Heterotic type I strings at high temperature

Shyamoli Chaudhuri*
Pennsylvania State University, State College, Pennsylvania 16801, USA

(Received 27 March 2018; published 22 August 2018)

We show that the high-temperature limits of the heterotic $E_8 \times E_8$ and Spin $32/Z2$ strings and their type I A/B superstring duals are finite and convergent. The Hagedorn growth of the degeneracies in the string mass level expansion is suppressed by an exponential that is linear in the mass level number for both heterotic strings, and suppressed by the exponential of the negative square root of the mass level number for the type IB superstring. However, in the massless gauge field-theoretic limit of the type IB open and closed superstring, we find clear evidence for the thermal deconfinement phase transition at the self-dual temperature by examining the annulus graph alone. Above the self-dual temperature, there is a discontinuity in the first derivative with respect to temperature of both the free energy, and the heavy quark potential, leading to a deconfined thermal gluon ensemble, with universal $1/r$ potential, and temperature-dependent corrections, as predicted by Lüscher and Weisz. A number of essential aspects of the worldsheet formalism of the heterotic strings are derived in an Appendix, deducing thereby the O8-D0-D8-brane type IA duals of all of the heterotic Chaudhuri-Hockney-Lykken island universe moduli spaces.

DOI: 10.1103/PhysRevD.98.046017

I. INTRODUCTION

The Nambu-Goto-Eguchi-Schild string has long been the prototype effective string theory description of QCD flux lines and flux tubes [1–4] and it is well known that the Nambu-Goto and Polyakov string actions are classically equivalent. While there has been an explosion of work applying supergravity/M-theory techniques to the study of nonperturbative strongly coupled supersymmetric large-N gauge theories in recent years using Maldacena's gauge-gravity dualities [5], our current investigation examines instead the anomaly-free perturbative non-Abelian gauge theories [6] derived from exactly known one-loop results for type I/heterotic superstring theory amplitudes—focusing in particular on the phase structure of the finite-temperature non-Abelian gauge theories appearing in the low-energy field theory limit of the open string sector of the type IB/IA superstrings.

Our starting point is finite-temperature string theory in the Euclidean-time Polyakov path-integral formulation, compactifying the (heterotic or type I) ten-dimensional (10D) $N = 1$ superstring theory. As was pointed out by Bernard, the finite-temperature quantization of gauge theory in the Feynman Euclidean path-integral prescription requires a physical gauge choice on the Yang-Mills one-form potential, such that the longitudinal degrees of freedom (d.o.f.) are eliminated, and the usual choice is axial gauge, $A_0 = 0$ [7,8]. The analog here is the quantization of the Neveu-Schwarz two-form field in the Polyakov path integral prescription. This B field lives in the same sector as the graviton and dilaton scalar, and is common to every superstring theory. To set a string theory background field to zero is to violently disrupt the moduli space symmetries [9,10], and we therefore choose the more benign path of requiring a relationship on moduli space between two moduli fields, which has the same consequence of reducing the number of physical d.o.f. Namely, we set a single component of the B field proportional to the radius, $\beta = 1/T$, of the Euclidean scalar, X^0_E: $B = B_{09} = \tanh(\pi\alpha)$, $\alpha = (\beta_C/\beta)$, which is therefore linear in the temperature, at low temperatures, asymptoting to unity at the self-dual point. That this is the appropriate analog of the axial gauge choice on the finite-temperature Yang-Mills one-form gauge potential is apparent in its consequences on the result for the one-loop string free energy. Moreover, we can deduce that this is the unique choice which breaks supersymmetry spontaneously, while being compatible with the required two-dimensional Diff × Weyl gauge invariances of the Polyakov path-integral formulation.[1]

The twist on the B field achieves the necessary physical gauge condition on the finite-temperature quantization of

[1]It occurred to us to try this ansatz soon after the String Math conference of 2011, upon meeting with Paul Aspinwall. That the result was unique was already known to us from our previous work on the renormalization of the noncommutative string theory obtained by quantizing with the background fields of the open and closed string theories. It takes some trial and error to verify that this is the unique answer for the type IIA and type IIB oriented closed string theories as well, and we give the result directly for the heterotic closed and oriented superstring in the paper.

*scplassmann@gmail.com

the (Neveu-Schwarz) two-form gauge potential which couples to the fundamental string, in every superstring theory, whether heterotic, type IA-IB, or type IIA-IIB. The two-torus is a complex manifold, and marginal deformations are parametrized by two real numbers[2]:

$$B = \frac{1}{2} B_{i\bar{j}} dx^i \wedge dx^{\bar{j}}, \qquad J = \frac{1}{2} g_{i\bar{j}} dx^i \wedge dx^{\bar{j}}, \qquad (1.1)$$

and shifts in B, J, appear in the mass level expansions through the, thermal and spatial, momentum modes and winding modes. Shifts of B by an integer leave the action, and all correlation functions, unchanged, due to the $SL(2,\mathrm{Z})$ symmetry generated by $\sigma = \frac{1}{4\pi\alpha'}(B + iJ)$. The moduli space of marginal deformations is the group $SL(2,\mathrm{Z}) \times SL(2,\mathrm{Z})$, where the additional $SL(2,\mathrm{Z})$ describes the complex structure of the torus: we have a rectangular domain of lengths, R_0, R_9, where $\eta = iR_9/R_0$ and $\sigma = \frac{i}{\alpha'} R_9 R_0$, which divide the complex plane by the translations $2\pi R_0$ and $2\pi R_9$. The thermal duality transformation, $\beta \leftrightarrow 1/\beta$, quite remarkably, can be identified as an Abelian subset of the mirror map [11] for the two-torus, the simplest Calabi-Yau manifold of complex dimension one. Here, $\beta = R_0/2\pi$. Thus, $R_0 \leftrightarrow 1/R_0$ generates the Z_2 mirror map exchanging the two $SL(2,\mathrm{Z})$ factors. The additional Z_2 symmetry is generated by complex conjugation, interchanging the two real coordinates, plus a change in the sign of B, namely, inversion in the upper half-plane, $(\sigma, \eta) \leftrightarrow (-\sigma, -\eta)$.

Supersymmetry is spontaneously broken at low temperatures by the gravitino and gaugino masses [12], provided by a scalar Kähler modulus, the inverse of the radius of Euclidean time. It should be noted that supersymmetry breaking occurs at very low temperatures, quite distinct from the thermal duality transition at the self-dual temperature. Nevertheless, our work provides a continuous parametrization in terms of (β, B, R), for both of these phenomena. It should be emphasized that nothing changes if the $D = 10$, $N = 1$ superstring theories analyzed here are replaced by, e.g., an orbifold compactification, yielding a $D = 4$, $N = 1$ superstring vacuum state. Our analysis can thus be said to provide direct evidence by demonstration in favor of the "continuity" conjecture made by Poppitz, Schäfer, and Unsal [13], that low-temperature soft supersymmetry breaking in a $N = 1$ non-Abelian gauge theory mediated by a gaugino mass—what has been named the SYM* theory in Ref. [13]—can be seen to be continuously connected to the thermal deconfinement phase transition in thermal Yang-Mills gauge theory. The results we present here would likely hold for any K3 compactification [11], e.g., on $R^3 \times K3 \times (S^1 \times S^1/Z_2)$, where the Z_2 is chosen so as to give a 4D $N = 1$ supersymmetric gauge theory with four supercharges for generic radii and the B field on the $(S^1 \times S^1/Z_2)$, since they hold at the orbifold points of the K3.

We begin in Sec. II by establishing the finiteness of the finite-temperature one-loop vacuum energy density of the heterotic $E_8 \times E_8$ and Spin $32/Z2$ strings in both the low-temperature supergravity and Yang-Mills field-theoretic limit, and in the high-temperature limit, both at, and beyond, the self-dual temperature, T_C. We show in particular how to complete the integral over both worldsheet moduli of the one-loop torus vacuum graph of closed superstrings (correcting some errors in previous papers that might mislead the reader), which is a *tour de force* that has important implications for the applications of string scattering amplitudes to problems in particle and astroparticle physics [14]. An analogous demonstration is carried out in Sec. III for the open unoriented and closed string vacuum graphs of the type IB superstring theory, except that the Hagedorn growth is suppressed by an exponential of the square root of the mass level number at the self-dual temperature. For clarity, we show that the thermal duality transition of the full string theory is benign: in both the heterotic and type I string theories, it is a Kosterlitz-Thouless phase transition characterized by an infinite tower of finite, and analytic, thermodynamic potentials, including the Helmholtz free energy, Gibbs free energy, entropy and specific heat.

In Refs. [15–17], we formulated the Polyakov string path-integral prescription [18–21] for macroscopic incoming and outgoing string states at finite separation in an embedding target spacetime. $\mathcal{M}_{\mathrm{IB}}(\mathbf{x})$ is the expectation value in the type IB string theory for the insertion of a macroscopic boundary loop on the worldsheet mapped to a fixed loop $\mathcal{C}$ at location $\mathbf{x}$ in the embedding target spacetime. The mapping must preserve the worldsheet super Diff × Weyl invariances, both in the bulk, and on the boundaries, of the worldsheet, due to its fixed spatial location. The observable $\mathcal{M}_f(\mathcal{C})$ transforms in the fundamental representation f of the non-Abelian gauge group, and the trace over the Chan-Paton index ensures that the observable is gauge invariant. Since the end points of the open strings carry color charge, the boundary of the hole in the worldsheet is mapped to a Wilson loop in the low-energy non-Abelian gauge theory limit, describing the world history of an infinitely massive probe carrying color charge. In Sec. IV, we analyze the insertion of a pair of spacelike macroscopic loop observables, mapped to the fixed spacelike loops in the target spacetime, $\mathcal{C}_2$, $\mathcal{C}_1$, spatially separated by a distance R, which directly yields the potential between two massive charged color sources in the gauge field theory limit—rather than the exponentiated potential which appears in the corresponding Wilson loop two-point function of the non-Abelian gauge theory, as in Ref. [13].

[2]As has been noted by Aspinwall for $K3$ compactifications [11], a constant, and periodic, B field on a K3 surface cannot be strictly taken to zero, without approaching a singularity at finite distance in the moduli space. The nonvanishing B field is necessary in order to approach several of the enhanced gauge symmetry points of interest, related to the orbifold points on the K3 moduli space. This is not to be confused with the B field on the T_9-dualized two-torus, $S^1 \times S^1/Z_2$, where the constant B field, $B_{09} = -B_{90}$, must vanish in the supersymmetric large-radius limits.

For open string end-point Chan-Paton wave functions transforming in the fundamental representation f of the gauge group, we shall derive an expression for the macroscopic pair correlation function. In this analysis, we suppress the full content of the unoriented open and closed type I superstring theories, in order to draw attention to the properties of the massless gauge theory limit in and of itself [16][3]:

$$\mathcal{W}^{(2)}_{\rm IB}(R) = \langle {\rm Tr}_f \mathcal{M}_f(\mathcal{C}_2)\mathcal{M}_f(\mathcal{C}_1)\rangle = -\beta V(R,\beta), \qquad (1.2)$$

which interpolates neatly between the low-temperature, large spatial separation, confinement regime of the non-Abelian gauge theory, derived from the T_9, and T_0, dual, type IA string, and dominated by a linear term in the heavy quark potential, and the high-temperature, small spatial separation, deconfinement regime, derived from the type IB string, which is dominated by the $1/R$ Lüscher potential. We will show that the expression for the heavy quark potential we derive is in qualitative agreement with the original Cornell phenomenological potential model [22], the Nambu-Goto-Eguchi-Schild-Polyakov QCD effective strings [2–4,23], and lattice gauge theory measurements [24]. At low type IB temperatures, and for large type IB spatial separations of the infinitely heavy quarks, in addition, we derive from the expectation value of two Polyakov-Susskind loops the next-to-leading-order thermal (field-dependent) corrections to the universal $1/R$ potential in the deconfined phase, thereby confirming Lüscher and Weise's conjecture that the leading correction to the universal $1/R$ term is $O(1/R^3)$ [2,23].

The Appendix contains a series of significant developments in the worldsheet formalism of the heterotic string theories that lend insight to the results in this paper, and on the string/M-theory strong-weak duality web more generally. In particular, we deduce in Appendix A D the type IA strong coupling duals of the heterotic Chaudhuri-Hockney-Lykken (CHL) orbifold island universes, using the formalism for the gauge group in O8-D0-D8-brane compactifications, with 16 pairs of D0-D8-branes and their images at each of two orientifold planes at the end points of the interval in $(S^1\times S^1/Z_2)$ compactifications of the type IA $O(16)\times(16)$ superstring [25]. The spinor of $O(16)$ is given by the solitonic fundamental strings created at the intersection of D0-branes with D8-branes [25,26], and the configuration we suggest gives the full gauge group $E_8\times E_8$. It is straightforward to then identify all of the type IA duals of the CHL orbifolds of the $E_8\times E_8$ heterotic string, and breaking the supersymmetry further by orbifold compactification gives three generations of chiral fermions in the Standard Model embedded within the spinor of $SO(16)$, a technique well known to string and grand unified theory phenomenologists [27,28].

II. FINITE-TEMPERATURE HETEROTIC STRING VACUUM FUNCTIONAL

The finite-temperature one-loop vacuum functional of the $E_8\times E_8$ heterotic string is given in the Euclidean time prescription by the compactification of the heterotic string on the twisted two-torus with radii, (β_H, R_H), and constant background B field parametrized as, $B_{09} = -B_{90} = |B| = \tanh(\pi\alpha)$, where $\alpha = T/T_C$, and T_C is the self-dual temperature:

$$\begin{aligned} W_{\rm H}(\beta) &= \mathcal{N}\beta_H(2\pi R_H)L^8(4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2^2}\cdot(\tau_2)^{-4}[\eta(\tau)\bar\eta(\bar\tau)]^{-6}\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\tau)}{\Theta_{11}(\alpha,\tau)}\right]\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\bar\tau)}{\bar\Theta_{11}(\alpha,\bar\tau)}\right] \\ &\times\frac{1}{4}\left[\frac{\bar\Theta_{00}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\bar\eta}\left(\frac{\bar\Theta_{00}}{\bar\eta}\right)^3 - \frac{\bar\Theta_{01}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\bar\eta}\left(\frac{\bar\Theta_{01}}{\bar\eta}\right)^3 - \frac{\bar\Theta_{10}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\bar\eta}\left(\frac{\bar\Theta_{10}}{\bar\eta}\right)^3\right] \\ &\times\frac{1}{4}\left[\left(\frac{\Theta_{00}}{\eta}\right)^8+\left(\frac{\Theta_{01}}{\eta}\right)^8+\left(\frac{\Theta_{10}}{\eta}\right)^8\right]^2 \\ &\times\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{\alpha' n_9^2}{R_H^2}\right)\right] \\ &\times\exp\left[-\pi\tau_2(1+\tanh(\pi\alpha))^2\left(\frac{w_9^2\beta_H^2}{4\pi^2\alpha'}+\frac{w_0^2R_H^2}{\alpha'}\right)\right] \\ &\times\exp[i\pi\tau_1(n_0w_9+n_9w_0)(1+\tanh(\pi\alpha))]. \end{aligned} \qquad (2.1)$$

[3]This remarkable feature of type IA-IB open and closed superstrings follows from the relation of the couplings; at tree level, it is simply $g_{\rm closed} = g^2_{\rm open}$. We see that the annulus with macroscopic loops can be analyzed in the non-Abelian gauge theory alone, as with the finite-temperature vacuum energy density, since the supergravity is decoupled at tree level. The supergravity multiplet belongs in the closed string sector of the type IB-IA superstrings. Upon computing the renormalized couplings and string mass scale, this tree relation receives loop corrections [14,17]. Very recent progress has been made in the computation of unambiguous multiloop superstring amplitudes, at two loops and beyond, which lends promise to the systematic extension of our worldsheet analysis.

Note the presence of the holomorphic one-loop $E_8 \times E_8$ vacuum functional in the third line of this formula, which we leave unchanged by a possible Wilson line since we wish to keep the gauge group fixed.

In the high mass level number regime as we approach the self-dual temperature, the parameters $\alpha \to$, $\tanh(\pi\alpha)$, will asymptote to unity, giving pure numerical factors; we leave them as parameters of the background B field in this preliminary expression valid for all temperatures and mass levels. Expressing the theta functions, and eta functions, in terms of mass level number, namely, positive integer powers of $|q\bar{q}|$, we can expand in this variable to obtain

$$\begin{aligned} W_{\mathrm{H}}(\beta) &= \mathcal{N}\beta_H(2\pi R_H)L^8(4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2}\cdot(\tau_2)^{-5}\sum_{m=0}^{\infty}e^{-4m\pi\tau_2}f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha)) \\ &\times\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{\alpha' n_9^2}{R_H^2}\right)\right] \\ &\times\exp\left[-\pi\tau_2(1+\tanh(\pi\alpha))^2\left(\frac{w_9^2\beta_H^2}{4\pi^2\alpha'}+\frac{w_0^2R_H^2}{\alpha'}\right)\right] \\ &\times\exp\left[i\pi\tau_1(n_0w_9+n_9w_0)(1+\tanh(\pi\alpha))\right], \end{aligned} \tag{2.2}$$

where $f^{(m)}(0)$, and $f^{(m)}(2)$, are, respectively, the numerical mass degeneracies of the partition function of the $E_8 \times E_8$ heterotic string in the supersymmetric zero-temperature limit, and in the vicinity of the high-temperature self-dual critical point. Solving for the Helmholtz free energy at one-loop order in string perturbation theory, we obtain

$$\begin{aligned} F_{\mathrm{H}}(\beta) &= -\mathcal{N}V\frac{1}{4}(4\pi^2\alpha')^{-5}\int_{-1/2}^{1/2}d\tau_1\int_{(1-\tau_1^2)^{1/2}}^{\infty}d\tau_2 \\ &\times(\tau_2)^{-6}\sum_{m=0}^{\infty}e^{-4m\pi\tau_2}f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha)) \\ &\times\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{\alpha' n_9^2}{R_H^2}\right)\right] \\ &\times\exp\left[-\pi\tau_2(1+\tanh(\pi\alpha))^2\left(\frac{w_9^2\beta_H^2}{4\pi^2\alpha'}+\frac{w_0^2R_H^2}{\alpha'}\right)\right] \\ &\times\exp\left[i\pi\tau_1(n_0w_9+n_9w_0)(1+\tanh(\pi\alpha))\right]. \end{aligned} \tag{2.3}$$

More familiar to a particle physicist, the Helmholtz free energy at one-loop order is nothing but the one-loop vacuum energy density of the finite-temperature string vacuum, $F = V_9\rho_H = -W_H/\beta_H$. Substituting $y = 1/\tau_2$, we can express the one-loop vacuum energy density at finite temperature as

$$\begin{aligned} \rho_{\mathrm{H}}(\beta) &= -\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty}f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha)) \\ &\times\int_{-1/2}^{1/2}d\tau_1\exp\left[i\pi\tau_1(n_0w_9+n_9w_0)\right]\int_0^{(1-\tau_1^2)^{-1/2}}dy\,y^4e^{-A/y}, \end{aligned} \tag{2.4}$$

where the function in the exponent, A, is the mass formula for the finite-temperature heterotic string spectrum

$$A_{\mathrm{H}}(\beta,\alpha;m) = m+\frac{1}{4}\left[\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{\alpha' n_9^2}{R_H^2}+(1+\tanh(\pi\alpha))^2\left(\frac{w_9^2\beta_H^2}{4\pi^2\alpha'}+\frac{w_0^2R_H^2}{\alpha'}\right)\right]. \tag{2.5}$$

Note that mass level number, m, by mass level number, there is an infinite tower of thermal momentum and thermal winding modes in the finite-temperature spectrum, in addition to the tower of possible spatial momenta and windings, as a consequence of the generalized axial gauge condition necessitating compactification on the Neveu-Schwarz B_{09}-field twisted two-torus.

The integral over the variable y in the expression for the vacuum energy density can be recognized as a standard integral representation of the Whittaker function, $\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(Au)$ [29],

$$\rho_{\rm H}(\beta) = -\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))$$
$$\times\int_{-1/2}^{1/2} d\tau_1 \exp\left[i\pi\tau_1(n_0w_9+n_9w_0)\right]A^{\frac{\nu-1}{2}}u^{\frac{\nu+1}{2}}e^{-\frac{1}{2}A/u}\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A/u),$$
$$\text{where } \nu = 5, \quad \text{and} \quad u\equiv(1-\tau_1^2)^{-1/2}, \tag{2.6}$$

and upon substitution, it is helpful to make the change of variable $x^2 = 1-\tau_1^2 = 1/u^2$. The one-loop vacuum energy density therefore takes the form

$$\rho_{\rm H}(\beta) = -2\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))$$
$$\times\int_0^{\sqrt{3}/2} dx\mathrm{Cos}[\pi\sqrt{1-x^2}(n_0w_9+n_9w_0)]A^{\frac{\nu-1}{2}}x^{-\frac{\nu+1}{2}+1}e^{-\frac{1}{2}Ax}\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(Ax). \tag{2.7}$$

Alternatively, we can use the integral representation in terms of the inverse variable, u:

$$\rho_{\rm H}(\beta) = -2\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))\int_0^{2/\sqrt{3}} du(1-u^{-2})^{-1/2}$$
$$\times\mathrm{Cos}[\pi(1-u^{-2})^{1/2}(n_0w_9+n_9w_0)]A^{\frac{\nu-1}{2}}u^{-3+\frac{\nu+1}{2}}e^{-\frac{1}{2}A/u}\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A/u). \tag{2.8}$$

We will be interested in the low-temperature, power-law, and high-temperature asymptotics of the Whittaker function. Prior to that step, notice that the cosine function, and its argument, can be further simplified by replacing each by their Taylor expansions, both of which are completely valid in the domain of the integral over x. This is the form of the expression for the one-loop vacuum energy density at finite temperature which we will analyze in the low-temperature limit in what follows. With this substitution, we get the expression

$$\rho_{\rm H}(\beta) = -2\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))$$
$$\times\int_0^{\sqrt{3}/2} dx\left[\sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{s!}{l!(s-l)!}\frac{(-1)^{l+2s}}{(2s)!}(n_0w_9+n_9w_0)^{2s}(x)^{2l}\right]$$
$$\times A^2x^{-2}e^{-\frac{1}{2}Ax}\mathcal{W}_{-3,\frac{5}{2}}(Ax). \tag{2.9}$$

Alternatively, we use the inverse variable, u. This is the form of the expression for the one-loop vacuum energy density at finite temperature which we will analyze in the high-temperature limit in what follows. With the two substitutions, we get the alternative expression

$$\rho_{\rm H}(\beta) = -2\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))$$
$$\times\int_0^{2/\sqrt{3}} du\left[\sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{s!}{l!(s-l)!}\frac{(-1)^{l+2s}}{(2s)!}(n_0w_9+n_9w_0)^{2s}(u)^{-2l}\right]$$
$$\times\sum_{r=0}^{\infty}(-1)^r\frac{u^{-2r}}{r!}\left(\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\cdot 3\right)\cdots\left(r-\frac{1}{2}\right)\right)A^{\frac{\nu-1}{2}}u^{-3+\frac{\nu+1}{2}}e^{-\frac{1}{2}A/u}\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A/u). \tag{2.10}$$

A. Low-temperature supergravity-super-Yang-Mills theory limit

We now take the low-temperature field-theoretic limit of this expression, verifying that it has the expected properties of a ten-dimensional finite-temperature field theory. We substitute the power-series expansion of the Whittaker function, prior to performing the integral over the variable x (the τ_1 worldsheet modulus). The expansion in powers of A, has as its leading term at low temperatures, the thermal spectrum of the massless modes of the supersymmetric string:

$$\begin{aligned}\rho_{\mathrm{H}}(\beta) = -2\mathcal{N}\frac{(-1)^5}{5!}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty}A^5 f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))\\ \times\left[\sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{s!}{l!(s-l)!}\frac{(-1)^{l+2s}}{(2s)!}(n_0w_9+n_9w_0)^{2s}\right]\\ \times\left\{\sum_{k=0}^{\infty}A^{2l+2k}\gamma\left(2l+k,\frac{1}{2}\sqrt{3}A\right)\left[\frac{\Psi(k+1)-\ln A}{k!}\right]\right.\\ +\sum_{k=0}^{4}(-1)^{5+k}\Gamma(5-k)A^{2l+2k-10}\gamma\left(2l+k-5,\frac{1}{2}\sqrt{3}A\right)\\ \left.-\sum_{p=1}^{\infty}\sum_{k=0}^{\infty}\frac{A^{2l+2k+p}}{k!}\sum_{r=0}^{p}\frac{p!}{r!(p-r)!}\gamma\left(2l+k+p,\frac{1}{2}\sqrt{3}A\right)\right\},\end{aligned}\tag{2.11}$$

where the function A gives the full thermal, and spatial, momenta and windings, of the thermal spectrum, for any given mass level number, m. Expanding about the massless modes

$$A_{\mathrm{H}}(T,\alpha;0)=\frac{1}{4}\left[4\pi^2\alpha' n_0^2T^2+\frac{\alpha' n_9^2}{R_9^2}+(1+2\tanh\pi(T/T_C))\left(\frac{w_9^2}{4\pi^2\alpha' T^2}+\frac{w_0^2R_9^2}{\alpha'}\right)\right],\tag{2.12}$$

it is apparent that at low temperatures, the leading term is the tower of thermal Kaluza-Klein modes, or Matsubara frequencies, in the language of thermal field theories. For low temperature, or large radius, the spatial coordinate is in the large-radius limit, the spatial Kaluza-Klein modes tend towards a continuum, and no spatial windings are excited. Thus, we first extract the thermal momentum modes, and the power A^5 instantly gives the expected field-theoretic T^{10} in the one-loop string free energy. The correction from the tower of thermal winding and spatial Kaluza-Klein modes is a purely string-theoretic artifact:

$$A^5(T,\alpha;0)\simeq\frac{1}{4}(4\pi^2\alpha' n_0^2)^5T^{10}\times\left\{1+5\left\{1+\frac{\alpha' n_9^2}{R_9^2 4\pi^2\alpha' n_0^2T^2}+[1+2\pi(T/T_C)]\frac{w_0^2R_9^2}{4\pi^2\alpha'^2n_0^2T^2}\right\}\right\}.\tag{2.13}$$

Suppressing the winding modes at the lowest temperatures, the string finite-temperature vacuum energy density takes the simple form

$$\begin{aligned}\rho_{\mathrm{H}}(\beta)\simeq -2\mathcal{N}\frac{(-1)^5}{5!}(4\pi^2\alpha')^{-5}\sum_{n_0,w_0=-\infty}^{\infty}f^{(0)}_{E_8\times E_8}\\ \times\frac{1}{4}(4\pi^2\alpha' n_0^2)^5T^{10}\times\left\{1+5\left[1+\frac{\alpha' n_9^2}{R_9^2 4\pi^2\alpha' n_0^2T^2}\right]\right\}.\end{aligned}\tag{2.14}$$

B. High mass level asymptotics at the self-dual temperature

Finally, we substitute the asymptotic expansion of the Whittaker function into the expression derived above for the one-loop free energy of the canonical ensemble of thermal $E_8\times E_8$ heterotic strings, valid for large mass level number. The asymptotic series expands in negative powers of A, namely, large m, and arbitrary temperature, although we will be interested in the behavior of this sum over mass levels in the vicinity of the self-dual temperature. Note that the expression is finite, and convergent, for the full temperature range, even beyond the self-dual temperature. We begin with the inverse variable representation

$$\rho_{\mathrm{H}}(\beta) = -2\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(1+\tanh(\pi\alpha))$$
$$\times\int_0^{2/\sqrt{3}} du\left[\sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{s!}{l!(s-l)!}\frac{(-1)^{l+2s}}{(2s)!}(n_0w_9+n_9w_0)^{2s}(u)^{-2l}\right]$$
$$\times\sum_{r=0}^{\infty}(-1)^r\frac{u^{-2r}}{r!}\left(\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\cdot 3\right)\cdots\left(r-\frac{1}{2}\right)\right)A^{\frac{\nu-1}{2}}u^{-3+\frac{\nu+1}{2}}e^{-\frac{1}{2}A/u}\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A/u), \tag{2.15}$$

and substitute the asymptotic expansion for the Whittaker function, keeping its leading term, $k=0$:

$$F_{\mathrm{H}}(\beta) \simeq -2\mathcal{N}V(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty} f^{(m)}_{E_8\times E_8}(2)$$
$$\times\left[\sum_{s=0}^{\infty}\sum_{l=0}^{s}\frac{s!}{l!(s-l)!}\frac{(-1)^{l+2s}}{(2s)!}(n_0w_9+n_9w_0)^{2s}\right]$$
$$\times\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}\left(\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\cdot 3\right)\cdots\left(r-\frac{1}{2}\right)\right)$$
$$\times\left[1+\sum_{k=1}^{\infty}O(k)\right]$$
$$\times\left\{e^{-2A/\sqrt{3}}A^{l+r}\left(\frac{1}{2}\sqrt{3}\right)^{-l-r-5/2}\mathcal{W}_{4-2l-2r,2-l-r}(2/\sqrt{3})\right\}. \tag{2.16}$$

Note that the free energy is exponentially damped as a linear power of m, the mass level number, correcting the Hagedorn growth of the numerical degeneracies [30] as a square root of the mass level number in the vicinity of the self-dual temperature. The free energy is finite, and the expression given above is strongly convergent at the critical point. The function in the exponential, A, gives the full thermal, and spatial, momenta and windings, of the thermal spectrum, expanding about the asymptotic mass level number, m, and setting $\tanh(\pi\alpha)$ to unity

$$A_{\mathrm{H}}(\beta,\alpha;m)=m+\left[\left(\frac{\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{\alpha' n_9^2}{4R_H^2}\right)+\left(\frac{w_0^2R_H^2}{\alpha'}+\frac{w_9^2\beta_H^2}{4\pi^2\alpha'}\right)\right], \tag{2.17}$$

where we have rearranged the formula to highlight the symmetry linking Kaluza-Klein and winding modes, with the twist having mixed spatial and thermal coordinates. For large m, and at high temperatures of order the self-dual temperature, we find that all of the thermal and spatial winding modes are excited, and winding modes dominate the expression for the free energy, due to the presence of the exponential, in the small-radius, high-temperature limit. The $f^m(2)$ in the one-loop free energy are the numerical degeneracies at the critical temperature T_C.

Finally, in passing, we recall that our starting point was the generating functional for connected vacuum diagrams in one-loop string perturbation theory, W, and we do not face the usual problems associated with taking the thermodynamic limit of the canonical partition function, Z. The Helmholtz free energy, F, and the finite-temperature vacuum energy density, ρ, are related to this as follows:

$$W\equiv\ln Z=-\beta_H V\rho,\qquad F\equiv -T_H Z=-W/\beta_H=V\rho,$$
$$P=-\left(\frac{\partial F}{\partial V}\right)_{T_H}=-\rho, \tag{2.18}$$

where P is the pressure of the string canonical ensemble at fixed temperature, and V is its spatial volume. Note that P equals the negative of ρ for the string canonical ensemble. The next few entries in the list of thermodynamic potentials are the internal or Gibbs free energy, the entropy, and the specific heat at constant volume:

$$U=-T_H^2\left(\frac{\partial W}{\partial T_H}\right)_V,\qquad S=-\left(\frac{\partial F}{\partial T_H}\right)_V,$$
$$C_V=T_H\left(\frac{\partial S}{\partial T_H}\right)_V. \tag{2.19}$$

It is evident by inspection of the expressions for the Helmholtz free energy and the detailed dependence on β that the results for an infinity of partial derivatives are completely analytic and finite, and we identify the thermal duality transition as being of Kosterlitz-Thouless type. There is no divergence in the expressions at any order in the thermodynamic potentials. We leave further discussion of this intriguing observation to the future.

III. FINITE-TEMPERATURE TYPE IA STRING VACUUM FUNCTIONAL

We now perform the analysis of the ultraviolet limit of the unoriented graphs of the $O(32)$ type IA superstring at finite temperature. If the $N = 32$ D8-branes are all on a single O8-plane, the Dirichlet string measures the potential energy of the string stretched between the D8-brane stack on an O8-plane, with the O8-plane defect at the other end of the interval of length, R. We will show at the conclusion of our derivation that it is in fact possible to take R to zero, and recover the result for the finite-temperature type IB $O(32)$ vacuum, *without* the Dirichlet stretched string. Note that we have the constant mode of the Neveu-Schwarz (NS) sector antisymmetric tensor gauge potential, B, which remains after the orientation transformation which eliminates the propagation of the NS two-form field. We shall set the constant background field, $|B_{09}| = -|B_{90}| = |\tanh(\pi\alpha)|$, where $\alpha \equiv (\beta_C/\beta) = \alpha'^{1/2}T$, is linear in temperature, measured in units of the inverse string scale.

We analyze the small-t (high-temperature) behavior of the three individual open and unoriented one-loop type IB superstring graphs. We begin with the result for the oriented open string sector, or annulus graph, in terms of Jacobi theta functions, where N denotes the number of D8-branes or, equivalently, the Chan-Paton factor carried by the end points of the open string. We have expressed the Jacobi theta functions in the integrand as the modular transformed functions of $1/t$, as appropriate in the small-t limit. Dividing by the spatial volume, and the circumference of Euclidean time, we have the following expression for the vacuum energy density of the type IB superstring with N D9-branes, thermal duality transformed after compactification on the twisted torus with B field $|B_{09}| = \tanh(\pi\alpha)$, and $\alpha = \beta_C/\beta$. A T_9-duality transformation likewise enables analysis of the short-distance limit of the Dirichlet vacuum, with a stretched string extending along the interval $X^{9\mathrm{A}}$, of length $R = R^9_{\mathrm{IA}}$. The argument of the B field asymptotes to its high-temperature value, $\tanh(\pi\alpha) \to 1$, and the high-temperature asymptotic expansion of the Jacobi theta functions is in integer powers of $q = e^{-\pi/t}$, exposing the $t \to 0$ limit of the integrand:

$$
\begin{aligned}
F^{(\mathrm{IA})}_{\mathrm{ann}} = &-N^2 V_9 (8\pi^2\alpha')^{-5}(1+\tanh(\pi\alpha)) \int_0^\infty \frac{dt}{t} \cdot t^{-5} e^{-R^2 t/2\pi\alpha'} \times [\eta(it)^{-6}] \left[\frac{e^{-\pi\alpha^2 t}\eta(it)}{\Theta_{11}(\alpha, it)}\right] \\
&\times \sum_{w_0=-\infty}^{\infty} \exp\left[-\frac{4\pi^2 w_0^2 \beta_{\mathrm{IA}}^2}{\alpha'} t\right] \\
&\times \left[\frac{\Theta_{00}(\alpha, it)}{e^{-\pi\alpha^2 t}\eta(it)} \left(\frac{\Theta_{00}(0, it)}{\eta(it)}\right)^3 - \frac{\Theta_{01}(\alpha/, it)}{e^{-\pi\alpha^2 t}\eta(it)} \left(\frac{\Theta_{01}(0, it)}{\eta(it)}\right)^3\right] \\
&+ N^2 (8\pi^2\alpha')^{-5}(1+\tanh(\pi\alpha)) \int_0^\infty \frac{dt}{t} \cdot t^{-5} e^{-R^2 t/2\pi\alpha'} \times [[\eta(it)]^{-6}] \\
&\times \frac{1}{4} \left[\frac{e^{-\pi\alpha^2 t}\eta(it)}{\Theta_{11}(\alpha, it)}\right] \left[\frac{\Theta_{10}(\alpha, it)}{e^{-\pi\alpha^2 t}\eta(it)} \left(\frac{\Theta_{10}(0, it)}{\eta(it)}\right)^3 + \frac{\Theta_{11}(\alpha, it)}{e^{-\pi\alpha^2 t}\eta(it)} \left(\frac{\Theta_{11}(0, it)}{\eta(it)}\right)^3\right] \\
&\times \sum_{w_0=-\infty}^{\infty} \exp\left[-\frac{4\pi^2 w_0^2 \beta_{\mathrm{IA}}^2}{\alpha'} t\right],
\end{aligned}
\tag{3.1}
$$

where the last term from the Ramond-Ramond (R-R) sector is only formal, since $\Theta_{11}(0, 1t) = 0$.

Moving on to the corresponding results for the Möbius strip and Klein bottle, we express each worldsheet modular integral in terms of the variable t, where t is the intrinsic length of either holes or crosscaps on the one-loop unoriented type IB string worldsheets [31]. For the Möbius strip topology, we have

$$
\begin{aligned}
\rho^{(\mathrm{IA})}_{\mathrm{mob}} = &-2N(2^5)(8\pi^2\alpha')^{-5}(1+|B_{09}|) \int_0^\infty \frac{dt}{t} \cdot t^{-5} e^{-R^2 t/2\pi\alpha'} \times [\eta(it)^{-6}] \left[\frac{e^{-\pi\alpha^2 t}\eta(it)}{\Theta_{11}(\alpha, it)}\right] \\
&\times \sum_{w_0=-\infty}^{\infty} \exp\left[-\frac{4\pi^2 w_0^2 \beta_{\mathrm{IA}}^2}{\alpha'} t\right] \\
&\times \left[\frac{\Theta_{01}(\alpha, it)}{e^{-\pi\alpha^2 t}\eta(it)} \left(\frac{\Theta_{01}(0, it)}{\eta(it)}\right)^3 \frac{\Theta_{10}(\alpha/, it)}{e^{-\pi\alpha^2 t}\eta(it)} \left(\frac{\Theta_{10}(0, it)}{\eta(it)}\right)^3\right] \\
&\times \frac{e^{-\pi\alpha^2 t}\eta(it)}{\Theta_{00}(\alpha, it)} \left(\frac{\eta(it)}{\Theta_{00}(0, it)}\right)^3,
\end{aligned}
\tag{3.2}
$$

and likewise, summing unoriented type IB worldsheets with the topology of a Klein bottle, we have

$$\rho_{\rm kb}^{\rm (IA)} = 2^{10}(8\pi^2\alpha')^{-5}(1+|B_{09}|)\int_0^\infty \frac{dt}{t}\cdot t^{-5}e^{-R^2t/2\pi\alpha'}\times[\eta(it)^{-6}]\left[\frac{e^{-\pi\alpha^2 t}\eta(it)}{\Theta_{11}(\alpha,it)}\right]$$
$$\times\sum_{w_0=-\infty}^{\infty}\exp\left[-\frac{4\pi^2w_0^2\beta_{\rm IA}^2}{\alpha'}t\right]$$
$$\times\left[\frac{\Theta_{00}(\alpha,it)}{e^{-\pi\alpha^2t}\eta(it)}\left(\frac{\Theta_{00}(0,it)}{\eta(it)}\right)^3-\frac{\Theta_{01}(\alpha,it)}{e^{-\pi\alpha^2t}\eta(it)}\left(\frac{\Theta_{01}(0,it)}{\eta(it)}\right)^3\right]$$
$$-2^{10}(8\pi^2\alpha')^{-5}(1+|B_{09}|)\int_0^\infty\frac{dt}{t}\cdot t^{-5}e^{-R^2t/2\pi\alpha'}\times[[\eta(it)]^{-6}]$$
$$\times\frac{1}{4}\left[\frac{e^{-\pi\alpha^2t}\eta(it)}{\Theta_{11}(\alpha,it)}\right]\left[\frac{\Theta_{10}(\alpha,it)}{e^{-\pi\alpha^2t}\eta(it)}\left(\frac{\Theta_{10}(0,it)}{\eta(it)}\right)^3+\frac{\Theta_{11}(\alpha,it)}{e^{-\pi\alpha^2t}\eta(it)}\left(\frac{\Theta_{11}(0,it)}{\eta(it)}\right)^3\right]$$
$$\times\sum_{w_0=-\infty}^{\infty}\exp\left[-\frac{4\pi^2w_0^2\beta_{\rm IA}^2}{\alpha'}t\right]. \tag{3.3}$$

A. Low-temperature massless limit of the type I $O(32)$ string

We begin with the low-temperature limit of the annulus amplitude by making the change of variable $y = A/t$ in order to make the integral representation of the Whittaker function evident:

$$\rho_{\rm ann}^{\rm (IB)} = -(8\pi^2\alpha')^{-5}(1+\pi\alpha)\int_0^\infty\frac{dt}{t}\cdot t^{-5}\sum_{m=0}^{\infty}\sum_{n_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}f_m^{\rm (IB)}(\alpha)$$
$$\times\exp\left[-\pi mt+t\left(\frac{\alpha' n_9^2}{R_{\rm IB}^2}+\frac{4\pi^2n_0^2\alpha'}{\beta_{\rm IB}^2}\right)\right]$$
$$=-(8\pi^2\alpha')^{-5}(1+\pi\alpha)\sum_{n_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\sum_{m=0}^{\infty}f_m^{\rm (IB)}(\alpha)$$
$$\times\left[m+\frac{4\pi^2n_0^2\alpha'}{\beta_{\rm IB}^2}+\frac{\alpha' n_9^2}{R_{\rm IB}^2}\right]^{-5}\int dyy^4e^{-1/y}$$
$$=-(8\pi^2\alpha')^{-5}(1+\pi\alpha)\sum_{n_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\sum_{m=0}^{\infty}f_m^{\rm (IB)}(\alpha)$$
$$\times\left[m+\frac{4\pi^2n_0^2\alpha'}{\beta_{\rm IB}^2}+\frac{\alpha' n_9^2}{R_{\rm IB}^2}\right]^{-5}A^2\Gamma(-5)e^{A/2}\mathcal{W}_{3,-5/2}(A). \tag{3.4}$$

Substituting into this expression the power-series expansion of the Whittaker function gives the result

$$\rho_{\rm ann}^{\rm (IB)} = -(8\pi^2\alpha')^{-5}(1+\pi\alpha)\sum_{n_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\sum_{m=0}^{\infty}f_m^{\rm (IB)}(\alpha)\left[m+\frac{4\pi^2n_0^2\alpha'}{\beta_{\rm IB}^2}+\frac{\alpha' n_9^2}{R_{\rm IB}^2}\right]^{-5}$$
$$\times\frac{\Gamma(-5)(-1)^5}{\Gamma(1)\Gamma(-5)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(k-5)}{k!(k-5)!}A^k(\psi(k+1)+\psi(k-4)-\psi(k-5)-\ln(A))\right.$$
$$\left.+(-A)^5\sum_{k=0}^{4}\frac{\Gamma(5+k)\Gamma(k)}{k!}(-A)^k\right\}. \tag{3.5}$$

Expanding about the massless limit, and setting the degeneracies to the massless bosonic spacetime modes alone, we can sum the thermal momentum modes to extract the zeta function, $\zeta(-2,0)$, and the T^{10} leading behavior of the low-energy finite-temperature gauge theory:

$$\begin{aligned}\rho_{\mathrm{ann}}^{(\mathrm{IB})} &\simeq -(8\pi^2\alpha')^{-5}\sum_{n_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty} b_0^{(\mathrm{IB})}\left[\frac{4\pi^2 n_0^2\alpha'}{\beta_{\mathrm{IB}}^2}+\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}\right]^{-5}\\ &\quad\times\frac{\Gamma(-5)(-1)^5}{\Gamma(1)\Gamma(-5)}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(k-5)}{k!(k-5)!}A^k(\psi(k+1)+\psi(k-4)-\psi(k-5)-\ln(A))\right.\\ &\quad\left.+(-A)^5\sum_{k=0}^{4}\frac{\Gamma(5+k)\Gamma(k)}{k!}(-A)^k\right\}\\ &\simeq -(8\pi^2\alpha')^{-5}T^{10}\zeta(-2,0)496(4\pi^2\alpha')^{-5}\{1\}. \end{aligned} \tag{3.6}$$

Finally, we note that the normalization of the heterotic $\mathrm{Spin}(32)/\mathrm{Z}_2$ string vacuum energy density can be determined from this result by matching with the corresponding graph of the type IB superstring, since the massless 496 gauge bosons of the spacetime gauge group are restricted to the oriented open string sector. Comparing with the analogous zero-temperature spacetime bosonic massless mode limit of the heterotic string one-loop amplitude,

$$\rho_{\mathrm{H}}(\beta)\simeq -2\mathcal{N}496\frac{1}{5!}(4\pi^2\alpha')^{-5}\times\frac{1}{4}(4\pi^2\alpha')^5T^{10}, \tag{3.7}$$

we find the simple result

$$2^{-5}=2\mathcal{N}\frac{1}{5!}\frac{1}{4}. \tag{3.8}$$

B. High mass level limit of the type IB $\mathbf{Spin(32)/Z_2}$ string

We begin with the high-temperature limit of the one-loop vacuum energy density of the open oriented sector derived above, recognizing in that expression the integral representation of the modified Bessel function:

$$\begin{aligned}\rho_{\mathrm{ann}}^{(\mathrm{IB})} &= \left(8\pi^2\alpha')^{-5}(1+\tanh(\pi\alpha))\int_0^\infty\frac{dt}{t}\cdot t^{-5}\sum_{m=0}^{\infty}\sum_{w_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\right.\\ &\quad\times f_m^{(\mathrm{IB})}(\alpha)\exp\left[-\frac{\pi m}{t}-t\left(\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}+\frac{4\pi^2 w_0^2\beta_{\mathrm{IB}}^2}{\alpha'}\right)\right]\\ &\simeq (8\pi^2\alpha')^{-5}(1+\tanh(\pi\alpha))\sum_{m=0}^{\infty}f_m^{(\mathrm{IB})}(\alpha)\sum_{w_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\\ &\quad\times(m\pi)^{5/2}\left[\frac{4\pi^2 w_0^2\beta_{\mathrm{IB}}^2}{\alpha'}+\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}\right]^{-5/2}K_5(z). \end{aligned} \tag{3.9}$$

The Bessel function can be replaced by its asymptotic expansion (GR 8.446.1) in the limit of high mass level numbers, and we can also set the tanh function to unity, and the f_m to their values at the self-dual temperature:

$$\begin{aligned}\rho_{\mathrm{ann}}^{(\mathrm{IB})} &= 2(8\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}f_m^{(\mathrm{IB})}(1)\sum_{w_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\\ &\quad\times(m\pi)^{5/2}\left[\frac{4\pi^2 w_0^2\beta_{\mathrm{IB}}^2}{\alpha'}+\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}\right]^{-5/2}K_5(z). \end{aligned} \tag{3.10}$$

Restricting to the degeneracies of the bosonic spacetime modes alone, $b_m^{(\mathrm{IB})}$, the result is a damping of the Hagedorn growth of the numerical degeneracies for large level number by the exponential of the square root of the mass level number with a coefficient which is always large at high mass level numbers and high temperature. It is helpful to T dualize to the thermal dual large radius, β_{IA}, since the thermal IB coordinate is approaching the small-radius limit:

$$\begin{aligned}\rho_{\mathrm{ann}}^{(\mathrm{IB})} &= 2(8\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}b_m^{(\mathrm{IB})}(1)\sum_{w_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\\ &\quad\times(m\pi)^{5/2}\left[\frac{4\pi^2 w_0^2\beta_{\mathrm{IB}}^2}{\alpha'}+\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}\right]^{-5/2}\\ &\simeq 2(8\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}b_m^{(\mathrm{IB})}(1)\sum_{w_0=-\infty}^{\infty}\sum_{n_9=-\infty}^{\infty}\\ &\quad\times m^{9/4}2^6\pi^{-5/2}e^{-4\pi w_0 T_{IA}\alpha'^{1/2}[1+\frac{n_9^2\beta_{\mathrm{IA}}^2}{w_0^2R_{IB}^2}]^{1/2}\sqrt{m}}. \end{aligned} \tag{3.11}$$

This completes our demonstration of the finiteness of the type IB open and closed superstring theory. It should be noted that the high mass level number limit of the unoriented graphs are also integral representations of the modified Bessel function, and their asymptotic growth can be analyzed similarly.

IV. TYPE I PAIR CORRELATOR OF SPACELIKE WILSON LOOPS

In the massless mode, field-theoretic, limit of the type IB superstring amplitude annulus graph, the spacelike Wilson loop expectation value [15–17,32] is the change in the internal energy of the finite-temperature gauge theory vacuum due to the introduction of an infinitely massive quark in the presence of the external NS two-form field.[4]

[4]A shift of the NS two-form potential by an external Abelian gauge field strength gives the result in the presence of an external constant chromoelectric field, with slow-moving heavy quarks, or, in an external chromomagnetic field, with static heavy quarks [15,16,32].

The spacelike Wilson loop is the world history of a semiclassical heavy charged color source living in the fundamental representation of an $O(4)$ subset of the $O(16)$ gauge group. The coincidence of two D8-branes, and their orientifold images, gives four additional massless, zero-length, open string modes, which are states that complete the $\mathbf{3} \oplus \mathbf{3}$ of the $O(4) \simeq SU(2) \times SU(2)$ gauge group. Thus, the single spacelike Polyakov-Susskind loop at spatial coincidence is the world history of a heavy quark in the $\mathbf{2} \oplus \mathbf{2}$ representation of $O(4)$. Namely, the parallel stack of two D8-branes, and their two orientifold image D8-branes, at one of the orientifold planes of the $O(16) \times O(16)$ type IA string compactified on a twisted torus, coincide to give all of the massless zero-length open strings in the adjoint representation of $O(4)$, and the Chan-Paton factor for the end points themselves, i.e., the end-point wave function, transforms in the $\mathbf{2} \oplus \mathbf{2}$ fundamental irrep of $SU(2) \times SU(2)$. Note that the Wilson loop operator always contains a trace over the representation of the non-Abelian gauge group.

The pair correlator of spacelike Polyakov-Susskind loops, $\mathcal{W}^{(2)}$, can be derived from first principles. Incorporating the changes required by finite temperature for the type IB superstring compactified on the twisted torus, and using the results of Refs. [15,16] for superstring amplitudes with macroscopic incoming and outgoing strings, gives the following expression for the open oriented contribution to the pair correlator of parallel spacelike loops spatially separated by a distance $R_{\rm IB}$:

$$
\begin{aligned}
\mathcal{W}^{(2)}_{\rm IB} = {}& (1 + |\tanh(\pi\alpha)|) \int_0^\infty \frac{dt}{2t} (2t)^{1/2} \frac{e^{-R^2_{\rm IB} t/2\pi\alpha'}}{\eta(it)^6} \left[\frac{e^{i\pi t\alpha^2}\eta(it)}{\Theta_{11}(\alpha, it)}\right] \\
& \times \sum_{n_i=-\infty}^{\infty} \exp\left[-\pi\left(\frac{\alpha' n_9^2}{R^2_{\rm IB}} + \frac{4\pi^2\alpha' n_0^2}{\beta^2_{\rm IB}}\right) t\right] \\
& \times \left[\frac{\Theta_{00}(\alpha, it)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{00}(0, it)}{\eta(it)}\right)^3 - \frac{\Theta_{01}(\alpha, it)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{01}(0, it)}{\eta(it)}\right)^3 - \frac{\Theta_{10}(\alpha, \tau)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{10}(0, it)}{\eta(it)}\right)^3\right] \\
& - (1 + |\tanh(\pi\alpha)|) \int_0^\infty \frac{dt}{2t} \cdot (2t)^{1/2} [\eta(it)]^{-6} e^{-R^2_{\rm IB} t/2\pi\alpha'} \\
& \times \frac{1}{4}\left[\frac{e^{i\pi t\alpha^2}\eta(it)}{\Theta_{11}(\alpha, it)}\right]\left[\frac{\Theta_{11}(\alpha, it)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{11}(0, it)}{\eta(it)}\right)^3\right] \\
& \times \sum_{n_i=-\infty}^{\infty} \exp\left[-\pi\left(\frac{\alpha' n_9^2}{R^2_{\rm IB}} + \frac{4\pi^2\alpha' n_0^2}{\beta^2_{\rm IB}}\right) t\right]. \qquad (4.1)
\end{aligned}
$$

Note that the expression above is valid for all values of $T = 1/\beta_{\rm IB}$, where B and β are both target spacetime moduli, and $|B|$ is linear for small T, and asymptotes to unity at temperatures approaching the string deconfinement scale.

The massless Yang-Mills gauge field theory limit of $\mathcal{W}^{(2)}_{\rm IB}$ yields the potential between two heavy color-charged sources at spatial separations $R_{\rm IB} > \alpha'^{1/2}$. We work in the large-radius limit, and at type IB temperatures much below the string scale. Note that the amplitude $\mathcal{W}^{(2)}$ is dimensionless. We will find that this is a good paradigm for a non-Abelian gauge theory in the *deconfinement* regime; all of the thermal excitations are Matsubara modes, namely, thermal momenta, and the type IB superstring has no winding modes. The heavy quark pair can be pulled out to spatial separations larger than the string scale, giving clear evidence for the inverse linear term which is universal—the Lüscher term, common to all effective Nambu-Goto-Eguchi-Schild-Polyakov QCD strings [1,2,23]. In addition, we can also derive the systematic thermal corrections to the Lüscher potential.

Retaining the leading terms in the q expansion, dominated by thermal momentum modes at low temperatures far below the string mass scale, and performing an explicit term-by-term integration over the worldsheet modulus, t, isolates the leading terms in the massless string spectrum, $m = 0$, and thermal and spatial Kaluza-Klein modes. In the low-temperature regime, the inverse temperature lies within the range, $2\pi\alpha'^{1/2} \ll R_{\rm IB} \ll \beta_{\rm IB}$, in string scale units [16], and we substitute the power-law expansion for the gamma function, after a change of variable, $t \to At$. The argument of the gamma function, $\Gamma(z)$, takes the form, $|z| < 1$:

$$
A = \left[m + \frac{n_9^2\alpha'}{R^2_{\rm IB}} + \frac{R^2_{\rm IB}}{4\pi^2\alpha'} + \frac{n_0^2 4\pi^2\alpha'}{\beta^2_{\rm IB}}\right], \qquad (4.2)
$$

expanding about $m = 0$. The result of the modular integral can be expressed in terms of the power-series expansion of the gamma function, with argument $z = \frac{1}{2}$:

$$\Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k, \qquad c_0 = 1, \qquad c_1 = -\mathrm{C}, \qquad c_{n+1} = \sum_{k=0}^{n} \frac{(-1)^{k+1} s_{k+1} c_{n-k}}{n+1}, \qquad s_1 = \mathrm{C}, \qquad s_n = \zeta(n), \quad (4.3)$$

which gives the result

$$\begin{aligned}
\mathcal{W}_{\mathrm{IA}}^{(2)} &= 2^{-1/2}(1 + |\tanh(\pi\alpha)|) \sum_{m=0}^{\infty} f_m^{(\mathrm{IB})}(\alpha) \sum_{n_0=-\infty}^{\infty} \sum_{n_9=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t} t^{1/2} e^{-t} \\
&\quad \times \left(\frac{R^2}{2\pi^2\alpha'} + \frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2} + \frac{n_0^2 4\pi^2\alpha'}{\beta_{\mathrm{IB}}^2} \right)^{-1/2} \\
&\simeq 2^{-1/2}\Gamma(1/2)(1 + \tanh(\pi\alpha)) f_0^{\mathrm{IB}}(\alpha) \times \left(\frac{R^2}{2\pi^2\alpha'} + \frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2} + \frac{n_0^2 4\pi^2\alpha'}{\beta_{\mathrm{IB}}^2} \right)^{-1/2} \\
&\simeq 2^{-1/2}\Gamma(1/2)(1 + \pi\alpha) \times \left(\frac{R^2}{2\pi^2\alpha'} + \frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2} + \frac{n_0^2 4\pi^2\alpha'}{\beta_{\mathrm{IB}}^2} \right)^{-1/2} \\
&\quad \times [2(2\mathrm{Cosh}(2[\pi \tanh(\pi\alpha)]) + 6) - 16\mathrm{Cosh}([\pi \tanh(\pi\alpha)])] \\
&= \Gamma(1/2)(1 + \pi\alpha)[16 - 16(1 + (\pi\alpha)^2)] \\
&\quad \times \left[\frac{\pi\alpha'^{1/2}}{R_{\mathrm{IB}}} - \zeta(-2,0) \left(\frac{4\pi^5 \alpha'^{5/2}}{\beta_{\mathrm{IB}}^2} \right) \frac{1}{R_{\mathrm{IB}}^3} + O(\alpha'^{9/2}/R_{\mathrm{IB}}^5 \beta_{\mathrm{IB}}^4) \right], \qquad (4.4)
\end{aligned}$$

where we recall that the inverse temperature lies within the range, $2\pi\alpha'^{1/2} < \beta_{\mathrm{IB}} \ll R_{\mathrm{IB}}$, in string scale units [16]. Our result shows that the leading correction to the inverse linear attractive potential, namely, the universal Lüscher term, in the zero-temperature static heavy quark potential, is $O(1/R^3)$, taking the form of a systematic series expansion in powers of $(\alpha'^2/\beta_{\mathrm{IB}}^2 R^2)$ at type IB temperatures far above the thermal duality transformation temperature, namely, the string mass scale, $T_C = \frac{1}{2\pi}\alpha'^{-1/2}$.[5]

We now perform a thermal duality transformation on the expression for the type IB pair correlator of spacelike Wilson loops, in addition to a spatial T_9-duality transformation. Expressing the result in terms of type IA string variables, the target spacetime geometry is that of a stack of 32 thermal D8-branes in the 10D type IA $O(32)$ superstring compactified on R^8, with a $S^1/Z_2 \times S^1/Z_2$ orthogonal to the worldvolume of the thermal D8-brane stack.[6] We consider a pair of heavy colored sources whose world histories are loops winding along X_E^0, which is now an interval of length β_{IA}; hence the world histories of the infinitely heavy "quarks" are stretched parallel to the Euclidean time interval. In addition, they are spatially separated by a Dirichlet-string of length R, stretched parallel to the interval X_9'. Note that upon T dualizing both the X^9 and X^0 coordinates, we obtain a type IA superstring theory, with a tower of spatial, and thermal, winding modes, replacing the thermal momentum modes of the type IB superstring. Thus, the infinitely heavy color sources are now confined in a bound state with spatial separation within a type IA string length; remarkably, we will show that we can nevertheless derive analytical expressions for the binding energy.

We find that this novel type TA phase is a good model for the *confinement* phase of non-Abelian gauge theories, with its tower of thermal and spatial winding string modes. Comparing with the expression for the pair correlator of spacelike Wilson loops in the finite-temperature type IB superstring theory given in Eq. (4), we can repeat the steps taken above, and extract the massless level $m = 0$ low-energy gauge theory limit. The result is

$$\begin{aligned}
\mathcal{W}_{\mathrm{Linear}}^{(2)} &\simeq \alpha'^{-1/2}\Gamma(1/2)[2(2\mathrm{Cosh}(2[\pi \tanh(\pi\alpha)]) + 6) \\
&\quad - 16\mathrm{Cosh}([\pi \tanh(\pi\alpha)])] \\
&\quad \times R\left[1 - \frac{1}{2}\sum_{w_0=-\infty}^{\infty} \left(\frac{4\pi^2}{\alpha'^2}\right)(w_0^2 \beta_{\mathrm{IA}}^2 R^2)\right] \\
&= \alpha'^{-1/2}\Gamma(1/2)[2(2\mathrm{Cosh}(2[\pi \tanh(\pi\alpha)]) + 6) \\
&\quad - 16\mathrm{Cosh}([\pi \tanh(\pi\alpha)])] \\
&\quad \times R[1 - \zeta(-2,0)\alpha'^{-2}\beta_{\mathrm{IA}}^2 R^2]. \qquad (4.5)
\end{aligned}$$

This expression gives the subleading thermal correction to the linear potential for a pair of semiclassical heavy quarks,

[5]Note the remarks by Lüscher and Weisz in Ref. [23] on the absence of a $1/R^2$ term in the heavy quark potential, presciently arguing in favor of the leading $1/R^3$ correction. It should be noted their argument is valid on general grounds, and is not specific to the finite-temperature gauge theory, but it is exactly what we too find in our derivation from string theory of the static heavy quark potential at finite temperature.

[6]A thermal Dp-brane has only p noncompact coordinates, and a p-dimensional Euclidean, spatial, worldvolume. The gauge fields supported on this brane are finite-temperature supersymmetric gauge theories in $(p+1)$ dimensions.

with spatial separation R, and with a Dirichlet spatial winding mode string stretched between them. Pulling apart the heavy colored sources, gives a potential that grows linearly with R and the Dirichlet string is the confining string. There is a potential energy cost to separating the color-charged sources, and the temperature-dependent term is a further suppression, pointing to confinement.

Comparing with Eq. (17), note that the binding energy density is a continuous function of temperature at the string scale critical temperature, but the first derivative with respect to temperature has a discontinuity. Remarkably, precise computations can nevertheless be carried out on either side of the phase boundary in temperature by, respectively, working in the respective low-energy gauge theory limits of thermal and spatial dual string theories (both type IB and type IA). Thus, our results are clearly suggestive of a thermal deconfinement phase *transition* at $T_C = 1/2\pi\alpha'^{1/2}$ in the type IA gauge theory, and this deconfining phase transition can be identified as *first order*. In particular, we note that taking $R \to 0$ smoothly gives a vanishing expectation value for the single Polyakov-Susskind loop in the confinement regime, as was conjectured for the order parameter of the thermal deconfinement transition—for the $O(2n)$ groups the center symmetry is $Z_2 \times Z_2$. This completes our discussion of the thermal deconfinement transition and its order parameter for the $O(32-2n)$ anomaly-free non-Abelian gauge theory limit of the finite-temperature type I superstring. In closing, we should point out that we have barely touched the considerable information in the full string pair correlation function, and the thermal spectrum with massive winding modes, which remains for future analysis.

V. CONCLUSIONS

The original suggestion that the Polyakov string path integral might provide a renormalizable analytic description of the expectation value of a Wilson loop valid to arbitrarily short distances was made by Alvarez in Ref. [3], although its implementation at the time was stymied by many technical, and conceptual, problems. The extension of the string path-integral formalism for on-shell scattering amplitudes to those for the off-shell closed string tree propagator, incorporating the modified Dirichlet, or Wilson loop, boundary conditions proposed by Alvarez in Ref. [3], was given by Cohen, Moore, Nelson, and Polchinski [19]. The suggestive sketch of the computation given in Ref. [19] was subsequently reformulated by us in collaboration with our students Chen and Novak [15–17], giving a proper implementation of the super boundary reparametrization invariance, that was also Weyl invariant, and for macroscopic Wilson loops. We incorporated also the modern framework of Dirichlet strings, and D-branes and orientifold planes in background two-form field strengths [17,25,32]. Most importantly, a discussion of thermal deconfinement required the development of a consistent Euclidean time quantization of finite-temperature superstring theory, settling also the troubling issue of the Hagedorn divergence of the degeneracies of the string mass level expansion, which is suppressed by a compensating exponential suppression arising from the integral over worldsheet moduli that preserves worldsheet reparametrization invariance.

Our results show that the Polyakov macroscopic string path-integral framework [15,16,19] provides not merely an effective QCD string model for the computation of the expectation value of the spacelike history of a semiclassical heavy quark, but also strong evidence for the veracity of heterotic type I superstring theories as accurate descriptions of the real world at particle-accelerator-scale short distances and high energies, with a precise derivation of their low-energy gauge theory limit. In particular, our work expands upon the original analyses by Brink, Green, and Schwarz, and others [6,21,33], by performing in closed form the integrals over worldsheet moduli, and thereby preserving the worldsheet super Diff $\times$ Weyl gauge symmetries, when taking both the low-energy supergravity and non-Abelian gauge theory limits [15,16], in the presence of background two-form field strengths [17], and at finite temperature. Our considerations have been restricted to one-loop string amplitudes, and it is fascinating to observe the wealth of new physics which can be extracted with the inclusion of macroscopic superstring amplitudes. In particular, it has been interesting to compare our results with the analytic gauge theory methodology pursued by Poppitz *et al.* [13].

Perhaps the most remarkable result of our analysis is the light it sheds on the nature of the relationship between the type IB and heterotic strings. While the former has a low-energy field theory limit which is the closest approximation to pure gauge theory as we know it, the latter carries the fuller insight into nonperturbative string/M-theory. The type IB–heterotic strong-weak duality is the most striking insight we have into the as yet unknown M-theory, and our results further strengthen a rather benign conclusion: the well-established fact that M-theory on $S^1 \times S^1/Z_2$ is dual to a string theory, or a field theory, in every one of its low-energy limits, continues to hold at high temperatures, and high string mass levels! We have made this behavior rather explicit in terms of the precise exponential suppression, or exponential balance, of the Hagedorn growth of the string d.o.f. at high temperature. Most pertinently, while there is a clear match to the physics of a thermal deconfinement transition in the low-energy field theory limit, there are no infinities in the full string theory, and we have a finite description of the phase transition. Furthermore, our results in the Appendix suggest a novel approach to studying the strong-weak heterotic–type I duality by relating it to the dualities of the type IIA and type IIB on $K3 \times T^2/Z_2$ string, which would simplify our understanding of which is the most fundamental of the underlying string dualities.

ACKNOWLEDGMENTS

This research was funded in part by the National Science Foundation CAREER Program (1997–2001), and completed with personal funds. I would like to acknowledge Paul Plassmann for his unstinting encouragement and patience in bringing this work to completion. I would like to acknowledge my collaborators, Yujun Chen and Eric Novak, for their helpful participation in the early stages of this research. I thank Erich Poppitz for his interest, and useful comments on the manuscript. Paul Aspinwall, David Gross, Andreas Kronfeld, Juan Maldacena, Stephen Shenker, Herman Verlinde, and Edward Witten have been helpful at various stages during this research, which was brought to completion in the stimulating environs of the Lewis-Fine and Firestone Libraries at Princeton University.

APPENDIX A: ASPECTS OF HETEROTIC STRING THEORY

In this Appendix, we clarify certain aspects of the worldsheet formalism of the heterotic strings, showing their intimate relation to the basic building blocks—the type IIB and the type IIA superstrings—and shedding light on the strong-weak dualities of the superstrings and M-theory.

1. Chiral $N_S = 1$ orbifolds of the 10D type II superstrings

Recall that the type II superstring theories are named by the parity of their 32-component 10D Majorana-Weyl spinors. In the type IIA string theory, the 10D spinors, and massless spin-3/2 gravitinos, have opposite spacetime parity, whereas in the chiral IIB string theory, they have identical spacetime parity. This property distinguishes the two type II superstrings. Recall that spacetime parity is given by the product of left and right worldsheet parities. We will show that the Hilbert space of the type IIA superstring admits *two* inequivalent $N_S = 1$ chiral projections, where the subscript denotes the 10D target spacetime supersymmetry. One of which eliminates the higher-rank p-form potentials of the Ramond-Ramond sector, whereas the other gives the T_9-dual of the familiar type IB orientifold.

We will follow the pedagogical derivation of the sum over spin structures for the type IIA and type IIB superstrings, given in Sec. 10.6 of Ref. [34]. Each has two equivalent, and self-consistent, ways to sum over spin structures, which result in the Gliozzi-Scherk-Olive (GSO) projection to states with even spacetime G parity (respectively, chiral and nonchiral ten-dimensional type II superstrings). We begin with the $N_S = 1$ chiral projection of the Hilbert space of the type IIIB superstring leading to the well-known type IB orientifold [21]:

$$\begin{aligned}\text{type IIB}:&\ ((NS+,NS-)\oplus(NS-,NS+))_s,(R-,NS+),(NS+,R-),(R-,R-),\\ \text{type IB}:&\ ((NS+,NS-)\oplus(NS-,NS+))_s,((NS+,R-)\oplus(R-,NS+))_s,(R-,R-).\end{aligned}\tag{A1}$$

Notice that the alternative choice of Hilbert space and worldsheet parity assignments in Ref. [34] gives the same result with an Ω projection, since the type IIB superstring is a chiral theory:

$$\begin{aligned}\text{type IIB}':&\ (NS+,NS+),(R+,NS+),(NS+,R+),(R+,R+),\\ \text{type IB}':&\ (NS+,NS+),((NS+,R+)\oplus(R+,NS+))_s,(R+,R+).\end{aligned}\tag{A2}$$

Note that the 10D vector spinor now has positive spacetime parity, but the theory is identical to that above, being a mere rewriting of the type IB orientifold projection.

Let us now contrast this with the inequivalent $N_S = 1$ chiral projections of the 10D nonchiral type IIA superstring, which can be written as (see Chapter 10.6 of Ref. [34])

$$\begin{aligned}\text{type IIA}:&\ (NS+,NS+),(R+,NS+),(NS+,R-),(R+,R-),\\ \text{type IIA}':&\ (NS+,NS+),(NS+,R+),(R-,NS+),(R-,R+).\end{aligned}\tag{A3}$$

Notice that we have the freedom to symmetrize the type IIA superstring Hilbert space over both choices of GSO convention for $\tilde{F}$: $e^{\pi i F} = 1$; with $e^{\pi i \tilde{F}} = +1(R)$, $-1(NS)$, and $e^{\pi i \tilde{F}} = -1(R)$, $+1(NS)$:

$$\begin{aligned}\text{type IIA}:&\ ((NS+,NS-)\oplus(NS-,NS+))_s,((NS+,R+)\oplus(R+,NS+))_s,\\ &\ ((R-,NS+)\oplus(NS+,R-))_s,((R-,R+)\oplus(R+,R-))_s,\\ \text{type IIA}':&\ (NS+,NS+),((NS+,R+)\oplus(R+,NS+))_s,\\ &\ ((R-,NS+)\oplus(NS+,R-))_s,((R-,R+)\oplus(R+,R-))_s.\end{aligned}\tag{A4}$$

Under an Ω projection on the former, only states symmetric under the interchange of left and right movers remain, which eliminates one of the 10D vector spinors, giving the 10D $N=1$ type I′, or type IA, string:

$$\text{type IA}:\ ((NS+,NS-)\oplus(NS-,NS+))_s,((R-,NS+)\oplus(NS+,R-))_s,((R-,R+)\oplus(R+,R-))_s. \qquad \text{(A5)}$$

Note that the left and right worldsheet parity of the states in the Virasoro tower that contains the 10D spacetime vector-spinor are *opposite*, giving a spacetime spinor with *negative* 10D parity.

The Hilbert space of the type IIA superstring allows an even simpler N_S chiral truncation which eliminates the Ramond-Ramond states and does not break Poincaré invariance. This chiral projection can therefore be identified as the heterotic string: the superconformal gauge-fixed $N_s=(1,0)$ conformal field theory has central charge $c=(12,8)$, and all closed string states with *mixed* left- and right-moving worldsheet parity are absent. We have

$$\text{type IA}':\ (NS+,NS+),((R+,NS+)\oplus(NS+,R+))_s, \qquad \text{(A6)}$$

with the following bosonic massless particle spectrum:

$$(\mathbf{8}_v+\mathbf{8})\times\mathbf{8}_v=(\mathbf{1},\mathbf{1})+(\mathbf{28},\mathbf{1})+(\mathbf{35},\mathbf{1})+(\mathbf{56},\mathbf{1})+(\mathbf{8}',\mathbf{1}). \qquad \text{(A7)}$$

It is helpful to restate our result for the two distinct freely acting asymmetric orbifolds of the type IIA string as follows. Modding by a spacetime reflection on a target space coordinate $X^9\to-X^9$, $\psi^9\to\psi^9$, $\tilde\psi^9\to-\tilde\psi^9$, maps the IIA to the equivalent IIA′ sum over spin structures. Modding in addition by the discrete groups, and setting $(-1)^{F_L}=+1$, $(-1)^{F_R}=+1$, in *both* the Ramond and Neveu-Schwarz sectors, defines the truncation to the type IA′ orbifold, an anomalous 10D $N_S=1$ theory that we will show can be extended to either of the two ultraviolet finite and infrared unambiguous, exact renormalized heterotic string theories.

In closing, it should be noted that our derivation of the heterotic strings as $N_S=1$ chiral projections of the 10D type IIA superstring has the following important consequence: under the chiral projection, the zero-momentum states in the Hilbert space of the heterotic descendant, for both physical and ghost d.o.f., will be unchanged from those deduced from a Bechi-Rouet-Stora-Tyutin analysis of the type IIA superstring. In other words, as reviewed in the Appendix, the measure in the string path integral is the same as that for the type IIA superstring, namely, it preserves the Wess-Zumino gauge-fixed $N=(1,1)$ local worldsheet supersymmetry, except for choices of spin structure which imply the presence of supermoduli, or conformal Killing spinors. Neither is present at genus one, except in the Ramond-Ramond sector.

Note that the type IIA R-R p-form potentials are projected out of the Hilbert space of the descendant, since the chiral projection removes all states in the type IIA Hilbert space with mixed left and right worldsheet parity. The parity projection, however, retains the *constant* modes of the odd rank R-R potentials of the type IIA theory. This suggests that heterotic strings can be formulated in backgrounds with constant R-R type IIA p-form potentials of definite parity. Note that R-R fluxes, and, consequently, D-brane sources, are always absent in the heterotic descendants.

Notice that because of our identification of spacetime parity with the *product* of worldsheet parities, following the projection to positive spacetime parity, Ramond worldsheet fermions only appear in the Hilbert space of the right-moving superconformal field theory. We no longer have the ingredients to build the spinorial **8** or **8′** in the Hilbert space of the left-moving conformal field theory.[7]

2. Heterotic string descendants of the type IIA string

An alternative means of fulfilling the infrared consistency conditions on a closed string theory with massless chiral fermions in the **8′** is available for the chiral projection of the type IIA superstring. Notice that the projection to states with positive spacetime parity eliminates the R-R sector of the worldsheet superconformal field theory *in its entirety*. One consequence, of course, is that the heterotic string theory therefore cannot accommodate D-branes and, based on our discussion in the previous section, it is clear that there is no consistent extension incorporating open string sectors.

In addition, as mentioned earlier, the necessary ingredients for building a spinorial vacuum no longer exist in the left-moving conformal field theory. We wish to extend the spectrum of the closed string theory in such a way that the low-energy limit yields additional supermultiplets with chiral fermions. Such chiral fermions can

[7]It is conventional to refer to the superconformal half of the heterotic string theory as right-moving, or antiholomorphic, listing the boundary conditions on *right-moving* worldsheet fermions before those on fermions in the left-moving, or holomorphic, sector [35]: ([right],[left]), as in the equation above. Thus, it is the right-movers that will provide a realization of the SO(8) subgroup of the 10D Lorentz group in the heterotic string theory, giving rise to the $SO(8)_{\rm spin}$ representations listed. Note that right-moving worldsheet fields are distinguished by tildes.

contribute the necessary compensating terms to the anomaly polynomials. The analysis of the hexagon anomaly in the chiral ten-dimensional $N = 1$ supergravity reveals that coupling to the chiral fermions of a 10D super-Yang-Mills theory with precisely 496 massless vector bosons meets the conditions for the cancellation of all anomalies: gauge, gravitational, and mixed. What extension to the chiral projection of the type IIA string theory can account for this massless field content in the low-energy field theory limit? Recall that as a consequence of the chiral projection, we begin with the bosonic massless particle spectrum:

$$(NS+, NS+) \oplus (R+, NS+):\ (\mathbf{8}_v + \mathbf{8}) \times \mathbf{8}_v = (\mathbf{1},\mathbf{1}) + (\mathbf{28},\mathbf{1}) + (\mathbf{35},\mathbf{1}) + (\mathbf{56},\mathbf{1}) + (\mathbf{8}',\mathbf{1}). \tag{A8}$$

We wish to augment this bosonic massless particle spectrum with a $(\mathbf{8}_v, \mathbf{496})$ of vector bosons, and their (8,496) chiral superpartners under the 10D $N = 1$ supersymmetry. The two Yang-Mills gauge groups with an adjoint representation of dimension 496 are $SO(32)$ and $E_8 \times E_8$.

The clue towards uncovering the nature of the fully consistent heterotic string lies in the peculiar mismatch in the properties of the Hilbert spaces of left- and right-moving conformal field theories of the type IIA string following the chiral projection to physical states with positive spacetime parity. Note that the worldsheet local superconformal algebra following the chiral projection remains the familiar (1,1) superconformal field theory (SCFT) underlying the type IIA superstring, except that the superconformal generators of the left-moving (super) conformal field theory, which belongs in the $(\mathrm{NS}+, \mathrm{R}-)$ sector of the IIA string, can no longer contribute to the physical Hilbert space of the heterotic string theories because of the restriction to states of positive spacetime parity! We emphasize that the total central charge of the worldsheet superconformal field theory is (15,15), just as in the type II superstrings, and the transverse d.o.f. that remain after superconformal gauge fixing, following the elimination of timelike and longitudinal worldsheet (1,1) supermultiplets together with compensating bosonic and fermionic ghosts, are also the familiar (12,12) of the type II superstrings.[8] Notice that the physical Hilbert space has only states of positive definite norm. Thus, without having any impact on the target spacetime Lorentz and supersymmetry algebra, we can self-consistently extend the *left-moving* conformal field theory with a unitary compact chiral conformal field theory, subject to the overall constraints of modular invariance on the expression for the string vacuum amplitude. Recall that invariance of the one-loop vacuum amplitude under the modular group of the torus also determines the physical state spectrum.

From our earlier discussion on the 10D type II superstrings, the one-loop vacuum amplitude for a heterotic string theory will therefore take the general form

$$\begin{aligned} W_{\mathrm{het}} = L^{10}(4\pi^2\alpha')^{-5} \int_{\mathcal{F}} \left\{ \frac{d^2\tau}{4\tau_2^2} \cdot (\tau_2)^{-4} [\eta(\tau)\bar{\eta}(\bar{\tau})]^{-8} \right\} \\ \times \frac{1}{4} \left[\left(\frac{\tilde{\Theta}_{00}}{\tilde{\eta}}\right)^4 - \left(\frac{\tilde{\Theta}_{01}}{\tilde{\eta}}\right)^4 - \left(\frac{\tilde{\Theta}_{10}}{\tilde{\eta}}\right)^4 - \left(\frac{\tilde{\Theta}_{11}}{\tilde{\eta}}\right)^4 \right] \times Z_{\mathrm{chiral}}(\tau). \end{aligned} \tag{A9}$$

Recall that the factor within curly brackets is already modular invariant. The holomorphic sum over Jacobi theta functions is the remnant contribution from the eight transverse worldsheet fermions of the type IIA string following the projection to states of positive spacetime parity. Under a $\tau \to \tau + 1$ transformation, this function transforms as

$$\tau \to \tau + 1:\ \left[\left(\frac{\Theta_{00}}{\eta}\right)^4 - \left(\frac{\Theta_{01}}{\eta}\right)^4 - \left(\frac{\Theta_{10}}{\eta}\right)^4 - \left(\frac{\Theta_{11}}{\eta}\right)^4 \right] \to e^{2\pi i/3} \left[\left(\frac{\Theta_{00}}{\eta}\right)^4 - \left(\frac{\Theta_{01}}{\eta}\right)^4 - \left(\frac{\Theta_{10}}{\eta}\right)^4 - \left(\frac{\Theta_{11}}{\eta}\right)^4 \right] \tag{A10}$$

[8]To the best of our knowledge, the perspective on the heterotic string theories offered here is completely new. The traditional approach, including that followed in the original papers [35], has been to invoke the Hamiltonian quantization of independent left-moving and right-moving 2D conformal field theories with total central charge (15,26): namely, the left-moving half of a 26D bosonic string theory and the right-moving half of a 10D type II superstring. Eight of the 24 transverse bosonic left-moving modes are subsequently paired with the eight transverse right-moving bosonic modes of the superstring, and identified as the transverse "coordinates" of a 10D target spacetime. Our goal here is to point out that there exists an alternative derivation of the heterotic string theories as chiral projections of the type IIA superstring that can reproduce the results of the traditional construction. To be precise, the *physical* Hilbert spaces and *on-shell* scattering amplitudes derived in either approach will be indistinguishable.

and it is invariant under the transformation $\tau \to -1/\tau$. Thus, the function $Z_{\rm chiral}(\tau)$ must transform with the compensating phase under the $\tau \to \tau + 1$ transformation, and is required to be *invariant* under a $\tau \to -1/\tau$ transformation. The latter property is very significant. In terms of the vertex operator construction for the physical states in the chiral conformal field theory, namely, those counted by the level expansion of the function $Z_{\rm chiral}$, it implies the self-consistency, or *closure*, of the vertex operator algebra.

Closed vertex operator algebras with central charge $c > 1$ are known to exist for only special values of the central charge. Vertex operator algebras with c taking *integer* values are characterized by the properties of Euclidean, even self-dual lattices of dimensionality c [35]. The smallest integer solutions with $c > 1$ are 8 and 16, and the corresponding Euclidean even self-dual lattices contain, respectively, the direct sum of the root and weight lattices of the simply laced Lie algebras E_8, and either $E_8 \times E_8$ or $SO(32)$.[9] Recall that the contribution to the vacuum energy from a chiral vertex operator algebra with central charge c is $c/24$. For $c = 16$, we have $E_{\rm vac} = +2/3$, and either of the given 16-dimensional Euclidean lattices contains precisely 480 lattice vectors of squared length two. Thus, either choice is a consistent candidate that can provide the 496 massless vector bosons in the $({\rm NS}+, {\rm NS}+; \mathbf{r}) = (\mathbf{8}_v, 1; 496)$ sector, as was required by the infrared consistency conditions mandating the absence of gauge, gravitational, and mixed anomalies:

$$\frac{1}{4}\alpha'({\rm mass})^2_L = N_b^\mu + N_b^I + N_f^\mu - 1 + \frac{1}{2}k_L^2 = \frac{1}{4}\alpha'({\rm mass})^2_R = \tilde{N}_b^\mu + \tilde{N}_f^\mu - \frac{1}{2}. \tag{A11}$$

Here, μ runs from 1 to 8, labeling the transverse modes of the 10D type IIA superstring. The index I runs from 1 to 16, labeling the 16 orthogonal directions of the Euclidean even self-dual lattice that self-consistently extends the left-moving conformal field theory, following the chiral projection to type IIA states with positive spacetime parity. Note that the lattice vector $\mathbf{k}_L$ has 16 components, and states in the CFT with $\mathbf{k}_L^2 = 2$ correspond to massless physical states in the closed string spectrum. The function $Z_{\rm chiral}(\tau)$ takes the form

$$Z_{\rm chiral}(\tau) = [\eta(\tau)]^{-16} \sum_{\mathbf{k}_L \in \Lambda_{16}} q^{\frac{1}{2}\mathbf{k}_L^2}, \tag{A12}$$

where Λ_{16} is an even self-dual lattice of rank 16. Namely, for every pair of vectors $\mathbf{k}$, $\mathbf{k}'$ in Λ_{16}, $k \cdot k'$ is an even integer, and both the vector, $\mathbf{k}$, and its dual, k^*, where $\mathbf{k} \cdot \mathbf{k}^* = 1$, belong in the lattice Λ_{16}. The transformation $\tau \to -1/\tau$ simply interchanges the root lattice with its dual lattice, the direct sum of the weight lattices of the irreducible representations of the Lie algebra. The overall multiplicative factor $(-i\tau)^8$ is canceled by the corresponding transformation of the eta function. Under the $\tau \to \tau + 1$ transformation, the lattice summation instead transforms by an overall phase which is unity for rank 16. Thus, the only factor of relevance is the overall phase $e^{-2\pi i/3}$ in the transformation of $[\eta(\tau)]^{-16}$; this phase is canceled by the corresponding transformation of the antiholomorphic sum over spin structures, namely, that for fermions in the right-moving superconformal field theory. Invoking boson-fermion equivalence in two dimensions, it is sometimes convenient to write the result for the one-loop vacuum amplitudes of the two heterotic string theories, respectively, in the alternative form [35]

$$Z_{\rm SO(32)}(\tau) = \frac{1}{2}\left[\left(\frac{\Theta_{00}}{\eta}\right)^{16} + \left(\frac{\Theta_{01}}{\eta}\right)^{16} + \left(\frac{\Theta_{10}}{\eta}\right)^{16} + \left(\frac{\Theta_{11}}{\eta}\right)^{16}\right],$$
$$Z_{E_8\times E_8}(\tau) = \frac{1}{4}\left[\left(\frac{\Theta_{00}}{\eta}\right)^{8} + \left(\frac{\Theta_{01}}{\eta}\right)^{8} + \left(\frac{\Theta_{10}}{\eta}\right)^{8} + \left(\frac{\Theta_{11}}{\eta}\right)^{8}\right]^2, \tag{A13}$$

inferred from the equivalent fermionic representation of the respective chiral vertex operator algebras with $c = 16$ by 16 complex (Weyl) worldsheet fermions. It is helpful to summarize the full massless spectrum of the two heterotic string theories. For the SO(32) and $E_8 \times E_8$ theories, respectively, we have

$$(\mathbf{8}_v + \mathbf{8}) \times (\mathbf{8}_v, \mathbf{1}) = (\mathbf{1}, 1) + (\mathbf{28}, 1) + (\mathbf{35}, 1) + (\mathbf{56}, 1) + (\mathbf{8}', 1)(10{\rm D}\, N = 1 \text{ supergravity}),$$
$$(\mathbf{8}_v + \mathbf{8}) \times (\mathbf{1}, \mathbf{496}) = (\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}, \mathbf{496}), \quad \text{or}$$
$$(\mathbf{8}_v + \mathbf{8}) \times (\mathbf{1}, \mathbf{496}) = (\mathbf{8}_v, \mathbf{120}, 1) + (\mathbf{8}_v, 1, \mathbf{120}) + (\mathbf{8}_v, \mathbf{128}, 1) + (\mathbf{8}_v, 1, \mathbf{128}) + (\mathbf{8}, \cdots). \tag{A14}$$

[9]More precisely, the even self-dual lattice obtained in the latter case pertains to the Lie algebra ${\rm Spin}(32)/Z_2$; the Z_2 projection removes the root vectors of squared length unity, so that the massless vector bosons in the string spectrum live in a 496 of the simply laced algebra SO(32).

We have spelled out the $SO(16) \times SO(16)$ decomposition of the 496 states in the adjoint representation of $E_8 \times E_8$ in the last equation. Note that the $\mathbf{56} + \mathbf{8}'$ of the 10D $N = 1$ supergravity multiplet is generated by the $\mathbf{8} \times \mathbf{8}_v$; the gauginos live in an 8, precisely as in the type IB unoriented string, and as required by the anomaly cancellation conditions. However, unlike the type IB supergravity, there appears to be no consistent extension to the spectrum of antisymmetric supergravity potentials in the heterotic string theories because the Ramond-Ramond sector of the "parent" type IIA superstring, namely $(\mathrm{R}+, \mathrm{R}-)$, has *negative* spacetime parity.

3. Compact Lie algebras and the R-R ten-form

We begin with a simple explanation of the realization of $E_8 \times E_8 \times \mathrm{Z}_2$ in the type IIA string with parallel stacks of eight D8-branes—and their eight orientifold image branes —separated by the interval, X_9', obtained by T_9 dualizing the circle X_9 in the type IB superstring with 16 space-filling D9-branes, plus their 16 orientifold image branes, and the gauge group $O(32)$. The Dirichlet coordinate separating the O8-planes is of length R_9. The gauge group realized on each of the D8-brane stacks is clearly $O(16)$. Are there any additional massless open string states? If we introduce a D0-brane on the stack of D8-branes, including its image D0-brane, the D0-brane pair can freely explore all of the 16 D8-branes plus images. Zero-length D0-D0, or D0-D8 strings in the Neveu-Schwarz sector are massive, due to the vacuum energy. However, in the Ramond sector of the open string spectrum, we can describe the vacuum state of the D0-brane pair as follows: its intersection with each D8-brane is a two-state Ramond vacuum, of charge $\pm\frac{1}{2}$, and vacuum energy, $E_0 = \frac{1}{2}(\frac{1}{2})^2$.

In nine dimensions, and below, it is well known that the two, apparently inequivalent, ten-dimensional electrically charged heterotic string theories with $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathrm{Z}_2$ gauge symmetry [35], are, in fact, related by a T_9-duality transformation upon compactification on a circle. A Wilson line background in the $E_8 \times E_8$ string continuously interpolates between the two stable and supersymmetric heterotic string vacua, leaving unbroken all 16 conserved supercharges. On the other hand, it is well known that the only allowed perturbative type IB gauge groups obtained by an analysis of Chan-Paton factors are the classical groups A_n, B_n, C_n, and D_n, as was proven by Marcus and Sagnotti [36]. This leaves us with the following puzzle: by the Polchinski-Witten type IB heterotic string-string duality map, it would have to be true that the strong-coupling dual of the 9D heterotic $E_8 \times E_8 \times U(1)$ vacuum with 16 unbroken supersymmetries should be a stable, massless-tadpole and tachyon-free, *nonperturbative* background of the open and closed unoriented type IB string [37]. It was suggested in the early work [38], that in the presence of D0-branes, in addition to the 16 D8-branes on either of the two orientifold planes bounding the interval in the type IA string with $SO(16) \times SO(16)$ gauge fields [37], the non-Abelian gauge symmetry might extend to the elusive $E_8 \times E_8$. Many authors subsequently attempted to solve this problem with partial success [26], but without pointing out that the 9D type I $E_8 \times E_8$ vacuum is tachyon and tadpole free, an exact renormalized background with 16 conserved supercharges. We will fill in the gaps in that sketchy presentation in what follows, in response to questions since put to me, while also completing the details for type I realizations of all of the simply-laced, and non-simply-laced, compact Lie algebras in the Cartan-Weyl classification.

My original goal was to provide a realization of the exceptional Lie algebras in 9D D-brane backgrounds of the type IB and type IA strings. We will now show that, in fact, D-branes cover all of the Cartan-Weyl classification A_n, B_n, C_n, D_n, E_6, E_7, and E_8, including both simply-laced, exceptional, and non-simply-laced Lie algebras. This fact, long elusive, becomes rather obvious, once we establish a detailed isomorphism, namely, a one-to-one mapping, between the standard root and weight systems of the Lie algebras and the sequence of jumps in the ten-form when a D0-brane crosses a D8-brane in a generic type IA orientifold. In addition to reviewing this analysis of positive, and negative, vacuum energy contributions from the intersection, or crossing, of D0- and D8-branes, we will give an even simpler derivation, directly in terms of the allowed "no-force" configurations of D0-branes in the worldvolume of the stack of D8-branes, such that the state conserves 16 supersymmetries.

Recall that, unlike the origin of gauge symmetry in affine Lie algebras embedded in the bulk worldsheet conformal field theory, non-Abelian gauge symmetry in the type IB string open and closed string has a completely different origin [21,34]. The Chan-Paton wave functions labeling the end points of open strings provide a representation of a Lie group, rather than a Lie algebra, and the massless lowest-lying mode in the open string spectrum lives in the adjoint representation of this group. This counting gives rise to what were known to be the list of possible perturbative type I gauge groups: $U(n)$, $O(2n)$, $Sp(2n)$, and $O(2n+1)$ [36].

We remind the reader that the root and weight lattice and Dynkin diagram representations of Lie algebras do not distinguish between the classical and exceptional algebras in the Cartan-Weyl classification of the compact Lie algebras. In Ref. [25], we noticed that a precise counting of states in the $SO(16)$ spinor weight lattice is given rather easily by an isomorphism to the sequence of jumps in the Ramond-Ramond ten-form field upon D0- and D8-brane crossings along the interval between the two O8-planes [37]. We will derive the massless Ramond-Ramond tadpole cancellation conditions that single out the stable 9D type IA background with $E_8 \times E_8$ non-Abelian gauge symmetry. Note that this is a nonperturbative background of the type IA, or type IB, string [37,38].

Let us begin with type IA backgrounds with 32 D8-branes. The gauge group on the 9D worldvolume of eight coincident D8-branes, and images, is given by the counting of zero-length strings stretched between any pair of D8-branes on either O8-plane. Note that for realizations of the classical groups, we can count massless gauge bosons by the traditional counting for the (classical) $SO(2n)$ group: $2n$ choices of D8-branes (or image) at one end point, $2n-1$ choices for the other end point, with a factor of 2 for symmetry under interchange. This method of counting is predicted by T_9 duality, from the usual counting of Chan-Paton wave functions in the type IB vacuum [21,34,36].

Fortunately, in the T_9 dual type IA backgrounds, we have an equivalent prescription that extends to cover exceptional algebras as was discovered by us in Ref. [25]: the isomorphism of ten-form profiles on D8-D8-brane crossings to a Lie algebra root lattice. We include all profile vectors with net vanishing ten-form background on the O8-plane; the compensating ten-form profile with an overall negative sign exists for the image D8-D8 crossings. In addition, we impose the cancellation of both dilaton tadpoles and anomalies on each O8-plane. It is easy to verify the correctness of this prescription for the $SO(16)$ lattice when $n=\bar{n}=8$:

$$2n(n-1)/2 + 2\bar{n}(\bar{n}-1)/2 = 56 + 56. \tag{A15}$$

We have either $(+,+)$ or $(+,-)$ ten-form profiles for zero-length strings on any pair of coincident D8-branes. The factor of 2 counts the negative of the profile, and the factor of 1/2 corrects for overcounting, since this is an interchange of branes and images: the pair of image D8-branes *always* has the compensating ten-form background so that there is no net ten-form on the O8-plane. This is consistent with our requiring the absence of dilaton tadpoles in the absence of a gradient in the ten-form on either O8-plane. The eight gauge bosons transforming in the $U(1)^8$ subalgebra arise from the (null) ten-form background due to a soliton between the D8-brane and its own image, giving a total of 120 massless gauge bosons.

Let us move on to the $2m$D0-$2n$D8-O8 background; m is an integer, and we hope to find a solution for $n=8$, since we wish to keep the 120 $SO(16)$ gauge bosons. The massless R-R tadpole cancellation for $2m$D0-$2n$D8 strings can be expressed by combining the conditions for $2n$D8-$2n$D8, $2m$D0-$2n$D8, and $2m$D0-$2m$D0 strings stretched between O8-planes. It is obvious that there is no solution unless $m=8$, since the two O8-planes each contribute-16 to these equations:

$$(2n)(2n) - 2^4[2n+2n] + 2^8 = 0,$$
$$(2m)(2n) - 2^4[2m+2n] + 2^8 = 0,$$
$$(2m)(2m) - 2^4[2m+2m] + 2^8 = 0. \tag{A16}$$

4. Sign of the R-R ten-form and SU(2) anomaly cancellation

The counting of zero-length D0-D8 strings on each O8-plane is as follows [25]. A D0-D8 crossing results in soliton string creation, and there are eight D0-D8 soliton strings, plus images, at each O8-plane. We can represent the ten-form profile at the eight D0-D8 crossings by an eight-component vector; the profiles take the form

$$\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\right)$$
$$\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \oplus \text{permutations}$$
$$\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \oplus \text{permutations}. \tag{A17}$$

This gives a total of $2\cdot(1/1!\oplus(8\cdot 7)/2!)\oplus(8\cdot 7\cdot 6\cdot 5)/4!2! = 128$ vectors. Each ten-form profile corresponds to a degenerate vacuum: the distinct profiles differ by permutations of the eight D0-D8 solitons, and are indistinguishable in the coincidence limit. The factor of 2 accounts for the interchange of branes and images; the factorials correct for permuting indistinguishable solitons. Recall the standard definition of the spinor lattice of $O(16)$: we include all vectors of squared length two, with each component normalized to $\pm\frac{1}{2}$, and an *even* total number of plus signs. Namely, we have either a (8,0), (6,2), or (4,4), split of plus and minus signs. The restriction to an odd total number of plus signs will give the conjugate spinor lattice.

In the spirit of tracing all string consistency requirements to infrared target space physics [21,34], we require the absence of gauge, gravitational, and mixed anomalies. The above-mentioned "sign" rule for consistent ten-form profile vectors at the eight D0-D8 crossings has a simple low-energy field theory origin in the SU(2) anomaly first noticed in Ref. [39]. E8 contains $SO(16)$, and $SO(16)=(SO(4))^4=(SU(2)\times SU(2))^4$. The SU(2)'s come in *pairs* in all consistent backgrounds; a single, unpaired, SU(2), in the low-energy gauge group indicates an anomalous vacuum, and it is well known that the Kalb-Ramond field of string theory entering the Green-Scwarz mechanism can only correct for an Abelian anomaly [34]. Thus, an infrared consistency condition is the cancellation of all $SU(2)$ anomalies, leading to the following consequence: the D8-branes can only be moved into the bulk spacetime in pairs, each with its image. These rules tell us what gauge groups can arise in nonanomalous vacua by moving D8-branes in pairs off the O8-planes.

Note that we can invoke a T_9 duality transformation, from type IA with eight coincident D0-D8 to eight coincident type IB D1-D9 solitons. How do we deduce the number of gauge bosons and the gauge group for zero-size coincident D-string–D9 solitons in the T-dual-type IB state? After all, this is a nonperturbative type IB background, and we need a method other than the counting of Chan-Paton wave functions. Fortunately, the reversed T_9-duality provides the answer. Recall that there is a Wilson line, $A_9 = ((\frac{1}{2})^8; 0^8)$, responsible for breaking the original $O(32) \times U(1)$ to $O(16) \times O(16) \times U(1)$ in the 9D type IB string, labeling the 32 D9-branes as two identical stacks of 16 D9-branes, coincident with the O9 plane. Thus, we can deduce via T-duality, the existence of a 9D type IB D1-D9 background with 16 supercharges and a Yang-Mills gauge group *extended to* $E_8 \times E_8 \times U(1)$; the necessary D-strings are wrapped around the circle.

5. Affine Lie algebras and type I duals of the CHL strings

A related puzzle also having to do with enhanced gauge symmetry, and with fundamental consequences for the connectivity of the landscape, arises as follows. The non-simply-laced algebras $Sp(2n)$, $SO(2n+1)$, F_2, and G_4 are known to arise at enhanced symmetry points (ESPs) in the CHL supersymmetry-preserving Abelian Z_N orbifold moduli spaces, each with 16 supercharges [28], including the 8D $Sp(20)$ and $E_8 \times SO(5)$ ESPs in the moduli space of the Z_2 orbifold, obtained by modding by the order-two outer automorphism exchanging the identical E_8 Euclidean self-dual lattices, accompanied by a Z_2 shift in the 2D momentum lattice, and at the fermionic radius [28]: $\mathbf{p} = (p_L|p_R) = (\frac{1}{2}, 0|\frac{1}{2}, 0)$. The resulting shift in masses of string states leaves massless gauge bosons in the diagonal subgroup of $E_8 \times E_8$, while those in the orthogonal E_8 acquire masses of order the string scale. In 9D and below, the shift vector can be chosen to preserve target spacetime supersymmetry [28]. Thus we have a new 9D half-BPS state with 16 unbroken supercharges and the E_8 gauge group, realized at level two. Note that there is no further enhancement of the gauge symmetry at this radius: if R is tuned to the self-dual radius, the Kaluza-Klein $U(1)$ current algebra is enhanced to a $SU(2)$ [28].

It may be helpful to point out that it is possible to find sporadic examples of $c > 24$ self-consistent holomorphic conformal field theories which meet the prerequisites for the closure and completeness of the chiral algebra. As was shown by Lykken and Chung in Ref. [27], using results by Verlinde, holomorphic conformal field theories of twisted Majorana fermions with self-consistent closed operator algebras occur at only specific values of the central charge, namely, 8, 12, 14, 16, 18, 20, 24, 32, and beyond, deduced by imposing the requirements of a self-consistent fusion algebra on the tensor product of an even number of twisted $c = \frac{1}{2}$ Majorana (real) fermions [27]. Such an analysis cannot provide an exhaustive classification, but suffices to establish the consistency of sporadic self-consistent holomorphic CFTs with $c > 24$, a useful complement to lattice classifications.

It should be noted that the single E_8 current algebra is realized at level two [28]. A fermionic realization of an 8D ESP with 16 unbroken supercharges and gauge group $Sp(20)$ was discovered in Ref. [28]. It turns out to belong in the E_8 moduli space, as shown by us in Ref. [28]. The full structure of the moduli spaces, and the intriguing appearance of a systematic sequence of electric-magnetic dual enhanced gauge symmetry points with non-simply-laced groups, was uncovered by us, using the orbifold technique. This is important, since unlike the simply laced cases, where electric and magnetic groups are the same, the electric and magnetic dual groups differ in the case of any non-simply-laced Lie group. It turns out that it is indeed true that an ESP with non-simply-laced gauge symmetry can appear, without the magnetic dual ESP, in the moduli spaces in Ref. [28] in spacetime dimensions $9 \geq D \geq 5$. Remarkably, precisely as required by the self-duality of the 4D $N = 4$ supergravity coupled to super-Yang-Mills gauge theory, it is only in four dimensions that the moduli space contains *both* of the necessary enhanced symmetry points, with electric-magnetic dual groups interchanged. This last observation is due to Polchinski.

Not surprisingly, we discover the $Sp(20)$ ESP in its moduli space, but in the T_9-dual regime of large type IB radius. It is quite easy to identify the type IA dual of 9D $E_8 \times (U(1))^2$ moduli space once we observe the analogy between D0-D8 crossings and the vectors in the $E_8 \times E_8$ gauge lattice. To begin with, it is helpful to write the ten-form vectors for eight D0-D8 crossings, and their eight image crossings, as 16-component profile vectors. We label the slots in the 16-component profile vectors as follows: $(1, 2, 3, 4, 5, 6, 7, 8|\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8})$. It can be confirmed that this global pairing of D0-D8-branes, and images, is compatible with all 128 ten-form profiles listed above, now written in a 16-component basis. There is no new information in the last eight slots of these vectors; they are the negatives of the first eight:

$$\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\middle|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$$
$$\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\middle|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\right)$$
$$\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\middle|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\right), \tag{A18}$$

plus all vectors equivalent up to permutations of the first eight components. Is it possible to find additional sets of ten-form profile vectors that meet infrared consistency, namely, dilaton tadpole cancellation and the absence of $SU(2)$ anomalies, but *without* a global mutually compatible pairing of all eight D0-D8 crossings and eight image D0-D8 crossings?

For readers familiar with the realization of Lie algebras by Majorana fermions—the worldsheet framework for the fermionic ESPs in the moduli spaces in Refs. [27,28]—it should be obvious that many solutions to this problem are already known. The minimal block of $2n$-component vectors that does not admit a mutually compatible pairing, or complexification, has $n = 8$, as was proven in Ref. [27]:

$$\begin{aligned}
&\left(+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}\middle|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right)\\
&\left(+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\middle|+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right)\\
&\left(+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\middle|+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right)\\
&\left(+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}\middle|+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}\right).
\end{aligned}\tag{A19}$$

The analogy between eight branes, and eight images, and 16 Majorana worldsheet fermions is as follows. The Ramond vacuum of a Majorana fermion exists in one of two possible states which we denote $\pm\frac{1}{2}$. In the worldsheet framework underlying the fermionic ESPs of the moduli spaces in Ref. [28], the vacuum amplitude is a sum over sectors with distinct Ramond, or Neveu-Schwarz, boundary conditions for the 32 worldsheet fermions in the bosonic CFT with total central charge 16. A mutually compatible pairing of Majorana fermions among all sectors summed in the vacuum amplitude provides a complexification of Majorana fermions, $\psi^i + i\psi^{\bar{i}}$, $i = 1, \ldots, n$, where $n \leq 16$. Such a complexification gives n complex fermions, each with central charge one, and, n U(1)'s in the gauge group, since all n lowest excitations in the NS vacuum are retained in the string spectrum: $\psi^i_{-1}\psi^{\bar{i}}_{-1}|0>$.

Conversely, if a complexification of $2n$ worldsheet Majorana fusion algebras does not exist, the gauge group in the type IA vacuum will have n fewer U(1)'s. It was proven in Ref. [27] that the minimal solution has $n = 8$. Moreover, the basic rules for the overlap of common signs among vectors in the block of ten-forms listed above originate as follows: any pair of profile vectors is required to have an overlap of 0 mod 4, while any triad must have an overlap of 0 mod 2. Both of these conditions originate in the ambiguity in the fusion rules of a 2D Majorana fermion conformal field theory; the full derivation can be found in Ref. [27].

The 9D type IA string with 16 unbroken supercharges but eight fewer U(1)'s is a new stable half-BPS state; we introduce the ten-form profiles above in Eqs. (A18) and (A19), with O8-planes at the two end points of the interval. The tadpole cancellation conditions are identical to those in the previous section. The absence of a global pairing of eight D0-D8 crossings, and eight image crossings, on either O8-plane implies a gauge group with eight fewer U(1)'s. The 9D gauge group is $E_8 \times U(1)$. This theory is the type IB dual of the 9D heterotic CHL string inferred in Ref. [28] as an asymmetric orbifold. In 8D, it contains both $E_8 \times SO(5)$ and $Sp(20)$ ESPs [28].

Our identification of an isomorphism between the Ramond-Ramond ten-form field in the bulk between the O8-planes and the root and weight lattices of a Lie algebra in the full Cartan-Weyl classification also leads to a rule for the *sign* of the ten-forms, necessitated by the cancellation of all $SU(2)$ anomalies in the generic D0-D8-O8 background. Finally, we make the following important observation. The type IB-heterotic duality map with $R_{\rm H}$ set to the Dirac fermion radius also establishes that the fermionic CHL strings [28], and type IA-IB duals [25], are exact renormalized conformal field theory backgrounds describing *weakly coupled* ESPs in both the heterotic and the weak-strong dual type IA-IB string moduli spaces, in nine and lower target spacetime dimensions [25]. As pointed out in the main text, while the type IB dual is strongly coupled, a T-duality gives a type IA string that is weakly coupled, so long as the heterotic string does not approach the self-dual compactification radius, $R_{\rm H} = \alpha'^{1/2}$ [37]. We evade this regime by matching the normalization of the heterotic and type IA string vacuum functionals in the small-volume, sub-string-scale, weakly coupled, regime of the type IA string, restricting the compactification radii of the dual $O(32)$ heterotic string to the large-volume regime, $R_{\rm H} \gg \alpha'^{1/2}$, $R_{\rm IB} \gg \alpha'^{1/2}$.

[1] Y. Nambu, QCD and the string model, Phys. Lett. B **80**, 372 (1979); J. Gervais and A. Neveu, The quantum dual string wave functional in Yang-Mills theories, Phys. Lett. B **80**, 255 (1979); A. Schild, Classical null strings, Phys. Rev. D **16**, 1722 (1977); T. Eguchi, New Approach to the Quantized String Theory, Phys. Rev. Lett. **44**, 126 (1980).

[2] M. Luscher, Symmetry-breaking aspects of the roughening transition in gauge theories, Nucl. Phys. **B180**, 317 (1981).

[3] O. Alvarez, The static potential in string models, Phys. Rev. D **24**, 440 (1981); Theory of strings with boundaries, Nucl. Phys. **B216**, 125 (1983).

[4] B. Durhuus, P. Olesen, and J. Petersen, On the static potential in Polyakov's theory of the quantized string, Nucl. Phys. **B232**, 291 (1984).

[5] J. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998); Wilson Loops in Large N Field Theories, Phys. Rev. Lett. **80**, 4859 (1998); S. J. Brodsky and G. de Teramond, AdS/QCD, light-front holography, and sublimated gluons, Proc. Sci., QCD-TNT-II2011 (**2011**) 008, arXiv:1112.4212.

[6] L. Brink, M. Green, and J. Schwarz, $N = 4$ SYM and $N = 8$ supergravity as limits of string theories, Nucl. Phys. **B198**, 474 (1982).

[7] R. Jackiw, Functional evaluation of the effective potential, Phys. Rev. D **9**, 1686 (1974); L. Dolan and R. Jackiw, Phys. Rev. D Symmetry behavior at finite temperature, **9**, 3320 (1974); C. Bernard, Feynman rules for gauge theories at finite temperature, Phys. Rev. D **9**, 3312 (1974).

[8] D. Gross, L. Yaffe, and M. Perry, Instability of flat space at finite temperature, Phys. Rev. D **25**, 330 (1982).

[9] P. Ginsparg, On toroidal compactification of heterotic superstrings, Phys. Rev. D **35**, 648 (1987).

[10] K. Narain, M. Sarmadi, and E. Witten, A note on toroidal compactification of heterotic string theory, Nucl. Phys. **B279**, 369 (1987).

[11] P. Aspinwall and D. Morrison, in *Mirror Symmetry II* (International Press, Cambridge, MA, 1997) p. 703; P. Aspinwall, The moduli space of $N = 2$ superconformal field theories, arXiv:hep-th/9412115.

[12] A useful brief review of supersymmetric gauge theories at finite temperature and finite N can be found in E. Witten, Anti de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998).

[13] E. Poppitz, T. Schaefer, and M. Unsal, Continuity, deconfinement and super Yang-Mills theory, J. High Energy Phys. 10 (2012) 115; E. Poppitz, T. Schaefer, and M. Unsal, Universal mechanism of semi-classical deconfinement and theta-dependence for all simple groups, J. High Energy Phys. 03 (2013) 087; M. Anber, E. Poppitz, and B. Teeple, Deconfinement and continuity between thermal and (super) Yang-Mills theory for all gauge groups, J. High Energy Phys. 09 (2014) 040.

[14] S. Chaudhuri, The normalization of perturbative string amplitudes: Weyl covariance and zeta function regularization, arXiv:hep-th/0409031.

[15] S. Chaudhuri, Y. Chen, and E. Novak, Pair correlation function of Wilson loops, Phys. Rev. D **62**, 026004 (2000).

[16] S. Chaudhuri and E. Novak, Supersymmetric pair correlation function of Wilson loops, Phys. Rev. D **62**, 046002 (2000).

[17] S. Chaudhuri and E. Novak, Effective string tension and renormalizability, J. High Energy Phys. 08 (2000) 027.

[18] J. Polchinski, Evaluation of the one loop string path integral, Commun. Math. Phys. **104**, 37 (1986).

[19] A. Cohen, G. Moore, P. Nelson, and J. Polchinski, An off-shell propagator for string theory, Nucl. Phys. **B267**, 143 (1986); Strings and supermoduli, Phys. Lett. B **169**, 47 (1986).

[20] A. M. Polyakov, Quantum geometry of bosonic strings, Phys. Lett. B **103**, 207 (1982).

[21] J. Schwarz, Superstring theory, Phys. Rep. **89**, 223 (1982).

[22] E. Eichten, K. Gottfried, T. Kinoshita, K. Lane, and T.-M. Yan, Charmonium: The model, Phys. Rev. D **17**, 3090 (1978); Erratum, Phys. Rev. D **21**, 313(E) (1980); M. Peskin, Aspects of the dynamics of heavy quark systems, in *Dynamics and Spectroscopy at High Energy: Proceedings of the 11th SLAC Summer Institute on Particle Physics, Stanford, CA, 1983*, edited by P. M. McDonough (SLAC, Stanford, CA, 1984), p. 151.

[23] M. Luscher and P. Weisz, Quark confinement and the bosonic string, J. High Energy Phys. 07 (2002) 049; String excitation energies in $SU(N)$ gauge theories beyond the free-string approximation, J. High Energy Phys. 07 (2004) 014.

[24] K. J. Juge, J. Kuti, and C. J. Morningstar, Gluon excitations of the heavy quark potential and the hybrid quarkonium spectrum, Nucl. Phys. B, Proc. Suppl. **63**, 326 (1998).

[25] S. Chaudhuri, Confinement and the short flux tube, Nucl. Phys. **B591**, 243 (2000).

[26] C. Bachas and M. Green, On the anomalous creation of branes, arXiv:hep-th/9705074; G. Ferretti and I. Klebanov, Creation of Fundamental Strings by Crossing Dirichlet Branes, Phys. Rev. Lett. **79**, 1984 (1997); O. Bergmann, M. Gaberdiel, and G. Lifschytz, Branes, orientifolds, and the creation of elementary strings, Nucl. Phys. **B509**, 194 (1998); String creation and heterotic-type I' duality, Nucl. Phys. **B524**, 524 (1998); D. Lowe, Bound states of type I' D-particles and gauge symmetry, Nucl. Phys. **B501**, 134 (1997).

[27] S. Chaudhuri, S. Chung, G. Hockney, and J. Lykken, String consistency for unified model building, Nucl. Phys. **B456**, 89 (1995); E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, Nucl. Phys. **B300**, 360 (1988).

[28] S. Chaudhuri, G. Hockney, and J. Lykken, Maximally supersymmetric theories in four dimensions, Phys. Rev. Lett. **75**, 2264 (1995); S. Chaudhuri and J. Polchinski, Moduli space of CHL strings, Phys. Rev. D **52**, 7168 (1995).

[29] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965).

[30] M. Bowick and L. C. R. Wijewardhana, Superstrings at High Temperatures, Phys. Rev. Lett. **54**, 2485 (1985).

[31] M. Bianchi, A note on toroidal compactifications of the type I superstring and other superstring vacuum configurations, arXiv:hep-th/9711201; M. Bianchi and A. Sagnotti, Twist symmetry and open-string Wilson lines, Nucl. Phys. **B361**, 519 (1991); M. Bianchi, G. Pradisi, and A. Sagnotti, Planar duality in the discrete series, Phys. Lett. B **273**, 389 (1991);

C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti, and Y. S. Stanev, Comments on Gepner models and type I vacua in string theory, Phys. Lett. B **387**, 743 (1996).

[32] M. Douglas, D. Kabat, P. Pouliot, and S. Shenker, D-branes and short distances in string theory, Nucl. Phys. **B485**, 85 (1997); S. Shenker, Another length scale in string theory, arXiv:hep-th/9509132.

[33] E. Fradkin and A. Tseytlin, Non-linear electrodynamics from quantized strings, Phys. Lett. B **163**, 123 (1985); A. Abouesaood, C. Callan, C. Nappi, and S. Yost, Open string theory in background gauge fields, Nucl. Phys. **B280**, 599 (1987); See citations in C. Callan and L. Thorlacius, Sigma models and string theory, in *Particles, String, and Supernovae*, edited by A. Jevicki and C.-I. Tan (World Scientific, Singapore, 1989).

[34] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998).

[35] D. Gross, J. Harvey, E. Martinec, and R. Rohm, Heterotic string theory I: The free heterotic string, Nucl. Phys. **B256**, 253 (1985); Heterotic string theory II: The interacting heterotic string, Nucl. Phys. **B267**, 75 (1985).

[36] N. Marcus and A. Sagnotti, Tree level constraints on gauge groups for type I superstrings, Phys. Lett. B **119**, 97 (1982).

[37] J. Polchinski and E. Witten, Evidence for heterotic-type I string duality, Nucl. Phys. **B460**, 525 (1995).

[38] J. Polchinski, TASI lectures on D-branes, arXiv:hep-th/9611050.

[39] E. Witten, An $SU(2)$ anomaly, Phys. Lett. B **117**, 324 (1982).