Quasi-two-body decays $B_{(s)} \to Pf_2(1270) \to P\pi\pi$ in the perturbative QCD approach

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In this work, we calculate the *CP*-averaged branching ratios and direct *CP*-violating asymmetries of the quasi-two-body decays $B_{(s)} \to Pf_2(1270) \to P\pi\pi$ with the two-pion distribution amplitude $\Phi_{\pi\pi}^D$ by using the perturbative QCD factorization approach, where *P* represents a light pseudoscalar meson K, π, η and η'. The relativistic Breit-Wigner formula for the *D*-wave resonance $f_2(1270)$ is adopted to parameterize the timelike form factor F_π, which contains the final state interactions between the pions in the resonant regions. The consistency of theoretical results with data can be achieved by determining the Gegenbauer moments of the *D*-wave two-pion distribution amplitudes. The decay rates for the considered decay modes are generally in the order of 10^{-9} to 10^{-6}. The integrated direct *CP* asymmetries for the charged modes agree with the *BABAR* and Belle measurements. As a by-product, we extract the branching ratios of $B_{(s)} \to Pf_2(1270)$ from the corresponding quasi-two-body decay modes, which still need experimental tests at the ongoing and forthcoming experiments.

I. INTRODUCTION

Three-body hadronic *B* meson decays are a rich field for studying the direct *CP* violation and testing the standard model and QCD. Recent measurements by *BABAR* [1–6] and Belle [7–9] Collaborations of a number of $B \to \pi\pi\pi$, $B \to K\pi\pi$, or $B \to J/\psi\pi\pi$ decays have triggered considerable theoretical interests in understanding three-body hadronic *B* decays. These three-body decays are known experimentally to be dominated by the low energy resonances on $\pi\pi$, KK and $K\pi$ channels. As the LHCb Collaboration reported, the $\pi\pi$ final states are found

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to comprise the decay products of the $\rho(770)$, $f_0(500)$, $f_0(980)$, $f_2(1270)$,[1] and $f_0(1370)$ (etc.) mesons in case of B^0 or B_s^0 decays [10–15].

It is known that such three-body *B* decay modes are more intractable than those two-body decays due to the entangled resonant and nonresonant contributions, as well as the possible final-state interactions (FSIs) [16–18], whereas the relative strength of these contributions vary significantly for different decay modes. The analysis of these three-body decays utilizing the Dalitz plots [19,20] enables one to investigate the properties of various tensor, vector, and scalar resonances with the isobar model [21,22] in terms of the usual Breit-Wigner (BW) model [23]. Unfortunately, no proof of factorization has been given for the decays of the *B* into three mesons. However, we can restrict ourselves to specific kinematical configurations, in which two energetic final state mesons almost collimate to each other. Then the three-body interactions are expected to be suppressed strongly in such conditions. It seems reasonable to assume the validity of factorization for these quasi-two-body *B*

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[1]For the sake of simplicity, we generally use the abbreviation $f_2 = f_2(1270)$ in the following sections.

decays. In the "quasi-two-body" mechanism, the two-body scattering and all possible interactions between the two involved particles are included but the interactions between the third particle and the pair of mesons are neglected.

There are several theoretical approaches for describing hadronic three-body decays of B mesons based on the symmetry principles and factorization theorems. U-spin and flavor $SU(3)$ symmetries were adopted in Refs. [24–29]. The QCD-improved factorization (QCDF) [30] has been widely applied to the studies of three-body hadronic B meson decays in Refs. [31–40]. The authors investigated the detailed factorization properties of the $B^+ \to \pi^+\pi^+\pi^-$ mode in different regions of phase space [31]. In Ref. [35], the authors studied the decays of $B^\pm \to \pi^\pm\pi^\mp\pi^\pm$ within a quasi-two-body QCD factorization approach and introduced the scalar and vector form factors for the S and P waves, as well as a relativistic BW formula for the D wave to describe the meson-meson final state interactions. In Ref. [38], for instance, the authors studied the nonresonant contributions using heavy meson chiral perturbation theory [41–43] with some modifications and analyzed the resonant contributions with the isobar model by using the usual BW formalism.

Relying on the perturbative QCD (PQCD) approach, furthermore, some three-body B meson decays have also been investigated in Refs. [44–47]. The authors of Ref. [48] studied the direct CP asymmetries of $B^\pm \to \pi^\pm\pi^+\pi^-$ and $K^\pm\pi^+\pi^-$ decays by fitting the time-like form factors and the rescattering phases contained in the two-pion distribution amplitudes $\Phi_{h_1 h_2}$ [49–55] to relevant experimental data. Very recently, in the PQCD approach, we studied the S-wave resonance contributions to the decays $B^0_{(s)} \to J/\psi\pi^+\pi^-$ [56], $B^0_{(s)} \to \eta_c(1S, 2S)\pi^+\pi^-$ [57,58], $B^0_s \to \psi(2S, 3S)\pi^+\pi^-$ [59] and $B^0_{(s)} \to J/\psi(\psi(2S))K\pi$ [60], as well as the P-wave resonance contributions to the decays $B \to P(\rho, \rho(1450), \rho(1700)) \to P\pi\pi$ [61–63], $B \to D(\rho, \rho(1450), \rho(1700)) \to D\pi\pi$ [64–66] and $B \to \eta_c(1S, 2S)(\rho, \rho(1450), \rho(1700)) \to \eta_c(1S, 2S)\pi\pi$ [67]. One of the aims for studying such three-body B meson decays is to test the usability of our PQCD approach. The above works do support our general expectation: the PQCD factorization is universal for exclusive hadronic three-body B meson decays.

In the PQCD factorization approach, the contribution from the dynamical region, where there is at least one pair of the final state light mesons having an invariant mass below $O(\bar{\Lambda}m_B)$ [44], $\bar{\Lambda} = m_B - m_b$ being the B meson and b quark mass difference, is dominant. Because the hard b-quark decay kernels containing two hard gluons at leading order is not important due to the power-suppression, the configuration involving two energetic mesons almost parallel to each other may provide the dominant contribution. Then it's reasonable that the dynamics associated with the pair of mesons can be factorized into a two-meson

distribution amplitude $\Phi_{h_1 h_2}$. The typical PQCD factorization formula for the $B \to h_1 h_2 h_3$ decay amplitude can be described as the form of [44]

$$\mathcal{A} = \Phi_B \otimes H \otimes \Phi_{h_1 h_2} \otimes \Phi_{h_3}, \qquad (1)$$

where the hard kernel H contains only one hard gluon and describes the dynamics of the strong and electroweak interactions in the three-body hadronic decays as in the formalism for the two-body B meson decays. The Φ_B and Φ_{h_3} are the wave functions for the B meson and the final state h_3, which absorb the non-perturbative dynamics in the relevant processes. In the PQCD approach based on the k_T factorization theorem, we adopt the widely used wave function for B meson [68], which includes the intrinsic b dependence with b being a variable conjugate to k_T. In Ref. [69], the authors pointed out that the operator-level definition of the transverse-momentum-dependent (TMD) hadronic wave functions is highly nontrivial in order to avoid the potential light-cone divergence and the rapidity singularity. A well-defined TMD can be found in Ref. [70]. Meanwhile, the Sudakov factors from the k_T resummation have been included to suppress the long-distance contributions from the large b region in this work. The more precise joint resummation derived in [71] can be included in the future. For the QCD resummation, one can include its effect as going beyond the tree level in PQCD analysis, which will be done in the future by taking into account the results as given in Refs. [70,71].

In this work, we will extend our previous work on S and P- wave resonances to the D-wave ones in the PQCD framework. Taking the decays $B_{(s)} \to Pf_2(1270) \to P\pi\pi$, $P = (\pi, K, \eta \text{ or } \eta')$ as examples, the relevant Feynman diagrams are illustrated in Figs. 1 and 2. Since the tensor resonance cannot be created from the $(V \pm A)$, $(S \pm P)$ or tensor current, the diagrams with a D-wave $\pi\pi$ pair emitted in Fig. 2 are prohibited in naive factorization. Phenomenologically there are growing appeals for the two-body charmless hadronic B decays involving a light tensor meson in the final states in the past few years [72–79]. More recently, one of us has investigated the two-body decays of the B_c meson into the tensor charmonium using the PQCD approach [80]. Experimentally the CP-averaged branching ratios and direct CP-violating asymmetries of quasi-two-body decays $B \to \pi f_2(1270) \to \pi\pi\pi$ [2] and $B \to Kf_2(1270) \to K\pi\pi$ [4,5,8,9] have been measured. One can see that the measured CP violation is just a number in two-body B decays, while the CP asymmetry depends on the invariant mass displayed in the Dalitz plot in the three-body modes [81]. It is meaningful to study the decays $B \to Pf_2(1270) \to P\pi\pi$ in the three-body framework, which provide useful information for understanding the CP-violation mechanisms. For the D-wave resonant state $f_2(1270)$, the relativistic BW formula is adopted to parametrize the timelike form factors F_π, which contains
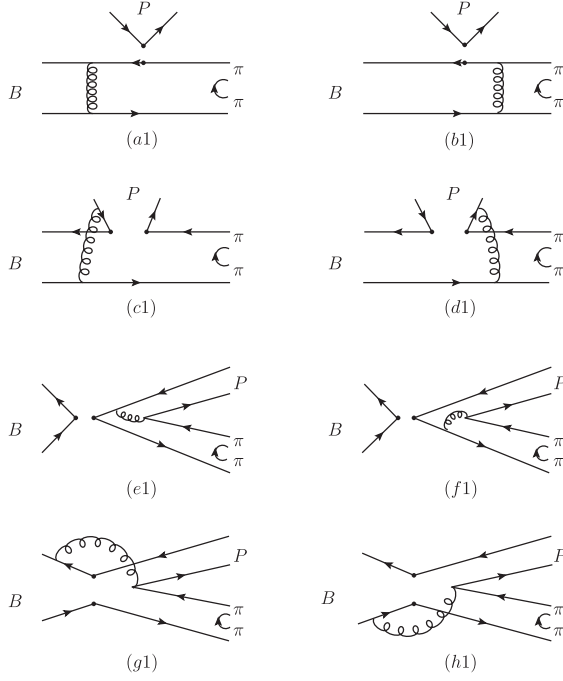
FIG. 1. Typical Feynman diagrams for the quasi-two-body decays $B \to P(f_2(1270) \to)\pi\pi$, where B stands for the $B^{\pm}$, B^0 or B_s meson and P denotes K, π, η or η'. With $\alpha = a - d$ and $\beta = e - h$, the diagrams $(\alpha 1)$ for the $B \to f_2(1270) \to \pi\pi$ transition and the diagrams $(\beta 1)$ for annihilation contributions.

the final state interactions between the pions in the resonant regions. The agreement of theoretical results with data can be achieved by determining the appropriate Gegenbauer moments of the D-wave two-pion distribution amplitudes. Just like the $\eta - \eta'$ mixing in the pseudoscalar case, the isoscalar tensor states $f_2(1270)$ and $f'_2(1525)$ also have a
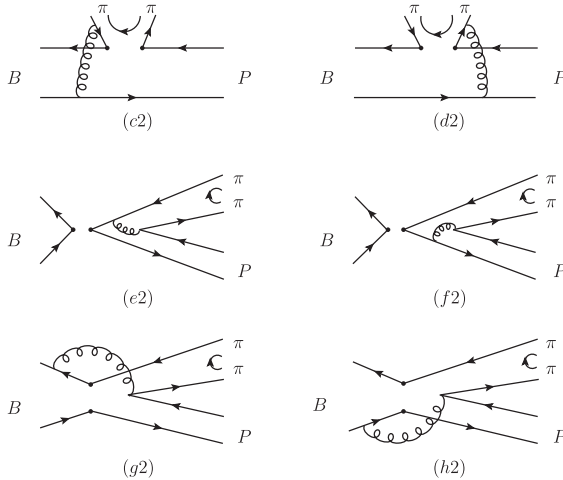

FIG. 2. Typical Feynman diagrams for the quasi-two-body decays $B \to P(f_2(1270) \to)\pi\pi$, where B stands for the $B^{\pm}$, B^0 or B_s meson and P denotes K, π, η or η'. With $\alpha = c - d$ and $\beta = e - h$, the diagrams $(\alpha 2)$ for the $B \to P$ transition and the diagrams $(\beta 2)$ for annihilation contributions.

similar mixing. The mixing angle between the $f_2(1270)$ and $f'_2(1525)$ is really small due to a fact that the $f_2(1270)(f'_2(1525))$ predominantly decays into $\pi\pi$ ($K\bar{K}$). In our paper, we focus on the resonances on $\pi\pi$ channel and leave the detailed discussion about mixtures of $f_2(1270) - f'_2(1525)$ for future studies associated with precise experimental measurements.

The present paper is organized as follows. In Sec. II, we give a brief introduction for the theoretical framework. The numerical values, some discussions and the conclusions will be given in last two sections. The explicit PQCD factorization formulas for all the decay amplitudes are collected in the Appendix.

II. FRAMEWORK

In the light-cone coordinates, the B meson momentum p_B, the total momentum of the pion pair, $p = p_1 + p_2$, the momentum p_3 of the final state meson P, and the momentum k_B of the spectator quark in the B meson, the momentum k for the resonant state f_2, k_3 for the final state P are in the form of

$$p_B = \frac{m_B}{\sqrt{2}}(1, 1, 0_{\mathrm{T}}), \qquad p = \frac{m_B}{\sqrt{2}}(1, \eta, 0_{\mathrm{T}}),$$

$$p_3 = \frac{m_B}{\sqrt{2}}(0, 1 - \eta, 0_{\mathrm{T}}),$$

$$k_B = \left(0, x_B \frac{m_B}{\sqrt{2}}, k_{B\mathrm{T}}\right), \qquad k = \left(z\frac{m_B}{\sqrt{2}}, 0, k_{\mathrm{T}}\right),$$

$$k_3 = \left(0, (1-\eta)x_3\frac{m_B}{\sqrt{2}}, k_{3\mathrm{T}}\right), \tag{2}$$

where m_B is the mass of B meson, the variable η is defined as $\eta = \omega^2/m_B^2$ with the invariant mass squared $\omega^2 = p^2 = (p_1 + p_2)^2$. The parameters x_B, z, x_3 are chosen as the momentum fraction of the positive quark in each meson and run from zero to unity. $k_{B\mathrm{T}}, k_{\mathrm{T}}$, and $k_{3\mathrm{T}}$ denote the transverse momentum of the positive quark, respectively. We define $\zeta = p_1^+/p^+$ as one of the pion pair's momentum fraction, the two pions momenta $p_{1,2}$ can be described as

$$p_1 = \left(\zeta\frac{m_B}{\sqrt{2}}, (1-\zeta)\eta\frac{m_B}{\sqrt{2}}, p_{1\mathrm{T}}\right),$$

$$p_2 = \left((1-\zeta)\frac{m_B}{\sqrt{2}}, \zeta\eta\frac{m_B}{\sqrt{2}}, p_{2\mathrm{T}}\right), \tag{3}$$

where $p_{1\mathrm{T}}^2 = p_{2\mathrm{T}}^2 = \zeta(1-\zeta)\omega^2$.

We here adopt the D-wave two-pion distribution amplitude similar as the one being used in Ref. [61],

$$\Phi_{\pi\pi}^{\mathrm{D}} = \frac{1}{\sqrt{2N_c}} \left[\not{p}\Phi_{v\nu=-}^{I=0}(z, \zeta, \omega^2) + \omega\Phi_s^{I=0}(z, \zeta, \omega^2) \right.$$
$$\left. + \frac{\not{p}_1\not{p}_2 - \not{p}_2\not{p}_1}{w(2\zeta - 1)}\Phi_{t\nu=+}^{I=0}(z, \zeta, \omega^2) \right]. \tag{4}$$

For $I = 0$, the distribution amplitude $\Phi_{v\nu=-}^{I=0}$ contributes at twist-2, $\Phi_s^{I=0}$ and $\Phi_{t\nu=+}^{I=0}$ contribute at twist-3. It is worthwhile to stress that this $\pi - \pi$ system has similar asymptotic distribution amplitudes as the ones for a tensor meson [75–77], but replacing the tensor decay constants with the timelike form factor:

$$\Phi_{v\nu=-}^{I=0} = \phi_0 = \frac{6F_\pi(s)}{2\sqrt{2N_c}} z(1-z)[3a_1^0(2z-1)]P_2(2\zeta-1),$$
(5)

$$\Phi_s^{I=0} = \phi_s = -\frac{9F_s(s)}{4\sqrt{2N_c}}[a_1^0(1-6z+6z^2)]P_2(2\zeta-1),$$
(6)

$$\Phi_{t\nu=+}^{I=0} = \phi_t = \frac{9F_t(s)}{4\sqrt{2N_c}}[a_1^0(1-6z+6z^2)(2z-1)]P_2(2\zeta-1),$$
(7)

where the Legendre polynomial $P_2(2\zeta-1) = 1-6\zeta(1-\zeta)$. The twist-3 distribution amplitudes should be fixed by the equations of motion [82,83] related to twist-2 ones. The moment a_1^0 is regarded as a free parameter and determined in this work.

Now, we focus on the dipion electromagnetic form factor. Taking the resonance contribution to the pion form factor into account, the pion electromagnetic form factor is defined in the standard way $\langle \pi^+(p_1)\pi^-(p_2)|j_\mu^{em}|0\rangle = (p_1 - p_2)_\mu F_\pi(s)$, where $s = (p_1 + p_2)^2$ is the timelike momentum transfer squared and $s \geq 4m_\pi^2$ [84]. The form factor $F_\pi(s)$ can be analytically continued to the spacelike region $s < 0$, corresponding to the hadronic matrix element $\langle \pi^+(p_1)|j_\mu^{em}|\pi^+(-p_2)\rangle$ by crossing-symmetry. Even so, the continuation from the timelike to spacelike region does not work well for the resonance J/ψ as shown in Ref. [84]. As well known, the electromagnetic form factor of pion at large (spacelike) momentum transfer on the basis of one-pion distribution amplitude has been computed in Ref. [85] with the PQCD approach at NLO. Applying the analytical continuation of the obtained result in the kinetic variable of momentum-transfer squared, one should be able to compute the timelike pion electromagnetic form factor directly without resorting to fitting the experimental measurements [86]. However, it is difficult to make the analytical continuation from the spacelike to timelike region for the dipion form factor. When we start from the spacelike region, it is not easy to identify the decay width piece, which can be interpreted as contribution of multihadron states to the imaginary part of the form factor in the resonant contribution [84,87]. In other word, it is impossible to generate the pole from the spacelike region by the analytical continuation.

Certainly, an alternative way to account for the hadronic resonance effect is that the electromagnetic form factor of pion at large (timelike) momentum transfer can be computed from perturbative QCD factorization approach at large momentum transfer with the parton-hadron duality ansatz. In order to account for the hadronic resonance effect, one can apply the hadronic dispersion relation for the electromagnetic form factor of the pion in the entire kinematic region and then implement the constraints from QCD calculation at large Q^2 for the determination of the unknown hadronic parameters entering the nonperturbative parametrization of the dispersion form. We will make efforts to calculate the electromagnetic form factor of pion at large (timelike) momentum transfer in the future. In this work, the relativistic BW formula for the D-wave resonance $f_2(1270)$ is adopted to parametrize the timelike form factor $F_\pi(s)$ and the explicit simplified expressions are in the following form,

$$F_\pi(s) = \frac{m_{f_2}^2}{m_{f_2}^2 - s - im_{f_2}\Gamma(s)},$$
(8)

$$\Gamma(s) = \Gamma_{f_2}\left(\frac{\sqrt{s-4m_\pi^2}}{\sqrt{m_{f_2}^2-4m_\pi^2}}\right)^5 \frac{m_{f_2}}{\sqrt{s}}\frac{X_{J=2}(\frac{1}{2}\sqrt{s-4m_\pi^2})}{X_{J=2}(\frac{1}{2}\sqrt{m_{f_2}^2-4m_\pi^2})},$$
(9)

$$X_{J=2}(z) = \frac{1}{(zr_{BW})^4 + 3(zr_{BW})^2 + 9},$$
(10)

with the two-pion invariant mass squared $s = \omega^2 = m^2(\pi\pi)$ and the $m_{f_2} = 1.276$ GeV and $\Gamma_{f_2} = 0.187$ GeV are the pole mass and width of resonance state $f_2(1270)$, respectively. We find that the variations of radius parameter $r_{BW} = 4$ [2] do not significantly change the values in our calculations. Hence, it is reasonable to set $r_{BW} = 0$ in our latter numerical calculations. The approximate relations $F_{s,t}(s) \approx (f_{f_2}^T/f_{f_2})F_\pi(s)$ [61] will also be used in the following section.

III. NUMERICAL RESULTS

For the numerical calculations, those parameters such as meson masses, the Wolfenstein parameters, decay constants, and the lifetimes of $B_{(s)}$ mesons are given in Table I, while the B meson and kaon (pion) distribution amplitudes are the same as widely adopted in the PQCD approach [62,88–90].

For the decays $B \to Pf_2(1270) \to P\pi\pi$, the differential branching ratio is written as [91],

$$\frac{d\mathcal{B}}{ds} = \tau_B \frac{|\vec{p}_\pi||\vec{p}_3|}{64\pi^3 m_B^3}|\mathcal{A}|^2.$$
(11)

TABLE I. The decay constants of $f_2(1270)$ meson is from [74], while other parameters are adopted in PDG [91] in our numerical calculations.

Masses (GeV)	$m_B = 5.280$	$m_{B_s} = 5.367$	$m_b = 4.66$	$m_{f_2(1270)} = 1.276$	$m_{\pi^\pm} = 0.140$
	$m_{\pi^0} = 0.135$	$m_{K^\pm} = 0.494$	$m_{K^0} = 0.498$	$m_\eta = 0.548$	$m_{\eta'} = 0.958$
The Wolfenstein parameters		$A = 0.811$	$\lambda = 0.22506$	$\bar\rho = 0.124$	$\bar\eta = 0.356$
Decay constants (MeV)	$f_B = 190.9 \pm 4.1$	$f_{B_s} = 227.2 \pm 3.4$		$f_{f_2(1270)} = 102 \pm 6$	$f^T_{f_2(1270)} = 117 \pm 25$
Lifetimes (ps)		$\tau_{B_s} = 1.51$		$\tau_{B_0} = 1.52$	$\tau_{B^+} = 1.638$

The kinematic variables $|\vec{p}_\pi|$ and $|\vec{p}_3|$ denote one of the pion pair's and P's momentum in the center-of-mass frame of the pion pair,

$$|\vec{p}_\pi| = \frac{1}{2}\sqrt{s - 4m_\pi^2},$$

$$|\vec{p}_3| = \frac{1}{2}\sqrt{[(m_B^2 - M_3^2)^2 - 2(m_B^2 + M_3^2)s + s^2]/s}. \quad (12)$$

By using the differential branching fraction in Eq. (11), and the decay amplitudes in the Appendix, we calculate the CP averaged branching ratios ($\mathcal{B}$) and direct CP-violating asymmetries ($\mathcal{A}_{CP}$) for the decays $B_{(s)} \to P(f_2 \to)\pi\pi$. In this work, we consider two scenarios for the tensor meson $f_2(1270)$. In scenario I, it is assumed that $f_2(1270)$ is a pure nonstrange isospin singlet state $(u\bar{u} + d\bar{d})/\sqrt{2}$, while in scenario II, the strange state $s\bar{s}$ enters the contributions with a nonvanishing mixing angle just like the $\eta - \eta'$ mixing in the pseudoscalar sector. Thus, the isoscalar tensor states can be written as

$$f_2(1270) = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\cos\theta_{f_2} + (s\bar{s})\sin\theta_{f_2},$$

$$f'_2(1525) = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\sin\theta_{f_2} - (s\bar{s})\cos\theta_{f_2}. \quad (13)$$

The detailed discussions about the mixing angle could be found in Refs. [92,93]. Here we employ the most recent updated value $(9 \pm 1)°$ from PDG2016 [91]. The predictions on the CP averaged branching ratios and direct CP-violating asymmetries from scenario I and II are enumerated distinctly in Tables II and III, as well as the current available data, respectively. The fit to the data [91] determines the Gegenbauer moment $a_1^0 = 0.40$, which differs from that in the distribution amplitudes for a longitudinally polarized $f_2(1270)$ meson [75–77].

In our numerical calculations, the first theoretical uncertainty results from the variations of the shape parameter $\omega_{B_{(s)}}$ of the $B_{(s)}$ meson distribution amplitude. We adopt the value $\omega_B = 0.40 \pm 0.04$ GeV or $\omega_{B_s} = 0.50 \pm 0.05$ GeV and vary its value with a 10% range, which is supported by intensive PQCD studies [68,94–96]. It is shown that the

TABLE II. CP averaged branching ratios of $B_{(s)} \to P(f_2(1270) \to)\pi^+\pi^-$ decays calculated in PQCD approach together with experimental data [91]. The theoretical errors correspond to the uncertainties due to the shape parameters $\omega_{B_{(s)}}$ in the wave function of $B_{(s)}$ meson, the Gegenbauer moment a_1^0 and the next-to-leading-order effects (the hard scale t and the QCD scale $\Lambda_{\rm QCD}$), respectively.

	Quasi-two-body $\mathcal{B}$ (in 10^{-7})		
Modes	Scenario I	Scenario II	Experiment[a]
$B^+ \to K^+(f_2(1270) \to)\pi^+\pi^-$	$11.09^{+1.60+6.23+2.82}_{-1.45-4.85-3.18}$	$12.77^{+1.80+7.18+3.22}_{-1.62-5.59-3.61}$	6.01 ± 1.52
$B^0 \to K^0(f_2(1270) \to)\pi^+\pi^-$	$8.81^{+1.35+4.95+2.48}_{-1.09-3.86-2.56}$	$10.30^{+1.54+5.79+2.85}_{-1.21-4.51-2.90}$	$15.16^{+7.30}_{-6.74}$
$B_s^0 \to \bar{K}^0(f_2(1270) \to)\pi^+\pi^-$	$0.37^{+0.04+0.20+0.06}_{-0.04-0.16-0.11}$	$0.42^{+0.05+0.24+0.10}_{-0.04-0.18-0.12}$	–
$B^+ \to \pi^+(f_2(1270) \to)\pi^+\pi^-$	$10.55^{+2.06+5.93+0.90}_{-1.70-4.62-0.89}$	$10.49^{+2.05+5.89+0.87}_{-1.70-4.59-0.88}$	$8.98^{+3.93}_{-2.25}$
$B^0 \to \pi^0(f_2(1270) \to)\pi^+\pi^-$	$0.30^{+0.03+0.17+0.03}_{-0.04-0.13-0.06}$	$0.33^{+0.04+0.18+0.03}_{-0.05-0.15-0.06}$	–
$B_s^0 \to \pi^0(f_2(1270) \to)\pi^+\pi^-$	$0.003^{+0.000+0.002+0.001}_{-0.001-0.001-0.001}$	$0.008^{+0.002+0.002+0.001}_{-0.001-0.003-0.001}$	–
$B^0 \to \eta(f_2(1270) \to)\pi^+\pi^-$	$0.52^{+0.08+0.29+0.07}_{-0.07-0.23-0.09}$	$0.52^{+0.08+0.29+0.06}_{-0.07-0.23-0.09}$	–
$B_s^0 \to \eta(f_2(1270) \to)\pi^+\pi^-$	$1.35^{+0.43+0.75+0.23}_{-0.33-0.59-0.46}$	$1.78^{+0.63+1.01+0.32}_{-0.45-0.78-0.50}$	–
$B^0 \to \eta'(f_2(1270) \to)\pi^+\pi^-$	$0.61^{+0.12+0.35+0.08}_{-0.09-0.27-0.11}$	$0.63^{+0.12+0.35+0.08}_{-0.10-0.28-0.13}$	–
$B_s^0 \to \eta'(f_2(1270) \to)\pi^+\pi^-$	$2.70^{+0.73+1.52+0.48}_{-0.58-1.18-0.88}$	$4.83^{+1.33+2.72+0.88}_{-1.03-2.11-1.33}$	–

[a]The experimental results are obtained by multiplying the relevant measured two-body branching ratios according to the Eq. (16).

TABLE III.   Direct CP-violating asymmetries of $B_{(s)} \to P(f_2(1270) \to)\pi^+\pi^-$ decays calculated in PQCD approach together with experimental data [91]. The theoretical errors correspond to the uncertainties due to the shape parameters $\omega_{B_{(s)}}$ in the wave function of $B_{(s)}$ meson and the next-to-leading-order effects (the hard scale t and the QCD scale Λ_{QCD}), respectively.

| Modes | $\mathcal{A}_{\mathrm{CP}}(\%)$ | | |
	Scenario I	Scenario II	Experiment
$B^+ \to K^+(f_2(1270) \to)\pi^+\pi^-$	$-48.2^{+1.3+12.4}_{-0.9-13.9}$	$-45.6^{+1.4+11.6}_{-0.8-12.3}$	-68^{+19}_{-17}
$B^0 \to K^0(f_2(1270) \to)\pi^+\pi^-$	$1.1^{+0.7+0.9}_{-0.6-0.0}$	$1.1^{+0.5+0.8}_{-0.4-0.1}$	$-$
$B^0_s \to \bar{K}^0(f_2(1270) \to)\pi^+\pi^-$	$-39.3^{+4.9+1.4}_{-5.1-2.9}$	$-37.0^{+4.5+0.8}_{-5.2-3.0}$	$-$
$B^+ \to \pi^+(f_2(1270) \to)\pi^+\pi^-$	$28.6^{+1.2+5.9}_{-3.1-4.4}$	$28.9^{+1.1+6.0}_{-3.2-4.7}$	41 ± 30
$B^0 \to \pi^0(f_2(1270) \to)\pi^+\pi^-$	$-19.7^{+2.1+17.8}_{-5.1-11.7}$	$-17.2^{+1.6+16.4}_{-4.5-12.3}$	$-$
$B^0_s \to \pi^0(f_2(1270) \to)\pi^+\pi^-$	$-0.2^{+0.0+25.9}_{-8.0-37.3}$	$-13.5^{+6.1+24.6}_{-0.0-0.7}$	$-$
$B^0 \to \eta(f_2(1270) \to)\pi^+\pi^-$	$-65.1^{+2.5+10.2}_{-0.1-20.2}$	$-64.4^{+1.5+10.5}_{-0.6-19.7}$	$-$
$B^0_s \to \eta(f_2(1270) \to)\pi^+\pi^-$	$-1.6^{+0.2+2.1}_{-0.3-1.5}$	$2.0^{+0.0+1.4}_{-0.9-0.8}$	$-$
$B^0 \to \eta'(f_2(1270) \to)\pi^+\pi^-$	$-28.3^{+2.6+0.2}_{-2.8-2.1}$	$-28.0^{+3.1+0.6}_{-2.3-1.2}$	$-$
$B^0_s \to \eta'(f_2(1270) \to)\pi^+\pi^-$	$2.1^{+0.3+1.9}_{-0.0-0.0}$	$4.6^{+1.0+0.7}_{-0.6-0.2}$	$-$

shape parameter ω_B can reach about 20% in magnitude for the main uncertainties. We note that another value $\omega_B(1 \text{ GeV}) = 0.354^{+0.038}_{-0.030}$ GeV implied by the light-cone sum rules calculations of the semileptonic $B \to \pi$ form factors with B-meson DAs on the light-cone [97] has been taken in the Refs. [98,99]. This number is very close to our error range, which result in the branching ratios changing 20 percents as mentioned above. Model-independent determinations of the inverse moment of the B-meson light-cone distribution amplitude ω_B in HQET have been discussed extensively from the radiative leptonic B-meson decays with the QCD factorization approach and the dispersion relations in Refs. [98,99]. The opportunity of constraining the inverse moment ω_B should be explored with the improvement of better measurements at the Belle II experiment in the near future. The second error comes from the Gegenbauer moment $a_1^0 = 0.40 \pm 0.10$. The last one is caused by the variation of the hard scale t from $0.75t$ to $1.25t$ (without changing $1/b_i$) and the QCD scale $\Lambda_{\mathrm{QCD}} = 0.25 \pm 0.05$ GeV, which characterizes the effect of the NLO QCD contributions. For the CP averaged branching ratios, the second error contributes the main uncertainties in our approach, while the other two errors are comparable and less than 30%. For the direct CP-violating asymmetries, the error from the Gegenbauer moment is largely cancelled between the numerator and denominator, and the main uncertainty refer to the hard scale. The uncertainties from $\tau_{B^\pm}$, τ_{B^0}, τ_{B_s}, and the Wolfenstein parameters in [91] are small and have been neglected. The significance of the radius parameter r_{BW} has been verified in our calculations. Taking the decay channel $B^+ \to K^+(f_2 \to)\pi^+\pi^-$ in scenario I as an example, the branching ratio $\mathcal{B} = 11.09 \times 10^{-7}$ and direct CP asymmetry $\mathcal{A}_{CP} = -48.2\%$ for $r_{BW} = 0$, while for $r_{BW} = 4.0$, the

corresponding values are $\mathcal{B} = 10.77 \times 10^{-7}$ and $\mathcal{A}_{CP} = -48.4\%$. One can find that the difference between the results with two choices is really small.

It is observed that the branching ratios from the two scenarios are comparable for the same B meson decay modes, while in the case of B_s decays, the two scenarios are rather different. For example, the value of $\mathcal{B}(B_s \to \eta'(f_2 \to)\pi^+\pi^-)$ in scenario II are nearly twice as that in scenario I. The main reason is that the factorizable contributions from the P emission diagrams [see Fig. 1(a1) and Fig. 1(b1)] in $s\bar{s}$ component enhance the corresponding decay amplitudes. However, for the B meson decays, such factorizable contributions come from the $n\bar{n}$ $(n = u, d)$ component, while the $s\bar{s}$ will contribute to the annihilation diagrams or nonfactorizable emission diagrams, which are power suppressed when compared with the factorizable emission diagrams according to the power counting rules in the factorization assumption.

It is shown that $f_2(1270)$ is primarily an $(u\bar{u} + d\bar{d})/\sqrt{2}$ state for the case of B meson decay modes, whereas for the B_s decays, the $s\bar{s}$ component makes a significant contributions albeit with suffering a large suppression from the mixing angle, especially for the process $B_s \to \eta'(f_2 \to)\pi^+\pi^-$. Therefore, we recommend the LHCb and/or Belle-II experiments to measure this mode to probe the precise structures of $f_2(1270)$. In addition, from Table II, it is found that $\mathcal{B}(B^0_{(s)} \to \eta(f_2 \to)\pi^+\pi^-) < \mathcal{B}(B^0_{(s)} \to \eta'(f_2 \to)\pi^+\pi^-)$. Since both η_q and η_s will contribute in these modes, but the relative sign of the η_q state with respect to the η_s state is negative for η and positive for η', which leads to destructive interference between η_q and η_s for the former, but constructive interference for the latter.

Combined with the Clebsch-Gordan Coefficients, we can describe the relation

$$|\pi\pi\rangle = \frac{1}{\sqrt{3}}|\pi^+\pi^-\rangle + \frac{1}{\sqrt{3}}|\pi^-\pi^+\rangle - \frac{1}{\sqrt{3}}|\pi^0\pi^0\rangle. \quad (14)$$

Isospin conservation is assumed for the strong decays of an $I = 0$ resonance f_2 to $\pi\pi$ when we compute the branching fraction of the quasi-two-body process $B \to Pf_2$, namely,

$$\frac{\Gamma(f_2 \to \pi^+\pi^-)}{\Gamma(f_2 \to \pi\pi)} = 2/3. \quad (15)$$

According to the relation of the decay rates between the quasi-two-body and the corresponding two-body decay modes

$$\mathcal{B}(B_{(s)} \to P(f_2 \to)\pi^+\pi^-)$$
$$= \mathcal{B}(B_{(s)} \to Pf_2) \cdot \mathcal{B}(f_2 \to \pi\pi) \cdot \frac{2}{3}, \quad (16)$$

with $\mathcal{B}(f_2 \to \pi\pi) = (84.2^{+2.9}_{-0.9})\%$ [91], we further obtain the PQCD predictions for $\mathcal{B}(B/B_s \to Pf_2)$ from the values as listed in the second column of Table II:

$$\mathcal{B}(B^+ \to K^+ f_2) = [19.76^{+12.51}_{-10.65}] \times 10^{-7},$$
$$\mathcal{B}(B^0 \to K^0 f_2) = [15.69^{+10.15}_{-8.48}] \times 10^{-7},$$
$$\mathcal{B}(B^0_s \to \bar{K}^0 f_2) = [0.66^{+0.37}_{-0.36}] \times 10^{-7},$$
$$\mathcal{B}(B^+ \to \pi^+ f_2) = [18.79^{+11.29}_{-8.91}] \times 10^{-7},$$
$$\mathcal{B}(B^0 \to \pi^0 f_2) = [0.53^{+0.32}_{-0.27}] \times 10^{-7},$$
$$\mathcal{B}(B^0_s \to \pi^0 f_2) = [0.005 \pm 0.004] \times 10^{-7},$$
$$\mathcal{B}(B^0 \to \eta f_2) = [0.93^{+0.55}_{-0.46}] \times 10^{-7},$$
$$\mathcal{B}(B^0_s \to \eta f_2) = [2.40^{+1.59}_{-1.46}] \times 10^{-7},$$
$$\mathcal{B}(B^0 \to \eta' f_2) = [1.09^{+0.68}_{-0.55}] \times 10^{-7},$$
$$\mathcal{B}(B^0_s \to \eta' f_2) = [4.81^{+3.12}_{-2.81}] \times 10^{-7}, \quad (17)$$

where the individual errors have been added in quadrature.

One can see that some channels have both large branching ratios and direct CP asymmetries, letting the corresponding measurement appear feasible. In fact, some physical observables (such as the CP averaged branching ratios and direct CP violations) of the two charged decay modes like $B^+ \to K^+(f_2 \to)\pi^+\pi^-$ and $B^+ \to \pi^+(f_2 \to)\pi^+\pi^-$ have been searched by BABAR and Belle Collaborations [2,4,8,9]. For example, BABAR Collaboration [2] reported a measurement, $\mathcal{B}(B^+ \to \pi^+(f_2 \to)\pi^+\pi^-) = (0.90^{+0.37}_{-0.24}) \times 10^{-6}$, which agrees with our calculations in both scenarios I and II. Furthermore, for the K mode, the measurements from BABAR [4] and Belle [8] are the following

$$\mathcal{B}(B^+ \to K^+(f_2 \to)\pi^+\pi^-)$$
$$= \begin{cases} (0.50^{+0.21}_{-0.19}) \times 10^{-6}, & BABAR, \\ (0.75^{+0.21}_{-0.25}) \times 10^{-6}, & \text{Belle.} \end{cases}$$

Their weighted average, enter the numbers given in Table II, are typically smaller than our prediction. None the less, taking the errors into consideration, the theoretical prediction and experimental data can still agree with each other. For the direct CP asymmetries, although the error bars from the data are still large, we are happy to see that all these measured entries have the same sign as our theoretical calculations [see Table III].

From Table II, one can see that the branching ratio of $B^+ \to K^+(f_2 \to)\pi^+\pi^-$ decay is a little larger than that of $B^0 \to K^0(f_2 \to)\pi^+\pi^-$ decay due to the extra contribution from the tree diagrams [see Eq. (A1) and (A2)] and the larger lifetime of the B^+ meson for the former decay mode. The similar situations also appear in the previous calculations from QCDF [75] and PQCD [77]. However, the data [91] shows that the latter decay mode has a relative large decay rate. It is worth of noting that this mode also has much larger relative errors because of limited statistics and the Dalitz-plot signal model dependence [5]. Such a difference should be clarified in the forthcoming experiments based on much larger data samples.

According to the full Dalitz-plot analysis to the $B^\pm \to \pi^\pm\pi^\pm\pi^\mp$ decay by the BABAR experiment [2], the dominant contributions come from the P-wave resonance $\rho(770)$ and nonresonant contributions. The relative rate between the contributions from the P-wave resonance $\rho(770)$ and D-wave resonance $f_2(1270)$ was measured to be

$$R_{\text{exp}} \equiv \frac{\mathcal{B}(B^\pm \to \pi^\pm(\rho^0(770) \to)\pi^+\pi^-)}{\mathcal{B}(B^\pm \to \pi^\pm(f_2(1270) \to)\pi^+\pi^-)} = 9.00^{+0.59}_{-1.48}. \quad (18)$$

For a more direct comparison with this available experimental data, we use our previous PQCD calculation of the P-wave resonance contribution $\mathcal{B}(B^+ \to \pi^+(\rho^0(770) \to)\pi^+\pi^-) = (8.84^{+1.91}_{-1.69}) \times 10^{-6}$, where all errors are combined in quadrature, as an input. Combined with the prediction on $\mathcal{B}(B^+ \to \pi^+(f_2(1270) \to)\pi^+\pi^-)$ in Table II, we obtain the ratio

$$R_{\text{PQCD}} = \frac{\mathcal{B}(B^+ \to \pi^+(\rho^0(770) \to)\pi^+\pi^-)}{\mathcal{B}(B^+ \to \pi^+(f_2(1270) \to)\pi^+\pi^-)} = 8.38^{+4.50}_{-2.02}, \quad (19)$$

which is consistent with above BABAR data quite well. For the similar ratio for the K counterpart, the calculated value is $3.64^{+2.91}_{-0.93}$, which is compatible with BABAR data $R_{\text{exp}} = 7.12^{+2.30}_{-1.08}$ [4] within about two standard deviations and Belle data $R_{\text{exp}} = 5.19^{+1.35}_{-0.47}$ [8] within about one standard
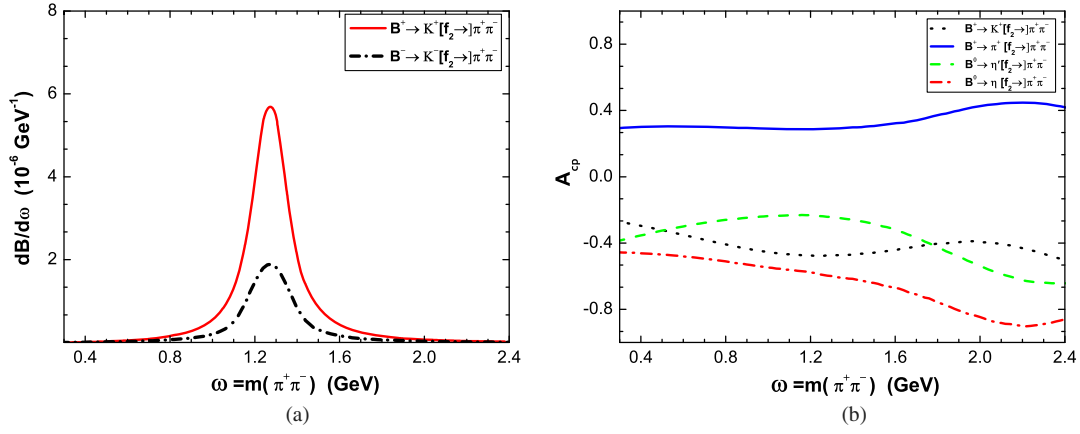
FIG. 3. (a) Differential branching ratios for the $B^{\pm} \to K^{\pm} f_2(1270) \to K^{\pm}\pi^+\pi^-$ decays. (b) Differential distribution of $\mathcal{A}_{cp}$ in ω for the decay modes $B^+ \to K^+[f_2 \to]\pi^+\pi^-$, $B^+ \to \pi^+[f_2 \to]\pi^+\pi^-$, $B^0 \to \eta'[f_2 \to]\pi^+\pi^-$ and $B^0 \to \eta[f_2 \to]\pi^+\pi^-$.

deviation. These results suggest that the PQCD factorization approach is suitable for describing the quasi-two-body B meson decays through analyzing various resonances by reconstructing $\pi\pi$ final states and reproducing the invariant mass spectra of Dalitz plots.

Different from the fixed kinematics of the two-body decays, the decay amplitudes of the quasi-two-body decays depends on the $\pi\pi$ invariant mass, which resulting in the differential distribution of branching ratios and direct CP asymmetries. In Fig. 3(a), we plot the differential branching ratios of the $B^{\pm} \to K^{\pm}f_2 \to K^{\pm}\pi^+\pi^-$ decays. One can see that the differential branching ratios of the $B^{\pm} \to K^{\pm}f_2 \to K^{\pm}\pi^+\pi^-$ decays exhibit peaks at the f_2 meson mass. Thus, the main portion of the branching ratios lies in the region around the pole mass of the $f_2(1270)$ resonance as expected. The central values of the branching ratio $\mathcal{B}$ are 6.03×10^{-7} and 8.51×10^{-7} when the integration over ω is limited in the range of $\omega = [m_{f_2} - 0.5\Gamma_{f_2}, m_{f_2} + 0.5\Gamma_{f_2}]$ or $\omega = [m_{f_2} - \Gamma_{f_2}, m_{f_2} + \Gamma_{f_2}]$ respectively, which amount to 54% and 77% of the total branching ratio $\mathcal{B} = 11.09 \times 10^{-7}$ as listed in Table II. In Fig. 3(b), we display the differential distributions of $\mathcal{A}_{CP}$ for the four decay modes $B^+ \to K^+[f_2 \to]\pi^+\pi^-$ (black dotted line), $B^+ \to \pi^+[f_2 \to]\pi^+\pi^-$ (blue solid line), $B^0 \to \eta'[f_2 \to]\pi^+\pi^-$ (green dashed line), and $B^0 \to \eta[f_2 \to]\pi^+\pi^-$ (red dash-dotted line), respectively. One can find a falloff of $\mathcal{A}_{CP}$ with ω for $B^+ \to K^+[f_2 \to]\pi^+\pi^-$, $B^0 \to \eta'[f_2 \to]\pi^+\pi^-$, and $B^0 \to \eta[f_2 \to]\pi^+\pi^-$. It implies that the direct CP asymmetries in the above three quasi-two-body decays, if calculated as the two-body decays with the f_2 resonance mass being fixed to m_{f_2}, may be overestimated. The ascent of the differential distribution of $\mathcal{A}_{CP}$ with ω for $B^+ \to \pi^+[f_2 \to]\pi^+\pi^-$ suggests that its direct CP asymmetry, if calculated in the two-body formalism, may be underestimated. In two-body B decays, the measured CP violation is just a number. But in three-body decays, one can measure the distribution of CP asymmetry in the Dalitz plot.

Hence, the Dalitz-plot analysis of $\mathcal{A}_{CP}$ distributions can reveal very rich information about CP violation. In the future, we will make more efforts to describe the distributions of CP asymmetries for various resonances in the Dalitz plot.

IV. CONCLUSION

In this work, we calculated the quasi-two-body decays $B_{(s)} \to Pf_2(1270) \to P\pi\pi$ with $P = (\pi, K, \eta, \eta')$ by utilizing the timelike form factor $F_\pi(s)$ within the PQCD factorization framework. The relativistic Breit-Wigner formula for the D-wave resonance $f_2(1270)$ was adopted to parametrize the timelike form factors F_π, which contains the final state interactions between the pions in the resonant regions. Using the determined Gegenbauer moments of the D-wave two-pion distribution amplitudes, we have predicted the branching ratios and the direct CP asymmetries of the $B_{(s)} \to Pf_2 \to P\pi\pi$ channels, and compared their differential branching ratios with currently available data. General agreements between the PQCD predictions and the data could be achieved, although there is no enough data at present. We have taken two scenarios of constituents of $f_2(1270)$ into consideration and found that the interference between $(u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ part can result in remarkable effects on some decay modes. The branching ratios of the corresponding two-body decays have been extracted from the quasi-two-body decay modes. More precise data from the LHCb and the future Belle II will test our predictions.

APPENDIX: DECAY AMPLITUDES

When the meson $f_2(1270)$ is treated as an pure $\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$ state, the total decay amplitude for each considered decay mode in this work are given as follows:

$$
\begin{aligned}
\mathcal{A}(B^+ \to K^+(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} &\Bigg\{ V_{ub}^* V_{us}\left[\left(\frac{C_1}{3} + C_2\right)(F_{ef_2}^{LL} + F_{af_2}^{LL}) + C_1(M_{ef_2}^{LL} + M_{af_2}^{LL}) + C_2 M_{eP}^{LL}\right] \\
&- V_{tb}^* V_{ts}\left[\left(\frac{C_3}{3} + C_4 + \frac{C_9}{3} + C_{10}\right)(F_{ef_2}^{LL} + F_{af_2}^{LL}) + \left(\frac{C_5}{3} + C_6 + \frac{C_7}{3} + C_8\right)(F_{ef_2}^{SP} + F_{af_2}^{SP})\right. \\
&+ (C_3 + C_9)(M_{ef_2}^{LL} + M_{af_2}^{LL}) + (C_5 + C_7)(M_{ef_2}^{LR} + M_{af_2}^{LR}) + \left(2C_4 + \frac{C_{10}}{2}\right)M_{eP}^{LL} \\
&\left.+ \left(2C_6 + \frac{C_8}{2}\right)M_{eP}^{SP}\right]\Bigg\},
\end{aligned} \tag{A1}
$$

$$
\begin{aligned}
\mathcal{A}(B^0 \to K^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} &\Bigg\{ V_{ub}^* V_{us}[C_2 M_{eP}^{LL}] - V_{tb}^* V_{ts}\left[\left(\frac{C_3}{3} + C_4 - \frac{1}{2}\left(\frac{C_9}{3} + C_{10}\right)\right)(F_{ef_2}^{LL} + F_{af_2}^{LL})\right. \\
&+ \left(\frac{C_5}{3} + C_6 - \frac{1}{2}\left(\frac{C_7}{3} + C_8\right)\right)(F_{ef_2}^{SP} + F_{af_2}^{SP}) + \left(C_3 - \frac{C_9}{2}\right)(M_{ef_2}^{LL} + M_{af_2}^{LL}) \\
&\left.+ \left(C_5 - \frac{C_7}{2}\right)(M_{ef_2}^{LR} + M_{af_2}^{LR}) + \left(2C_4 + \frac{C_{10}}{2}\right)M_{eP}^{LL} + \left(2C_6 + \frac{C_8}{2}\right)M_{eP}^{SP}\right]\Bigg\},
\end{aligned} \tag{A2}
$$

$$
\begin{aligned}
\mathcal{A}(B_s^0 \to \bar{K}^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} &\Bigg\{ V_{ub}^* V_{ud}[C_2 M_{eP}^{LL}] - V_{tb}^* V_{td}\left[\left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)M_{eP}^{LL} + \left(2C_6 + \frac{C_8}{2}\right)M_{eP}^{SP}\right. \\
&+ \left(C_5 - \frac{C_7}{2}\right)(M_{eP}^{LR} + M_{aP}^{LR}) + \left(\frac{C_3}{3} + C_4 - \frac{1}{2}\left(\frac{C_9}{3} + C_{10}\right)\right)F_{aP}^{LL} \\
&\left.+ \left(\frac{C_5}{3} + C_6 - \frac{1}{2}\left(\frac{C_7}{3} + C_8\right)\right)F_{aP}^{SP} + \left(C_3 - \frac{C_9}{2}\right)M_{aP}^{LL}\right]\Bigg\},
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
\mathcal{A}(B^+ \to \pi^+(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} &\Bigg\{ V_{ub}^* V_{ud}\left[\left(\frac{C_1}{3} + C_2\right)(F_{ef_2}^{LL} + F_{af_2}^{LL} + F_{aP}^{LL}) + C_1(M_{ef_2}^{LL} + M_{af_2}^{LL} + M_{aP}^{LL}) + C_2 M_{eP}^{LL}\right] \\
&- V_{tb}^* V_{td}\left[\left(\frac{C_3}{3} + C_4 + \frac{C_9}{3} + C_{10}\right)(F_{ef_2}^{LL} + F_{af_2}^{LL} + F_{aP}^{LL}) + \left(\frac{C_5}{3} + C_6 + \frac{C_7}{3} + C_8\right)(F_{ef_2}^{SP}\right. \\
&+ F_{af_2}^{SP} + F_{aP}^{SP}) + (C_3 + C_9)(M_{ef_2}^{LL} + M_{af_2}^{LL} + M_{aP}^{LL}) + (C_5 + C_7)(M_{ef_2}^{LR} + M_{af_2}^{LR} + M_{aP}^{LR}) \\
&\left.+ \left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)M_{eP}^{LL} + \left(C_5 - \frac{C_7}{2}\right)M_{eP}^{LR} + \left(2C_6 + \frac{C_8}{2}\right)M_{eP}^{SP}\right]\Bigg\},
\end{aligned} \tag{A4}
$$

$$
\begin{aligned}
\mathcal{A}(B^0 \to \pi^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}} &\Bigg\{ V_{ub}^* V_{ud}\left[\left(C_1 + \frac{C_2}{3}\right)(F_{ef_2}^{LL} + F_{af_2}^{LL} + F_{aP}^{LL}) + C_2(M_{ef_2}^{LL} + M_{af_2}^{LL} - M_{eP}^{LL} + M_{aP}^{LL})\right] \\
&- V_{tb}^* V_{td}\left[\left(-\frac{C_3}{3} - C_4 - \frac{3}{2}\left(C_7 + \frac{C_8}{3}\right) + \frac{5C_9}{3} + C_{10}\right)(F_{ef_2}^{LL} + F_{af_2}^{LL} + F_{aP}^{LL})\right. \\
&+ \left(-C_3 + \frac{C_9}{2} + \frac{3C_{10}}{2}\right)(M_{ef_2}^{LL} + M_{af_2}^{LL} + M_{aP}^{LL}) + \frac{3C_8}{2}(M_{ef_2}^{SP} + M_{af_2}^{SP} + M_{aP}^{SP}) \\
&- \left(\frac{C_5}{3} + C_6 - \frac{C_7}{6} - \frac{C_8}{2}\right)(F_{ef_2}^{SP} + F_{af_2}^{SP} + F_{aP}^{SP}) - \left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)M_{eP}^{LL} \\
&\left.- \left(C_5 - \frac{C_7}{2}\right)(M_{ef_2}^{LR} + M_{af_2}^{LR} + M_{eP}^{LR} + M_{aP}^{LR}) - \left(2C_6 + \frac{C_8}{2}\right)M_{eP}^{SP}\right]\Bigg\},
\end{aligned} \tag{A5}
$$

$$\mathcal{A}(B_s^0 \to \pi^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}} \left\{ V_{ub}^* V_{us} \left[\left(C_1 + \frac{C_2}{3}\right)(F_{af_2}^{LL} + F_{aP}^{LL}) + C_2(M_{af_2}^{LL} + M_{aP}^{LL}) \right] \right.$$
$$\left. - V_{tb}^* V_{ts} \left[\frac{3}{2} \left(C_9 + \frac{C_{10}}{3} - C_7 - \frac{C_8}{3}\right)(F_{af_2}^{LL} + F_{aP}^{LL}) + \frac{3C_{10}}{2}(M_{af_2}^{LL} + M_{aP}^{LL}) + \frac{3C_8}{2}(M_{af_2}^{SP} + M_{aP}^{SP}) \right] \right\},$$
$$(A6)$$

$$\mathcal{A}(B^0 \to \eta_q(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}} \left\{ V_{ub}^* V_{ud} \left[\left(C_1 + \frac{C_2}{3}\right)(F_{ef_2}^{LL} + F_{af_2}^{LL} + F_{aP}^{LL}) + C_2(M_{ef_2}^{LL} + M_{af_2}^{LL} + M_{eP}^{LL} + M_{aP}^{LL}) \right] \right.$$
$$- V_{tb}^* V_{td} \left[\left(\left(\frac{7C_3}{3} + \frac{5C_4}{3} - 2\left(C_5 + \frac{C_6}{3}\right) - \frac{1}{2}\left(C_7 + \frac{C_8}{3} - \frac{2}{3}(C_9 - C_{10})\right)\right)(F_{ef_2}^{LL} + F_{af_2}^{LL} + F_{aP}^{LL}) \right.$$
$$+ \left(\frac{C_5}{3} + C_6 - \frac{1}{2}\left(\frac{C_7}{3} + C_8\right)\right)(F_{ef_2}^{SP} + F_{af_2}^{SP} + F_{aP}^{SP})$$
$$+ \left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)(M_{ef_2}^{LL} + M_{af_2}^{LR} + M_{eP}^{LL} + M_{aP}^{LL})$$
$$\left. + \left(C_5 - \frac{C_7}{2}\right)(M_{ef_2}^{LR} + M_{af_2}^{LR} + M_{eP}^{LR} + M_{aP}^{LR}) + \left(2C_6 + \frac{C_8}{2}\right)(M_{ef_2}^{SP} + M_{af_2}^{SP} + M_{eP}^{SP} + M_{aP}^{SP}) \right] \right\},$$
$$(A7)$$

$$\mathcal{A}(B^0 \to \eta_s(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} \left\{ -V_{tb}^* V_{td} \left[\left(C_3 + \frac{C_4}{3} - C_5 - \frac{C_6}{3} + \frac{1}{2}\left(C_7 + \frac{C_8}{3} - C_9 - \frac{C_{10}}{3}\right)\right) \right] F_{ef_2}^{LL} \right.$$
$$\left. + \left(C_4 - \frac{C_{10}}{2}\right)M_{ef_2}^{LL} + \left(C_6 - \frac{C_8}{2}\right)M_{ef_2}^{SP} \right] \right\},$$
$$(A8)$$

$$\mathcal{A}(B^0 \to \eta(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B^0 \to f_2\eta_q)\cos\phi - \mathcal{A}(B^0 \to f_2\eta_s)\sin\phi, \qquad (A9)$$

$$\mathcal{A}(B^0 \to \eta'(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B^0 \to f_2\eta_q)\sin\phi + \mathcal{A}(B^0 \to f_2\eta_s)\cos\phi, \qquad (A10)$$

$$\mathcal{A}(B_s^0 \to \eta_q(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}} \left\{ V_{ub}^* V_{us} \left[\left(C_1 + \frac{C_2}{3}\right)(F_{af_2}^{LL} + F_{aP}^{LL}) + C_2(M_{af_2}^{LL} + M_{aP}^{LL}) \right] \right.$$
$$- V_{tb}^* V_{ts} \left[\left(2C_3 + \frac{2C_4}{3} - 2C_5 - \frac{2C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6} + \frac{C_9}{2} + \frac{C_{10}}{6}\right)(F_{af_2}^{LL} + F_{aP}^{LL}) \right.$$
$$\left. + \left(2C_4 + \frac{C_{10}}{2}\right)(M_{af_2}^{LL} + M_{aP}^{LL}) + \left(2C_6 + \frac{C_8}{2}\right)(M_{af_2}^{SP} + M_{aP}^{SP}) \right] \right\},$$
$$(A11)$$

$$\mathcal{A}(B_s^0 \to \eta_s(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} \left\{ V_{ub}^* V_{us}[C_2 M_{eP}^{LL}] - V_{tb}^* V_{ts} \left[\left(2C_4 + \frac{C_{10}}{2}\right)M_{eP}^{LL} + \left(2C_6 + \frac{C_8}{2}\right)M_{eP}^{SP} \right] \right\}, \qquad (A12)$$

$$\mathcal{A}(B_s^0 \to \eta(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B_s^0 \to f_2\eta_q)\cos\phi - \mathcal{A}(B_s^0 \to f_2\eta_s)\sin\phi, \qquad (A13)$$

$$\mathcal{A}(B_s^0 \to \eta'(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B_s^0 \to f_2\eta_q)\sin\phi + \mathcal{A}(B_s^0 \to f_2\eta_s)\cos\phi, \qquad (A14)$$

On the other hand, the meson $f_2(1270)$ is more like an pure $s\bar{s}$ state, the total decay amplitude for each considered decay mode can be written as:

$$\mathcal{A}(B^+ \to K^+(f_2 \to)\pi^+\pi^-) = \frac{G_F}{\sqrt{2}} \left\{ V_{ub}^* V_{us} \left[\left(\frac{C_1}{3} + C_2 \right) F_{aP}^{LL} + C_1 M_{aP}^{LL} \right] - V_{tb}^* V_{ts} \left[\left(C_3 + C_4 - \frac{1}{2}(C_9 + C_{10}) \right) M_{eP}^{LL} \right. \right.$$
$$+ \left(C_5 - \frac{C_7}{2} \right) M_{eP}^{LR} + \left(C_6 - \frac{C_8}{2} \right) M_{eP}^{SP} + \left(\frac{C_3}{3} + C_4 + \frac{C_9}{3} + C_{10} \right) F_{aP}^{LL}$$
$$\left. \left. + \left(\frac{C_5}{3} + C_6 + \frac{C_7}{3} + C_8 \right) F_{aP}^{SP} + (C_3 + C_9) M_{aP}^{LL} + (C_5 + C_7) M_{aP}^{LR} \right] \right\}, \tag{A15}$$

$$\mathcal{A}(B^0 \to K^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{\sqrt{2}} \left\{ -V_{tb}^* V_{ts} \left[\left(C_3 + C_4 - \frac{1}{2}(C_9 + C_{10}) \right) M_{eP}^{LL} + \left(C_5 - \frac{C_7}{2} \right) M_{eP}^{LR} \right. \right.$$
$$+ \left(C_6 - \frac{C_8}{2} \right) M_{eP}^{SP} + \left(\frac{C_3}{3} + C_4 - \frac{1}{2} \left(\frac{C_9}{3} + C_{10} \right) \right) F_{aP}^{LL}$$
$$\left. \left. + \left(\frac{C_5}{3} + C_6 - \frac{1}{2} \left(\frac{C_7}{3} + C_8 \right) \right) F_{aP}^{SP} + \left(C_3 - \frac{C_9}{2} \right) M_{aP}^{LL} + \left(C_5 - \frac{C_7}{2} \right) M_{aP}^{LR} \right] \right\}, \tag{A16}$$

$$\mathcal{A}(B_s^0 \to \bar{K}^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{\sqrt{2}} \left\{ -V_{tb}^* V_{td} \left[\left(\frac{C_3}{3} + C_4 - \frac{1}{2} \left(\frac{C_9}{3} + C_{10} \right) \right) (F_{ef_2}^{LL} + F_{af_2}^{LL}) \right. \right.$$
$$+ \left(\frac{C_5}{3} + C_6 - \frac{1}{2} \left(\frac{C_7}{3} + C_8 \right) \right) (F_{ef_2}^{SP} + F_{af_2}^{SP}) + \left(C_3 - \frac{C_9}{2} \right) (M_{ef_2}^{LL} + M_{af_2}^{LL})$$
$$\left. \left. + \left(C_5 - \frac{C_7}{2} \right) (M_{ef_2}^{LR} + M_{af_2}^{LR}) + \left(C_4 - \frac{C_{10}}{2} \right) M_{eP}^{LL} + \left(C_6 - \frac{C_8}{2} \right) M_{eP}^{SP} \right] \right\}, \tag{A17}$$

$$\mathcal{A}(B^+ \to \pi^+(f_2 \to)\pi^+\pi^-) = \frac{G_F}{\sqrt{2}} \left\{ -V_{tb}^* V_{td} \left[\left(C_4 - \frac{C_{10}}{2} \right) M_{eP}^{LL} + \left(C_6 - \frac{C_8}{2} \right) M_{eP}^{SP} \right] \right\}, \tag{A18}$$

$$\mathcal{A}(B^0 \to \pi^0(f_2 \to)\pi^+\pi^-) = -\frac{G_F}{2} \left\{ -V_{tb}^* V_{td} \left[\left(C_4 - \frac{C_{10}}{2} \right) M_{eP}^{LL} + \left(C_6 - \frac{C_8}{2} \right) M_{eP}^{SP} \right] \right\}, \tag{A19}$$

$$\mathcal{A}(B_s^0 \to \pi^0(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} \left\{ V_{ub}^* V_{us} \left[\left(C_1 + \frac{C_2}{3} \right) F_{ef_2}^{LL} + C_2 M_{ef_2}^{LL} \right] \right.$$
$$\left. - V_{tb}^* V_{ts} \left[\frac{3}{2} \left(C_9 + \frac{C_{10}}{3} - C_7 - \frac{C_8}{3} \right) F_{ef_2}^{LL} + \frac{3C_{10}}{2} M_{ef_2}^{LL} + \frac{3C_8}{2} M_{ef_2}^{SP} \right] \right\}, \tag{A20}$$

$$\mathcal{A}(B^0 \to \eta_q(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} \left\{ -V_{tb}^* V_{td} \left[\left(C_4 - \frac{C_{10}}{2} \right) M_{eP}^{LL} + \left(C_6 - \frac{C_8}{2} \right) M_{eP}^{SP} \right] \right\}, \tag{A21}$$

$$\mathcal{A}(B^0 \to \eta_s(f_2 \to)\pi^+\pi^-) = \frac{G_F}{\sqrt{2}} \left\{ -V_{tb}^* V_{td} \left[\left(C_3 + \frac{C_4}{3} - C_5 - \frac{C_6}{3} + \frac{1}{2} \left(C_7 + \frac{C_8}{3} - C_9 - \frac{C_{10}}{3} \right) \right) (F_{af_2}^{LL} + F_{aP}^{LL}) \right. \right.$$
$$\left. \left. + \left(C_4 - \frac{C_{10}}{2} \right) (M_{af_2}^{LL} + M_{aP}^{LL}) + \left(C_6 - \frac{C_8}{2} \right) (M_{af_2}^{SP} + M_{aP}^{SP}) \right] \right\}, \tag{A22}$$

$$\mathcal{A}(B^0 \to \eta(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B^0 \to f_2\eta_q) \cos\phi - \mathcal{A}(B^0 \to f_2\eta_s) \sin\phi, \tag{A23}$$

$$\mathcal{A}(B^0 \to \eta'(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B^0 \to f_2\eta_q) \sin\phi + \mathcal{A}(B^0 \to f_2\eta_s) \cos\phi, \tag{A24}$$

$$\mathcal{A}(B_s^0 \to \eta_q(f_2 \to)\pi^+\pi^-) = \frac{G_F}{2} \left\{ V_{ub}^* V_{us} \left[\left(C_1 + \frac{C_2}{3} \right) F_{ef_2}^{LL} + C_2 M_{ef_2}^{LL} \right] - V_{tb}^* V_{ts} \left[\left(2C_3 + \frac{2C_4}{3} - 2C_5 - \frac{2C_6}{3} \right. \right. \right.$$
$$\left. \left. \left. - \frac{C_7}{2} - \frac{C_8}{6} + \frac{C_9}{2} + \frac{C_{10}}{6} \right) F_{ef_2}^{LL} + \left(2C_4 + \frac{C_{10}}{2} \right) M_{ef_2}^{LL} + \left(2C_6 + \frac{C_8}{2} \right) M_{ef_2}^{SP} \right] \right\}, \tag{A25}$$

$$\mathcal{A}(B_s^0 \to \eta_s(f_2 \to)\pi^+\pi^-) = \frac{G_F}{\sqrt{2}}\left\{-V_{tb}^*V_{ts}\left[\left(\frac{4}{3}\left(C_3+C_4-\frac{C_9}{2}-\frac{C_{10}}{2}\right)-C_5-\frac{C_6}{3}+\frac{C_7}{2}+\frac{C_8}{6}\right)(F_{ef_2}^{LL}+F_{af_2}^{LL}+F_{aP}^{LL})\right.\right.$$

$$+\left(\frac{C_5}{3}+C_6-\frac{C_7}{6}-\frac{C_8}{2}\right)(F_{ef_2}^{SP}+F_{af_2}^{SP}+F_{aP}^{SP})+\left(C_3+C_4-\frac{1}{2}(C_9+C_{10})\right)(M_{ef_2}^{LL}+M_{af_2}^{LL}$$

$$+M_{eP}^{LL}+M_{aP}^{LL})+\left(C_5-\frac{C_7}{2}\right)(M_{ef_2}^{LR}+M_{af_2}^{LR}+M_{eP}^{LR}+M_{aP}^{LR})$$

$$\left.\left.+\left(C_6-\frac{C_8}{2}\right)(M_{ef_2}^{SP}+M_{af_2}^{SP}+M_{eP}^{SP}+M_{aP}^{SP})\right]\right\}, \tag{A26}$$

$$\mathcal{A}(B_s^0 \to \eta(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B_s^0 \to f_2\eta_q)\cos\phi - \mathcal{A}(B_s^0 \to f_2\eta_s)\sin\phi, \tag{A27}$$

$$\mathcal{A}(B_s^0 \to \eta'(f_2 \to)\pi^+\pi^-) = \mathcal{A}(B_s^0 \to f_2\eta_q)\sin\phi + \mathcal{A}(B_s^0 \to f_2\eta_s)\cos\phi, \tag{A28}$$

where G_F is the Fermi coupling constant. V_{ij}'s are the Cabibbo-Kobayashi-Maskawa matrix elements. The functions $(F_{ef_2}^{LL}, F_{af_2}^{LL}, M_{ef_2}^{LL}, M_{af_2}^{LL}, \cdots)$ appeared in above equations are the individual decay amplitudes corresponding to different currents, and their explicit expressions can be found in the Appendix of Ref. [61].

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