B^0-$\bar{B}^0$ mixing: Matching to HQET at NNLO

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We compute perturbative matching coefficients to the heavy quark effective theory (HQET) representation for the QCD effective local $\Delta B = 2$ Hamiltonian that determines the mass difference in the B^0-$\bar{B}^0$ system of states. We report on the results at next-to-next-to-leading order in the strong coupling constant for matching coefficients of two physical operators in HQET. Our results provide firm confirmation that the recent next-to-leading order sum rules analysis of the bag parameter B_q is stable with regard of inclusion of higher order radiative corrections. As a byproduct of our calculation we give a fully analytical solution for the one-loop QCD-to-HQET matching problem: we present the explicit formulas for the renormalization of four-quark operators of the full bases in both QCD and HQET and the expressions for matching coefficients in a closed form.

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I. INTRODUCTION

The accuracy of theoretical predictions for the neutral meson mixing within the standard model (SM) has been steadily improving over the last years [1–3]. The main reason for this progress is due to better numerical precision achieved for the numerical values for hadronic matrix elements in the lattice simulations (e.g., [4,5]). Nevertheless, recently some new results with a competitive level of accuracy appeared in the domain of analytical computation based on the sum rules approach [6,7]. The latter calculation has become possible due to the technical advance of integral computation at three loops [8]. At present, the high precision of experimental material provides for good opportunities for searches of physics beyond the SM [9], and these new theoretical predictions for the mixing within the SM are important since the search for new physics depends heavily on the accurate knowledge of numerical values of low energy parameters [10,11]. The mixing of the neutral B mesons is dominated by the top-quark contribution and hence looks directly to the ultraviolet (UV) new physics having long distance effects under control. The concept of effective theories allows for getting better precision for the theoretical predictions due to separation of scales [12–14] that essentially improves on old electroweak results [15]. Clearly, the most crucial point of such an improvement is perturbation theory (PT) corrections at the lowest scales involved in the analysis. This motivates our research—we compute next-to-next-to-leading order (NNLO) corrections to the matching of QCD to heavy quark effective theory (HQET) that is applicable at the scales of the order of few $\Lambda_{\rm QCD}$. Some partial results have already been published [16].

The paper is organized as follows. In the next section we briefly introduce notation just for the paper to be self-contained (the details can be found in [16]) and give our main results. In Sec. III we describe the technique of the computation. In Sec. IV we briefly discuss implications of our calculation for phenomenology. In the summary section we give our conclusions.

II. OPERATORS AND WILSON COEFFICIENTS

In the SM the B^0-$\bar{B}^0$ transition is described by a nonlocal expression of quark scattering at loop level. The most important corrections to the leading electroweak term are the contributions of strong interactions. They can be computed within QCD perturbatively as the relevant scale is of the order of the meson mass m_B and is much larger than QCD infrared scale $\Lambda_{\rm QCD}$.

In this section we repeat some formulas from [16] to introduce notation. More details are given in [16].

A. QCD: Below m_W

A relevant scale for the description of B^0-$\bar{B}^0$ physics is around the B-meson mass m_B that is kinematically saturated by the b-quark mass m_b. After integrating out particles of the SM that are heavier than b quark the effective Hamiltonian for the process of B^0-$\bar{B}^0$ mixing reads

$$\mathcal{H} = C(m_W, m_t, \mu, \alpha_s(\mu))Q(\mu), \tag{2.1}$$

with

$$Q(\mu) = \bar{q}_L \gamma_\alpha b \bar{q}_L \gamma_\alpha b(\mu). \tag{2.2}$$

The quantity $Q(\mu)$ is a local renormalized operator with $\Delta B = 2$. One can choose a low normalization point $\mu \sim m_b$ for the operator Q while large logarithms of the scales ratio, $\ln(m_W/m_b)$, in the Wilson coefficient $C(m_W, m_t, \mu, \alpha_s(\mu))$ can be regularly resummed with renormalization group techniques. At present the coefficient $C(m_W, m_t, \mu, \alpha_s(\mu))$ is known at the next-to-leading order (NLO) of the expansion in the strong coupling constant that gives the accuracy of a few percent [12–14]. The renormalization properties of the operator Q are a bit peculiar but well understood and intensively discussed in the literature. The point is that the calculations are usually and dominantly performed in dimensional regularization and one has to close up the algebra of Dirac gamma matrices, which is infinitely dimensional in general d-dimensional space. This is an outstanding problem and has been much discussed in the literature [17]. Dimensional reduction has been used for the computation of corrections to the $\Delta F = 1$ Hamiltonian [18]. A naive dimensional regularization prescription is introduced in [19]. As a result of using dimensional regularization for the integral evaluation one requires a special treatment of Dirac structure and, eventually, the extension of the operator basis. For the four-quark operator in question, i.e., $Q(\mu)$ in Eq. (2.2), the procedure was discussed by Buras *et al.* [20]. The method has been further analyzed in [21,22]. Note that the anomalous dimensions of baryon or three quark operators have been computed earlier within a similar approach [23,24]. A clear presentation of the techniques is given in [25].

Thus, the effective Hamiltonian in Eq. (2.1) should, in fact, contain additional evanescent operators and should explicitly read as

$$\mathcal{H} = C(m_W, m_t, \mu, \alpha_s(\mu))Q(\mu) + C_E E, \tag{2.3}$$

where E is a general notation for a string of evanescent operators in QCD. Their matrix elements vanish but their presence in the basis influences the renormalization pattern and, therefore, evolution of physical operators $Q(\mu)$. The renormalized operator $Q(\mu)$ depends on the choice of the evanescent ones. By choosing $\mathrm{E}' = \mathrm{E} + a\epsilon\mathrm{Q}$ one obtains a different renormalized physical operator $Q'(\mu)$. At the one-loop level one gets the relation between the two,

$$Q'(\mu) = \left[1 - a z_{QE} \frac{\alpha_s(\mu)}{4\pi}\right] Q(\mu). \tag{2.4}$$

Physical predictions stay independent of the choice of evanescent operators. We discuss how it is achieved later in the text.

B. HQET: Below m_b

Since the scale m_b is still QCD perturbative, $m_b \gg \Lambda_{\mathrm{QCD}}$, one can go lower in scales and remove the explicit dependence on m_b from the matrix element or the mixing amplitude at low energy. This is achieved by using HQET [26–28].

The low scale for the operators involved in the mixing used to be necessary for the lattice computation; however at present there is sufficient power for lattice simulations directly at the scale m_b. In a computational framework within analytical methods one matches the theory of QCD on to HQET where considerable technical simplifications occur for subsequent computation of sum rules. Thus the matching is an unavoidable part of the whole computation. Though there is an approach based on calculation at m_b [29]. The results need an update as the definition of the operator has been different.

The heavy quark expansion (HQE) of the operator $Q(\mu)$ goes

$$Q(\mu) = 2\sum_{i=1}^{2} C_i(\mu)\tilde{O}_i(\mu) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right), \tag{2.5}$$

with $\tilde{O}_1(\mu) = O_l(\mu)$, $\tilde{O}_2(\mu) = O_s(\mu)$. The HQET operators $O_{l,s}(\mu)$ are defined as

$$O_l = (\bar{q}_L \gamma_\mu h_+)(\bar{q}_L \gamma_\mu h_-), \quad O_s = (\bar{q}_L h_+)(\bar{q}_L h_-). \tag{2.6}$$

The field h_+ annihilates the heavy quark in HQET (moving with the four velocity v), and h_- creates the heavy antiquark (again moving with the four velocity v), which is a completely separate particle in the HQET framework.

In HQET one can define its own set of physical and evanescent operators (see [16]). The general basis is

$$O_n = (\bar{q}\gamma_\perp^n h_+)(\bar{q}\gamma_\perp^n h_-), \qquad O'_n = (\bar{q}_i \gamma_\perp^n h_+^j)(\bar{q}_j \gamma_\perp^n h_-^i), \tag{2.7}$$

and q is a light fermion that is usually a chiral one, $q \equiv q_L$. A choice for a basis in HQET is an antisymmetrized product of transverse gamma's,

$$\gamma_\perp^\mu = \gamma^\mu - v^\mu \not{v}. \tag{2.8}$$

Then $\gamma_\perp^n$ is a notation for an antisymmetrized product of n transverse gamma matrices. We sometimes call the number n a rank of the product and, therefore, of the corresponding operator.

For our two-loop calculation it is more convenient to change the operator basis in the physical sector from the standard $\{O_l, O_s\}$ [16] to $\{O_l, O_p\}$ with

$$O_p = O_s + \frac{1}{4} O_l. \tag{2.9}$$

The operators $\{O_p, O_l\}$ do not mix under renormalization at the one-loop level [30]. The expression of operators $\{O_l, O_p\}$ through the basis operators $\{O_n, O'_n\}$ is

$$O_l = O_1 - O_0, \qquad O_p = \frac{3}{4} O_0 + \frac{1}{4} O_1. \tag{2.10}$$

The matching pattern of the QCD operator $Q(\mu)$ at $\mu = m_b$ then reads

$$Q(m_b) = 2\{C_l(m_b) O_l(m_b) + C_p(m_b) O_p(m_b) + \cdots\}, \tag{2.11}$$

where dots denote the contribution of evanescent operators.

We define a PT expansion of the matching coefficients $C_{l,p}(m_b)$ as

$$\begin{aligned} C_l(m_b) &= 1 + \frac{\alpha_s(m_b)}{4\pi} C_l^{(1)} + \left(\frac{\alpha_s(m_b)}{4\pi}\right)^2 C_l^{(2)}, \\ C_p(m_b) &= \frac{\alpha_s(m_b)}{4\pi} C_p^{(1)} + \left(\frac{\alpha_s(m_b)}{4\pi}\right)^2 C_p^{(2)}. \end{aligned} \tag{2.12}$$

Both $\{O_l, O_s\}$ and $\{O_l, O_p\}$ bases are convenient since at the leading order (LO) there is a single operator O_l in the matching relation and the other operator (O_s or O_p) first appears at NLO.

In the HQET limit the theory has n_l massless flavors and the coupling constant in Eq. (2.12) is defined accordingly as $\alpha_s^{(n_l)}$.

At NLO the values of the matching coefficients in $\{O_l, O_p\}$ basis are [30–32]

$$\begin{aligned} C_l^{(1)}(m_b) &= -\frac{(N-1)(7N+15)}{2N}, \\ C_p^{(1)}(m_b) &= -2(N+1), \end{aligned} \tag{2.13}$$

where N is a number of colors for the $SU(N)$ gauge group. Note the difference with the $\{O_l, O_s\}$ basis that is traditionally used at NLO in the literature,

$$C_l^{(1)}(\text{ls-basis}) = \frac{-8N^2 - 9N + 15}{2N}. \tag{2.14}$$

In the present paper we have computed the NNLO contributions to the coefficients $\{C_l, C_p\}$, which is the main result of the paper.

The coefficient $C_p^{(2)}(m_b)$ reads

$$\begin{aligned} C_p^{(2)} = (N+1)\Bigg[&\frac{38}{9} n_l - \frac{8}{3} I_0 - \frac{2}{9} \frac{4N^2 + 9N - 29}{N} \pi^2 \\ &- \frac{686N^3 - 563N^2 + 1599N + 18}{36N^2}\Bigg]. \end{aligned} \tag{2.15}$$

Here I_0 is one of the master integrals of the computation that reads

$$I_0 = \pi^2 \log(2) - \frac{3}{2}\zeta(3) = 5.038\ldots$$

Note that the PT expansion in HQET goes over the n_l-flavored coupling constant since there are just n_l flavors in the low energy theory. In QCD we have in addition an active heavy quark b and, therefore, $n_l + 1$ flavors.

For the number of colors $N = 3$ and the number of massless flavors $n_l = 4$, the numerical values of the expansion coefficients are

$$C_p \sim \{0, -8, -311.166\}.$$

One sees that the convergence of PT series for quantity $C_p(m_b)$ in the renormalized coupling constant $\alpha_s(m_b)$ with $n_l = 4$ is marginal.

The coefficient $C_l^{(2)}$ is

$$\begin{aligned} C_l^{(2)} = (N-1)\Bigg[&\left(\frac{N+3}{3N}\pi^2 + \frac{123N + 211}{24N}\right) n_l \\ &- 2\frac{N^2 + N + 1}{N^2} I_0 - 2(N-1)\frac{N^2 + 2N + 2}{N^2}\zeta(3) \\ &- \frac{43N^3 + 111N^2 - 111N - 275}{48N^2}\pi^2 \\ &- \frac{13518N^3 + 8456N^2 - 7981N + 35037}{576N^2}\Bigg]. \end{aligned} \tag{2.16}$$

For $N = 3$ and $n_l = 4$ the numerical values of the coefficients of the consecutive powers of the coupling constant are

$$C_l \sim \{1, -12, -175.559\ldots\}.$$

The coefficient of the n_l structure is different from the one given in our early paper [16] for two reasons. First, the mixing of operators $\{O_l, O_s\}$ has not been accounted for, and second, the expansion of $\Gamma(\epsilon)$ at NLO has not been

done to a necessary order in ε [up to $O(\epsilon)$, in fact] that has caused a finite shift proportional to π^2 in the NNLO coefficient. The operators $\{O_l, O_p\}$ do not mix with each other at the α_s order.

The results (2.15) and (2.16) can be rewritten at $N = 3$, $n_l = 4$ as

$$C_p^{(2)} = -25.333\beta_0 - 100.055 = -211.111 - 100.055 = -311.166,$$
$$C_l^{(2)} = -43.906\beta_0 + 190.323 = -365.882 + 190.323 = -175.560.$$

So, the naive non-Abelianization [33] works moderately well: the β_0 terms are about two times larger than those without β_0.

The results for the matching coefficients correspond to the normalization point $m_b \equiv m_b^{\rm pole}$, $\mu = m_b^{\rm pole}$. One sees that the NNLO corrections are rather large. Their calculation is crucial for estimating the reliability of the results at NLO in PT. We discuss some physical implications of the obtained results later.

A note on subtraction scheme dependence is in order here. The actual form of the matching coefficients depends on the renormalization scheme used in the calculations. Generally, the $\overline{\rm MS}$ subtraction scheme is standard for dimensionally regularized integrals. The operators in Eqs. (2.1) and (2.5) are defined within a minimal subtraction scheme in QCD and HQET correspondingly, and the coefficients in Eqs. (2.1) and (2.5) are obtained in $\overline{\rm MS}$-scheme as well. While the whole mixing amplitude in the SM does not depend on any scheme used for constructing the operator product expansion (OPE) the coefficients in Eqs. (2.1) and (2.5) do depend simply because the corresponding operators are defined in a particular subtraction scheme. This is a standard situation within the OPE analysis as one expands a physical amplitude over the UV subtracted operators that are chosen conventionally with the coefficients that correspond to a particular definition of the operators. Only a product of the coefficients and the matrix elements of the operators has a unique physical meaning and is scheme independent. However, in our case there is an additional dependence/freedom related to the extension of the basis of four-quark operators and introduction of evanescent operators. It is a new feature compared to just the change of UV subtraction procedure. The renormalization of four-quark operators in dimensional regularization has been discussed at length in the original papers on the subject [20–23]. We use the standard basis of four-quark operators in QCD with evanescent operators as defined in [20]. As for HQET, our extended basis with evanescent operators has been introduced in Ref. [16]. We follow the choice of Ref. [16] in the present paper. The scheme dependence, i.e., the sensitivity to both a standard UV subtraction and the choice of evanescent operators, disappears after one computes properly the matrix elements (ME) of the operators used in the OPE. The values of the matrix elements depend on the definition of the operators and this freedom cancels the dependence on the scheme in the matching coefficients. In Ref. [7] there has been given an explicit example of such a cancellation with MEs evaluated within the sum rules technique. In case the MEs are computed on the lattice one must supply the transition coefficients (matching) to the lattice operators that again cancel the dependence on the basis; an explicit discussion can be found in Ref. [31]. Thus, theoretically the dependence on the operator basis is completely under control. Our matching coefficients are given in the well-defined fixed basis and can be used with MEs of the paper [7].

III. DESCRIPTION OF COMPUTATION

QCD operators can be expanded in $1/m$ in terms of HQET operators,

$$Q(\mu) = C(\mu)O(\mu) + \frac{1}{m}B(\mu)P(\mu) + \mathcal{O}\left(\frac{1}{m^2}\right), \quad (3.1)$$

where $Q(\mu)$ is the column of renormalized QCD operators, $O(\mu)$ is the column of renormalized HQET operators, and $C(\mu)$ is the matrix of matching coefficients. For example, the leading matching coefficients C for heavy-light quark currents were calculated in [33–36].

Here we follow the same procedure but for the four-quark operators. First we calculate the bare matching coefficients

$$Q_0 = C_0 O_0, \qquad Q_0 = Z(\mu)Q(\mu), \qquad O_0 = \tilde{Z}(\mu)O(\mu),$$
$$C(\mu) = Z^{-1}(\mu)C_0\tilde{Z}(\mu), \quad (3.2)$$

where $Z(\mu)$ and $\tilde{Z}(\mu)$ are the matrices of renormalization constants in QCD and HQET (we omit $1/m$ corrections). This is done by equating the on-shell matrix element of Q_0,

$$\langle \bar{b}d|Q_0|b\bar{d}\rangle = Z_Q^{\rm os} Z_q^{\rm os} \Gamma_0, \quad (3.3)$$

where $Z_Q^{\rm os}$ is the on-shell heavy-quark field renormalization constant, $Z_q^{\rm os}$ is the on-shell light-quark renormalization constant, and Γ_0 is the vertex function of Q_0, to C_0 times the corresponding on-shell matrix element of O_0. If all light quarks (including the charmed quark c) are considered massless, all loop corrections to the HQET quantities $\tilde{Z}_Q^{\rm os}$, $\tilde{Z}_q^{\rm os}$, $\tilde{\Gamma}_0$ vanish in dimensional regularization because they contain no scale.

The two-loop results for $Z_Q^{\rm os}$ [37] and $Z_q^{\rm os}$ [35] are known. We calculate Γ_0 up to two loops using the REDUCE package RECURSOR [38], similarly to [33]. The basis of antisymmetrized products of $\gamma_\perp$ allows us to project onto individual

HQET operators easily. The calculation is done in an arbitrary covariant gauge; we check that the on-shell matrix element (3.3) is gauge invariant ($Z_Q^{\rm os}$ and $Z_q^{\rm os}$ are gauge invariant up to two loops). If we keep only the (gauge-invariant) subset of diagrams in which only b quark and the light quark belonging to the same color-singlet current interact, we successfully reproduce the two-loop matching coefficients [33,35] of the heavy-light currents with the Dirac structures 1, $\not{v}$, $\gamma_\perp$.

The quantities Γ_0, $Z_Q^{\rm os}$, $Z_q^{\rm os}$ are calculated via the n_f-flavor bare coupling $g_0^{(n_f)}$; we reexpress them via the renormalized $\alpha_s^{(n_f)}(\mu)$. The renormalization constant matrix $Z(\mu)$ is also expressed via $\alpha_s^{(n_f)}(\mu)$. However, the renormalization constant matrix $\tilde{Z}(\mu)$ is expressed via $\alpha_s^{(n_l)}(\mu)$, $n_l = n_f - 1$. We have to use the decoupling relation between $\alpha_s^{(n_f)}(\mu)$ and $\alpha_s^{(n_l)}(\mu)$; see, e.g., an introductory review [39]. It is most convenient to perform matching at $\mu = m$, the on-shell mass of the b quark. Then

$$\alpha_s^{(n_f)}(m) = \alpha_s^{(n_l)}(m)\left[1 + T_F\frac{\pi^2}{9}\varepsilon\frac{\alpha_s^{(n_l)}(m)}{4\pi}\right] \tag{3.4}$$

with $T_F = T(R) = 1/2$ for fermions in fundamental representation. We need this $\mathcal{O}(\varepsilon)$ term because the one-loop QCD matrix element contains poles in $1/\varepsilon$.

The computation has been done in leading logs in [40,41] where LO anomalous dimension for O_l has been found. It happens to be equal to that of the simple product of two heavy-light currents. In higher orders this factorization does not necessarily persist. The standard result for the coefficients $C_{l,p}$ at NLO has been obtained in [30–32].

In our paper [16] we have calculated the NNLO contribution of the leading order in n_l only. This allows one to estimate the full results for the two-loop matching coefficients using the approximate method of naive non-Abelianization [33]. Presently available techniques allow for the analytical calculation of any number of fermionic bubbles (e.g., chapter 8 in [28], also available as [42]) that can be converted into the β_0-dominance estimates of the matching coefficients at any order of perturbative expansion. While the estimate is technically feasible, the quantitative validity of the approximation for a phenomenological analysis is not immediately clear (e.g., see some discussion in [43]).

General description of matching calculations given above is well known. In our case of matching four-quark operators of QCD onto four-quark operators in HQET there is a subtlety of using dimensional regularization caused by the presence of spurious operators that formally vanish in four-dimensional space. Therefore, one organizes the basis of operators in both QCD and HQET in physical and evanescent sectors. In QCD the operators are symbolically $Q = \{Q, E\}$ (do not confuse the concrete operator Q and general notation for the whole set in QCD). In HQET the operators are $O = \{O_l, O_p, e_i\}$. The matching is by necessity a relation between whole basis sets of the operators in QCD and in HQET and it reads in general matrix form

$$Q = CO. \tag{3.5}$$

Here CO denotes a matrix multiplication of the matrix of matching coefficients C and the string (one-dimensional matrix) of operators of the basis.

The renormalization pattern of the operators in Eq. (3.5) (including evanescent ones) is

$$Q = Z^{-1}Q^B, \qquad O = \tilde{Z}^{-1}O^B, \tag{3.6}$$

where $Z, \tilde{Z}$ are the renormalization matrices and Q^B, O^B are bare images of renormalized operators in QCD and HQET. One obtains for the matching in Eq. (3.5)

$$Q^B = ZC\tilde{Z}^{-1}O^B. \tag{3.7}$$

The matching coefficients can be found by taking the matrix elements on shell in HQET from both sides of this relation. And for HQET one has

$$\langle O^B\rangle = \text{tree level values only} \tag{3.8}$$

because all loops are scaleless and vanish in dimensional regularization. This is independent of whether the operator is a physical one or evanescent one. Let $O|_j$ be an operation of projecting on a particular state P_j (operator) and one chooses a complete system of P_j for the HQET basis such that $O_i^B|_j = \delta_{ij}$. For the basis of four-quark operators this operation has a simple realization: one takes traces in both Dirac strings. Since the bare operator of the basis has a structure of a direct product $O_n^B = A_n \otimes B_n$, symbolically,

$$O_i^B|_j = \mathrm{tr}(\gamma_\perp^{(j)}A_i)\mathrm{tr}(B_i\gamma_\perp^{(j)}) \sim \delta_{ij}. \tag{3.9}$$

The matching coefficients C become

$$C_{mn} = (Z^{-1})_{np}(Q_p^B|_j)\tilde{Z}_{jm}. \tag{3.10}$$

This is a working formula. The quantity $(Q_p^B|_j)$ is basically a bare matching coefficient of the bare operator Q_p^B from the QCD basis to the bare operator O_j^B in a HQET basis. The bare matching coefficient depends on one scale m_b and is represented by loop integrals on shell. At NNLO they are two-loop integrals.

At one-loop level the renormalization matrix in QCD reads

$$Z = 1 + \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} z_{QQ} & z_{QE} \\ \varepsilon z_{EQ} & z_{EE}\end{pmatrix}, \tag{3.11}$$

where z_{EQ} is obtained from the requirement that matrix elements of the renormalized evanescent operator $E(\mu)$ vanish, $z_{EQ} = 3(1-1/N)(3N+14-22/N)$. Here $z_{QQ} = -3(1-1/N)$ is related to an anomalous dimension of the physical operator Q. It is independent of details of the basis. The quantity $z_{QE} = -T_F$ describes the admixture of the evanescent operator E to the physical one. The quantity z_{EE} is the anomalous dimension of evanescent operators. Note that there is more than one evanescent operator in the basis; however for our computation we need only one independent combination of those, and z_{EE} is irrelevant altogether. At the two-loop level we need only one additional entry to the renormalization constant Z, which is $z^{(2)}_{QQ}$,

$$Z = 1 + \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} z_{QQ} & z_{QE} \\ \varepsilon z_{EQ} & z_{EE} \end{pmatrix} + \left(\frac{\alpha_s}{4\pi\varepsilon}\right)^2 \begin{pmatrix} z^{(2)}_{QQ} & * \\ * & * \end{pmatrix}. \tag{3.12}$$

The value of $z^{(2)}_{QQ}$ is reconstructed from a two-loop anomalous dimension of the physical operator Q. The anomalous dimension of the operator $Q(\mu)$ is

$$-\frac{1}{2}\gamma_Q = z_{QQ}\frac{\alpha_s}{4\pi} + \left(\frac{\alpha_s}{4\pi}\right)^2 \times \left\{\frac{1}{\epsilon}\left(\beta_0 z_{QQ} + 2z^{(2)}_{QQ} - z^2_{QQ}\right) - z_{QE}z_{EQ}\right\}, \tag{3.13}$$

with

$$\beta_0 = \frac{11}{3}N - \frac{2}{3}n_f, \tag{3.14}$$

and $n_f = n_l + 1$ is the number of flavors. For the perturbative expansion of the anomalous dimension of the form

$$\gamma(\alpha_s) = \frac{\alpha_s}{4\pi}\gamma^0 + \left(\frac{\alpha_s}{4\pi}\right)^2\gamma^1, \tag{3.15}$$

the coefficient γ^1 in the basis described reads [20]

$$\gamma^1 = \frac{N-1}{2N}\left(-21 + \frac{57}{N} - \frac{19}{3}N + \frac{4}{3}n_f\right). \tag{3.16}$$

In QCD one needs only one evanescent operator for the whole computation of matching in two loops, just that one that admixes to Q at NLO even if it can be composed of several basis operators Q_n.

In HQET the renormalization matrix for relevant operators looks similar to that one in QCD,

$$\tilde{Z} = 1 + \frac{\alpha_s}{4\pi\varepsilon}\begin{pmatrix} z_{OO} & z_{Oe} \\ \varepsilon z_{eO} & z_{ee} \end{pmatrix} + \left(\frac{\alpha_s}{4\pi\varepsilon}\right)^2 \begin{pmatrix} z^{(2)}_{OO} & * \\ * & * \end{pmatrix}, \tag{3.17}$$

but now the quantity z_{OO} is a 2×2 matrix in the subspace of physical operators $\{O_l, O_p\}$.

In HQET two evanescent operators are necessary for renormalization of physical operators, one for O_l and one for O_p. For computing the matching coefficients the whole set of evanescent operators is relevant or at least the one that Q can match onto at NLO (in fact, there is one operator that is multiplied by the poles of the matching coefficient and the other with only finite parts). It shows that, indeed, in general the whole operator basis should be matched onto.

As for bare coefficients we need two-loop values for $C(Q \to O_l, O_p)$ and one-loop values for $C(Q \to j)$ for any operator O_l, O_p, e_i and $C(E \to l, p)$. One more ingredient is Z_Q on shell at NNLO from [37]. The calculation contains rather delicate cancellations of infinities (poles in ϵ) and fixing the finite parts according to sophisticated conventions. To give just an example, the renormalization matrix Z is expanded, by convention, over the renormalized coupling $\alpha_s^{(n_l+1)}(\mu)$ while the renormalization matrix $\tilde{Z}$ is expanded, by convention, over the renormalized coupling $\alpha_s^{(n_l)}(\mu)$. In the course of computation of matching coefficients the poles in ϵ cancel.

We have computed the necessary quantities. The quantity $z^{(2)}_{QQ}$ can be extracted from earlier calculations through anomalous dimensions of Q at two loops. The matrix $z^{(2)}_{OO}$ is related to the anomalous dimension matrix of the physical pair (l, p). One of the entries has been considered in [44] where the result for the anomalous dimension of O_l has been given. The whole basis of operators was not explicitly specified in [44]. We do not compute the corresponding anomalous dimension independently. It can be extracted from our calculations though. In fact, one can extract only the difference $\gamma^1 - \tilde{\gamma}^1$ that reads

$$\tilde{\gamma}^1 - \gamma^1 = \left(1 - \frac{1}{N}\right)\left(n_l\left(3 + \frac{5}{3}N\right) + \pi^2\frac{2}{3N}(N-1)(2+2N+N^2) - \frac{177 + 161N - 3N^2 + 83N^3}{12N}\right).$$

Assuming the value for γ_1 from Eq. (3.16) with $n_f = n_l + 1$ we have extracted the anomalous dimension $\tilde{\gamma}_1$ of the HQET operator O_l,

$$\tilde{\gamma}^{(1)} = \left(1 - \frac{1}{N}\right)\left(-\frac{-165 + 279N + 35N^2 + 83N^3}{12N} + n_l\frac{11+5N}{3} + \pi^2\frac{2(N-1)(2+2N+N^2)}{3N}\right).$$

The coefficient n_l agrees with our paper [16]. The whole expression disagrees with the results quoted in Ref. [44]. Numerically we find

$$\tilde{\gamma}^{(1)} = \frac{2}{27}(-807 + 78n_l - 68\pi^2) = -86.3802\ldots$$

while the result of Ref. [44] reads

$$\tilde{\gamma}^{(1)}|_{\mathrm{G}} = \frac{2}{27}(-1212 + 96n_l - 26\pi^2) = -80.3415\ldots$$

for $n_l = 4$ and QCD with $N = 3$. The irony of life is that the two quantities $\tilde{\gamma}^{(1)}$ and $\tilde{\gamma}^{(1)}|_{\mathrm{G}}$ are rather close numerically though the analytical expressions for them are quite different.

As for the calculation at one-loop level the whole program of renormalization of bases of four-quark operators in QCD and HQET and matching onto one another has been explicitly realized in a closed form. The corresponding formulas are given in the Appendix.

IV. IMPLICATIONS FOR PHENOMENOLOGY OF THE MIXING

The splitting between the mass eigenstates of the B^0-$\bar{B}^0$ system is determined, in the SM, by the nondiagonal matrix element of the effective Hamiltonian (2.1),

$$\Delta m = \langle B^0|\mathcal{H}|\bar{B}^0\rangle = C(m_W, m_t, \mu, \alpha_s(\mu))\langle B^0|Q(\mu)|\bar{B}^0\rangle, \tag{4.1}$$

or, more specifically, if the concrete calculations are made within dimensional regularization, by the expression (2.3). After the coefficient functions have been determined in QCD perturbation theory the task is the computation of hadronic matrix elements of four-quark operators, Q, E. While for evanescent operators E this task can be trivially accomplished since by construction $\langle B^0|E(\mu)|\bar{B}^0\rangle = 0$, an accurate determination of numerical values for physical operators is a genuine QCD-bound-states problem. The nonperturbative techniques required for the computation can be QCD sum rules and direct lattice simulation. The sum rules results for the matrix element $\langle B^0|Q(m_b)|\bar{B}^0\rangle$ have been presented earlier [29,45–47] within different approximation schemes for the Green functions used in the OPE. The obtained precision has been rather limited by modern standards though. Until recently the lattice analysis with fully relativistic heavy quarks was impossible due to insufficient computational power; therefore the matching to HQET (2.5) has been a necessity. The NLO results for matching coefficients to both HQET and to the lattice representation for the operators were obtained more than a quarter of a century ago.

The technical breakthrough with computation of three-loop HQET integrals in Ref. [8] allowed for a NLO analysis of the mixing matrix element using sum rules in HQET. In our recent paper [6] we have computed the bag parameter B_d for B_d^0-$\bar{B}_d^0$ mixing at the next-to-leading order of perturbative expansion for matching coefficient and for the Green functions entering the sum rules analysis. To evaluate the matrix element of the mixing we use a vertex (three-point) correlation function [48]. The analysis uses the splitting of the whole Green function necessary for the calculation within OPE and for the sum rule approach [46–48] into factorizable and nonfactorizable parts. It happens that the nonfactorizable part starts only at NLO of perturbation theory and turns out to be small. These features allow for getting a numerical result of high precision for the bag parameter. The techniques have also been used for other four-quark operators in [7].[1] The computation of Ref. [7] is pinning down the important uncertainty for lifetime differences of the heavy mesons with both b and c quarks. It is rather a complete analysis but it is limited to only NLO of the perturbation theory for HQET Green functions. Future experimental data may require even more accurate theoretical predictions. For obtaining still better theoretical accuracy, the NNLO perturbative corrections to matching coefficients can be useful. The first step in this direction has been made in our recent paper [16] where the NNLO corrections proportional to the n_l factor have been computed that allowed us to perform an approximate evaluation of the coefficients within the naive non-Abelianization (β_0-dominance) approach [33]. In the present paper we have computed the full NNLO results for matching coefficients. They read numerically,

$$C_l(m_b) = 1 - 12a_s - 175.6a_s^2, \tag{4.2}$$

$$C_p(m_b) = -8a_s - 311.2a_s^2, \tag{4.3}$$

where $a_s = \alpha_s^{(4)}(m_b)/(4\pi)$. Our calculation of matching coefficients is an important step in the program of NNLO description of mixing within analytical methods of computation. From our results we see that NLO approximation for matching coefficient may not be reliable since the NNLO corrections are large. On the other hand, we also see that the values of corrections to the coefficients by themselves do not lead to immediate physical conclusions. One has to add corrections to corresponding Green functions, which determine the OPE for sum rule analysis. Thus, the NLO correction to C_l requires NLO correction to the correlator

$$K = \int d^dx_1 d^dx_2 e^{ip_1x_1 - ip_2x_2}\langle 0|T\tilde{j}_2(x_2)O_l(0)\tilde{j}_1(x_1)|0\rangle \tag{4.4}$$

of the operator O_l for consistent NLO analysis (see Ref. [16] for more details). In the case of the O_p operator the

[1]Note a minor misrepresentation of the results in [7]: the right-hand side of Eq. (3.17) should include the factor $NC_F/4$ for general values of N. The results are correct in QCD for $N = 3$ since $NC_F/4 = 1$. A. Lenz has confirmed this finding in his private communication to us.

corresponding correlator of the O_p operator can be taken at tree-level approximation for obtaining the NLO result for the QCD matrix element. With the NNLO correction included to C_p, one needs the NLO correction to the correlators with insertion of O_p; this turns out to be feasible [8] and it is even available for some cases [7]. In case of the NNLO corrections to C_l one has to compute the NNLO corrections to the correlator (4.4) as well that leads to four-loop integrals, which are currently beyond known technology.

Note that at NNLO there appear new evanescent operators that we do not specify explicitly. However, one can try to choose the basis of evanescent operators such that $C_l^{(2)}$ becomes smaller or even vanishing (that changes the numerical value of $C_p^{(2)}$ accordingly) and one can hope to obtain a chance to avoid the necessity of the computation of NNLO corrections to the correlator (4.4).

The other possibility to get the most of our NNLO calculation of matching coefficients even without having the NNLO results for the correlator (4.4) is to perform a direct comparison between physical quantities. While most observables in QCD contain nonperturbative contributions, there are, in fact, some observables for which one can show that one may construct a perturbative relation between the two observables. In such a case the perturbative expansion acquires an immediate physical meaning and statements about the size of coefficients as well as on the convergence of the perturbative series become meaningful. To this end, while for the individual observables large coefficients may appear (depending on the definition of the operator matrix elements), in a relation between these observables the large coefficients my cancel, once the matrix elements are defined in the same way in both observables.

As we pointed out above, the bag factor B_B turns out to be small and hence to a good approximation we expect a perturbative relation between ΔM_d and the B-meson decay constant f_B. In fact, it is known that the matching coefficients of the axial current are also large [33], and we may expect that in a direct comparison of the axial vector current with ΔM_d a well-behaved perturbation theory results.

The matching coefficient of the axial current to HQET interpolating operator is [33,35]

$$J = C_J \tilde{j}, \qquad J = v_\mu J_5^\mu = \bar{q}\not{v}\gamma_5 b, \qquad \tilde{j} = \bar{q}\gamma_5 h_v,$$

$$\begin{aligned} C_J(m_b) &= 1 - 2C_F a_s + C_F\left[C_F\left(4I_0 - 8\zeta_3 - \frac{5}{2}\pi^2 + \frac{255}{16}\right)\right. \\ &\quad + C_A\left(-2I_0 + 2\zeta_3 + \frac{5}{6}\pi^2 - \frac{871}{48}\right) \\ &\quad \left. + T_F n_l\left(\frac{2}{3}\pi^2 + \frac{47}{12}\right) + T_F\left(-4\pi^2 + \frac{727}{18}\right)\right] a_s^2 \\ &= 1 - \frac{8}{3}a_s - 31.6 a_s^2 \end{aligned} \tag{4.5}$$

with a rather sizable coefficient at NNLO.

The HQET matching (2.5) has a great deal of arbitrariness in distributing the contributions between coefficients and operators. One concrete choice of fixing the $\overline{\text{MS}}$-scheme for the definition of the operators is a current standard. If we call $Q(m_b)$ our physical quantity determining δm (again up to the freedom of redefinition the matrix elements and the coefficients in QCD but we set this aside now), then the expansion in HQET is not unique in a variety of aspects: choosing a physical pair of the operators, changing renormalization scheme, etc. Also clearly, the freedom of the definition of evanescent operators is not only the choice of a physical pair but deviation from minimality. By adding a physical operator to an evanescent one with a coefficient vanishing at $\epsilon = 0$ we obtain different renormalized physical operators; see (2.4) and therefore a different matching coefficient.

While physics, i.e., predictions for observables, does not depend on this reshuffling of the operator bases, the independence restores only after adding matrix elements computed within the same scheme up to the same accuracy. The magnitude of the correction to a particular coefficient is not an invariant characteristic of PT expansion and one should really collect Wilson coefficients and matrix elements together, which is difficult in QCD since there is no quantitative scheme for computing hadronic matrix elements. Lattice simulation may be an exception, but then the perturbative short distance part of the matrix element cannot be easily identified.

The full NNLO analysis of the mixing seems to be feasible, and the most intriguing part is the possibility to find a basis of evanescent operators in which the NNLO correction to the matching coefficient of the operator O_l turns out to be small. This assertion deserves to be validated in future work.

Note also that the two-loop anomalous dimensions of the operators $\{O_l, O_p\}$ are not very important quantitatively. The point is that the difference of scales is not large and summation of logarithms of the scale differences is not crucial numerically. Indeed, matching is done at m_b with $m_b^{pole} \sim 4.8$ GeV (e.g., [49,50]) while the Green functions necessary for sum rules analysis are computed at the scale of the order of $w_0 \sim 1$ GeV. The leading logs of the form $(\alpha_s(w_0)\ln(w_0/m_b))^n$, with $n > 0$ can be summed with the leading anomalous dimensions of the operators $\{O_l, O_p\}$ which are known while the subleading logs of the form $\alpha_s(w_0)(\alpha_s(w_0)\ln(w_0/m_b))^n$ with $n \geq 1$ are not large and can be retained in an expanded form.

Here we check that our results for matching coefficients allow for a reasonable perturbative expansion of physical parameters. Recall that the matrix element in QCD is represented by the expression

$$\langle B^0|Q(\mu)|\bar{B}^0\rangle = 2\left(1 + \frac{1}{N}\right) B_q(\mu) f_B^2 \tag{4.6}$$

and therefore $B_q(\mu)$ gives a relation between physical parameters Δm and f_B^2 [we factor out the redefinition freedom of the operator $Q(\mu)$ in QCD considering it fully under control in perturbation theory]. The perturbation theory series for the relation between Δm and f_B^2 is unambiguous and can depend only on N_c and α_s (with all possible reservations that they are not completely PT quantities). The realization of this idea is given in Eq. (2.18) of Ref. [16] in the form

$$C_J^2 B_q(m_b) = C_l(m_b) B_l + C_p \left(\frac{(-2N+1)}{2(N+1)} B_s + \frac{1}{4} B_l \right) \tag{4.7}$$

or

$$B_q(m_b) = \left\{ C_l(m_b) B_l + C_p \left(-\frac{5}{8} B_s + \frac{1}{4} B_l \right) \right\} / C_J^2. \tag{4.8}$$

Here $B_{l,s}$ are bag parameters for the HQET operators $O_{l,s}$ (see Ref. [16] for more detail). The perturbative corrections to the parameter B_l have the form [6]

$$B_l \to B_l \left(1 - a_s \frac{N-1}{2N} \left(\frac{4}{3}\pi^2 - 5 \right) - x_l^{(2)} a_s^2 \right), \tag{4.9}$$

where $x_l^{(2)}$ represents a yet-unknown NNLO correction to B_l from HQET sum rules (compare to the parameter X used in [6] for estimating higher order contributions). We have extracted the correction to the parameter B_s from Ref. [7] with the result

$$B_s \to B_s \left(1 - a_s \frac{8}{15} \left(9 - \frac{2\pi^2}{3} \right) \right). \tag{4.10}$$

Substituting our results for the coefficients C_l, C_p and the expression for C_J, we finally obtain the relation between physical observables in the form

$$\Delta m = \text{const}(1 - 6.4 a_s - (4.9 + x_l^{(2)}) a_s^2) f_B^2. \tag{4.11}$$

The quantity $x_l^{(2)}$ emerges from NNLO corrections to HQET sum rules and it is expected to be in the range of NLO correction, which is just of order unity (compare assumptions on the parameter X in [6], $|X| < 20$). The perturbative expansion in Eq. (4.11) has reasonably small coefficients as we had expected. We see that all large coefficients in C_l, C_p, C_J mutually cancel each other and numbers of order unity survive in the final expression rending the PT expansion reliable.

The conjecture about the above pattern of perturbative expansion is quite feasible and can be explicitly checked once $x_l^{(2)}$ is determined. Note that the idea of reexpressing the physical quantities through one another with a resulting PT factor is rather old and has been widely used. It is often applied as well to observables that are not quite fully perturbatively related (see, e.g., [14,51]).

Note that the lattice results are based on NLO analysis for the matching coefficients between continuum and lattice representations of the operators. While that matching is different, our computation shows that PT corrections at NNLO to the matching coefficients can be important at this level of precision.

If there is no particularly large NNLO contribution to the sum rule determination of $x_l^{(2)}$, our numbers show that the precision of the matrix element at the level of a few percent even in the presence of NNLO matching can be obtained. The large corrections are mainly hidden in C_J^2. In other words, singling out the B parameter is very efficient physically since it has a reasonable perturbative expansion *a posteriori* and a bunch of perturbative correction to the matrix element simply reproduces the correct value of f_B.

Note in passing that with the NNLO accuracy of the leading term one may need to account also for nonleading terms of HQE [52] while small power corrections have been accounted in [47].

In general, up-to-date lattice results turn out to be more precise than sum rule estimates for classical quantities like decay constants [53,54]. Only due to a special structure of the observables occurring in the B^0-$\bar{B}^0$ mixing one can still use the sum rules to obtain predictions that are still competitive with lattice computations.

Extension of our results to the case of $B_s^0 \bar{B}_s^0$ has been discussed in [16]. As the mass m_s is not large it can be taken into account in expansion in m_s/w_0 (an interesting example is given in [55]). We have shown that the strange quark mass appears in nonfactorizable quantities only at NLO level while the leading order contributions are hidden in factorizable parameters like f_{B_s} [53]. Numerically the parameter $(m_s/w_0)\alpha_s(w_0)$ [56] corresponds to NNLO level but achieving this accuracy requires only one loop calculations. With formulas given in the Appendix it is rather a straightforward computation which we are going to present in future publications.

V. SUMMARY

We have calculated NNLO corrections to matching coefficients necessary for the analysis of mixing in the B^0-$\bar{B}^0$ system, i.e., for the calculation of bag parameters in sum rules at three-loop level in HQET [6,7]. The NNLO corrections happen to be large, however; to a rather large extent they cancel the large NNLO corrections in the matching coefficients for the axial current that determines the B-meson leptonic decay constant, f_B. The relation between experimentally measured quantities Δm and f_B^2 turns out to be rather well behaved as a perturbative series up to NNLO. This observation gives a strong ground to our

estimate of the uncertainties for the QCD bag parameter B_q along the lines of Refs. [6,16].

We also discuss possible ways of getting invariant physical predictions from our results independent of the introduction of evanescent operators in the physical sector due to renormalization. We have constructed and present the completely analytical framework for analyzing the renormalization and matching of four-quark operators at one-loop level.

ACKNOWLEDGMENTS

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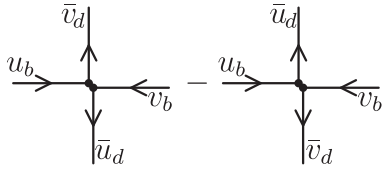
APPENDIX A: RENORMALIZATION OF FOUR-QUARK OPERATORS AT ONE LOOP: QCD

Let us consider a bare operator

$$O_0 = \frac{1}{2}(\bar{d}_{L0i}\Gamma b_0^i)(\bar{d}_{L0j}\Gamma b_0^j), \tag{A1}$$

where Γ is an arbitrary Dirac matrix. In the Born approximation its matrix element is

$$<\bar{b}d|O_0|b\bar{d}> =$$



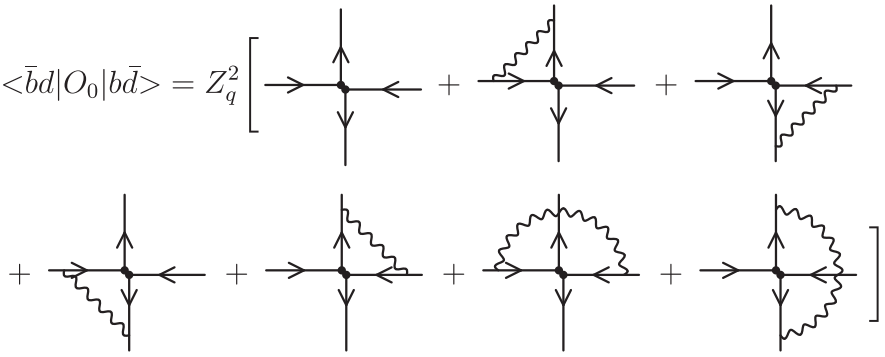
$$= \left(\bar{v}_{di}\Gamma u_b^i\right)\left(\bar{u}_{dj}\Gamma v_b^j\right) - \left(\bar{u}_{dj}\Gamma u_b^i\right)\left(\bar{v}_{di}\Gamma v_b^j\right). \tag{A2}$$

For brevity we write it as $M_0 = T_1\Gamma\otimes\Gamma$ where

$$\Gamma_1\otimes\Gamma_2 \equiv (\bar{v}_{di_1}\Gamma_1 u_b^{j_1})(\bar{u}_{di_2}\Gamma_2 v_b^{j_2}) - (\bar{u}_{di_1}\Gamma_1 u_b^{j_1})(\bar{v}_{di_2}\Gamma_2 v_b^{j_2}), T_1 \equiv \delta_{j_1}^{i_1}\delta_{j_2}^{i_2}, T_2 \equiv \delta_{j_2}^{i_1}\delta_{j_1}^{i_2}. \tag{A3}$$

The one-loop matrix element is

$$<\bar{b}d|O_0|b\bar{d}> = Z_q^2\Big[$$



$$\Big] = Z_q^2\left[M_0 + M_1 + M_2 + M_3 + M_4 + M_5 + M_6\right]. \tag{A4}$$

We are interested only in the UV $1/\varepsilon$ divergent terms. Therefore we may treat all quarks as massless and set all external momenta to 0. Then we need some IR regulator, say, replacing all massless denominators by the ones with some nonzero mass, or a hard IR cutoff in Euclidean momentum space. Such a regulator is implied, not written explicitly. The $\overline{\text{MS}}$ quark field renormalization constant is

$$Z_q = 1 - C_F\frac{\alpha_s}{4\pi\varepsilon}(1-\xi), \tag{A5}$$

where ξ is the gauge fixing parameter.

Averaging over directions of the loop momentum k in the integrands, we easily obtain

$$M_1 = C_F T_1 \frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\gamma_\nu\gamma_\mu\otimes\Gamma - \xi\Gamma\otimes\Gamma\right],$$
$$M_2 = C_F T_1 \frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\Gamma\otimes\gamma^\mu\gamma^\nu\Gamma\gamma_\nu\gamma_\mu - \xi\Gamma\otimes\Gamma\right],$$
$$M_3 = T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\Gamma\gamma^\nu\gamma^\mu\otimes\gamma_\mu\gamma_\nu\Gamma - \xi\Gamma\otimes\Gamma\right],$$
$$M_4 = T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\otimes\Gamma\gamma_\nu\gamma_\mu - \xi\Gamma\otimes\Gamma\right],$$
$$M_5 = -T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\Gamma\gamma^\nu\gamma^\mu\otimes\Gamma\gamma_\nu\gamma_\mu - \xi\Gamma\otimes\Gamma\right],$$
$$M_6 = -T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\otimes\gamma_\mu\gamma_\nu\Gamma - \xi\Gamma\otimes\Gamma\right].$$

The matrix element (A4) is gauge invariant,

$$\begin{aligned}\langle\bar{b}d|O_0|b\bar{d}\rangle &= T_1\Gamma\otimes\Gamma\\ &+ C_F T_1\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\gamma_\nu\gamma_\mu\otimes\Gamma + \frac{1}{d}\Gamma\otimes\gamma^\mu\gamma^\nu\Gamma\gamma_\nu\gamma_\mu - 2\Gamma\otimes\Gamma\right]\\ &+ T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}(\Gamma\gamma^\nu\gamma^\mu - \gamma_\mu\gamma_\nu\Gamma)\otimes(\gamma_\mu\gamma_\nu\Gamma - \Gamma\gamma_\nu\gamma_\mu),\end{aligned} \tag{A6}$$

where only the UV $1/\varepsilon$ divergences are kept in the one-loop terms. The result for the operator

$$O_0' = \frac{1}{2}(\bar{d}_{L0i}\Gamma b_0^j)(\bar{d}_{L0j}\Gamma b_0^i) \tag{A7}$$

differs from Eq. (A6) only by the color factors,

$$\begin{aligned}\langle\bar{b}d|O_0'|b\bar{d}\rangle &= T_2\Gamma\otimes\Gamma\\ &+ C_F T_2\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\Gamma\gamma^\nu\gamma^\mu\otimes\gamma_\mu\gamma_\nu\Gamma + \frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\otimes\Gamma\gamma_\nu\gamma_\mu - 2\Gamma\otimes\Gamma\right]\\ &+ T_F\left(T_1 - \frac{T_2}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}[\gamma^\mu\gamma^\nu\Gamma\gamma_\nu\gamma_\mu\otimes\Gamma + \Gamma\otimes\gamma^\mu\gamma^\nu\Gamma\gamma_\nu\gamma_\mu\\ &- \Gamma\gamma^\nu\gamma^\mu\otimes\Gamma\gamma_\nu\gamma_\mu - \gamma^\mu\gamma^\nu\Gamma\otimes\gamma_\mu\gamma_\nu\Gamma].\end{aligned} \tag{A8}$$

Now we specifically consider the operators

$$O_{n0} = \frac{1}{2}(\bar{d}_{L0i}\Gamma_n b_0^i)(\bar{d}_{L0j}\Gamma_n b_0^j),$$
$$O_{n0}' = \frac{1}{2}(\bar{d}_{L0i}\Gamma_n b_0^j)(\bar{d}_{L0j}\Gamma_n b_0^i), \tag{A9}$$

where

$$\Gamma_n = \gamma^{[\mu_1}\cdots\gamma^{\mu_n]} \tag{A10}$$

is the antisymmetrized product of n γ matrices. We have [21]

$$\begin{aligned}\gamma^\mu\Gamma_n\gamma_\mu &= (-1)^n(d-2n)\Gamma_n,\\ \gamma^\mu\Gamma_n\otimes\gamma_\mu\Gamma_n &= \Gamma_{n+1}\otimes\Gamma_{n+1} + n(d-n+1)\Gamma_{n-1}\otimes\Gamma_{n-1},\\ \Gamma_n\gamma^\mu\otimes\gamma_\mu\Gamma_n &= (-1)^n[\Gamma_{n+1}\otimes\Gamma_{n+1} - n(d-n+1)\\ &\quad\times\Gamma_{n-1}\otimes\Gamma_{n-1}].\end{aligned} \tag{A11}$$

Using these relations twice, we can rewrite the matrix elements (A6) and (A8) as

$$\langle\bar{b}d|O_{n0}|b\bar{d}\rangle = \langle O_n\rangle + 2C_F\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{(d-2n)^2}{d} - 1\right]\langle O_n\rangle \tag{A12}$$

$$\begin{aligned}&- T_F\frac{\alpha_s}{4\pi\varepsilon}\frac{4}{d}\left[\langle O_{n+2}'\rangle - \frac{\langle O_{n+2}\rangle}{N}\right.\\ &\left.+ n(n-1)(d-n+1)(d-n+2)\left(\langle O_{n-2}'\rangle - \frac{\langle O_{n-2}\rangle}{N}\right)\right],\\ \langle\bar{b}d|O_{n0}'|b\bar{d}\rangle &= \langle O_n'\rangle - C_F\frac{\alpha_s}{4\pi\varepsilon}\frac{2}{d}[\langle O_{n+2}'\rangle - 2n(d-n)\langle O_n'\rangle\\ &+ n(n-1)(d-n+1)(d-n+2)\langle O_{n-2}'\rangle]\\ &- T_F\frac{\alpha_s}{4\pi\varepsilon}\frac{2}{d}\left[\langle O_{n+2}\rangle - \frac{\langle O_{n+2}'\rangle}{N}\right.\\ &+ (6n(d-n) - d(d-1))\left(\langle O_n\rangle - \frac{\langle O_n'\rangle}{N}\right)\\ &\left.+ n(n-1)(d-n+1)(d-n+2)\left(\langle O_{n-2}\rangle - \frac{\langle O_{n-2}'\rangle}{N}\right)\right],\end{aligned} \tag{A13}$$

where $\langle O\rangle$ in the right-hand side are the Born-level matrix elements.

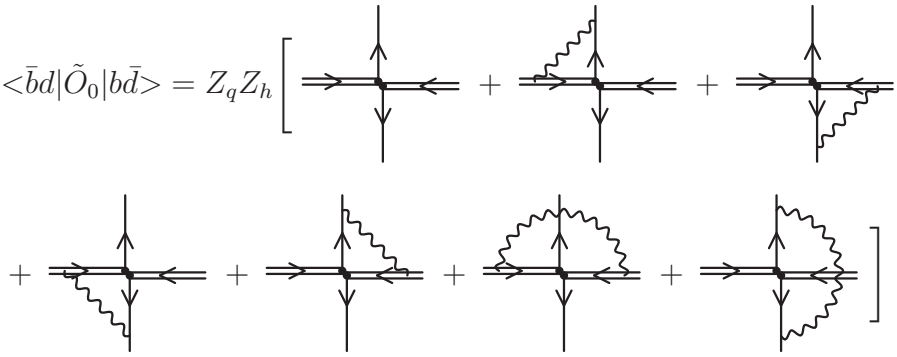
APPENDIX B: RENORMALIZATION OF FOUR-QUARK OPERATORS AT ONE LOOP: HQET

Let us consider the bare HQET operator

$$\tilde{O}_0 = (\bar{d}_{L0i}\Gamma h_{+0}^i)(\bar{d}_{L0j}\Gamma h_{-0}^j), \tag{B1}$$

where h_+ annihilates a heavy quark and h_- creates a heavy antiquark. The one-loop matrix element is

$$<\bar{b}d|\tilde{O}_0|b\bar{d}> = Z_q Z_h \Big[$$



$$= Z_q Z_h \big[M_0 + M_1 + M_2 + M_3 + M_4 + M_5 + M_6\big]\,. \tag{B2}$$

We are interested only in the UV $1/\varepsilon$ divergent terms. Therefore we may treat light quarks as massless, set their external momenta to 0, and set external residual momenta of HQET (anti-) quarks to 0. An IR cutoff is implied. The $\overline{\rm MS}$ HQET field renormalization constant is

$$Z_h = 1 + C_F \frac{\alpha_s}{4\pi\varepsilon}(2+\xi). \tag{B3}$$

Averaging the integrands over k directions [in particular, using $\overline{(k\cdot v)^{-2}} = -(d-2)(k^2)^{-1}$ [33,57]], we easily obtain

$$\begin{aligned}
M_1 &= M_2 = C_F T_1 \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\Gamma\otimes\Gamma,\\
M_3 &= M_4 = T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\Gamma\otimes\Gamma,\\
M_5 &= T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}(d-2+\xi)\Gamma\otimes\Gamma,\\
M_6 &= -T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\otimes\gamma_\mu\gamma_\nu\Gamma - \xi\Gamma\otimes\Gamma\right].
\end{aligned}$$

The matrix element (B2) is gauge invariant,

$$\begin{aligned}
&\langle\bar{b}d|\tilde{O}_0|b\bar{d}\rangle\\
&= \left[T_1 + 3C_F T_1\frac{\alpha_s}{4\pi\varepsilon} + T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}d\right]\Gamma\otimes\Gamma\\
&\quad - T_F\left(T_2 - \frac{T_1}{N}\right)\frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\gamma^\mu\gamma^\nu\Gamma\otimes\gamma_\mu\gamma_\nu\Gamma,
\end{aligned} \tag{B4}$$

where only the UV $1/\varepsilon$ divergences are kept in the one-loop terms. The matrix element of the operator

$$\tilde{O}'_0 = (\bar{d}_{L0i}\Gamma h^j_{+0})(\bar{d}_{L0j}\Gamma h^i_{-0}) \tag{B5}$$

differs from (B4) only by the interchange $T_1 \leftrightarrow T_2$.

Now we specifically consider the operators

$$\begin{aligned}
\tilde{O}_{n0} &= (\bar{d}_{L0i}\Gamma_{\perp n} h^i_{+0})(\bar{d}_{L0j}\Gamma_{\perp n} h^j_{-0}),\\
\tilde{O}'_{n0} &= (\bar{d}_{L0i}\Gamma_{\perp n} h^j_{+0})(\bar{d}_{L0j}\Gamma_{\perp n} h^i_{-0}),
\end{aligned} \tag{B6}$$

where

$$\Gamma_{\perp n} = \gamma_\perp^{[\mu_1}\cdots\gamma_\perp^{\nu_n]}, \qquad \gamma_\perp^\mu = \gamma^\mu - \not{v}v^\mu. \tag{B7}$$

Setting $\gamma^\mu = \not{v}v^\mu + \gamma^\nu_\perp$ and using the $(d-1)$-dimensional versions of (A11), we can rewrite the matrix element (B4) as

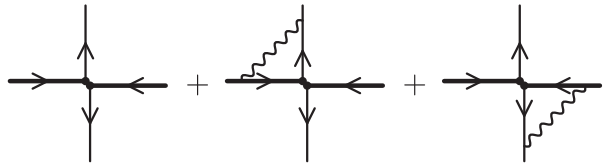
$$\begin{aligned}
&\langle\bar{b}d|\tilde{O}_{n0}|b\bar{d}\rangle\\
&= \langle\tilde{O}_n\rangle + 3C_F\frac{\alpha_s}{4\pi\varepsilon}\langle\tilde{O}_n\rangle\\
&\quad - T_F\frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\left[\langle\tilde{O}'_{n+2}\rangle - \frac{\langle\tilde{O}_{n+2}\rangle}{N} - 2\left(\langle\tilde{O}'_{n+1}\rangle - \frac{\langle\tilde{O}_{n+1}\rangle}{N}\right)\right.\\
&\quad + ((d-1)(2n-d) - 2n^2)\left(\langle\tilde{O}'_n\rangle - \frac{\langle\tilde{O}_n\rangle}{N}\right)\\
&\quad - 2n(d-n)\left(\langle\tilde{O}'_{n-1}\rangle - \frac{\langle\tilde{O}_{n-1}\rangle}{N}\right)\\
&\quad \left. + n(n-1)(d-n)(d-n+1)\left(\langle\tilde{O}'_{n-2}\rangle - \frac{\langle\tilde{O}_{n-2}\rangle}{N}\right)\right].
\end{aligned} \tag{B8}$$

The result for $\langle\bar{b}d|\tilde{O}'_{n0}|b\bar{d}\rangle$ differs only by the interchange of primed and nonprimed operators in the right-hand side.

APPENDIX C: MATCHING QCD ON HQET AT ONE LOOP

The one-loop on-shell matrix element of the QCD operator (A1) is

$$<\bar{b}d|O_0|b\bar{d}> = Z_Q^{\rm os}\Big[$$



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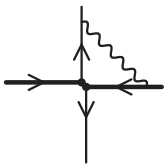
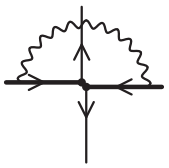


$$+$$



$$\Big]$$

$$= Z_Q^{\rm os}\big[M_0 + M_1 + M_2 + M_3 + M_4 + M_5\big]\,, \tag{C1}$$

where

$$Z_Q^{\rm os} = 1 - C_F\frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-1}{d-3}, \tag{C2}$$

m is the on-shell mass, and $Z_q^{\rm os} = 1$ at this order. The Born matrix element of the operator O_{n0} (A9) is $M_0 = T_1(\Gamma_{\perp n}\otimes\Gamma_{\perp n} - n\Gamma_{\perp n-1}\otimes\Gamma_{\perp n-1})$, where we have used

$$\Gamma_n\otimes\Gamma_n = \Gamma_{\perp n}\otimes\Gamma_{\perp n} + n\Gamma_{\perp n-1}\not{\bar{n}}\otimes\Gamma_{\perp n-1}\not{\bar{n}}. \tag{C3}$$

Each one-loop diagram is gauge invariant separately,

$$\begin{aligned}
M_1 = M_2 &= C_F T_1\frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{1}{(d-2)(d-3)}[(d-n-1)(d-2n-2)\langle\tilde{O}_n\rangle \\
&\quad + n(n-1)(d-2n+2)\langle\tilde{O}_{n-1}\rangle], \\
M_3 = M_4 &= T_F\left(T_2 - \frac{T_1}{N}\right)\frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{1}{2(d-2)(d-3)}[-\langle\tilde{O}_{n+2}\rangle + (d+n-2)\langle\tilde{O}_{n+1}\rangle \\
&\quad + n(3d-2n-4)\langle\tilde{O}_n\rangle - n(d-n)(d+2n-4)\langle\tilde{O}_{n-1}\rangle \\
&\quad - n(n-1)(d-n+1)(2d-n-2)\langle\tilde{O}_{n-2}\rangle \\
&\quad + n(n-1)(n-2)(d-n+1)(d-n+2)\langle\tilde{O}_{n-3}\rangle], \\
M_5 = T_F&\left(T_2 - \frac{T_1}{N}\right)\frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{1}{2(d-2)}\Big[-\langle\tilde{O}_{n+2}\rangle + (d+n-2)\langle\tilde{O}_{n+1}\rangle \\
&\quad + \frac{1}{d-3}(-[2(d-1)(d-2) + (d-3)(3d-4)n - 2(d-3)n^2]\langle\tilde{O}_n\rangle \\
&\quad + n[d^3 - 5d^2 + 6d + 4 + (d-3)(d+4)n - 2(d-3)n^2]\langle\tilde{O}_{n-1}\rangle) \\
&\quad - n(n-1)(d-n+1)(2d-n-2)\langle\tilde{O}_{n-2}\rangle \\
&\quad + n(n-1)(n-2)(d-n+1)(d-n+2)\langle\tilde{O}_{n-3}\rangle\Big].
\end{aligned}$$

The matrix element $\langle\bar{b}d|O'_{n0}|b\bar{d}\rangle$ is obtained by replacing $C_F \to T_F(T_1 - T_2/N)$ in $M_{1,2}$, $T_F(T_2 - T_1/N) \to C_F$ in $M_{3,4}$, and $T_F(T_2 - T_1/N) \to T_F(T_1 - T_2/N)$ in M_5.

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