Gravitational effects of disformal couplings

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We consider how a nearly massless scalar field conformally and disformally coupled to matter can affect the dynamics of gravitationally interacting bodies. We focus on the case of two interacting objects, and we obtain the effective metric driving the dynamics of the two-body system when reduced to one body in the center-of-mass frame. We then concentrate on the case of a light particle in the scalar and gravitational fields generated by a heavy object and find the effects of the conformal and disformal couplings on the body's trajectory such as the advance of the perihelion and the Shapiro time delay. The disformal coupling leads to a negligible contribution to the Shapiro effect and therefore no constraint from the Cassini experiment. On the other hand, it contributes to the perihelion advance leading to a weak bound on the strength of the disformal coupling itself. Finally, we remark that the disformal coupling gives rise to a contribution to the perihelion advance which varies quadratically with the mass of the heavy body, leading to possible strong effects for stars in the vicinity of astrophysical black holes. For neutron stars in a binary system, the disformal effects vary as the quartic power of the size of the orbit which might lead to interesting consequences in the inspiralling phase prior to a merger.

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I. INTRODUCTION

Nearly massless scalar fields are ubiquitous in cosmology [1–3]. They could play a role in generating the late-time acceleration of the expansion of the Universe. They could also belong to an extended gravitational sector of the theory describing the Universe [4–7]. In this work, we shall consider that such a scalar could be both conformally and disformally coupled to matter [8]. The effects of a conformal coupling are well known [9,10] and must be suppressed in the Solar System in order to comply with gravitational tests such as the ones performed by the Cassini probe [11] (existence of a fifth force) or the Lunar Laser Ranging experiment (test of the strong equivalence principle in the Earth-moon-Sun system) [12]. The resulting bounds on the coupling β are severe, and screening mechanisms have been invoked in order to comply with rather unnaturally small values of β [13–16]. Another type of interaction, the disformal coupling, could also play a role in the interactions between matter and the scalar field. This coupling has been constrained using numerous probes [17–28]. It can influence the dynamics of compact bodies as a one-loop effective interaction similar to the Casimir-Polder force can be generated between such objects [29]. It can also have effects on the atomic energy levels or even the burning rate of stars in astrophysics [30]. Finally, as a four-body interaction, it can be tested at accelerators such as the LHC [31]. In this paper, we will focus on the gravitational physics of such a disformal coupling [25,26], in conjunction with a conformal coupling, in the presence of celestial bodies. We will derive an effective one-body metric which describes the dynamics of two such interacting bodies at leading order in G_N. This will allow us to consider the disformal effects on the classical tests such as the advance of perihelion or the time delay of radio-wave signals. The effective one-body metric may also eventually allow us to consider the inspiralling emission of a gravitational wave by two rotating bodies, although we leave it for further work.

We find that the Shapiro time delay as probed by the Cassini experiment does not depend (or depends only slightly) on the disformal coupling. On the other hand, the perihelion advance of a light body in the presence of a heavy object is nonvanishing. We find that it varies quadratically with the mass of the heavy object and quartically with the size of the orbit. This may have consequences for the dynamics of stars in the vicinity of astrophysical black holes [32] or during the inspiralling phase of neutron star mergers.

In Sec. II, we study the solutions to the Klein-Gordon equation involving a conformal and disformal coupling to matter for point sources. In Sec. III, we consider the case of two interacting bodies, while in Sec. IV IV, we consider the dynamics of a light particle close to a heavy body.

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We discuss possible consequences for the dynamics of stars close to astrophysical black holes and binary systems of neutron stars in Sec. V.

II. DISFORMAL RADIATION

A. Ladder expansion

In this section, we consider the scalar emission from a moving body when the coupling between matter and the scalar field is mediated by the metric,

$$g_{\mu\nu} = A^2(\phi) g^E_{\mu\nu} + \frac{2}{M^4} \partial_\mu \phi \partial_\nu \phi, \tag{2.1}$$

where the conformal factor,

$$A(\phi) = e^{\beta\phi/m_{\rm Pl}}, \tag{2.2}$$

is characterized by the constant coupling β and the disformal interaction is specified by the suppression scale M. We could have chosen more complex function [33] such as a quadratic function $A(\phi)$, e.g., as for the environmentally dependent dilaton [34] and symmetron [35]. Here we consider the simplest case of a field independent coupling β. Similarly the disformal part could be more complex with $1/M^4 \to B(\phi, (\partial\phi)^2)$. In the following, we focus on the simplest case where the disformal coupling depends only on the constant coupling scale M. Matter couples minimally to $g_{\mu\nu}$ such that the total action reads

$$S = \int d^4x \sqrt{-g_E} \left(\frac{R_E}{16\pi G_N} - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right) + S_m(\psi_i, g_{\mu\nu}) \tag{2.3}$$

in the Einstein frame for the Einstein-Hilbert action and we have introduced a potential $V(\phi)$ for the scalar field. The matter fields are denoted by ψ_i and their action is S_m. In the following, we will focus of nearly massless scalar and take $V(\phi) = 0$. We could have considered the case of screened models with a nontrivial $V(\phi)$ [36]. Effectively in this case, and in a given environment such as the Solar System, the mass of the scalar field between the Sun and the planets is small; i.e., the scalar field is not Yukawa-screened, and in the screened models with either the chameleon or the Damour-Polyakov screenings, the scalar charge of these objects $\beta_{\rm eff}$ is reduced to pass the Solar System tests such as the Cassini bound [11]. We leave a detailed analysis of screened models for the future and concentrate on the case of a massless field with a small coupling β.

The gravitational dynamics are dictated by the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N (T_{\mu\nu} + T^\phi_{\mu\nu}) \tag{2.4}$$

where the matter energy-momentum tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g^E}} \frac{\delta S_m}{\delta g^{\mu\nu}_E} \tag{2.5}$$

and the corresponding one for the scalar field is

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{(\partial\phi)^2}{2} g^E_{\mu\nu}. \tag{2.6}$$

The dynamics of the scalar field are given by the Klein-Gordon equation

$$\Box\phi = -\beta \frac{T}{m_{\rm Pl}} + \frac{1}{M^4} D_\mu (A^{-2}(\phi) \partial_\nu \phi T^{\mu\nu}) \tag{2.7}$$

where D_μ is the covariant derivative for the Einstein metric. The Bianchi identity implies the nonconservation equation

$$D^\mu T_{\mu\nu} = \frac{\beta T}{m_{\rm Pl}} \partial_\nu \phi - \frac{1}{M^4} D_\mu (A^{-2}(\phi) \partial_\lambda \phi T^{\mu\lambda}) \partial_\nu \phi. \tag{2.8}$$

In the following, we will be interested in the leading terms at the $1/m_{\rm Pl}$ order in ϕ. Indeed, this leads to contributions to the interaction potential between two objects in $\beta\phi/m_{\rm Pl}$ proportional to G_N. As we are only focussing on the leading G_N contributions to the dynamics of interacting bodies and to leading order in $1/M^4$, we can safely consider that the matter energy momentum is conserved

$$D^\mu T_{\mu\nu} = 0 \tag{2.9}$$

at this order leading to the Klein-Gordon equation

$$\Box\phi = -\beta \frac{T}{m_{\rm Pl}} + \frac{1}{M^4} D_\mu \partial_\nu \phi T^{\mu\nu}. \tag{2.10}$$

The Klein-Gordon equation can be solved iteratively as

$$\phi = \phi^{(0)} + \delta\phi \tag{2.11}$$

where

$$\Box\phi^{(0)} = -\beta \frac{T}{m_{\rm Pl}} \tag{2.12}$$

is nontrivial when $\beta \neq 0$ and

$$\Box\delta\phi - \frac{1}{M^4} D_\mu \partial_\nu \delta\phi T^{\mu\nu} = \frac{1}{M^4} D_\mu \partial_\nu \phi^{(0)} T^{\mu\nu}. \tag{2.13}$$

Defining the retarded propagator G as

$$\Box G(x, x') = \delta^{(4)}(x - x') \tag{2.14}$$

we have

$$\phi^{(0)}(x) = -\frac{\beta}{m_{\rm Pl}}\int d^4x' G(x-x')T(x') \qquad (2.15)$$

and we can find a series representation of the solution corresponding to an expansion in ladder diagrams

$$\delta\phi = \sum_{n\geq 0}\delta\phi^{(n)} \qquad (2.16)$$

where

$$\Box\delta\phi^{(0)} = \frac{1}{M^4}D_\mu\partial_\nu\phi^{(0)}T^{\mu\nu}. \qquad (2.17)$$

and

$$\Box\delta\phi^{(n+1)} = \frac{1}{M^4}D_\mu\partial_\nu\delta\phi^{(n)}T^{\mu\nu}. \qquad (2.18)$$

This implies that

$$\delta\phi^{(0)} = \frac{1}{M^4}\int d^4x' G(x-x')D_\mu\partial_\nu\phi^{(0)}(x')T^{\mu\nu}(x') \qquad (2.19)$$

and

$$\delta\phi^{(n+1)}(x) = \frac{1}{M^4}\int d^4x' G(x-x')D_\mu\partial_\nu\delta\phi^{(n)}(x')T^{\mu\nu}(x'). \qquad (2.20)$$

Each iteration brings in another insertion of the energy-momentum tensor and is suppressed by a higher power of M^4. Hence, to be consistent with our approximation, we only consider the first two steps. Notice that the solution vanishes in the absence of a conformal coupling β.

We have neglected the possible effects coming from the cosmological background density. When the matter system is embedded in the cosmological background with an energy-momentum $T^{\mu\nu}_{\rm cosmo}$, one can separate the scalar field as $\phi = \phi_{\rm cosmo} + \bar\phi$, where

$$\Box\phi_{\rm cosmo} = -\beta\frac{T_{\rm cosmo}}{m_{\rm Pl}} + \frac{1}{M^4}D_\mu\partial_\nu\phi_{\rm cosmo}T^{\mu\nu}_{\rm cosmo}, \qquad (2.21)$$

and the background metric is now of the Friedmann-Lemaitre-Robertson-Walker type. The local matter density generates a scalar field such that

$$\Box\bar\phi = -\beta\frac{T}{m_{\rm Pl}} + \frac{1}{M^4}D_\mu\partial_\nu\bar\phi T^{\mu\nu} + \frac{1}{M^4}D_\mu\partial_\nu\bar\phi T^{\mu\nu}_{\rm cosmo} + \frac{1}{M^4}D_\mu\partial_\nu\phi_{\rm cosmo}T^{\mu\nu}. \qquad (2.22)$$

There are two new source terms which involve the cosmological energy-momentum tensor and the derivatives of the cosmological solution. As the cosmological matter density is negligible compared to the matter density in the moving objects we are considering and the variation of the cosmological solution is on time scales much larger than the rapid motion of the moving bodies, we can safely neglect the new source terms. Within this quasistatic approximation, the only effect of the cosmological background is to add to the solution generated by the local matter density a slowly varying background scalar field $\phi_{\rm cosmo}$.

B. Point source

We now focus on a point source of mass m whose energy-momentum tensor reads

$$T^{\mu\nu} = m\int d\tau A(\phi)u^\mu u^\nu\delta^{(4)}(x^\mu - x^\mu(\tau)) \qquad (2.23)$$

where τ is the proper time of the particle in the Einstein frame such that

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad (2.24)$$

and $u^\mu u_\mu = -1$. Notice that, as we work in the Einstein frame, the mass of the particle becomes $mA(\phi)$ which is field dependent. We will work in the case where $\beta\phi/m_{\rm Pl} \ll 1$ which will be valid as long as $\beta = \mathcal{O}(1)$ as, to leading order, $\beta\phi/m_{\rm Pl} \sim 2\beta^2\Phi_N$ for an object with Newtonian potential Φ_N. For the objects that we consider, such as the Sun, $\Phi_N \ll 1$, and we can therefore omit the $A(\phi) \sim 1$ in the mass. When the cosmological background is taken into account, at leading order, one can keep track of the effects of the slow variation of the background scalar field by retaining the slow time variation of the mass of the particles using $m \to A(\phi_{\rm cosmo})m$.

We will be interested in the effects of a point source on the geometry of space at the leading G_N order and its consequences on the effective geometry driving the motion of interacting particles. Hence, it is sufficient to consider the evolution of the point particle in Minkowski space and contract the tensors with $\eta_{\mu\nu}$. Defining the velocity

$$v^i = \frac{dx^i}{dx^0} \qquad (2.25)$$

for the particle, we find that

$$T = -m\sqrt{1-\vec{v}^2}\delta^{(3)}(x^i - x^i(x^0)) \qquad (2.26)$$

where we have $x^0 = x^0(\tau)$. Hence an ultrarelativistic particle has a traceless energy momentum, in agreement with the tracelessness of $T^{\mu\nu}$ for a relativistic fluid. Using the Green's function in Minkowski space,

$$G(x,x') \equiv G(x-x') = -\frac{1}{2\pi}\theta(x_0-x'_0)\delta((x-x')^2), \tag{2.27}$$

we find that

$$\phi^{(0)}(x) = -\frac{\beta m}{4\pi m_{\rm Pl}}\frac{1}{1-\vec{v}.\vec{n}'}\frac{\sqrt{1-\vec{v}^2}}{|\vec{x}-\vec{x}(x'^0)|}, \tag{2.28}$$

which matches the usual solution of the Poisson equation for a static scalar field sourced by a static point particle. We have defined the unit vector

$$\vec{n}' = \frac{\vec{x}-\vec{x}(x'^0)}{|\vec{x}-\vec{x}(x'^0)|} \tag{2.29}$$

and similarly

$$\vec{n} = \frac{\vec{x}-\vec{x}(x^0)}{|\vec{x}-\vec{x}(x^0)|}. \tag{2.30}$$

Here we have introduced the retarded time,

$$x'^0 = x^0 - |\vec{x}-\vec{x}(x'^0)|. \tag{2.31}$$

In the same vein, the first iteration of the ladder expansion reads

$$\delta\phi^{(0)}(x) = -\frac{m}{4\pi M^4\gamma}\frac{\partial_\mu\partial_\nu\phi^{(0)}u^\mu u^\nu}{||\vec{x}-\vec{x}(x'^0)| - \vec{v}.(\vec{x}-\vec{x}(x'^0))|}, \tag{2.32}$$

where $u^\mu = \gamma(1, v^i)$ and $\gamma = 1/\sqrt{1-\vec{v}^2}$. The higher-order terms can be deduced by iteration.

For small velocities, we can expand

$$x^0 - x'^0 = (1+\delta)|\vec{x}-\vec{x}(x^0)| \tag{2.33}$$

where

$$\delta \sim \vec{n}.\vec{v} + \frac{\vec{v}^2}{2} + \frac{(\vec{n}.\vec{v})^2}{2} \tag{2.34}$$

at second order in the velocity and the scalar field becomes

$$\phi^{(0)}(x) = -\frac{\beta m}{4\pi m_{\rm Pl}}\frac{1-\frac{\vec{v}^2}{2}+\frac{\vec{v}_\perp^2}{2}}{|\vec{x}-\vec{x}(x^0)|}, \tag{2.35}$$

where we have defined the projection of the velocity in the direction perpendicular to $\vec{n}$ as

$$\vec{v}_\perp = \vec{v} - (\vec{v}.\vec{n})\vec{n}. \tag{2.36}$$

This result can also be deduced using Lorentz invariance. In the frame where the particle is static, the solution is $-\frac{\beta m}{4\pi m_{\rm Pl}}\frac{1}{|\vec{x}'-\vec{x}'(x^0)|}$ where the distance in the static frame $|\vec{x}'-\vec{x}'(x^0)|$ is longer by a factor $(1+\frac{(\vec{v}.\vec{n})^2}{2})$. Notice that we always work at the $\vec{v}^2$ order as this is enough to deduce the form of the effective metric between two bodies in the leading G_N approximation.

As a side result, we obtain the Green's function for the spatial Klein-Gordon equation in the presence of a slowly moving particle

$$\Box G_0(x) = \delta^{(3)}(x^i - x^i(x^0)) \tag{2.37}$$

which is given by

$$G_0(x) = -\frac{1}{4\pi}\frac{1+\frac{\vec{v}_\perp^2}{2}}{|\vec{x}-\vec{x}(x^0)|}. \tag{2.38}$$

This will be useful when solving for the Newtonian potential.

At leading order, we have for the derivatives of the scalar field

$$\partial_0\phi^{(0)} = -\frac{\beta\gamma m}{4\pi m_{\rm Pl}}\frac{\vec{v}.(\vec{x}-\vec{x}(x^0))}{|\vec{x}-\vec{x}(x^0)|^3} \tag{2.39}$$

and

$$\partial_i\phi^{(0)} = \frac{\beta m}{4\pi\gamma m_{\rm Pl}}\frac{(x^i - x^i(x^0))}{|\vec{x}-\vec{x}(x^0)|^3} \tag{2.40}$$

which should also depend on the acceleration $a^i = \frac{dv^i}{dx^0}$. In the following, we will use the fact that the acceleration involves one power of G_N and therefore these terms appear at higher order in the G_N expansion, i.e., with the approximation that the acceleration

$$\vec{a}_A = -\frac{G_N m_A(\vec{x}-\vec{x}_A)}{|\vec{x}-\vec{x}_A|^3} \tag{2.41}$$

is of order G_N and induces corrections in G_N^2 that we have neglected. For the velocity dependent part we find that

$$\delta\phi^{(0)} = 0 \tag{2.42}$$

explicitly when only one particle is involved. This is also a result which follows from Lorentz invariance as in the frame where the particle is static, the solution $\phi^{(0)}$ is time independent. For two bodies, the solution does not vanish and will be given below.

III. TWO-BODY SYSTEM

A. The scalar field of moving particles

When two moving bodies are present, the solution to the Klein-Gordon equation at leading order in $1/M^4$ can be obtained in two steps. The first steps consist of solving

$$\Box\phi^{(0)} = -\beta\frac{T^A + T^B}{m_{\rm Pl}}, \tag{3.1}$$

where the energy-momentum tensor contains both the parts from particles A and B. The solution is simply given by the linear combination

$$\phi^{(0)} = \phi_A^{(0)} + \phi_B^{(0)} \tag{3.2}$$

where we have

$$\phi_{A,B}^{(0)}(x) = -\frac{\beta m_{A,B}(1 - \frac{\vec{v}_{A,B}^2}{2} + \frac{\vec{v}_{A,B\perp}^2}{2})}{4\pi m_{\rm Pl}|\vec{x} - \vec{x}_{A,B}|}. \tag{3.3}$$

This solution sources the next step in the iteration process where

$$\phi(x) = \phi^{(0)}(x) + \delta\phi^{(0)}(x) \tag{3.4}$$

is given by

$$\delta\phi^{(0)}(x) = \frac{1}{M^4}\int d^4x' G(x - x')\partial_\mu\partial_\nu(\phi_A^{(0)}(x') + \phi_B^{(0)}(x'))(T_A^{\mu\nu}(x') + T_B^{\mu\nu}(x')). \tag{3.5}$$

This leads to four contributions,

$$\delta\phi_{\alpha\beta}^{(0)}(x) = -\frac{m_\alpha}{4\pi M^4\gamma}\frac{\partial_\mu\partial_\nu\phi_\beta^{(0)}(x_\alpha)u_\alpha^\mu u_\alpha^\nu}{|\vec{x} - \vec{x}_\alpha|}, \tag{3.6}$$

where $\alpha, \beta = A$, B. Notice that here the $\vec{v}_{A,B}^2$ and $\vec{v}_{A,B\perp}^2$ corrections in (3.3) are negligible as we neglect the quartic terms in the velocities. It turns out then that $\delta\phi_{AA}^{(0)}$ and $\delta\phi_{BB}^{(0)}$ both vanish while

$$\delta\phi_{AB}^{(0)} = -\frac{\beta m_A m_B}{16\pi^2 m_{\rm Pl}}\frac{(\vec{v}_A - \vec{v}_B)^2 - 3(\vec{n}_{AB}.(\vec{v}_A - \vec{v}_B))^2}{M^4|\vec{x} - \vec{x}_A||\vec{x}_B - \vec{x}_A|^3}$$
$$\delta\phi_{BA}^{(0)} = -\frac{\beta m_A m_B}{16\pi^2 m_{\rm Pl}}\frac{(\vec{v}_A - \vec{v}_B)^2 - 3(\vec{n}_{AB}.(\vec{v}_A - \vec{v}_B))^2}{M^4|\vec{x} - \vec{x}_B||\vec{x}_B - \vec{x}_A|^3} \tag{3.7}$$

where $\vec{n}_{AB}$ is the unit vector between A and B. Notice that this involves only the Galilean invariant combination $(\vec{v}_A - \vec{v}_B)$.

B. The gravitational fields of moving particles

We now consider the interaction between two particles A and B which are conformally and disformally coupled to the scalar field. We shall work to leading order in G_N and $1/M^4$ and in the nonrelativistic limit where $\vec{v}_{A,B}^2 \ll 1$. The action for the two bodies can be obtained using

$$S = -m_A\int d\tau_A\sqrt{-g_{\mu\nu}^B u_A^\mu u_A^\nu} - m_B\int d\tau_B\sqrt{-g_{\mu\nu}^A u_B^\mu u_B^\nu} + \delta S_{AB}, \tag{3.8}$$

where the correction term δS_{AB} comes from the evaluation of the field action, both from general relativity and its scalar counterpart.

This calculation can be performed in a number of ways but we shall find convenient to work in the nonrelativistic limit of GR [37,38]. To do so, let us decompose the Einstein metric according to

$$ds^2 = -e^{2\Phi_N}(dt - A_i dx^i)^2 + e^{-2\Phi_N}\gamma_{ij}dx^i dx^j, \tag{3.9}$$

where Φ_N is the Newtonian potential and A_i is responsible for gravi-magnetism. We have chosen to treat the spatial metric $\gamma_{ij} = \delta_{ij}$ as flat. In this gauge, the Einstein-Hilbert action can be written as

$$S_{\rm EH} = -\frac{1}{16\pi G_N}\int d^4x\left(2(D_i\Phi_N)^2 - \frac{e^{2\Phi_N}}{4}F^2 + 4\dot{A}^i D_j\Phi_N + 6\dot{\Phi}_N e^{-4\Phi_N}\right) \tag{3.10}$$

in a $(3 + 1)$ decomposition of the Kaluza-Klein type. The covariant derivative is $D_i = \partial_i + A_i\partial_t$ and the field strength $F_{ij} = \partial_i A_j - \partial_j A_i$ where indices are raised and lowered with the flat δ_{ij}. The Einstein-Hilbert action must be complemented with a gauge fixing action which imposes the harmonic gauge $\partial_\nu(\sqrt{-g}g^{\mu\nu}) = 0$ and reads now

$$S_{\rm GF} = \frac{1}{32\pi G_N}\int d^4x((e^{2\Phi_N}D^i A_i + 4e^{-2\Phi_N}\dot{\Phi}_N)^2 - \dot{A}_i^2). \tag{3.11}$$

The equations of motion are then

$$\Box\Phi_N = 4\pi G_N(T^{00} + T_i^i) \tag{3.12}$$

for the Poisson equation and

$$\Box A_i = 16\pi G_N T^{0i} \tag{3.13}$$

for the Maxwell equation of gravi-magnetism. The Newtonian potential is modified compared to the static case by the fact that the sources are moving compared to a nearly Minkowski background. As a result, distances are effectively contracted by special relativistic effects. The solution to the Poisson equation for a single moving source of velocity $\vec{v}_A$ reads

$$\Phi_N(x) = -\frac{G_N m_A}{|\vec{x}-\vec{x}_A|}\left(1+\frac{3}{2}\vec{v}_A^2+\frac{\vec{v}_{A,\perp}^2}{2}\right), \qquad (3.14)$$

where the correction factor comes from the fact that T_i^i brings one factor of $\vec{v}_A^2$. Another factor of $\vec{v}_A^2/2$ comes from the time dilation factor $d\tau_A = \gamma^{-1}dx^0 = (\sqrt{1-\vec{v}_A^2})dx^0$ between proper time and background time. Finally, the Klein-Gordon equation must be solved with a spatial Dirac function as a source. We have already obtained this solution in the form of the Green's function G_0, i.e., (2.38). Here we have introduced

$$\vec{n} = \frac{\vec{x}-\vec{x}_A}{|\vec{x}-\vec{x}_A|} \qquad (3.15)$$

as the unit vector pointing towards $\vec{x}$ from $\vec{x}_A$ and defined the perpendicular velocity

$$\vec{v}_\perp = \vec{v} - (\vec{n}.v)\vec{n} \qquad (3.16)$$

such that $\vec{v}_\perp^2 = \vec{v}^2 - (\vec{v}.\vec{n})^2$. This result is nothing but the Newtonian potential after a boost as Lorentz invariance is preserved by the harmonic gauge [39]. Similarly the vector field is given by

$$A^i = -4\frac{G_N m_A v_A^i}{|\vec{x}-\vec{x}_A|} \qquad (3.17)$$

which is again the result of boosting the static Newtonian metric [39].

C. The two-body action

1. The gravitational action

The previous solutions to the field equations contribute to the gravitational action and can be expressed as a function of the velocities of the two moving bodies and their positions. Denoting by

$$g^E_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (3.18)$$

the Einstein metric, the gravitational action comprising both the Einstein-Hilbert term and the gauge fixing leads to

$$S_{\rm EH} + S_{\rm GF} = -\frac{1}{4}\int d^4x h_{\mu\nu}T^{\mu\nu}. \qquad (3.19)$$

Removing the infinite self-energies, the action for interacting particles is obtained as

$$S_{\rm EH} + S_{\rm GF} = -\frac{1}{4}\int d^4x (h^A_{\mu\nu}T^{\mu\nu}_B + h^B_{\mu\nu}T^{\mu\nu}_A), \qquad (3.20)$$

where $h^{A,B}_{\mu\nu}$ is the field generated by A (respectively B). It is useful to notice the identity (up to acceleration terms which are of higher order in G_N),

$$\frac{d}{dt}(\vec{n}_{AB}.\vec{v}_B) = \frac{\vec{v}_{B\perp}^2 - \vec{v}_{A\perp}.\vec{v}_{B\perp}}{|\vec{x}_B-\vec{x}_A|} \equiv 0, \qquad (3.21)$$

where $\vec{n}_{AB}$ is the unit vector between A and B and we have $\vec{v}_A^\perp = \vec{v}_A - (\vec{v}_A.\vec{n}_{AB})\vec{n}_{AB}$ (similarly for $\vec{v}_B^\perp$). The last equality is to be taken as integrated in an action where total derivatives are irrelevant. The corresponding gravitational Lagrangian becomes

$$\mathcal{L}_{\rm EH} + \mathcal{L}_{\rm GF} = -\frac{G_N m_A m_B}{|x_B - x_A|}\left(1+\frac{3}{2}(\vec{v}_A^2+\vec{v}_B^2) - 4\vec{v}_A.\vec{v}_B + \frac{1}{2}\vec{v}_{A\perp}.\vec{v}_{B\perp}\right). \qquad (3.22)$$

These terms appear as counterterms to avoid any double-counting in the overall action including matter.

2. The matter action

We can now expand the matter action to second order in the velocity field and get the Lagrangian for particle A in the fields generated by particle B

$$\mathcal{L}_A = -m_A e^{\Phi_N^B(x_A)}A(\bar{\phi}_B(x_A)) \times\sqrt{1-2A_i^B v_A^i - e^{-4\Phi_N^B(x_A)}\vec{v}_A^2 - \frac{2}{M^4}D_A} \qquad (3.23)$$

and symmetrically for particle B. We will use explicitly the fact that

$$\partial_0\phi_B^{(0)}(x) = -\vec{\partial}\phi_B^{(0)}(x).\vec{v}_B. \qquad (3.24)$$

and we will denote by

$$D_A = (\vec{\partial}\phi_B^{(0)}(x_A).\vec{v}_B)^2 + (\vec{\partial}\phi_B^{(0)}(x_A).\vec{v}_A)^2 - 2(\vec{\partial}\phi_B^{(0)}(x_A).\vec{v}_A)(\vec{\partial}\phi_B^{(0)}(x_A).\vec{v}_B). \qquad (3.25)$$

the part of the action which comes from the disformal term of the metric. The scalar field in this action is $\bar{\phi}_B(x)$ where the divergent self-energy contributions have been removed at $\vec{x} = \vec{x}_A$, i.e.,

$$\phi(x) = \bar{\phi}_B(x) + O\left(\frac{1}{|\vec{x}-\vec{x}_A|}\right). \qquad (3.26)$$

where explicitly

$$\bar{\phi}_B(x) = \phi_B^{(0)}(x) + \delta\phi_{BA}^{(0)}(x) \qquad (3.27)$$

is the field generated by the particle B evaluated at particle A. Notice that there is a component $\delta\phi_{BA}^{(0)}(x)$ which comes from the backreaction on the scalar field generated by B due to the motion of particle A; see (2.19). This contribution is not divergent and involves the second derivative of the field $\phi_B^{(0)}$ generated by particle B. Expanding the Lagrangian for particle A to second order in the velocities and to leading order in G_N and $1/M^4$, we obtain

$$\mathcal{L}_A = \frac{1}{2}m_A\left(1+\frac{\beta}{m_{\rm Pl}}\bar{\phi}_B(x_A)\right)\vec{v}_A^2 - m_A - m_A\Phi_N^B(x_A) - \frac{\beta m_A}{m_{\rm Pl}}\bar{\phi}_B(x_A) + m_A A_i^B v_A^i - \frac{3}{2}m_A\Phi_N^B(x_A)\vec{v}_A^2 + \frac{m_A}{M^4}D_A. \tag{3.28}$$

The terms involving the conformal coupling renormalize the kinetic energy and the potential energy of the particle.

Let us focus first on the terms coming from the kinetic energy and the Newtonian potential only. We get for the two particles,

$$\mathcal{L}_{\rm matter} \supset \frac{1}{2}m_A\vec{v}_A^2 + \frac{1}{2}m_B\vec{v}_B^2 - m_A - m_B - m_A\Phi_N^B(x_A) - m_B\Phi_N^A(x_B) - \frac{3}{2}m_A\Phi_N^B(x_A)\vec{v}_A^2 - \frac{3}{2}m_B\Phi_N^A(x_B)\vec{v}_B^2 - \frac{8G_N m_A m_B \vec{v}_A.\vec{v}_B}{|\vec{x}_A-\vec{x}_B|}. \tag{3.29}$$

We can add the counterterms $S_{\rm EH}+S_{\rm GF}$ to obtain the Lagrangian

$$\mathcal{L}_{\rm grav} = \frac{1}{2}m_A\vec{v}_A^2 + \frac{1}{2}m_B\vec{v}_B^2 - m_A - m_B + \frac{G_N m_A m_B}{|\vec{x}_B-\vec{x}_A|} + \frac{G_N m_A m_B}{2|\vec{x}_A-\vec{x}_B|}(3\vec{v}_A^2 + 3\vec{v}_B^2 - 8\vec{v}_A.\vec{v}_B + \vec{v}_{A\perp}.\vec{v}_{B\perp}). \tag{3.30}$$

We can now add the contributions of the scalar field to this Lagrangian in order to evaluate the effects of both the conformal and disformal interactions.

3. The scalar action

Here we collect all the scalar field expressions allowing one to complete the action for two moving particles. The scalar field contributes to the scalar Lagrangian $-\frac{1}{2}(\partial\phi)^2$. After integration by parts and upon using the equation of motion we find

$$\mathcal{L}_{\rm scalar} = -\frac{1}{2}\int d^3x\left(\frac{\beta\phi(x)}{m_{\rm Pl}}(T^A(x)+T^B(x)) + \frac{1}{M^4}\partial_\mu\phi(x)\partial_\nu\phi(x)(T^{A\mu\nu}+T^{B\mu\nu})(x)\right) \tag{3.31}$$

After expanding

$$T^{A,B}(x) = -m_{A,B}\left(1-\frac{\vec{v}_{A,B}^2}{2}\right)\delta^{(3)}(\vec{x}-\vec{x}_{A,B}) \tag{3.32}$$

and using (3.21) to replace $\frac{\vec{v}_{A,B\perp}^2}{|\vec{x}_A-\vec{x}_B|} \equiv \frac{\vec{v}_{A\perp}.\vec{v}_{B\perp}}{|\vec{x}_A-\vec{x}_B|}$, the first term becomes

$$-\frac{1}{2}\int d^3x\frac{\beta\phi(x)}{m_{\rm Pl}}(T^A(x)+T^B(x)) = -\frac{2\beta^2 G_N m_A m_B}{|x_B-x_A|}\left(1-\frac{\vec{v}_A^2}{2}-\frac{\vec{v}_B^2}{2}+\frac{\vec{v}_{A\perp}.\vec{v}_{B\perp}}{2}\right) + \frac{\beta}{2m_{\rm Pl}}m_A\delta\phi_{BA}^{(0)}(x_A) + \frac{\beta}{2m_{\rm Pl}}m_B\delta\phi_{AB}^{(0)}(x_B), \tag{3.33}$$

where the last term involves the fields from which the self energy divergences have been removed

$$\delta\phi^{(0)} \equiv \delta\phi_{AB,BA}^{(0)} + \mathcal{O}\left(\frac{1}{|\vec{x}-\vec{x}_{B,A}|}\right). \tag{3.34}$$

Expanding the last term of the action in terms of the regularized field with no self energy divergences $\phi_{A,B}^{(0)}$ in (3.3), we have finally as

$$\mathcal{L}_{\rm scalar} = -\frac{2\beta^2 G_N m_A m_B}{|x_B-x_A|}\left(1-\frac{\vec{v}_A^2}{2}-\frac{\vec{v}_B^2}{2}+\frac{\vec{v}_{A\perp}.\vec{v}_{B\perp}}{2}\right) + \frac{\beta}{2m_{\rm Pl}}m_A\delta\phi_{BA}^{(0)}(x_A) + \frac{\beta}{2m_{\rm Pl}}m_B\delta\phi_{AB}^{(0)}(x_B) - \frac{m_A}{2M^4}D_A - \frac{m_B}{2M^4}D_B \tag{3.35}$$

which acts as a counterterm preventing any double counting too. The end result for the scalar Lagrangian when adding the scalar contributions in the matter action and the scalar one is

$$\mathcal{L}_S = \frac{2\beta^2 G_N m_A m_B}{|x_B-x_A|}\left(1+\frac{\vec{v}_{A\perp}.\vec{v}_{B.\perp}}{2}\right) - \frac{\beta}{2m_{\rm Pl}}m_A\delta\phi_{BA}^{(0)}(\vec{x}_A) - \frac{\beta}{2m_{\rm Pl}}m_B\delta\phi_{AB}^{(0)}(\vec{x}_B) + \frac{\beta}{2m_{\rm Pl}}m_A\phi_B^{(0)}(\vec{x}_A))v_A^2 + \frac{\beta}{2m_{\rm Pl}}m_B\phi_A^{(0)}(\vec{x}_B))v_B^2 + \frac{m_A}{2M^4}D_A + \frac{m_B}{2M^4}D_B. \tag{3.36}$$

This can be explicitly evaluated and gives

$$\mathcal{L}_S = -\frac{\beta^2 G_N m_A m_B}{|x_B - x_A|}(\vec{v}_A^2 + \vec{v}_B^2) + \frac{2\beta^2 G_N m_A m_B}{|x_B - x_A|} \times \left(1 + \frac{\vec{v}_{A\perp}.\vec{v}_{B.\perp}}{2}\right) + \frac{\beta^2 G_N}{4\pi} m_A m_B (m_A + m_B) \times \frac{((\vec{v}_A - \vec{v}_B)_\perp)^2 - ((\vec{v}_A - \vec{v}_B).\vec{n}_{AB})^2}{M^4 |x_A - x_B|^4} \quad (3.37)$$

This contributes to the effective dynamics of the two-body system.

4. The complete action

The complete Lagrangian for the two-body system combines both the gravitational Lagrangian $\mathcal{L}_{\rm grav}$ and the contribution from the scalars $\mathcal{L}_S$

$$\mathcal{L}_{AB} \equiv \mathcal{L}_{\rm grav} + \mathcal{L}_S = \frac{1}{2} m_A \vec{v}_A^2 + \frac{1}{2} m_B \vec{v}_B^2 - m_A - m_B + \frac{G_N(1+2\beta^2) m_A m_B}{|\vec{x}_B - \vec{x}_A|} + \frac{G_N m_A m_B}{2|\vec{x}_A - \vec{x}_B|}((3-2\beta^2)\vec{v}_A^2 + (3-2\beta^2)\vec{v}_B^2 - 8\vec{v}_A.\vec{v}_B + (1+2\beta^2)\vec{v}_{A\perp}.\vec{v}_{B\perp}) + \frac{\beta^2 G_N}{4\pi} m_A m_B (m_A + m_B) \times \frac{((\vec{v}_A - \vec{v}_B)_\perp)^2 - ((\vec{v}_A - \vec{v}_B).\vec{n}_{AB})^2}{M^4 |\vec{x}_A - \vec{x}_B|^4}. \quad (3.38)$$

This Lagrangian is all that is required to obtain the effective metric in the center-of-mass frame at leading order in G_N and $1/M^4$.

D. The role of counterterms

The calculations of the previous section have been carried out for two particles interacting both gravitationally and via a scalar field. The Lagrangian for a two-body system has been obtained in several steps for which the role of counterterm played by the gravitational and scalar actions is crucial. For the gravitational and scalar interaction mediated by the conformal coupling, the counterterms serve only as book keeping devices which ensure that double counting does not occur. Let us illustrate this with the case of two bodies A and B with nonrelativistic velocities $\vec{v}_{A,B}$. At leading order the scalar field is the sum of two contributions, each sourced by the mass of the particles, so we find

$$\phi(x) = \phi_A^{(0)}(x) + \phi_B^{(0)}(x) \quad (3.39)$$

where

$$\phi_A^{(0)}(x) \sim -\frac{\beta m_{A,B}}{4\pi m_{\rm Pl} |\vec{x} - \vec{x}_{A,B}|}. \quad (3.40)$$

where we have dropped the velocity dependent parts as we are here only interested in the static interaction potential between the bodies. The contributions to the matter Lagrangian from this scalar field read simply

$$\mathcal{L}_{AB} \supset -m_A \frac{\beta\phi_B^{(0)}(x_A)}{m_{\rm Pl}} - m_B \frac{\beta\phi_A^{(0)}(x_B)}{m_{\rm Pl}} = \frac{4\beta^2 G_N m_A m_B}{|\vec{x}_A - \vec{x}_B|} \quad (3.41)$$

where we have removed the divergent self-energy parts. Notice that this is twice the interaction potential between particles A and B. This double counting which occurs as we have added the matter actions for both particles is, in fact, absent as the scalar field Lagrangian $-\frac{1}{2}(\partial\phi)^2$ gives a counterterm

$$\Delta\mathcal{L}_{AB} \supset m_A \frac{\beta\phi_B^{(0)}(x_A)}{2m_{\rm Pl}} + m_B \frac{\beta\phi_A^{(0)}(x_B)}{2m_{\rm Pl}} \quad (3.42)$$

such that the overall Lagrangian only contains one copy of the interaction potential. The same compensation occurs for the Newtonian potential between the two particles. Notice that one could have used a "symmetrization" principle and obtain the same result by taking the interaction potential obtained from the action of particle A and then, realizing that it is symmetric in $A \to B$, inferred that this must be the actual interaction potential between the two particles. We could have also symmetrized the result from the action of particle A by adding a contribution for which $A \to B$ and dividing the overall result by two. The proper and unambiguous way of obtaining the interaction potential is the one we have outlined, i.e., calculating both the matter, gravity and scalar actions.

For the disformal coupling the matter action for particles A and B involves four contributions which depend on the disformal coupling. The first two come from the fact that the moving particles source the scalar field in a velocity-dependent way leading to two terms

$$\mathcal{L}_{AB} \supset -\frac{\beta}{m_{\rm Pl}} m_A \delta\phi_{BA}^{(0)}(x_A) - \frac{\beta}{m_{\rm Pl}} m_B \delta\phi_{AB}^{(0)}(x_B) \quad (3.43)$$

where the self-energy parts have been removed. Notice that the two contributions involve different combinations of the masses, i.e., respectively, $m_A^2 m_B$ and $m_B^2 m_A$. Moreover they are Galilean invariant and involve only the difference $\vec{v}_A - \vec{v}_B$. The disformal part of the Jordan metric leads to two other terms

$$\mathcal{L}_{AB} \supset -\frac{m_A}{M^4}(\partial_\mu \phi_B^{(0)}(x_A) v_A^\mu)^2 - \frac{m_B}{M^4}(\partial_\mu \phi_A^{(0)}(x_B) v_B^\mu)^2 \quad (3.44)$$

where $v_{A,B}^\mu = (1, \vec{v}_{A,B})$ at this order, and we have $\partial_0 \phi_{A,B}^{(0)} = -\vec{\partial}\phi_{A,B}^{(0)}.\vec{v}_{A,B}$. The scalar field action plays the same role as in the gravitational and conformal cases and simply removes half of the previous terms

$$\Delta\mathcal{L}_{AB} = \frac{\beta}{2m_{\rm Pl}} m_A \delta\phi^{(0)}_{BA}(x_A) + \frac{\beta}{2m_{\rm Pl}} m_B \delta\phi^{(0)}_{AB}(x_B) + \frac{m_A}{2M^4}(\partial_\mu\phi^{(0)}_B(x_A)v^\mu_A)^2 + \frac{m_B}{2M^4}(\partial_\mu\phi^{(0)}_A(x_B)v^\mu_B)^2. \quad (3.45)$$

Overall the disformal contributions to the Lagrangian can be combined into two pairs

$$-\frac{\beta}{2m_{\rm Pl}} m_A \delta\phi^{(0)}_{BA}(x_A) + \frac{m_B}{2M^4}(\partial_\mu\phi^{(0)}_A(x_B)v^\mu_B)^2 = \frac{\beta^2 G_N}{4\pi} m_A^2 m_B \frac{((v_A - v_B)_\perp)^2}{M^4|x_A - x_B|^4} \quad (3.46)$$

where

$$\frac{m_B}{2M^4}(\partial_\mu\phi^{(0)}_A(x_B)v^\mu_B)^2 = \frac{m_B}{2M^4} D_B \quad (3.47)$$

and symmetrically for $A \to B$. It is important to notice that the two pairs have different origins. The term $-\frac{\beta}{2m_{\rm Pl}} m_A \delta\phi^{(0)}_{BA}(x_A)$ comes from the matter action of particle A and the corresponding counterterm. The term $\frac{m_B}{2M^4}(\partial_\mu\phi^{(0)}_A(x_B)v^\mu_B)^2$ comes from the matter action for B and its counterterm. It turns out that the contribution from the matter action for A combines with the term from the matter action for B, and vice versa.

This implies that one cannot isolate the action for either A or B in order to investigate the dynamics of the two-body system. This is particular to the disformal interaction compared to the gravitational and conformal ones. In hindsight, one could have taken the action for particle A minus the associated counterterm and obtained the correct action by symmetrizing the result in $A \to B$, i.e., by taking half the sum of the Lagrangian. On the contrary if one only selected the leading term in $m_A m_B^2$ in the matter action for A one would wrongly omit the term in $m_A m_B^2$ coming from the action for B which combine pairwise as in (3.46). Overall, a much more straightforward way of obtaining the complete action for the two-body system with the disformal interaction is to calculate the matter, gravitational and scalar actions as we have done.

E. Center-of-mass dynamics

The previous Lagrangian (3.38) involving the two bodies A and B can be projected onto a single particle Lagrangian by going to the center-of-mass frame. We will do this by first introducing the Newtonian center-of-mass frame coordinates

$$\vec{X} = \frac{m_A\vec{x}_A + m_B\vec{x}_B}{m_A + m_B}, \qquad \vec{x} = \vec{x}_A - \vec{x}_B \quad (3.48)$$

from which we get the velocities

$$\vec{v}_{A,B} = \vec{V} + \frac{\mu}{m_{A,B}}\vec{v}. \quad (3.49)$$

The total Lagrangian becomes the sum of a free Lagrangian ($x = |\vec{x}|$)

$$\mathcal{L}_0 = \frac{1}{2}\mu\vec{v}^2 + \frac{1}{2}\mathcal{M}\vec{V}^2 + \frac{G_N m_A m_B}{x} \quad (3.50)$$

where the reduced mass is

$$\mu = \frac{m_A m_B}{m_A + m_B} \quad (3.51)$$

and the total mass $\mathcal{M} = m_A + m_B$. The interaction Lagrangian is

$$\mathcal{L}_{\rm int} = \frac{G_N m_A m_B}{2x}\Big(-(1+2\beta^2)\vec{V}^2 + ((3-2\beta^2)\mu^{-2} + (1+2\beta^2)m_A^{-1}m_B^{-1})\mu^2\vec{v}^2 - (1+2\beta^2)(m_A^{-1} - m_B^{-1})\mu\vec{v}.\vec{V} - (1+2\beta^2)\Big((\vec{V}.\vec{n})^2 - \frac{\mu^2}{m_A m_B}(m_A^{-1} - m_B^{-1})(\vec{v}.\vec{n})^2 + \mu(\vec{v}.\vec{n})(\vec{V}.\vec{n})\Big)\Big) \quad (3.52)$$

together with the disformal term

$$\mathcal{L}_{\rm dis} = \frac{\beta^2 G_N}{4\pi M^4 x^4}\mu\mathcal{M}^2(\vec{v}_\perp^2 - (\vec{v}.\vec{n})^2) \quad (3.53)$$

We have identified $\vec{n} = \vec{n}_{AB}$ here. The absence of $\vec{X}$ dependence implies that

$$\vec{P} = \frac{\partial\mathcal{L}}{\partial\vec{V}} \quad (3.54)$$

is conserved. We set the center-of-mass momentum to zero $\vec{P} = 0$ and integrate out $\vec{V}$ at leading order in G_N

$$\mathcal{M}\vec{V} = \frac{G_N(m_A - m_B)\mu}{2x}(1+2\beta^2)(\vec{v} + (\vec{v}.\vec{n})\vec{n}). \quad (3.55)$$

The effective Lagrangian for the velocity $\vec{v}$ is then given by

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\mu\vec{v}^2 + \frac{G_N\mu\mathcal{M}(1+2\beta^2)}{x} + \frac{G_N\mu\mathcal{M}}{2x}((3-2\beta^2)\vec{v}^2 + (1+2\beta^2)\nu\vec{v}^2 + (1+2\beta^2)\nu(\vec{v}.\vec{n})^2) + \frac{\beta^2 G_N(\vec{v}_\perp^2 - (\vec{v}.\vec{n})^2)}{4\pi}\frac{\mu\mathcal{M}^2}{x^4M^4} \quad (3.56)$$

where we have introduced the parameter

$$\nu = \frac{m_A m_B}{(m_A+m_B)^2}. \quad (3.57)$$

Using the identity, at leading order in G_N,

$$\frac{d}{dt}(\vec{v}.\vec{n}) \equiv \frac{\vec{v}^2 - (\vec{v}.\vec{n})^2}{x} \quad (3.58)$$

the Lagrangian becomes equivalent to

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\mu\vec{v}^2 + \frac{G_N\mu\mathcal{M}(1+2\beta^2)}{x} + \frac{G_N\mu\mathcal{M}}{2x}((3-2\beta^2)\vec{v}^2 + 2(1+2\beta^2)\nu\vec{v}^2) + \frac{\beta^2 G_N(v_\perp^2 - (\vec{v}.\vec{n})^2)}{4\pi}\frac{\mu\mathcal{M}^2}{x^4M^4}. \quad (3.59)$$

Let us introduce the effective metric

$$g_{00}^{\rm eff} = -\left(1 - \frac{2G_N\mathcal{M}(1+2\beta^2)}{x}\right)$$
$$g_{ij}^{\rm eff} = \left(1 + \frac{2G_N\mathcal{M}(1-2\beta^2)}{x}\right)\delta_{ij} + \frac{\beta^2 G_N}{2\pi}\frac{\mathcal{M}^2}{x^4M^4}(\delta_{ij} - 2n_i n_j) \quad (3.60)$$

and the reduced Lagrangian

$$\mathcal{L}_{\rm red} = -g_{\mu\nu}^{\rm eff} v^\mu v^\nu \quad (3.61)$$

where $v^\mu = (1, v^i)$. Then we have to leading order that the center-of-mass Lagrangian can be reconstructed using

$$\mathcal{L}_{\rm eff} = -\mu\sqrt{\mathcal{L}_{\rm red} + 2\nu(\mathcal{L}_{\rm red} - 1)^2}. \quad (3.62)$$

The first term is the Lagrangian of a particle of mass μ subject to the effective metric created by the mass $\mathcal{M}$. The quadratic correction in the square root in ν is due to the fact that the masses are not light masses. The extrema of $\mathcal{L}_{\rm eff}$ can be obtained by extremizing $\mathcal{L}_{\rm red}$ which depends on the effective metric $g_{\mu\nu}^{\rm eff}$.

The effective metric (3.60) is known to provide an exact result in the post-Minkowskian limit [40], i.e., at leading order in G_N, and to all order in the velocity in the conformal case. Here we have retrieved this result using the low $\vec{v}^2$ expansion and we have extended it to include the first correction in $1/M^4$ due to the disformal coupling. Moreover the derivation of the effective metric is usually carried out in the Hamiltonian formalism whereas we have worked at the Lagrangian level and at the lowest order in $\vec{v}^2$ as it sufficient to reconstruct the effective metric.

Notice that the disformal correction has been assumed throughout to be the leading correction to the Newtonian case implying that we can consider this effective metric in situations where

$$\frac{\mathcal{M}}{x^3} \lesssim \frac{M^4}{\beta^2} \lesssim \frac{m_{\rm Pl}^2}{x^2}. \quad (3.63)$$

If the first inequality were nearly saturated then Newtonian orbits would be largely affected whereas if the second inequality were violated we would have to take into account the higher-order corrections to the metric in GR. Taking as an example the orbit of Mercury at an average distance of 6×10^7 km from the Sun this leads to

$$2\times10^{-3}\ {\rm MeV} \lesssim \frac{M}{\sqrt{\beta}} \lesssim 7.5\times10^{-2}\ {\rm MeV} \quad (3.64)$$

for the disformal interaction to play a relevant role. As the Cassini bound [11] leads to $\beta^2 \lesssim 10^{-5}$, this implies that gravitational effects of the disformal coupling could be relevant for planetary orbits when

$$M \lesssim 4\times10^{-3}\ {\rm MeV}. \quad (3.65)$$

Of course, the disformal interaction becomes relevant for larger values of M in situations where the Newtonian potential is larger, such as the orbits of two neutron stars in their inspiralling phase where objects of masses similar to the Sun's orbit are at a few hundreds of kilometers from each other. We will comment on this case below.

IV. THE DYNAMICS OF A LIGHT PARTICLE

In this section, we focus on the dynamics of a light particle and in particular the classical tests of general relativity such as the Shapiro time delay and the perihelion advance. In the following, we neglect the influence of the cosmological background which would result in a time variation of masses due to the conformal factor $A(\phi_{\rm cosmo})$. We focus on the effects due to the disformal coupling. The effects of the time drift of masses, or equivalently Newton's constant, in Brans-Dicke theories are well documented as can be found in [41]. Typically, the relative variation of masses should be less than one percent of the Hubble rate.

A. Violation of the equivalence principle

In this section, we focus on the light particle case which can be obtained from the two-body analysis by setting $\nu \to 0$. In this case, the light particle of mass $m_B \ll m_A$ evolves with the dynamical Lagrangian

$$\mathcal{L}_{\text{eff}} = -m_B \sqrt{\mathcal{L}_{\text{red}}}. \tag{4.1}$$

where the effective metric is given by

$$\begin{aligned} g_{00}^{\text{eff}} &= -\left(1 - \frac{2G_N m_A (1+2\beta^2)}{x}\right) \\ g_{ij}^{\text{eff}} &= \left(1 + \frac{2G_N m_A (1-2\beta^2)}{x}\right)\delta_{ij} \\ &\quad + \frac{\beta^2 G_N}{2\pi} \frac{m_A{}^2}{x^4 M^4} (\delta_{ij} - 2n_i n_j) \end{aligned} \tag{4.2}$$

The effective action is the one of a particle evolving in the background metric given by (4.2). Notice that the disformal part involves both the perpendicular and parallel velocities. This is different from the trajectories of photons which follow the null trajectories of the Jordan metric

$$ds_J^2 = -g_{\mu\nu} dx^\mu dx^\nu \equiv 0 \tag{4.3}$$

where the Jordan metric is given by

$$\begin{aligned} g_{00} &= -\left(1 - \frac{2G_N m_A (1+2\beta^2)}{x}\right) \\ g_{ij} &= \left(1 + \frac{2G_N m_A (1-2\beta^2)}{x}\right)\delta_{ij} \\ &\quad + \frac{\beta^2 G_N}{\pi} \frac{m_A{}^2}{x^4 M^4} n_i n_j. \end{aligned} \tag{4.4}$$

which involves the parallel velocity only. As a result the equivalence principle is violated between photons and matter. This follows from the fact that for nonrelativistic matter particles, the mass of a light particle cannot be neglected completely and generates a field contribution $\delta\phi_{AB}^{(0)}$ whose presence in the matter action of the massive particle is of the same order as the disformal terms in the matter action of the light particle; see the discussion in Sec. III D. We will analyze what differences this induces in both the Shapiro effect, i.e., the time delay of photons, and in the perihelion advance, i.e., the motion of a light particle.

B. Shapiro time delay

The study of the time delay of radio waves compared to its counterpart in GR is crucial as it gives direct access to modifications of GR in the environment of a massive object A, typically the Sun. It is convenient to introduce the metric potential $\Phi(r) = \Phi_N(r) + \beta \frac{\phi^{(0)}(r)}{m_{\text{Pl}}}$ such that

$$\Phi(r) = -\frac{G_{\text{eff}} m_A}{r}. \tag{4.5}$$

where the effective Newton constant is here

$$G_{\text{eff}} = (1 + 2\beta^2) G_N. \tag{4.6}$$

We are interested in the time delay of signals sent between two points such that to leading order the photon trajectory is a straight line with an impact parameter b, i.e., in terms of polar coordinates

$$r = \frac{b}{\cos\theta}. \tag{4.7}$$

Along this trajectory, the time delay compared to GR is due to the corrections to the metric felt by the photons. This reads

$$\begin{aligned} ds_J^2 = -(1+2\Phi)dt^2 + \Bigg(1 - 2(1+2\gamma)\Phi \\ + \sin^2\theta\cos^4\theta \frac{G_N \beta^2 m_A^2}{\pi M^4 b^4}\Bigg) dx^2 \end{aligned} \tag{4.8}$$

where $dx^2 = dr^2 + r^2 d\theta^2 = \frac{b^2}{\cos^4\theta} d\theta^2$ and $dr^2 = \sin^2\theta dx^2$. The last contribution to dx^2 comes from the disformal interaction. We have also introduced the parameter

$$\gamma = -\frac{2\beta^2}{1 + 2\beta^2}. \tag{4.9}$$

Let us apply this result to the trajectory of photons between two points C and D which can be taken to be the location of the Earth and of the Cassini satellite [11]. We have, therefore,

$$\frac{dt}{dx} = 1 - 2(1+\gamma)\Phi + \sin^2\theta\cos^4\theta \frac{G_N \beta^2 m_A^2}{2\pi M^4 b^4} \tag{4.10}$$

along the photon trajectory. The time delay due the modification of gravity is

$$\frac{d\delta t}{dx} = -2\gamma\Phi + \sin^2\theta\cos^4\theta \frac{G_N \beta^2 m_A^2}{2\pi M^4 b^4} \tag{4.11}$$

such that $\Phi = -\frac{G_{\text{eff}} m_A}{b}\cos\theta$. Using $dx = \frac{b}{\cos^2\theta} d\theta$, this implies that

$$\frac{d\delta t}{d\theta} = 2\gamma G_{\text{eff}} \frac{m_A}{\cos\theta} + \sin^2\theta\cos^2\theta \frac{G_N \beta^2 m_A^2}{2\pi M^4 b^3}. \tag{4.12}$$

Let us assume that the two massive bodies where the signal is received and emitted follow circular trajectories, for simplicity, around the massive body. This implies that, for instance, $\cos\theta_C = b/r_C$ and, therefore, a variation of the position of the emitter or the receiver by $d\theta_{C,D}$ corresponds to a change of impact parameter $db = -r_{C,D}\sin\theta_{C,D} d\theta_{C,D}$. As a result, the variation of the time delay due to a variation of the impact parameter is

$$\frac{d\delta t}{db} = -2\gamma G_{\rm eff}\frac{m_A}{b}\left(\frac{1}{\sin\theta_C}+\frac{1}{\sin\theta_D}\right) - \sin\theta_C\cos^3\theta_C\frac{G_N\beta^2 m_A^2}{2\pi M^4 b^4} - \sin\theta_D\cos^3\theta_D\frac{G_N\beta^2 m_A^2}{2\pi M^4 b^4}. \tag{4.13}$$

The probes such as Cassini which are used to investigate the time delay for radio waves are going behind the Sun for $\theta_{C,D}\sim\frac{\pi}{2}$. As a result, the contribution to the time delay from the disformal interaction is negligible. The time delay as measured between two positions of the emitter, after a round trip, with impact parameters b_1 and b_2 is, therefore, given by

$$\delta t_1 - \delta t_2 = -8\gamma G_{\rm eff} m_A \ln\frac{b_1}{b_2} \tag{4.14}$$

and does not depend on the disformal interaction.

C. The perihelion advance

One of the classical tests of general relativity is the advance of perihelion of mercury. Here we will calculate the advance of perihelion for a light particle around a heavy body when the conformal and disformal interactions are present. The study of the dynamics of such a light object is easier to carry out going back to the original Lagrangian (4.2). In particular, we consider that time and space are parametrized in proper time τ_B. The trajectories of massive objects are such that

$$g^{\rm eff}_{\mu\nu}u^\mu_B u^\nu_B = -1, \tag{4.15}$$

where

$$u^\mu_B = \frac{dx^\mu}{d\tau_B}. \tag{4.16}$$

In this section, we put $\tau=\tau_B$ and define $\dot{} = d/d\tau$. In polar coordinates, in the orbital plane, and using

$$d\Omega^2 = d\theta^2 \tag{4.17}$$

we have

$$g^{\rm eff}_{00} = -1 + 2\frac{G_N m_A(1+2\beta^2)}{r}$$
$$g^{\rm eff}_{rr} = 1 + 2\frac{G_N m_A(1-2\beta^2)}{r} - \frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4}$$
$$g^{\rm eff}_{\theta\theta} = r^2\left(1 + 2\frac{G_N m_A(1-2\beta^2)}{r} + \frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4}\right). \tag{4.18}$$

Notice that the disformal contribution appears both in the radial and tangential parts of the metric. As the Lagrangian is independent of θ, the angular momentum J is conserved, implying that

$$r^2\left(1 + 2\frac{G_N m_A(1-2\beta^2)}{r} + \frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4}\right)\dot\theta = \frac{J}{m_B}. \tag{4.19}$$

Similarly, the absence of any explicit time dependence in x_0 implies that

$$\left(1 - 2\frac{G_N m_A(1+2\beta^2)}{r}\right)\dot x^0 = k, \tag{4.20}$$

where k is a constant. The constraint (4.15) implies that

$$\frac{k^2}{(1-2\frac{G_N m_A(1+2\beta^2)}{r})} - \left(1+2\frac{G_N m_A(1-2\beta^2)}{r} - \frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4}\right)\dot r^2 - \frac{J^2}{m_B^2}\frac{1}{r^2(1+2\frac{G_N m_A(1-2\beta^2)}{r}+\frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4})} = 1. \tag{4.21}$$

The dynamics are most conveniently analyzed by changing coordinates and introducing the spherical distance,

$$\tilde r^2 = \left(1 + \frac{2G_N m_A}{r}(1-2\beta^2)\right)r^2, \tag{4.22}$$

which corresponds to writing the angular part of the metric as $g_{\theta\theta}=\tilde r^2$ in the absence of disformal interaction. We obtain that to leading order in G_N,

$$\tilde r = r + G_N m_A(1-2\beta^2), \tag{4.23}$$

and $\dot{\tilde r} = \dot r$. This implies that angular momentum conservation can be reformulated as

$$\tilde r^2\left(1 + \frac{G_N\beta^2 m_A^2}{2\pi M^4\tilde r^4}\right)\dot\theta = \frac{J}{m_B}. \tag{4.24}$$

At leading order in G_N and reverting to $\tilde r\to r$ for convenience, we have now

$$\frac{k^2}{(1-2\frac{G_N m_A(1+2\beta^2)}{r})} - \left(1+2\frac{G_N m_A(1-2\beta^2)}{r} - \frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4}\right)\dot r^2 - \frac{J^2}{m_B^2 r^2(1+\frac{G_N\beta^2 m_A^2}{2\pi M^4 r^4})} = 1. \tag{4.25}$$

Let us now introduce the Binet variable $u=1/r$ such that

$$\dot r = -\frac{J}{m_B}\frac{du}{d\theta}. \tag{4.26}$$

We then obtain the following differential equation,

$$\left(\frac{du}{d\theta}\right)^2\left(1-8G_N\beta^2 m_A u-\frac{G_N\beta^2 m_A^2}{2\pi M^4}u^4\right)+u^2 = \frac{k^2-1}{J^2}m_B^2+\frac{2G_N(1+2\beta^2)m_A m_B^2}{J^2}u + 2G_N m_A(1+2\beta^2)u^3+\frac{G_N\beta^2 m_A^2}{2\pi M^4}u^6. \tag{4.27}$$

This is the main equation for the dynamics of planar orbits involving both conformal and disformal interactions.

By taking the derivative of the previous relation, we deduce the generalized Binet equation,

$$\frac{d^2u}{d\theta^2}+u=\left(4G_N\beta^2 m_A+\frac{G_N\beta^2 m_A^2}{\pi M^4}u^3\right)\left(\frac{du}{d\theta}\right)^2 + \frac{G_N(1+2\beta^2)m_A m_B^2}{J^2} + G_N m_A(3-2\beta^2)u^2+\frac{G_N\beta^2 m_A^2}{\pi M^4}u^5, \tag{4.28}$$

which reduces to the one in general relativity when $\beta=0$. The new terms due to the conformal and disformal interactions modify the structure of the orbits.

One can construct solutions in perturbation theory around the classical trajectory,

$$u_0=\frac{G_N(1+2\beta^2)m_A m_B^2}{J^2}(1+e\cos\theta), \tag{4.29}$$

where the semilong axis is

$$a=\frac{J^2}{m_B^2 m_A G_N(1+2\beta^2)}\frac{1}{1-e^2}, \tag{4.30}$$

corresponding to

$$u_0=\frac{1}{a(1-e^2)}(1+e\cos\theta). \tag{4.31}$$

The first correction to the classical trajectory satisfies

$$\frac{d^2u_1}{d\theta^2}+u_1=4\pi G_N\beta^2 m_A\left(\frac{du_0}{d\theta}\right)^2 + G_N m_A(3-2\beta^2)u_0^2+\frac{G_N\beta^2 m_A^2}{\pi M^4}u_0^5. \tag{4.32}$$

Notice that the source terms are all proportional to G_N as befitting our expansion scheme.

We will not solve this equation in full generality. As we are only interested in the advance of the perihelion, we select the source terms on the right-hand side of the perturbed Binet equation (4.32) in $\cos\theta$. Higher harmonics are present and will not give rise to contributions to the advance of perihelion. As a result, we only need to select the $\cos\theta$ source terms which correspond to

$$\frac{d^2u_1}{d\theta^2}+u_1\supset\left(2\frac{e}{a^2(1-e^2)^2}G_N m_A(3-2\beta^2) + \frac{e}{a^5(1-e^2)^5}\frac{5G_N\beta^2 m_A^2}{\pi M^4}\right)\cos\theta, \tag{4.33}$$

whose solution is

$$u_1=\alpha\theta\sin\theta \tag{4.34}$$

with

$$\alpha=\left(\frac{e}{a^2(1-e^2)^2}G_N m_A(3-2\beta^2) + 5\frac{e}{a^5(1-e^2)^5}\frac{G_N\beta^2 m_A^2}{2\pi M^4}\right). \tag{4.35}$$

As a result, we have at this order,

$$u=u_0+u_1 \equiv\frac{1}{a(1-e^2)}\left(1+e\cos\left(\left(1-\frac{\alpha a}{e}(1-e^2)\right)\theta\right)\right), \tag{4.36}$$

and therefore the perihelion advance is given by

$$\Delta\theta=\frac{2\pi\alpha a}{e}(1-e^2) \tag{4.37}$$

or more directly

$$\Delta\theta=2\pi\frac{G_N m_A}{p}\left((3-2\beta^2)+5\frac{\beta^2 m_A}{2\pi M^4 p^3}\right), \tag{4.38}$$

where we have introduced

$$p=a(1-e^2). \tag{4.39}$$

The perihelion advance can be written as

$$\Delta\theta=2\pi\frac{G_{\rm eff}m_A}{p}\left(3+\gamma\left(4-\frac{5m_A}{4\pi M^4 p^3}\right)\right). \tag{4.40}$$

The first term is the result in GR corrected by the scalar-tensor coupling [42], the last term is new and comes from the disformal interaction. Notice that the GR and conformal cases have been retrieved while never going beyond the leading G_N corrections.

As the overall result depends on both β and M, no precise bound on M can be deduced. A reasonable requirement may be

$$M^4\gtrsim\frac{m_\odot}{p^3} \tag{4.41}$$

for planets orbiting around the Sun. This is, of course, not mandatory as β might be very small. For Mercury, this would simply require that

$$M \gtrsim 10^{-4}\ \mathrm{MeV}, \tag{4.42}$$

which is weaker than the Eötwash bound $M \geq 0.07$ MeV [30].

V. DISCUSSION AND CONCLUSION

We have analyzed the dynamics of bodies interacting both gravitationally and via a scalar field which can be exchanged between moving objects. We have seen that the disformal coupling has an effect only when combined with a conformal interaction. In this case, the disformal coupling leads to a change of the advance of perihelion for a light body and modifies the effective metric which governs the evolution of two interacting bodies. We have shown that contrary to GR and conformal couplings for which the advance of perihelion is proportional to the mass of the heavy object around which a light particle orbits, the disformal coupling leads to a quadratic dependence. Although we have not considered the case of black holes, as in particular the no-hair theorem implies that no scalar field is generated outside the horizon, we can certainly envisage that for astrophysical black holes of several million solar masses and with accretion discs [43,44], a large scalar field would be generated and therefore we expect that because of the large mass of the black holes, there could be large effects on the dynamics of stars in the vicinity of the center of a galaxy like the Milky Way. It would be worth analyzing this possibility and setting bounds on the disformal coupling from the advance of perihelion of such stars orbiting the Galactic center. In this paper, we have also shown that for a two-body system with conformal and disformal interactions, the center-of-mass dynamics can be captured by an effective metric at leading order in G_N. The effects of the disformal coupling in the case of two inspiralling neutron stars could be relevant for future observations and would give us indications on the existence of both conformal and disformal interactions between matter and a scalar field. We expect that the disformal interaction begins to induce large deviations from general relativity when

$$M^4 \lesssim \frac{\beta^2 m_\odot}{R^3} \tag{5.1}$$

for objects of masses around one solar mass at a distance R; see (4.28). Typically, the Cassini experiment implies that $\beta^2 \lesssim 10^{-5}$ [11], and therefore the typical order of magnitude of the upper bound for which disformal effects might be expected when two neutron stars are separated by $R \sim 100$ km is

$$M \lesssim 3\ \mathrm{MeV}. \tag{5.2}$$

This has to be compared with the constraints on disformal couplings in different environments (see Table 1 in [30]); i.e., the scale M could be different for physical processes involving various densities or energy scales. Typically, such a range for M is compatible with the lower bound on $M \geq 0.07$ MeV from the Eötwash experiment [30]. In conclusion, we find that disformal effects could play a role in the merging of two neutron stars when the disformal scale is $M \sim 1$ MeV. The details of this study are left for the future.

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