situation is due to the existence of the repulsive component of the AR-VTG modified gravity. As happens in a collapsing charged dust (see Ref. [17]), the motion of the outermost shell may not be independent of the evolution of the inner dust cloud. For instance, in the same way as in Ref. [17], the shell crossings may break down prior expectations. This possibility has to be analyzed.

Hereafter, it is assumed that the collapsing core is big enough and pressure gradients become negligible against gravitational forces. This core is an ideal fluid with $T^{\mu\nu}_{\rm GR} = \rho u^\mu u^\nu$, where ρ is the energy density, $u^\mu = dx^\mu/d\tau$ is the 4-velocity, and τ is the proper time. The nonvanishing components of the 4-velocity are u^0 and u^1.

A. Basic equations in Schwarzschild-like coordinates

Let us first consider Schwarzschild-like coordinates to write the interior nonstationary spherically symmetric metric in the form (5), with $\nu = \nu(r,t)$ and $\lambda = \lambda(r,t)$. In this way, the interior solution may be matched with the exterior vacuum solution given by Eqs. (7) and (8).

It has been proven (see above) that the scalar $\nabla \cdot A$ vanishes outside the collapsing core. Let us now assume that in the spherically symmetric case this scalar vanishes everywhere (also inside the collapsing object); so, there are no matching problems with $\nabla \cdot A$ at the core surface. This assumption will be proved to be consistent with the AR-VTG equations, and it is necessary to coherently match the inner and outer solutions (see below). In spite of the fact that $\nabla \cdot A$ vanishes everywhere, functions A_0 and A_1 do not simultaneously vanish, and AR-VTG does not coincide with GR in the spherically symmetric case (spherical collapse).

For $\nabla \cdot A = 0$, the field equation (2) reduces to

$$\nabla^\nu F_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial(\sqrt{-g}F^{\mu\nu})/\partial x^\nu = 0. \tag{11}$$

Inside the collapsing object, the nonvanishing components of the AR-VTG vector field are $A_0 = A_0(r,t)$ and $A_1 = A_1(r,t)$, and the nonvanishing components of $F_{\mu\nu}$ are $F_{01} = -F_{10} = \partial A_1/\partial t - \partial A_0/\partial r$. It is then easily proved that Eq. (11) reduces to

$$\partial(r^2 e^\sigma F^{01})/\partial r = 0 \tag{12}$$

and

$$\partial(r^2 e^\sigma F^{01})/\partial t = 0, \tag{13}$$

where $\sigma = (\lambda + \nu)/2$; hence, the $r^2 e^\sigma F^{01}$ quantity is a constant.

In the absence of electrical charges and currents, Eq. (11) is also valid in Einstein-Maxwell theory. In such a case, the relation $r^2 e^\sigma F^{01} = 0$ is satisfied (see Ref. [18]). In AR-VTG, without charges and currents, we can write $r^2 e^\sigma F^{01} \equiv D$, where D is a nonvanishing constant, and then the field equations (1) lead to

$$8\pi\left[\left(\varepsilon - \frac{\gamma}{2}\right)\frac{D^2}{r^4} - \rho u^0 u_0\right] = \frac{1}{r^2} + e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right), \tag{14}$$

$$8\pi\left[\left(\varepsilon - \frac{\gamma}{2}\right)\frac{D^2}{r^4} - \rho u^1 u_1\right] = \frac{1}{r^2} - e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right), \tag{15}$$

$$8\pi\rho u^1 u_0 = e^{-\lambda}\frac{\dot{\lambda}}{r}. \tag{16}$$

With the essential aim of matching metrics at the core surface, let us write the interior metric as follows:

$$e^{-\lambda} = 1 - \frac{2m(r,t)}{r} + 8\pi\left(\varepsilon - \frac{\gamma}{2}\right)\frac{D^2}{r^2}. \tag{17}$$

A similar procedure may be found in Ref. [18]. Since D is a constant, this last equation plus Eq. (14) gives

$$\frac{\partial m}{\partial r} = -4\pi\rho r^2 u_0 u^0, \tag{18}$$

and from Eqs. (16) and (17), one easily obtains

$$\frac{\partial m}{\partial t} = 4\pi\rho r^2 u_0 u^1. \tag{19}$$

Finally, from the 4-velocity definition plus Eqs. (18) and (19), it follows that

$$\frac{dm}{d\tau} = \frac{\partial m}{\partial r} u^1 + \frac{\partial m}{\partial t} u^0 = 0. \tag{20}$$

According to this equation, the proper-time derivative of m with respect to an observer comoving with the fluid vanishes, which means that the mass inside a sphere comoving with the collapsing matter is conserved. This is only true in the absence of pressure. The same conservation holds in Einstein-Maxwell theory [18]. It is not possible for $\nabla \cdot A \neq 0$. In particular, the total mass M will be conserved since it is the mass inside the comoving boundary. This fact is necessary to match the interior metric (17) and the exterior one; in fact, on the boundary, one has $m(r,t) = M$, and by choosing

$$8\pi\left(\varepsilon - \frac{\gamma}{2}\right) D^2 = \alpha^2 M^2, \tag{21}$$

the exterior metric has the same form as that defined by Eqs. (7) and (8).

B. Basic equations in comoving coordinates

Hereafter, we will try to use comoving coordinates (T, a, θ, ϕ) to get an analytical shell model inside the collapsing object; this will lead us to a revision of the shell crossings issue (an extensive analysis can be found in Ref. [19] within the framework of GR) that is the collision of two adjacent dust shells. As stated in Ref. [20], this fact is directly related to the absence of any coupling between the motion of the different dust shells. By using these coordinates, the line element inside the collapsing object has the form

$$ds^2 = -e^{\nu} dT^2 + e^{\lambda} da^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{22}$$

where ν, λ, and $R > 0$ are functions of a and T (see Refs. [16,18]).

For a collapsing spherically symmetric pressureless fluid, the conservation law $\nabla_\mu T^{\mu\nu}_{\rm GR} = 0$, which is valid in GR as well as in AR-VTG (see Sec. II), leads to

$$\dot{\lambda} + \frac{4\dot{R}}{R} = -2\frac{\dot{\rho}}{\rho}; \quad \nu' = 0. \tag{23}$$

Hereafter, the prime (dot) denotes a derivative with respect to coordinate a (T). Since function ν does not depend on a ($\nu' = 0$), the time T may be redefined to have $\nu = 0$, and the resulting coordinates are synchronous and comoving. In these coordinates, the line element reads as follows [16,21]:

$$ds^2 = -dT^2 + e^{\lambda} da^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{24}$$

Moreover, due to the remaining coordinate degree of freedom left, comoving coordinate a may be chosen in such a way that, inside the collapsing core, it takes values inside the interval $0 \leq a \leq 1$, with $a = 0$ at the center and $a = 1$ for the core boundary.

By using the line element (24), Eq. (11) gives

$$(R^2 e^{\lambda/2} F^{01})' = 0 \tag{25}$$

and

$$(R^2 e^{\lambda/2} F^{01})^{\cdot} = 0. \tag{26}$$

These two equations express that the quantity $R^2 e^{\lambda/2} F^{01}$ is a constant. Moreover, in Schwarzschild coordinates, it has been proven that the $r^2 e^{\sigma} F^{01}$ quantity is constant, too (see Sec. V B). Both expressions give $\sqrt{-g} F^{01}$ in the corresponding coordinates. Both constants are identical since $\sqrt{-g} F^{01}$ behaves as a scalar under coordinate transformations of the form $a = a(r,t)$ and $T = T(r,t)$ with fixed coordinates θ and ϕ, which has the form of the transformations between comoving synchronous and Schwarzschild coordinates. Hence, in comoving synchronous coordinates, we can write

$$R^2 e^{\lambda/2} F^{01} = D, \tag{27}$$

and Eq. (21) holds. On account of these equations, plus the above expressions for $T^{\mu\nu}_{GR}$ and $T^{\mu\nu}_{VT}$, Eq. (1) may be written as follows:

$$-8\pi\left[\left(\varepsilon - \frac{\gamma}{2}\right)\frac{D^2}{R^4} + \rho\right] = \frac{1}{R^2}[e^{-\lambda}(2RR'' + R'^2 - \lambda' RR') - (\dot{\lambda} R\dot{R} + \dot{R}^2) - 1], \tag{28}$$

$$-8\pi\left[\left(\varepsilon - \frac{\gamma}{2}\right)\frac{D^2}{R^4}\right] = \frac{1}{R^2}[e^{-\lambda} R'^2 - (2R\ddot{R} + \dot{R}^2) - 1], \tag{29}$$

$$-\frac{1}{R} e^{-\lambda}(2\dot{R}' - \dot{\lambda} R') = 0. \tag{30}$$

As is shown in Ref. [21], the solution of Eq. (30) is given by

$$e^{\lambda} = \frac{R'^2}{1 - \beta f^2(a)}, \quad (31)$$

where $\beta = 0, \pm 1$ and $f(a)$ is an arbitrary function of the radial coordinate a subject the sole condition $1 - \beta f^2(a) > 0$. To make the reading easier, let us mention that the function f we use and the ones used by Landau [16] and many other authors—say f_L—are related by the expression $f_L = -\beta f^2$.

Substituting expression (31) in Eq. (29), we find

$$2R\ddot{R} + \dot{R}^2 = -\beta f^2(a) + \frac{\alpha^2 M^2}{R^2}. \quad (32)$$

This equation reduces to the corresponding equation of GR [21] for $\alpha^2 = 0$. An integration gives

$$\dot{R}^2 = -\beta f^2(a) + \frac{F(a)}{R} - \frac{\alpha^2 M^2}{R^2}, \quad (33)$$

where $F(a)$ is a new arbitrary function.

By combining Eqs. (28), (31), and (33), one easily gets the following formula for the fluid energy density of the a shell at time T:

$$8\pi\rho(a, T) = \frac{F'(a)}{R^2(a, T)R'(a, T)}. \quad (34)$$

Equation (33) may be rewritten in the form

$$dT = \frac{RdR}{\sqrt{-\beta R^2 f^2(a) + RF(a) - \alpha^2 M^2}}, \quad (35)$$

and, evidently, the inequality

$$P_a(R) \equiv \beta R^2 f^2(a) - RF(a) + \alpha^2 M^2 < 0 \quad (36)$$

must be satisfied whatever the radial coordinate a value may be.

Assuming the weak energy condition ($T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$) in GR, it follows that $F' \geq 0$ [22]. It is worth noting that the same conclusion can be drawn in AR-VTG from the expression $T^{\mu\nu} = T^{\mu\nu}_{\rm GR} + T^{\mu\nu}_{\rm VT}$. In the same manner, $F(a)$ can be interpreted as twice the weighted mass [by the factor $\sqrt{1 - \beta f^2(a)}$] inside a volume V of coordinate radius a, so F must be positive everywhere; $\beta f^2 < 1$ must hold for a Lorentzian manifold, as mentioned previously, in relation with Eq. (31).

1. Elliptic regions

Let us first consider $\beta = 1$. In such a case, the above inequality (36) requires $F(a) > 0$ for any given shell labeled by the comoving coordinate a. If functions $F(a)$ and $f(a)$ are chosen in such a way that

$$\alpha^2 M^2 < F^2/4f^2, \quad (37)$$

the condition $0 < F^2(a) - 4f^2(a)\alpha^2 M^2 \equiv \Delta^2(a)$ is satisfied for any a, and the equation $P_a(R) = 0$ has two real solutions:

$$R_{\rm min} = [F(a) - \Delta(a)]/2f^2(a) > 0 \quad (38)$$

and

$$R_{\rm max} = [F(a) + \Delta(a)]/2f^2(a) > 0. \quad (39)$$

It may be trivially verified that the inequality (36) is satisfied by R values ranging inside the interval [$R_{\rm min}$, $R_{\rm max}$]. If a shell labeled as a arrives at $R = R_{\rm min}(a)$ at time $T = T_0(a)$, an integration of Eq. (35) leads to

$$T - T_0(a) = -\frac{1}{f^2}\sqrt{-R^2 f^2 + RF - \alpha^2 M^2} + \frac{F}{f^3}\sin^{-1}\sqrt{\frac{f^2 R}{\Delta} + \frac{\Delta - F}{2\Delta}}. \quad (40)$$

Denoting $\chi \equiv \frac{2\alpha M}{F}$, Eq. (40) is written in the parametric form

$$R = \frac{F}{2f^2}\left[1 - \sqrt{(1 - \chi^2)}\cos\eta\right],$$
$$T - T_0(a) = \frac{F}{2f^3}\left[\eta - \sqrt{(1 - \chi^2)}\sin\eta\right], \quad (41)$$

where η is the parameter.

As occurs with a test particle in the AR-VTG stationary spherically symmetric background, inner shells could move between two R values, which are turning points satisfying the conditions $\dot{R}(R_{\rm max}) = \dot{R}(R_{\rm min}) = 0$ as follows from Eq. (33). To avoid shell crossings in this motion, the arbitrary functions involved in the dynamic equations could be chosen to ensure the condition $R'(a, T) > 0$, which guarantees (strictly enforced) no shell crossings [19,23]. This concern will be reviewed in Sec. V C.

2. Parabolic regions

For $\beta = 0$ and $F(a) > 0$, inequality (36) reduces to $R > \alpha^2 M^2/F(a)$, which means that any admissible R value must be greater than

$$R_{\rm min} = \alpha^2 M^2/F(a). \quad (42)$$

According to Eq. (33), there is a turning point at $R = R_{\rm min}$, where $\dot{R}$ vanishes. Condition $F(a) < 0$ is not admissible, and now the resulting expression after integration of Eq. (35) is

$$T - T_0(a) = \frac{2\sqrt{FR - \alpha^2 M^2}}{3F^2}(FR + 2\alpha^2 M^2). \quad (43)$$

3. Hyperbolic regions

Finally, for $\beta = -1$, equation $P_a(R) = 0$ has only a positive solution given by

$$R_{\rm min} = [-F(a) + \tilde{\Delta}(a)]/2f^2(a) > 0, \tag{44}$$

where $\tilde{\Delta}(a)^2 \equiv F^2(a) + 4f^2(a)\alpha^2 M^2 > 0$ for any a. According to (36), any admissible R value must be greater than $R_{\rm min}$. By using Eq. (33), one easily concludes that $R = R_{\rm min}$ is a unique turning point where $\dot{R}$ vanishes. Now, the evolution equation for $R(a,T)$ takes the form

$$T - T_0(a) = \frac{1}{f^2}\sqrt{R^2 f^2 + RF - \alpha^2 M^2} - \frac{F}{f^3}\sinh^{-1}\sqrt{\frac{f^2 R}{\tilde{\Delta}} + \frac{F-\tilde{\Delta}}{2\tilde{\Delta}}}, \tag{45}$$

or parametrically can be written as

$$R = \frac{F}{2f^2}\left[\frac{\tilde{\Delta}}{F}\cosh\eta - 1\right],$$
$$T - T_0(a) = \frac{F}{2f^3}\left[\frac{\tilde{\Delta}}{F}\sinh\eta - \eta\right]. \tag{46}$$

Notice that Eqs. (40)–(46) describe an expanding or collapsing phase, in accordance with $T \geq T(a)$ or $T \leq T(a)$, respectively. We can parametrize this fact replacing $T - T_0(a)$ by $\epsilon[T - T_0(a)]$ in Eqs. (40)–(46), where the new ϵ parameter values are $\epsilon = +1$ for expansion and $\epsilon = -1$ for the collapse.

C. Shell crossings

For $\beta = 0$ and $\beta = -1$, just one turning point exists, and oscillations of shells are not possible. There is only a minimum $R_{\rm min}$, and consequently, falling shells bounce at $R = R_{\rm min}$ and then expand forever.

The above discussion of cases $\beta = 1$ (two turning points), $\beta = 0$ (one), and $\beta = -1$ (one) is consistent with the behavior of a test particle in the fixed AR-VTG stationary spherically symmetric background, which has two turning points for $E < 1/2$ and only one for $E \geq 1/2$. Let us discuss separately the existence of shell crossings for $\beta = 1$, $\beta = 0$, and $\beta = -1$.

In the first case, using the expression for R' [which is obtained from Eqs. (41)],

$$R' = \left(\frac{F'}{F} - \frac{2f'}{f}\right)R - \left[T_0' + \left(\frac{F'}{F} - \frac{3f'}{f}\right)(T - T_0)\right]\dot{R} + \frac{2\alpha^2 M^2}{\Delta^2}\left(\frac{F'}{F} - \frac{f'}{f}\right)\left(F - \frac{2\alpha^2 M^2}{R}\right), \tag{47}$$

it can be proved that, for any shell, quantities $R'[R_{\rm min}(a)]$ and $R'[R_{\rm max}(a)]$ have opposite signs and, consequently, $R'[R(a,T)]$ vanishes for some R between $R_{\rm min}(a)$ and $R_{\rm max}(a)$; hence, for $\beta = 1$, the use of comoving coordinates numbering shells is not a good choice to fully describe the internal shells collapse. A similar situation is analyzed in Refs. [17,24] for a spherically symmetric charged dust collapse and also in Ref. [25], in which the charged dust solution of Ruban [26], a generalization of the Datt [27] solution for a noncharged dust, is considered. It can be found in Ref. [28] that, in spite of the inclusion of a positive cosmological constant, this repulsion does not prevent the shell crossings. However, it is possible to describe a first bounce if the comoving time T origin is not chosen at $R_{\rm max}(a)$, say $R_N(a)$, in which case a set of functions $F(a)$, $f(a)$, and $T_0(a)$ so that $R'(a) > 0$, $F(a) > 0$, and $F'(a) > 0$ inside the interval $[R_N, R_{\rm min}]$ can be found, that is, our picture has begun with a certain $\dot{R} \neq 0$; in such a case, the following inequality has to be satisfied:

$$\frac{F'}{F} < \frac{f'}{f}\left(1 - \frac{\Delta}{F}\right). \tag{48}$$

When the $\beta = 0$ case is considered, taking the derivative of Eq. (43), it follows that

$$R' = \frac{1}{3}\frac{F'}{F}R + \frac{4}{3}\frac{F'}{F^2}\alpha^2 M^2\left(1 - 2\frac{\alpha^2 M^2}{FR}\right) - T'_0\dot{R}, \tag{49}$$

and therefore

$$\lim_{T\to T_0(a)} R'(a,T) = -F'\alpha^2 M^2/F^2 = R'_{\rm min}(a).$$

From Eq. (34), it is immediately evident that

$$\lim_{T\to T_0(a)} 8\pi\rho(a,T) = -F^4/\alpha^6 M^6 = 8\pi\rho_{\rm min}(a) < 0.$$

So, if the collapse were started at certain $R = R_N(a)$ with a certain energy density for the a shell defined by $8\pi\rho_N(a) = F'(a)/R_N^2(a)R_N'(a) > 0$, which means that the signs of $F'(a)$ and $R_N'(a)$ are the same, then the sign of $R'(a,T)$ would change at $R_{\rm min}$. Hence, we can conclude that the shell crossings are unavoidable (see Sec. II b in Ref. [19]).

And finally, let us see that in the case $\beta = -1$ the arbitrary functions involved in the model may be chosen to prevent shell crossings. To achieve this aim, first of all, R' is calculated using the set of Eqs. (45), which leads to

$$R' = \left(\frac{F'}{F} - \frac{2f'}{f}\right)R - \left[T_0' + \left(\frac{F'}{F} - \frac{3f'}{f}\right)(T - T_0)\right]\dot{R} + \frac{2\alpha^2 M^2}{\tilde{\Delta}^2}\left(\frac{F'}{F} - \frac{f'}{f}\right)\left(F - \frac{2\alpha^2 M^2}{R}\right). \tag{50}$$

Then, taking into account the condition $R'[R_{\min}(a)] > 0$, that is accomplished iff the inequality

$$\frac{F'}{F} < \frac{f'}{f}\left(1 - \frac{\tilde{\Delta}}{F}\right) \tag{51}$$

is satisfied for any a shell, we may select an appropriate function $T_0(a)$ in order to guarantee $R'(a, T) > 0$ in the interval $[R_N, R_{\min}[$. In the next section, other considerations that will limit this result will be taken into account.

For a general solution, numerical methods and codes similar to those used to study the collapse in GR [29–33] have to be adapted to AR-VTG. Nevertheless, this is a hard task that is out of this paper's scope.

D. Apparent horizons and trapped surfaces

Now, we analyze the apparent horizons for the AR-VTG, which reveal the boundary of the trapped surfaces specifying the region from which no light is allowed to escape. To obtain the condition $g^{\mu\nu}\partial_\mu R \partial_\nu R = 0$, Eqs. (31) and (33) are used. Table I shows the three different cases that can happen.

As can be appreciated, the existence and number of apparent horizons do not depend on the local type of time evolution defined by the β value and furthermore do not depend on the particular form of the f function. The $R = F(a)$ horizon of GR is recovered when setting $\alpha = 0$, as expected.

In the following, we present the different situations, depending on the β value, for the last entry in Table I, that is, when there are two apparent horizons. First, for $\beta = 0$, taking into account Eqs. (42) and (43) and the $R_{H_\pm}$ definition, it is easy to derive the inequalities $R_{\min} < R_{H_-} < R_{H_+}$ and $T_0 > T_{H_-} > T_{H_+}$, where $T_{H_\pm}$ represents the time when the a shell crosses the horizon $R_{H_\pm}$; the same results are obtained in the case of $\beta = -1$ when considering Eqs. (44) and (45). Second, for $\beta = 1$, in which case two turning points exist, using Eqs. (38)–(40) together with the fact that condition $f^2 < 1$ must be satisfied, the inequalities $R_{\min} < R_{H_-} < R_{H_+} < R_{\max}$ and $T_0 > T_{H_-} > T_{H_+} > T_{\max}$ are obtained, where $T_{\max} = T_0 - \pi F/2f^3$ represents time T value for an a shell at $R_{\max}$. In this case, the condition for the last row in Table I with the already-mentioned inequality $f^2 < 1$ provides the restriction (37) to be fulfilled. So, in any case, any signal emitted from $R_{\min}$ (that is, at $T = T_0$) is future trapped in the region above $T = T_{H_-}$.

TABLE I. AR-VTG apparent horizons in comoving coordinates. For simplicity, we define $\mathcal{M}(a) \equiv F(a)/2$.

Condition	Number of horizons	R Value
$\mathcal{M} < \alpha M$	No horizon	Not applicable
$\mathcal{M} = \alpha M$	One horizon	$R_H \equiv \mathcal{M} = \alpha M$
$\mathcal{M} > \alpha M$	Two horizons	$R_{H_\pm} \equiv \mathcal{M} \pm \sqrt{\mathcal{M}^2 - \alpha^2 M^2}$

VI. DISCUSSION AND CONCLUSIONS

Our starting point has been the vacuum static spherically symmetric metric of AR-VTG given by Eqs. (7) and (8). This metric and some values of $r_{\rm BH}$ corresponding to $\alpha^2 < 1$ were found in Ref. [6], in which the singularity was not considered at all. A very different behavior from GR is found in the introduction of Sec. V, in which a test particle on the AR-VTG fixed background is presented; it is revealed that oscillatory trajectories for the aforementioned particles can be obtained. One of the first published descriptions of the motion of a point on the surface of a charged collapsing ball of mass M and charge Q can be found in Ref. [34]. This description is fully equivalent to our case; however, in AR-VTG gravity, no charge is necessary to obtain a similar motion. The $\alpha^2 < 1$ condition in AR-VTG is analogous to $(Q/M)^2 < 1$ in the Reissner-Nordström spacetime, where Q/M is the ratio between charge and mass. In this latter case, the surface of the collapsing sphere crosses the inner horizon $r_{\rm inh} = M - \sqrt{M^2 - Q^2}$ [in AR-VTG, $r_{H_-} = M(1 - \sqrt{1 - \alpha^2})$]. An equivalent spacetime diagram can be found in Ref. [35].

The dynamic properties of AR-VTG are presented in Sec. V, divided in different subsections, where the dust core is considered. The study of the internal core in comoving coordinates leads to, as in GR, three possible families of solutions that classify the spacetime as bound, marginally bound, or unbound [22]. This fact has been characterized with $\beta = 1$, $\beta = 0$, and $\beta = -1$ respectively. In all three cases, a minimum value for R has been found, say $R_{\min}(a, T_0(a))$, where $\dot{R}(a, T_0(a)) = 0$ and which is reached after crossing the apparent horizon at T_{H_-}; then, the collapse is halted and reversed. The same fact was first found for a noncharged dust in an external electromagnetic field by Shikin [36]. So, under the premise that the real Universe and astrophysical objects (in general) have no net electric charge, AR-VTG provides an interesting alternative for preventing gravitational singularities as a pure classical gravitation effect.

The shell crossings issue has been analyzed, and several similarities can be observed when a charged dust is considered (see Secs. VI and VII in Ref. [24]). However, there are also differences; for instance, the absence of real charges eliminates fundamental equations relative to these and changes some of the regularity conditions at the center of symmetry (e.g. the charge has to be zero there). In AR-VTG, while using comoving coordinates, we have found the unavoidable shell crossings for $\beta = 0$, for any collapsing a shell before reaching $R_{\min}(a)$. So, the use of comoving coordinates does not allow the possibility of following further evolution. For $\beta = 1$, it is possible to find a set of appropriate free functions that should allow us to avoid shell crossings collapse during a first bounce at $R_{\min}$, although it will be unavoidable during the expansion before reaching $R_{\max}$. Finally, when $\beta = -1$, a configuration [defined by inequality (51)] avoiding the issue has been found, apparently, but there are extra constraints to be considered due to the regularity at the symmetry center. From inequality (51), it is immediately derived that the quotient f'/f has to be negative. Therefore, one of the conditions related as necessary in Ref. [37] for a charged dust, which is $f(a_c) = 0$ (where $a = a_c$ is the symmetry center), cannot be honored. Nevertheless, an initial cloud with a Minkowskian central bubble should not require the $f(a_c) = 0$ condition [17], and then inequality (51) may be achieved. Following the same procedure that Krasiński and Bolejko present in Ref. [24], it can be demonstrated that also in AR-VTG gravity shell crossing is a coordinate singularity, so inherent to the coordinates selection. According to Ref. [23], the degeneracy created in the metric (24) when $R' = 0$ can be saved through a C^0 extension of it. Once the fluid has been modeled into shells, this process represents the collisions of adjacent shells, and it is related to the fact that there is no coupling between the movement of different shells.

The fundamental symmetry of AR-VTG is $A^\mu \to A^\mu + \nabla^\mu \Phi$ with $\nabla_\mu \nabla^\mu \Phi = 0$, which is different from the standard U(1) gauge symmetry [5]; the quantities $\nabla \cdot A$ and $F_{\mu\nu}$ are invariant under these transformations. In cosmology, a quantity proportional to $(\nabla \cdot A)^2$ is a candidate to play the role of dark energy with an equation of state $w = p/\rho = -1$, and AR-VTG has been demonstrated to be in agreement with the background and perturbed spacetimes [3,5,7], while $F_{\mu\nu}$ vanishes. In this paper, it has been shown that a quantity related with F_{01} plays the role of the repulsive component of the gravitation and may prevent the gravitational collapse, while $\nabla \cdot A$ is null.

The influence of the repulsive component of AR-VTG on neutron star structure is also an open problem deserving attention. The presented dust collapse model provides an interesting alternative of a nonsingular collapse, such as might occur in a weakly charged dust. An interesting open issue is the study of the A_μ field perturbations that may save the unstable character found in a charged black hole where the innermost sections are unstable to electromagnetic perturbations [17].

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[1] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).
[2] C. M. Will, Living Rev. Relativity **9**, 3 (2006).
[3] R. Dale, J. A. Morales, and D. Sáez, arXiv:0906.2085.
[4] R. Dale and D. Sáez, Phys. Rev. D **85**, 124047 (2012).
[5] R. Dale and D. Sáez, Phys. Rev. D **89**, 044035 (2014).
[6] R. Dale, M. J. Fullana, and D. Sáez, Astrophys. Space Sci. **357**, 116 (2015).
[7] R. Dale and D. Sáez, J. Cosmol. Astropart. Phys. 01 (2017) 004.
[8] H. Stephani *et al.*, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
[9] F. Kottler, Ann. Phys. (Berlin) **361**, 401 (1918).
[10] K. Lake, Phys. Rev. D **19**, 421 (1979).
[11] J. B. Holberg, M. A. Barstow, F. C. Bruhweiler, A. M. Cruise, and A. J. Penny, Astrophys. J. **497**, 935 (1998).
[12] S. L. Shafiro and S. A. Teukolski, *Black Holes, White Dwarfs, and Neutron Stars* (WILEY-VCH Verlag, Weinheim, Germany, 2004).
[13] F. Ozel and P. Freire, Annu. Rev. Astron. Astrophys. **54**, 401 (2016).
[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[15] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).
[16] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Elsevier, Oxford, 1975).
[17] A. Ori, Phys. Rev. D **44**, 2278 (1991).
[18] J. D. Bekenstein, Phys. Rev. D **4**, 2185 (1971).
[19] C. Hellaby and K. Lake, Astrophys. J. **290**, 381 (1985).
[20] A. Ori, Classical Quantum Gravity **7**, 985 (1990).
[21] H. Stephani, *Relativity: An Introduction to Special and General Relativity* (Cambridge University Press, Cambridge, England, 2004).
[22] P. S. Joshi, *Gravitational Collapse and Spacetime Singularities* (Cambridge University Press, Cambridge, England, 2003).
[23] R. P. A. C. Newman, Classical Quantum Gravity **3**, 527 (1986).
[24] A. Krasiński and K. Bolejko, Phys. Rev. D **73**, 124033 (2006).
[25] A. Krasiński and G. Giono, Gen. Relativ. Gravit. **44**, 239 (2012).
[26] V. A. Ruban, Inhomogeneous cosmological models with planar and pseudospherical symmetries, in *Third Soviet Gravitational Conference* (Izdatel'stvo Erevanskogo Universiteta, Erevan, 1972), p. 348.
[27] B. Datt, Z. Phys. **108**, 314 (1938).
[28] S. M. C. V. Gonçalves, Phys. Rev. D **63**, 124017 (2001).
[29] J. A. Font, Living Rev. Relativity **6**, 4 (2003).
[30] C. R. Ghezzi, Phys. Rev. D **72**, 104017 (2005).

[31] C. R. Ghezzi and P. S. Letelier, Phys. Rev. D **75**, 024020 (2007).

[32] E. O'Connor and C. D. Ott, Classical Quantum Gravity **27**, 114103 (2010).

[33] D. Gerosa, U. Sperhake, and C. D. Ott, Classical Quantum Gravity **33**, 135002 (2016).

[34] I. D. Novikov, Astron. Zh. **43**, 911 (1966).

[35] J. Plebański and A. Krasiński, *An Introduction to General Relativity and Cosmology* (Cambridge University Press, Cambridge, England, 2006).

[36] I. S. Shikin, Commun. Math. Phys. **26**, 24 (1972).

[37] P. A. Vickers, Ann. Inst. Henri Poincaré **18**, 137 (1973).