AdS_2 dilaton gravity from reductions of some nonrelativistic theories

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We study dilaton-gravity theories in two dimensions obtained by dimensional reduction of higher-dimensional nonrelativistic theories. Focusing on certain families of extremal charged hyperscaling violating Lifshitz black branes in Einstein-Maxwell-scalar theories with an extra gauge field in four dimensions, we obtain AdS_2 backgrounds in the near-horizon throats. We argue that these backgrounds can be obtained in equivalent theories of two-dimensional dilaton-gravity with an extra scalar, descending from the higher-dimensional scalar, and an interaction potential with the dilaton. A simple subcase here is the relativistic black brane in Einstein-Maxwell theory. We then study linearized fluctuations of the metric, dilaton and the extra scalar about these AdS_2 backgrounds. The coefficient of the leading Schwarzian derivative term is proportional to the entropy of the (compactified) extremal black branes.

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I. INTRODUCTION

Gravity in two dimensions, trivial as such, is rendered dynamical in the presence of a dilaton scalar and additional matter. Such dilaton-gravity theories arise generically under dimensional reduction from higher-dimensional theories of gravity coupled to matter. There is interesting interplay with AdS_2 holography, which arises in the context of extremal black holes and branes: the near-horizon regions typically acquire an $AdS_2 \times X$ geometry, and a two-dimensional description arises after compactifying the transverse space X. Almheiri and Polchinski [1] considered toy models of two-dimensional dilaton-gravity of this sort, with backgrounds involving AdS_2 with a varying dilaton. Analyzing the backreaction of a minimally coupled scalar perturbation on the AdS_2 background reveals nontrivial scaling of boundary 4-point correlation functions thereby indicating the breaking of AdS_2 isometries in the deep IR. This breaking amounts to breaking of local reparametrizations of the boundary time coordinate (modulo global $SL(2)$ symmetries), which would have been preserved in the presence of exact conformal symmetry. In [2], as well as [3–5], it was argued that the leading effects describing such nearly AdS_2 theories are captured universally by a Schwarzian derivative action governing boundary time reparametrizations modulo $SL(2)$, which arises from keeping the leading nonconstant dilaton behavior. This picture dovetails with the absence of finite energy excitations in AdS_2 discussed previously in [6,7]. Parallel exciting developments involve various recent investigations of the SYK model [8–10], a quantum mechanical model of interacting fermions. This exhibits approximate conformal symmetry at low energies: the leading departures from conformality are governed by a Schwarzian derivative action for time reparametrizations modulo $SL(2)$, as above. A recent review is [11].

AdS_2 throats arise quite generally in the near-horizon regions of extremal black holes and black branes, where other fields acquire near constant "attractor" values. This attractor mechanism, first discussed in [12] for BPS black holes in $N = 2$ theories, arises from extremality rather than supersymmetry, as studied in [13,14]. In the last several years, this has been ubiquitous in the context of nonrelativistic generalizations of holography: a nice review is [15]. A large family of such theories is obtained by considering Einstein-Maxwell-scalar theories with a negative cosmological constant and potential: the $U(1)$ gauge field and scalar serve to support the nonrelativistic background, typically of the form of a Lifshitz, or hyperscaling violating (conformally Lifshitz) theory. The duals to the bulk uncharged black branes in these hvLif theories capture many features of finite density condensed matterlike systems. Towards studying extremal black branes, we note that charge can be added to these theories by adding an additional $U(1)$ gauge field, as discussed in, e.g., [16–18]. Now at extremality, the infrared region approaches an $AdS_2 \times X$ throat, with X typically of the form of an extended transverse plane R^d. The discussion above of AdS_2 holography now applies upon compactifying X taken as, e.g., a torus T^d. This was in fact the broad context for [1]: other recent discussions of reduction from higher-dimensional theories appear in, e.g., [19–25]; see also [26].

Towards studying such AdS_2 theories arising in this nonrelativistic context, we study effective gravity theories of the above form, with two $U(1)$ gauge fields and a scalar field Ψ with a negative cosmological constant and potential. We focus for concreteness on the charged hyperscaling violating Lifshitz black branes in four dimensions described in [17]. In the extremal limit, the near-horizon geometry of these charged hyperscaling violating Lifshitz black branes becomes $AdS_2 \times \mathbb{R}^2$. These charged hyperscaling violating Lifshitz attractors arise for certain regimes of the Lifshitz z and hyperscaling violating θ exponents allowed by the energy conditions, with the additional requirement that the theory exhibits hvLif boundary conditions in the ultraviolet: these are perhaps best regarded as intermediate infrared phases themselves in some bigger phase diagram. Then compactifying the two spatial directions as a torus T^2, we dimensionally reduce this charged hvLif extremal black brane to obtain a two-dimensional dilaton-gravity-matter theory. This theory is equivalent to gravity with a dilaton Φ and an additional scalar Ψ that descends from the hvLif scalar in the higher-dimensional theory, along with an interaction potential $U(\Phi, \Psi)$. The interaction potential raises the question of whether the extra scalar destabilizes the AdS_2 regime, possibly in some region of parameter space. Towards understanding this, we study small fluctuations about the extremal AdS_2 background in these theories and argue that these are in fact stable, the stability stemming from the restrictions imposed on z, θ stated above from energy conditions and asymptotic boundary conditions. Studying the action for small fluctuations up to quadratic order, it can be seen that the leading corrections to AdS_2 arise at linear order in $\delta\Phi$ leading again to a Schwarzian derivative action from the Gibbons-Hawking term, although there are subleading coupled quadratic corrections (Sec. III). The coefficient of the Schwarzian is proportional to the entropy of the compactified extremal black branes, which being the number of microstates of the background is akin to a central charge of the effective theory. In Sec. II A, we first describe in detail the simpler case of the relativistic black brane, which has $z = 1$, $\theta = 0$, arising in Einstein-Maxwell theory, the extra scalar being absent: at leading order this shows how the Jackiw-Teitelboim theory [27,28] arises, with subleading terms at quadratic order. We finally study, in Sec. V, a null reduction of the charged relativistic black brane: this results in charged hvLif black brane backgrounds with specific exponents, but with an extra scalar background profile (for the uncharged case, these coincide with [29]). Section VI contains a brief discussion and an Appendix contains some technical details.

II. EINSTEIN-MAXWELL THEORY IN FOUR DIMENSIONS

Einstein-Maxwell theory with a negative cosmological constant is a useful playground for various interesting physics: see, e.g., [15] for a review. We focus on four dimensions for simplicity: as a consistent truncation of M-theory on appropriate seven-manifolds, the bulk gauge field can be taken as the dual to the $U(1)_R$ current. The action is

$$S = \int d^4x\sqrt{-g^{(4)}}\left[\frac{1}{16\pi G_4}(\mathcal{R}^{(4)} - 2\Lambda) - \frac{1}{4}F_{MN}F^{MN}\right], \tag{2.1}$$

where $\Lambda = -3$ is the cosmological constant in four dimensions. The field equations are

$$\mathcal{R}^{(4)}_{MN} - \Lambda g_{MN} - 8\pi G_4\left(F_{MP}F_N{}^P - \frac{g_{MN}}{4}F^2\right) = 0,$$
$$\partial_M(\sqrt{-g}F^{MN}) = 0. \tag{2.2}$$

These equations have both electrically and magnetically charged black branes as solutions.

Magnetic branes: These are slightly simpler and we discuss them first, mostly reviewing discussions already in the literature. The metric and field strength [30] are

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2),$$
$$f(r) = 1 - \left(\frac{r_0}{r}\right)^3 + \frac{Q_m^2}{r^4}\left(1 - \frac{r}{r_0}\right),$$
$$F_{xy} = Q_m, \tag{2.3}$$

where Q_m is related to the magnetic charge of the black brane, r_0 is the location of the horizon and $r \to \infty$ is the boundary. In the extremal limit, the Hawking temperature vanishes, fixing the horizon location in relation to the charge,

$$T = \frac{3r_0}{4\pi}\left(1 - \frac{Q_m^2}{3r_0^4}\right) = 0 \Rightarrow Q_m^2 = 3r_0^4. \tag{2.4}$$

The near-horizon geometry of the magnetic black brane becomes $AdS_2 \times \mathbb{R}^2$,

$$ds^2 = -r_0^2 f(r)dt^2 + \frac{dr^2}{r_0^2 f(r)} + r_0^2(dx^2 + dy^2),$$
$$f(r)|_{r\to r_0} \simeq \frac{6}{r_0^2}(r - r_0)^2. \tag{2.5}$$

We compactify the two spatial dimensions x^i as T^2 and dimensionally reducing with an ansatz for the metric

$$ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \Phi^2(dx^2 + dy^2), \tag{2.6}$$

with $g^{(2)}_{\mu\nu}$ and Φ being independent of the compact coordinates $x, y \in T^2$. The action (2.1) for the magnetic black brane solution then reduces to

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g^{(2)}} \times \left[\Phi^2\mathcal{R}^{(2)} - 2\Lambda\Phi^2 - \frac{Q_m^2}{2\Phi^2} + 2\partial_\mu\Phi\partial^\mu\Phi\right], \tag{2.7}$$

where $G_2 = G_4/V_2$ is the dimensionless Newton constant in two dimensions. A Weyl transformation $g_{\mu\nu} = \Phi g_{\mu\nu}^{(2)}$ absorbs the kinetic term for the dilaton Φ in the Ricci scalar giving

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\left(\Phi^2\mathcal{R} - 2\Lambda\Phi - \frac{Q_m^2}{2\Phi^3}\right) \equiv \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}(\Phi^2\mathcal{R} - U(\Phi)). \tag{2.8}$$

The equations of motion from this action are

$$U(\Phi) = 2\Lambda\Phi + \frac{Q_m^2}{2\Phi^3}; \qquad \mathcal{R} - \frac{\partial U}{\partial\Phi^2} = 0,$$
$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}U(\Phi) = 0. \tag{2.9}$$

This two-dimensional dilaton-gravity theory admits AdS_2 as a solution with a constant dilaton. This constant dilaton, AdS_2 solution is just the near-horizon AdS_2 geometry of the extremal magnetic black brane in four dimensions (which asymptotically, as $r \to \infty$, is AdS_4).

The purpose of this section was to simply illustrate that the original theory with the gauge field is equivalent to a dilaton-gravity theory with an appropriate dilaton potential: this will be a recurrent theme. A simple toy model capturing many features of two-dimensional dilaton gravity is the Jackiw-Teitelboim theory [27,28]. In the discussion above, we have not been careful with length-scales: in the next section for the relativistic electric brane, we will reinstate various scales.

A. Relativistic electric black brane, reduction to two dimensions

The electric black brane solution to (2.1), (2.2), is

$$ds^2 = -\frac{r^2 f(r)}{R^2}dt^2 + \frac{R^2}{r^2 f(r)}dr^2 + \frac{r^2}{R^2}(dx^2 + dy^2),$$
$$f(r) = 1 - \left(\frac{r_0}{r}\right)^3 + \frac{Q_e^2}{r^4}\left(1 - \frac{r}{r_0}\right),$$
$$A_t = \frac{Q_e}{2\sqrt{\pi G_4}Rr_0}\left(1 - \frac{r_0}{r}\right), \qquad F_{rt} = \frac{Q_e}{2\sqrt{\pi G_4}R}\frac{1}{r^2}. \tag{2.10}$$

The gauge field A_t vanishes at the horizon. The charge parameter Q_e is related to the chemical potential μ and the charge density σ of the black brane as

$$\frac{Q_e}{2\sqrt{\pi G_4}Rr_0} = \mu, \qquad \sigma = \mu\frac{r_0}{R^2} = \frac{Q_e}{2\sqrt{\pi G_4}R^3}. \tag{2.11}$$

Reinstating the dimensionless gauge coupling e^2 in μ and σ as $\mu \to \frac{\mu}{e}$ and $\sigma \to \sigma e$ and using (2.11), we recover the expressions for the gauge field, field strength and the thermal factor in terms of r_0, μ, σ as given in Sec. 4. 2. 1 in [15]. Note that in (2.10) the charge parameter Q_e has dimensions of charge times length-squared, and the gauge field A_t has mass dimension one. In the extremal limit, the temperature vanishes giving

$$T = \frac{3r_0}{4\pi R^2}\left(1 - \frac{Q_e^2}{3r_0^4}\right) = 0 \Rightarrow Q_e^2 = 3r_0^4. \tag{2.12}$$

The near-horizon geometry of the electric black brane becomes $\mathrm{AdS}_2 \times \mathbb{R}^2$,

$$ds^2 = -\frac{r_0^2}{R^2}f(r)dt^2 + \frac{R^2}{r_0^2 f(r)}dr^2 + \frac{r_0^2}{R^2}(dx^2 + dy^2),$$
$$f(r)|_{r\to r_0} \simeq \frac{6}{r_0^2}(r - r_0)^2, \tag{2.13}$$

as in the magnetic case. The Bekenstein-Hawking entropy is the horizon area in Planck units

$$S_{\mathrm{BH}} = \frac{r_0^2}{R^2}\frac{V_2}{4G_4} = \frac{Q_e/\sqrt{3}}{R^2}\frac{V_2}{4G_4}. \tag{2.14}$$

With $V_2 = \int dxdy$ the area, this is finite entropy density for noncompact branes.

It is worth noting that asymptotically, these branes (2.10) give rise to an AdS_4 geometry, with scale R. In the near-horizon region, we obtain an AdS_2 throat with scale $\frac{R}{\sqrt{6}}$: this is a well-defined AdS_2 throat in the regime $\frac{r-r_0}{R} \gg 1$ and $\frac{r-r_0}{r_0} \ll 1$. The AdS_2 region is well-separated from the boundary of the AdS_4 geometry at $r \sim r_C \gg r_0$ if $\frac{r-r_0}{r_c} \ll 1$.

Compactifying the two spatial dimensions x^i as T^2 and dimensionally reducing with the metric ansatz (2.6) reduces the action (2.1) for the electric black brane solution to

$$S = \int d^2x\sqrt{-g^{(2)}}\left[\frac{1}{16\pi G_2}(\Phi^2\mathcal{R}^{(2)} - 2\Lambda\Phi^2 + 2\partial_\mu\Phi\partial^\mu\Phi) - \frac{V_2\Phi^2}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{2.15}$$

and we have suppressed a total derivative term which cancels with a corresponding term arising from the dimensional reduction of the Gibbons-Hawking boundary term (more on this later). Performing a Weyl transformation $g_{\mu\nu} = \Phi g_{\mu\nu}^{(2)}$ to absorb the kinetic term for the dilaton Φ^2 in the Ricci scalar, we get

$$S = \int d^2x\sqrt{-g}\left[\frac{1}{16\pi G_2}(\Phi^2\mathcal{R} - 2\Lambda\Phi) - \frac{V_2\Phi^3}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{2.16}$$

The Maxwell equations for the gauge field are

$$\partial_\mu(\sqrt{-g}\Phi^3F^{\mu\nu}) = 0. \tag{2.17}$$

The two components $b = t,\ r$ of (2.17), i.e., $\partial_t(\sqrt{-g}\Phi^3F^{tr}) = 0 = \partial_r(\sqrt{-g}\Phi^3F^{tr})$, imply

$$\sqrt{-g}\Phi^3F^{tr} = \text{const}. \tag{2.18}$$

Using the gauge field solution in (2.10) to fix this constant as $\frac{Q_e}{2\sqrt{\pi G_4 R^3}}$, we get

$$F^{\mu\nu} = \frac{Q_e}{2\sqrt{\pi G_4}R^3}\frac{1}{\sqrt{-g}\Phi^3}\varepsilon^{\mu\nu}, \tag{2.19}$$

where $\varepsilon^{\mu\nu}$ is defined as $\varepsilon^{tr} = 1 = -\varepsilon^{rt}$ and $\varepsilon_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}\varepsilon^{\rho\sigma}$. Substituting $F_{\mu\nu}F^{\mu\nu} = \frac{-Q_e^2}{2\pi G_4 R^6\Phi^6}$ and $F_{\mu\rho}F^{\rho}_{\nu} = \frac{-Q_e^2}{4\pi G_4 R^6\Phi^6}g_{\mu\nu}$ in Eq. (A1), we get

$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\left(2\Lambda\Phi + \frac{2Q_e^2}{R^6\Phi^3}\right) = 0,$$
$$\mathcal{R} - \frac{\Lambda}{\Phi} + \frac{3Q_e^2}{R^6\Phi^5} = 0. \tag{2.20}$$

These field equations can be obtained by varying the following equivalent action

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\left(\Phi^2\mathcal{R} - 2\Lambda\Phi - \frac{2Q_e^2}{R^6\Phi^3}\right)$$
$$\equiv \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}(\Phi^2\mathcal{R} - U(\Phi)), \tag{2.21}$$

This equivalent action is obtained by substituting the solution for $F^{\mu\nu}$ (in terms of the dilaton Φ^2) in the action (2.16) and changing the sign of the F^2 term which contains a minus sign for electric branes alone, arising from g_{tt} (a similar treatment appears also in, e.g., [5]). Note that this is also consistent with and expected from electric-magnetic duality $Q_e \to Q_m, Q_m \to -Q_e$, which would suggest that the effective dilaton potential for magnetic branes (2.8) is unchanged in going to electric branes. Now for instance the second equation in (2.20) becomes $R - \frac{\partial U}{\partial\Phi^2} = 0$. The constant dilaton, AdS_2 solution to the equations (2.20), consistent with the T^2 compactification of the near-horizon geometry in (2.13), is

$$ds^2 = L^2\left(-\frac{r_0^2}{L^4R^2}(r-r_0)^2dt^2 + \frac{dr^2}{(r-r_0)^2}\right),$$
$$\Phi = \frac{r_0}{R}, \qquad L^2 = \frac{Rr_0}{6}, \qquad Q_e^2 = 3r_0^4, \tag{2.22}$$

with L the AdS_2 scale. Changing the radial coordinate to $\rho = \frac{R^2}{6(r-r_0)}$, we write the metric in conformal gauge

$$ds^2 = e^{2\omega}(-dt^2 + d\rho^2) = e^{2\omega}(-dx^+dx^-), \qquad e^{2\omega} = \frac{L^2}{\rho^2}, \tag{2.23}$$

where the light-cone coordinates are $x^\pm = t \pm \rho$. To see that (2.21) admits the above AdS_2 solution, we compute $\frac{\partial U}{\partial\Phi^2}$ for the above solution, which gives

$$\frac{\partial U}{\partial\Phi^2} = -\frac{12}{Rr_0} = -\frac{2}{L^2} \Rightarrow \mathcal{R} = \frac{\partial U}{\partial\Phi^2} = -\frac{2}{L^2}, \tag{2.24}$$

using (2.20) for the Ricci scalar. This constant dilaton, AdS_2 solution (2.22) is just the compactification of the near-horizon AdS_2 geometry of the four-dimensional extremal electric black brane.

1. *Perturbations about the constant dilaton,* AdS_2 *background*

The four-dimensional theory has a large spectrum of tensor, vector and scalar perturbations, which upon reduction to two dimensions give a corresponding spectrum: we will discuss this briefly later, in Sec. III B 3. In this section, we focus on perturbations to only those fields that have nontrivial background profiles in the effective two-dimensional dilaton-gravity theory; thus, we turn on perturbations to the metric and the dilaton,

$$\Phi = \Phi_b + \phi(x^+, x^-), \qquad \omega = \omega_b + \Omega(x^+, x^-), \tag{2.25}$$

where Φ_b and ω_b denote the background (2.22). We expand the action (2.21) (in conformal gauge) about this background up to quadratic order to get

$$S = \frac{1}{16\pi G_2}\int d^2x\left(4\Phi^2\partial_+\partial_-\omega - \frac{e^{2\omega}}{2}U(\Phi)\right)$$
$$\equiv S_0 + S_1 + S_2, \tag{2.26}$$

where

$$S_0 = \frac{1}{16\pi G_2}\int d^2x\left(4\Phi_b^2\partial_+\partial_-\omega_b - \frac{e^{2\omega_b}}{2}U(\Phi_b)\right) \tag{2.27}$$

is the background action and S_1 is linear in perturbations and vanishes by equations of motion. S_2 is quadratic in perturbations given by

$$S_2 = \frac{1}{16\pi G_2}\int d^2x \times \left(\frac{4r_0^2}{3L^2}\phi\partial_+\partial_-\Omega + \frac{1}{(x^+-x^-)^2}\left(\frac{8r_0^2}{3L^2}\Omega\phi - 16\phi^2\right)\right). \tag{2.28}$$

Varying this action, we get the linearized equations of motion for the perturbations,

$$\partial_+\partial_-\phi + \frac{2}{(x^+-x^-)^2}\phi = 0,$$
$$\partial_+\partial_-\Omega + \frac{1}{(x^+-x^-)^2}\left(2\Omega - \frac{24L^2}{r_0^2}\phi\right) = 0. \tag{2.29}$$

These equations are consistent at linear order with the "constraint" equations for the $++$ and $--$ components of the Einstein equation in (2.20). From these linearized equations, we see that the dilaton fluctuation ϕ is decoupled from the metric fluctuation Ω. Solving the equation for ϕ in (2.29), we get

$$\phi = \frac{a + bt + c(t^2-\rho^2)}{\rho}, \tag{2.30}$$

where a, b, c are independent constants. Substituting the solution (2.30) for ϕ in the equation for Ω in (2.29), we can solve for the metric perturbation Ω, which implies that the AdS$_2$ metric gets corrected at the same order as the dilaton. The on-shell (boundary) action obtained then by using the linearized field equations in (2.28) gives terms at quadratic order in the perturbations,

$$S_2 = \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}n^\mu\left(\frac{2r_0^2}{3L^2}(\Omega\partial_\mu\phi - \phi\partial_\mu\Omega)\right), \tag{2.31}$$

where n^μ is the outward unit normal to the boundary.

2. The Schwarzian effective action

In this section, we switch to Euclidean time $\tau = it$. The Gibbons-Hawking boundary term in the two-dimensional theory arises from the reduction of the corresponding term in the higher-dimensional theory. The Gibbons-Hawking term on the three-dimensional boundary of the four-dimensional theories described by the Euclidean form of the action (2.1) is

$$S_{\rm GH}^{4d} = -\frac{1}{8\pi G_4}\int d^3x\sqrt{\gamma^{(3)}}K^{(4)}, \tag{2.32}$$

where the extrinsic curvature is defined as $K_{AB}^{(4)} = \frac{1}{2}(\nabla_A n_B + \nabla_B n_A)$, n^A being the outward unit normal to the three-dimensional boundary. Using the ansatz (2.6) for the T^2-compactification, dimensionally reducing and performing the Weyl transformation of the two-dimensional metric $g_{\mu\nu} = \Phi g_{\mu\nu}^{(2)}$, the Gibbons-Hawking term reduces to[1]

$$S_{\rm GH}^{4d} = -\frac{1}{16\pi G_2}\int d\tau\sqrt{\gamma}\left(2\Phi^2 K + \frac{3}{2}n_\mu\partial^\mu\Phi^2\right). \tag{2.33}$$

The Ricci scalar term in the bulk four-dimensional Euclidean action upon dimensional reduction and after the Weyl transformation becomes

$$-\sqrt{g^{(4)}}\mathcal{R}^{(4)} = -\sqrt{g}\left(\Phi^2\mathcal{R} - \frac{3}{2}\nabla^2\Phi^2\right). \tag{2.34}$$

Note also that $\sqrt{g^{(4)}} = \sqrt{g^{(2)}}\Phi^2$ and $\Phi^2 = g_{xx}$. We write the total derivative term (the second term) in (2.34) as a boundary term

$$-\frac{1}{16\pi G_2}\int d^2x\sqrt{g}\left(-\frac{3}{2}\nabla^2\Phi^2\right) = \frac{1}{16\pi G_2}\int d\tau\sqrt{\gamma}\left(\frac{3}{2}n_\mu\partial^\mu\Phi^2\right). \tag{2.35}$$

We see that this boundary term which comes from the dimensional reduction of the bulk action in four dimensions cancels the second term in (2.33), thereby giving the Gibbons-Hawking term on the boundary of the two-dimensional theory as

$$S_{\rm GH} = -\frac{1}{8\pi G_2}\int d\tau\sqrt{\gamma}\Phi^2 K. \tag{2.36}$$

Expanding the Gibbons-Hawking term in the perturbations (2.25) and adding it to the Euclidean form of S_2 (which is $S_2^E = -iS_2$, with $t = -i\tau$ in S_2), the leading term in the total boundary action $I_{\rm bdy} = S_2^E + S_{\rm GH}$ arises at linear order in the dilaton perturbation (with subleading terms at quadratic order). To illustrate this in greater detail, it is important that we define the dilaton perturbation in (2.25) in a physically appropriate manner. Since the background value Φ_b is constant, it is sensible to define the dilaton perturbation as

$$\Phi = \Phi_b(1+\tilde\phi), \qquad \Phi_b = \frac{r_0}{R} \Rightarrow \tilde\phi = \frac{\Phi - \Phi_b}{\Phi_b} \ll 1. \tag{2.37}$$

Thus, with this redefinition, the perturbation is reasonable since it automatically satisfies $\tilde\phi \ll 1$. In terms of the dilaton background value Φ_b, the entropy (2.14) is simply

[1]We have $K^{(4)} = \gamma^{(3)AB}K_{AB}^{(4)} = \gamma^{(3)\tau\tau}K_{\tau\tau}^{(4)} + 2\gamma^{(3)xx}K_{xx}^{(4)}$, with $K_{xx}^{(4)} = -\Gamma_{xx}^r n_r = \frac{1}{2}n_r\partial^r\Phi^2 = \frac{1}{2}n_\mu\partial^\mu\Phi^2$ becomes $K^{(4)} = K^{(2)} + \Phi^{-2}n_\mu\partial^\mu\Phi^2$. Then (2.32) gives (2.33) after the Weyl transformation.

$$S_{\rm BH} = \frac{\Phi_b^2 V_2}{4G_4} = \frac{\Phi_b^2}{4G_2}. \qquad (2.38)$$

This gives

$$S_{\rm GH}^{(1)} = -\frac{2\Phi_b^2}{8\pi G_2}\int d\tau\sqrt{\gamma}\tilde{\phi}K \to -\frac{\Phi_b^2}{4\pi G_2}\int du\,\phi_r(u)\{\tau(u),u\}. \qquad (2.39)$$

In evaluating the last term, we take the boundary of AdS_2 as a slightly deformed curve $(\tau(u),\rho(u))$ parametrized by the boundary coordinate u, and define $\tilde{\phi} = \frac{\phi_r(u)}{\epsilon}$, as discussed in [2] (reviewed in [11]). Now using the outward unit normal n^μ to the boundary, we expand the extrinsic curvature. Expanding $S_{\rm GH}^{(1)}$ then leads to a Schwarzian derivative action $Sch(\tau(u),u) = \{\tau(u),u\} = \frac{\tau'''}{\tau'} - \frac{3}{2}(\frac{\tau''}{\tau'})^2$. The integral above pertains only to AdS_2 does not contain any further scales besides the AdS_2 scale L which also appears in the extrinsic curvature giving the Schwarzian (also $\sqrt{\gamma} = \frac{L}{\epsilon}$). The various length scales in the original extremal brane have been absorbed into the AdS_2 scale L. Now we note that the coefficient of the Schwarzian is in fact proportional to the entropy (2.38) of the compactified extremal black brane with V_2 finite (the dependence on Φ_b is expected since it controls the transverse area). Since the entropy captures the number of microstates of the unperturbed background, this is akin to a central charge of the effective theory. Similar comments appear in [10] (see also [19], the Schwarzian arising in some cases from the conformal anomaly).

It is worth noting that the coefficient in the Schwarzian term above is proportional to the extremal entropy after the reasonable definition of the perturbation as (2.37) by scaling out Φ_b: apart from this, the Schwarzian term here is as in [2]. As discussed there, we note that the perturbation makes this nearly AdS_2 and contributes to the near-extremal entropy via the Schwarzian. This can be obtained as in the analysis there by a transformation $\tau(u) = \tan\frac{\tilde{\tau}(u)}{2}$ which gives $S_{\rm GH}^{(1)} = -\frac{\Phi_b^2}{4\pi G_2}\bar{\phi}_r\int du(\{\tilde{\tau}(u),u\} + \frac{1}{2}\tilde{\tau}'^2)$, treating $\bar{\phi}_r$ as constant. Solutions with $\tilde{\tau} = \frac{2\pi}{\beta}u$ have $\tilde{\tau} \sim \tilde{\tau} + 2\pi$, giving the action $S_{\rm GH}^{(1)} = -2\pi^2\frac{\Phi_b^2}{4\pi G_2}\bar{\phi}_r T = -\log Z$, giving the near-extremal correction to the entropy $\Delta S = 4\pi\frac{\Phi_b^2}{4G_2}\bar{\phi}_r T$ (which, being linear in temperature, can also be seen to be the specific heat): this again is proportional to the background entropy with the perturbation defined as (2.37).

The remaining terms in the expansion of $S_{\rm GH}$ and S_2^E are all quadratic in perturbations and, thus, subleading compared to $S_{\rm GH}^{(1)}$. See also, e.g., [19,22,24,25], for AdS_2 backgrounds obtained from reductions of higher-dimensional theories (see also [26]). In particular, there are parallels with some of the analysis on the reduction of near extremal black holes in [25].

Overall, expanding in the perturbations $\tilde{\phi}, \Omega$, we have $I = S^E + S_{\rm GH} = I_0 + I_1 + I_2 + \cdots$, with

$$I_0 = -\frac{\Phi_b^2}{16\pi G_2}\left(\int d^2x\sqrt{g}\mathcal{R} + 2\int_{\rm bndry}\sqrt{\gamma}K\right) \qquad (2.40)$$

is the background Euclidean action [see (2.27)]: it can be checked that $U(\Phi_b) = 0$. The action I_0 is a topological term and gives the extremal entropy $S_{\rm BH} = \frac{\Phi_b^2}{4G_2}$ after regulating this as a near-extremal background.[2] The linear terms are contained in

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2}\int d^2x\sqrt{g}\tilde{\phi}\left(\mathcal{R} - \frac{\partial U}{\partial\Phi^2}\right) - \frac{2\Phi_b^2}{8\pi G_2}\int_{\rm bndry}\sqrt{\gamma}\tilde{\phi}K, \qquad (2.41)$$

with $\frac{\partial U}{\partial\Phi^2}|_{\Phi_b} = -\frac{2}{L^2}$, which is the Jackiw-Teitelboim theory [27,28], which serves as a simple model for AdS_2 physics (with parallels with the SYK model). The bulk term vanishes by the $\tilde{\phi}$ equation giving the fixed background AdS_2 geometry, while the boundary term gives the Schwarzian as explained above. The analysis here of the higher-dimensional realization serves to recover the background entropy as expected and reveal the various subleading terms beyond the Jackiw-Teitelboim theory emerging from reduction: I_2 is second order in perturbations, from S_2^E [see (2.31)] and the second-order terms in the expansion of $S_{\rm GH}$,

[2]Here the Euclidean time periodicity, large for a small near-extremal temperature, precisely cancels the small regularized change in the extremal horizon. In more detail, expanding $f(r)$ in (2.10) about extremality, we have $f(r) \simeq \frac{6(r-r_0)}{r_0^2}(r - r_0 + \frac{r_0}{6}(3 - \frac{Q^2}{r_0^4})) \equiv \frac{6}{r_0^2}(r - r_0' - \frac{\delta}{2})(r - r_0' + \frac{\delta}{2})$, where $\delta = \frac{r_0}{6}(3 - \frac{Q^2}{r_0^4})$ and $r_0' = r_0 - \frac{\delta}{2}$. Then the nearly AdS_2 throat acquires a small horizon with metric $ds^2 \sim \frac{9\delta^2}{R^4}\rho^2 d\tau^2 + d\rho^2$ near the origin: the Euclidean time periodicity then is $\Delta\tau = \beta = \frac{2\pi R^2}{3\delta}$ consistent with (2.12). The horizon contribution to the action gives $I_0 = -\frac{\Phi_b^2}{16\pi G_2}\Delta\tau\frac{\delta}{2}(\frac{12}{R^2}) \equiv -\beta F$ and thereby the background extremal entropy $S_{\rm BH} = -I_0$. The boundary terms in the action above cancel: to elaborate, we have the AdS_2 metric $ds^2 = \frac{L^2}{\rho^2}(d\tau^2 + d\rho^2)$. The boundary at $\rho = \epsilon$ has outward unit normal $n_\rho = -\frac{L}{\rho}$. The extrinsic curvature defined as $K_{\mu\nu} = \frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ gives $K_{\tau\tau} = -\Gamma^\rho_{\tau\tau}n_\rho = \frac{L}{\rho^2}$ and $K = \gamma^{\tau\tau}K_{\tau\tau} = \frac{1}{L}$. Then the terms at the boundary cancel as $-\frac{\Phi_b^2}{16\pi G_2}(\int d\tau\frac{L^2 d\rho}{\rho^2}|_\epsilon^{\rm hrzn}(-\frac{2}{L^2}) + 2\int d\tau\frac{L}{\epsilon}(-\frac{1}{L}))$.

$$I_2 = -\frac{1}{16\pi G_2}\int d\tau\sqrt{\gamma}\left[\frac{2r_0^2}{3L^2}\Phi_b n^\rho(\Omega\partial_\rho\tilde{\phi} - \tilde{\phi}\partial_\rho\Omega) + 2\Phi_b^2(\tilde{\phi}^2 K - 2\tilde{\phi}e^{-\omega_b}\partial_\rho\Omega)\right], \tag{2.42}$$

expanding in conformal gauge.

III. CHARGED HYPERSCALING VIOLATING LIFSHITZ BLACK BRANES

Over the last several years, nonrelativistic generalizations of holography have been investigated extensively: see, e.g., [15] for a review of various aspects. A particular family of interesting theories comprises the so-called hyperscaling violating Lifshitz (hvLif) theories, which are conformal to Lifshitz theories. These arise as solutions to Einstein-Maxwell-scalar theories, the $U(1)$ gauge field and dilaton scalar necessary to support the nonrelativistic background. For the most part, we regard these as effective gravity theories: in certain cases these can be shown to arise from gauge/string realizations (see, e.g., [29]).

These nonrelativistic black branes are uncharged. A minimal way to construct charged black branes is to add an additional $U(1)$ gauge field, which serves to supply charge to the black brane: see, e.g., [16–18]. For these latter charged black branes, there exist extremal limits where the near-horizon geometry takes the form $AdS_2 \times X$, and contains an AdS_2 throat. Compactifying the transverse space now allows us to study the extremal limits of these theories in the context of a two-dimensional dilaton gravity theory with additional matter, notably the scalar descending from higher dimensions as well as gauge fields.[3]

A. Four-dimensional charged hvLif black brane

Consider Einstein-Maxwell-scalar theory with a further $U(1)$ gauge field, with action [17]

$$S = \int d^4x\sqrt{-g^{(4)}}\left[\frac{1}{16\pi G_4}\left(\mathcal{R}^{(4)} - \frac{1}{2}\partial_M\Psi\partial^M\Psi + V(\Psi) - \frac{Z_1}{4}F_{1MN}F_1^{MN}\right) - \frac{Z_2}{4}F_{2MN}F_2^{MN}\right], \tag{3.1}$$

where the scalar field dependent couplings and the scalar potential are

$$Z_1 = e^{\lambda_1\Psi}, \qquad Z_2 = e^{\lambda_2\Psi}, \qquad V(\Psi) = V_0 e^{\gamma\Psi}. \tag{3.2}$$

The field equations following from the above action are

$$\begin{aligned}
&\mathcal{R}^{(4)}_{MN} - \frac{1}{2}\partial_M\Psi\partial_N\Psi + g_{MN}\frac{V}{2} - \frac{Z_1}{2}\left(F_{1MP}F_{1N}{}^P - \frac{g_{MN}}{4}(F_1)^2\right) - 8\pi G_4 Z_2\left(F_{2MP}F_{2N}{}^P - \frac{g_{MN}}{4}(F_2)^2\right) = 0,\\
&\frac{1}{\sqrt{-g^{(4)}}}\partial_M\left(\sqrt{-g^{(4)}}\partial^M\Psi\right) + \gamma V - \frac{\lambda_1 Z_1}{4}F_{1MN}F_1^{MN} - 4\pi G_4\lambda_2 Z_2 F_{2MN}F_2^{MN} = 0,\\
&\partial_M\left(\sqrt{-g^{(4)}}Z_1 F_1^{MN}\right) = 0, \qquad \partial_M\left(\sqrt{-g^{(4)}}Z_2 F_2^{MN}\right) = 0.
\end{aligned} \tag{3.3}$$

The charged hvLif black brane solution to these equations is

$$\begin{aligned}
ds^2 &= \left(\frac{r}{r_{hv}}\right)^{-\theta}\left[-\frac{r^{2z}f(r)}{R^{2z}}dt^2 + \frac{R^2}{r^2 f(r)}dr^2 + \frac{r^2}{R^2}(dx^2 + dy^2)\right],\\
f(r) &= 1 - \left(\frac{r_0}{r}\right)^{2+z-\theta} + \frac{Q^2}{r^{2(1+z-\theta)}}\left(1 - \left(\frac{r}{r_0}\right)^{z-\theta}\right),\\
F_{1rt} &= \sqrt{2(z-1)(2+z-\theta)}e^{-\frac{\lambda_1\Psi_0}{2}}r_{hv}^2 R^{\theta-z-4}r^{1+z-\theta},\\
F_{2rt} &= \frac{Q\sqrt{2(2-\theta)(z-\theta)}e^{-\frac{\lambda_2\Psi_0}{2}}}{4\sqrt{\pi G_4}}R^{z-\theta-2}r_{hv}^{-z+\theta+1}r^{-(1+z-\theta)},\\
e^{\Psi} &= e^{\Psi_0}\left(\frac{r_{hv}r}{R^2}\right)^{\sqrt{(2-\theta)(2z-2-\theta)}},
\end{aligned} \tag{3.4}$$

[3]Note that in the AdS/CMT literature, these theories are referred to Einstein-Maxwell-dilaton theories: we here use Einstein-Maxwell-scalar since the two-dimensional dilaton Φ here is distinct from the hvLif scalar Ψ.

being explicit with length scales, and

$$V_0 = \frac{(2+z-\theta)(1+z-\theta)e^{-\gamma\Psi_0}}{R^{2-2\theta} r_{hv}^{2\theta}}, \qquad \gamma = \frac{\theta}{\sqrt{(2-\theta)(2z-2-\theta)}},$$
$$\lambda_1 = \frac{-4+\theta}{\sqrt{(2-\theta)(2z-2-\theta)}}, \qquad \lambda_2 = \sqrt{\frac{2z-2-\theta}{2-\theta}}. \tag{3.5}$$

Here r_{hv} is the hyperscaling violating scale arising in the conformal factor in the metric, and the charge parameter Q has dimensions of $r^{1+z-\theta}$: this is equivalent to absorbing factors of r_{hv}, R into Q. For $z=1$, $\theta=0$, this scaling coincides with that for the relativistic black brane in Sec. II A.

In these charged hyperscaling violating Lifshitz black brane solutions to the action (3.1), the gauge field A_1 and the scalar field Ψ source the hyperscaling violating Lifshitz background while the gauge field A_2 giving charge to the black brane, as mentioned above. This action (3.1) has also been defined by absorbing the Newton constant into the definition of the hyperscaling violating gauge field A_1 and scalar Ψ (which, thus, makes A_1 and Ψ dimensionless) while retaining the gauge field A_2 in F_2 as having mass dimension one. Thus, the field strength F_{2rt} in (3.4) has mass dimension two, as for the relativistic brane.

The null energy conditions for the metric follow from the asymptotic hvLif geometry [15] and are given by

$$(z-1)(2+z-\theta) \geq 0, \qquad (2-\theta)(2(z-1)-\theta) \geq 0. \tag{3.6}$$

In addition, we require the gauge field A_{2t} to vanish at the boundary ($r \to \infty$) so that the theory does not ruin the hvLif boundary conditions we have assumed: this is equivalent to assuming that these charged black branes represent finite temperature charged states in the boundary hvLif theory. The background profile $A_{2t} \sim 1 - (\frac{r_0}{r})^{z-\theta}$ then implies that

$$z - \theta \geq 0. \tag{3.7}$$

These conditions together constrain the range of z, θ for these extremal nonrelativistic black brane backgrounds, which will be important in the discussion of perturbations later. Specifically:

(i) First, the last condition (3.7) is specific to the charged case: using this, the first of the null energy conditions (3.6) implies that $z \geq 1$.

(ii) From the second of the conditions (3.6), we have either $2-\theta \geq 0, 2z-2-\theta \geq 0$, or $2-\theta < 0$, $2z-2-\theta < 0$. Considering the second possibility, we obtain $z \geq \theta \geq 2$, but this implies $2z-2-\theta = z-2+z-\theta > 0$, which is a contradiction. This forces $2-\theta \geq 0, 2z-2-\theta \geq 0$.

Overall, this gives the conditions

$$z \geq 1, \qquad 2z-2-\theta \geq 0, \qquad 2-\theta \geq 0, \tag{3.8}$$

for the regime of validity of the z, θ exponents of the charged hvLif background above. For the special case of $z=1$, the NEC becomes $(2-\theta)(-\theta) \geq 0$, which forces $\theta \leq 0$ by (3.8).

The relativistic limit of this charged hvLif black brane gives the relativistic electric black brane discussed previously in Sec. II A. From the constraint (3.8), we see that the correct relativistic limit is to take first $\theta = 0$ and then $z = 1$. In this limit, we get

$$\gamma = 0, \quad \lambda_1 \to -\infty, \quad \lambda_2 = 0, \quad V_0 = 6/R^2, \quad \Psi = \Psi_0. \tag{3.9}$$

With this the Einstein-Maxwell-scalar action (3.1) reduces to the Einstein-Maxwell action (2.1), where F_2 and V_0 in (3.1) are identified with F and -2Λ in (2.1).

1. Extremality and attractors

In the extremal limit,

$$T = \frac{(2+z-\theta)r_0^z}{4\pi R^{z+1}}\left(1 - \frac{(z-\theta)Q^2 r_0^{-2(1+z-\theta)}}{(2+z-\theta)}\right) = 0 \Rightarrow Q^2 = \frac{(2+z-\theta)}{(z-\theta)} r_0^{2(1+z-\theta)}, \tag{3.10}$$

and the near-horizon geometry becomes $AdS_2 \times \mathbb{R}^2$,

$$ds^2 = \left(\frac{r_0}{r_{hv}}\right)^{-\theta}\left[-\frac{r_0^{2z} f(r)}{R^{2z}}dt^2 + \frac{R^2}{r_0^2 f(r)}dr^2 + \frac{r_0^2}{R^2}(dx^2+dy^2)\right],$$
$$f(r)|_{r\to r_0} \simeq \frac{(2+z-\theta)(1+z-\theta)}{r_0^2}(r-r_0)^2, \tag{3.11}$$

the AdS_2 scale being $R(\frac{r_0}{r_{hv}})^{-\theta/2}$. The Bekenstein-Hawking entropy is the horizon area in Planck units

$$S_{\mathrm{BH}} = \left(\frac{r_0^2}{R^2}\right)\left(\frac{r_0}{r_{hv}}\right)^{-\theta}\frac{V_2}{4G_4} = \left(\frac{z-\theta}{2+z-\theta}\right)^{\frac{2-\theta}{2(1+z-\theta)}}\frac{r_{hv}^{\theta}V_2}{4G_4}\frac{Q^{(2-\theta)/(1+z-\theta)}}{R^2}, \tag{3.12}$$

where $V_2 = \int dxdy$ is the transverse area of the brane. For $z=1$, $\theta=0$, this coincides with the relativistic brane.

It is worth noting that the full metric in (3.4) is asymptotically of hvLif form, for $r \gg r_0$. The boundary of the theory could be taken as $r \sim r_{hv}$; i.e., the theory flows to hvLif below this scale, in some bigger phase diagram. The AdS_2 throat, well-defined if $\frac{r-r_0}{r_0} \ll 1$ and $\frac{r-r_0}{R} \gg 1$, is well separated from the asymptotic hvLif region if $\frac{r-r_0}{r_{hv}} \ll 1$ and the AdS_2 scale satisfies $R(\frac{r_0}{r_{hv}})^{-\theta/2} \ll r_{hv}$, i.e., $R \ll r_{hv}(\frac{r_0}{r_{hv}})^{\theta/2}$. Note that this is not vacuous since $r_0 \ll r_{hv}$ so that $\frac{r_0}{r_{hv}} \ll 1$ is a small factor.

Along the lines of the attractor mechanism discussion in [13], we would like to convert this theory to a dilatonic gravity theory in four dimensions with a potential (and no gauge fields). Towards this end, we integrate Maxwell's equations in (3.3) and use the solutions for field strengths in (3.4) to get

$$F_1^{tr} = \frac{\sqrt{2(z-1)(2+z-\theta)}e^{\frac{\lambda_1\Psi_0}{2}}r_{hv}^{\theta-2}R^{1-\theta}}{\sqrt{-g}e^{\lambda_1\Psi}},$$

$$F_2^{tr} = \frac{Q\sqrt{2(2-\theta)(z-\theta)}e^{\frac{\lambda_2\Psi_0}{2}}r_{hv}^{z-1}}{4\sqrt{\pi G_4}R^{2z+1-\theta}\sqrt{-g}e^{\lambda_2\Psi}}. \tag{3.13}$$

Substituting (3.13) in (3.3), we obtain equations of motion for the metric and the scalar field Ψ, which can be derived from the following equivalent action

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}(\partial\Psi)^2 - V_{\mathrm{eff}}(\Psi)\right),$$

$$V_{\mathrm{eff}}(\Psi) = -\frac{(2+z-\theta)(1+z-\theta)}{R^{2-2\theta}r_{hv}^{2\theta}}e^{\gamma(\Psi-\Psi_0)} + \frac{1}{g_{xx}^2}\left(\frac{(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{(2-\theta)(z-\theta)Q^2r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\right). \tag{3.14}$$

The explicit scales show that the potential term-by-term has mass dimension two. This equivalent action is obtained by substituting the solutions for F_1^{tr} and F_2^{tr} in the action (3.1) and changing the signs of F_1^2, F_2^2 terms, as earlier. At the critical point (extremality),

$$g_{xx} = \left(\frac{r_0}{r_{hv}}\right)^{-\theta}\left(\frac{r_0}{R}\right)^2, \qquad e^{\Psi} = e^{\Psi_0}\left(\frac{r_{hv}r_0}{R^2}\right)^{\sqrt{(2-\theta)(2z-2-\theta)}}, \qquad Q^2 = \frac{(2+z-\theta)}{(z-\theta)}r_0^{2(1+z-\theta)}, \tag{3.15}$$

the first and second derivatives of V_{eff} [(A2), (A3)] are

$$\left.\frac{\partial V_{\mathrm{eff}}}{\partial\Psi}\right|_{\mathrm{ext}} = 0, \qquad \left.\frac{\partial^2 V_{\mathrm{eff}}}{\partial\Psi^2}\right|_{\mathrm{ext}} = \frac{4(z-1)(2+z-\theta)(1+z-\theta)}{2z-2-\theta}\frac{r_0^{\theta}}{r_{hv}^{\theta}R^2} > 0, \tag{3.16}$$

which imply that the extremal point is stable for all values of z, θ allowed by the conditions (3.8). It is worth mentioning that for $z=1$ and θ nonzero, these and all higher derivatives of V_{eff} in fact vanish [see (A5)]: thus, we obtain no insight into the stability of these attractors in this case and we will not discuss this subcase in what follows.

B. Dimensional reduction to two dimensions

Compactifying the two spatial dimensions, x^i as T^2, we dimensionally reduce with the metric ansatz (2.6), taking the lower dimensional fields $g_{\mu\nu}^{(2)}, \Phi, \Psi, A_1, A_2$, to be T^2-independent: then the action (3.1) reduces to (A6). Performing a Weyl transformation, $g_{\mu\nu} = \Phi g_{\mu\nu}^{(2)}$ to absorb the kinetic term for the dilaton Φ in the Ricci scalar, the two-dimensional action (A6) becomes

$$S = \int d^2x\sqrt{-g}\left[\frac{1}{16\pi G_2}\left(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}\partial_\mu\Psi\partial^\mu\Psi + V\Phi - \frac{\Phi^3}{4}Z_1F_{1\mu\nu}F_1^{\mu\nu}\right) - \frac{V_2\Phi^3}{4}Z_2F_{2\mu\nu}F_2^{\mu\nu}\right]. \tag{3.17}$$

We only retain fields with nontrivial background profiles: more general comments appear later. The Maxwell equations for the gauge fields are

$$\partial_\mu(\sqrt{-g}\Phi^3 Z_1 F_1^{\mu\nu}) = 0, \qquad \partial_\mu(\sqrt{-g}\Phi^3 Z_2 F_2^{\mu\nu}) = 0. \tag{3.18}$$

Integrating and using F_{1rt}, F_{2rt} from (3.4) to fix the integration constants gives

$$F_1^{\mu\nu} = \frac{\sqrt{2(z-1)(2+z-\theta)}e^{\frac{\lambda_1\Psi_0}{2}} r_{hv}^{\theta-2} R^{1-\theta}}{\sqrt{-g} Z_1 \Phi^3}\varepsilon^{\mu\nu},$$
$$F_2^{\mu\nu} = \frac{Q\sqrt{2(2-\theta)(z-\theta)}e^{\frac{\lambda_2\Psi_0}{2}} r_{hv}^{z-1}}{4\sqrt{\pi G_4}R^{2z+1-\theta}\sqrt{-g}\, Z_2\Phi^3}\varepsilon^{\mu\nu}, \tag{3.19}$$

where $\varepsilon^{\mu\nu}$ satisfies $\varepsilon^{tr} = 1 = -\varepsilon^{rt}$ and $\varepsilon_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}\varepsilon^{\rho\sigma}$. We substitute the solutions (3.19) in the remaining field equations obtained by varying the action (3.17) [i.e., Eq. (A7)] to obtain

$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\left(\frac{\Phi^2}{2}(\partial\Psi)^2 + U\right) - \frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi = 0,$$
$$\mathcal{R} - \frac{1}{2}(\partial\Psi)^2 - \frac{\partial U}{\partial(\Phi^2)} = 0,$$
$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Phi^2\partial^\mu\Psi) - \frac{\partial U}{\partial\Psi} = 0, \tag{3.20}$$

where $U(\Phi,\Psi)$ is an effective interaction potential. These equations can then be obtained from the following equivalent action

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\left(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}(\partial\Psi)^2 - U(\Phi,\Psi)\right),$$
$$U(\Phi,\Psi) = -\frac{(2+z-\theta)(1+z-\theta)}{R^{2-2\theta}r_{hv}^{2\theta}}e^{\gamma(\Psi-\Psi_0)}\Phi + \frac{1}{\Phi^3}\left(\frac{(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\right), \tag{3.21}$$

where $V_0, \gamma, \lambda_1, \lambda_2$ are given in (3.5). This equivalent action is obtained by substituting the solutions for $F_1^{\mu\nu}$, $F_2^{\mu\nu}$ in terms of the dilaton Φ^2 and the scalar Ψ in the action (3.17) and changing the signs of F_1^2, F_2^2 terms, as discussed in the case for relativistic electric black brane, Sec. II A. Also note that the relativistic electric black brane is a special case of the dilaton-gravity-matter theory, considered here, for $\theta = 0$ and $z = 1$.

We note that the scalar Ψ that descends from the hyperscaling violating scalar in higher dimensions is not minimally coupled in the two-dimensional theory. The potential $U(\Phi,\Psi)$ contains nontrivial interactions between the dilaton Φ and the hvLif scalar Ψ. Thus, the small fluctuation spectrum of the dilaton and Ψ are coupled, and one might worry about the stability of the two-dimensional attractor. This is reminiscent of multifield inflation models, where one scalar field provides a slow-roll phase while another scalar provides a waterfall phase, ending inflation. In the current context, stability would require that no tachyonic modes arise from the interaction induced by $U(\Phi,\Psi)$ between Φ and Ψ. We will address this soon.

The field equations (3.20) admit a constant dilaton, AdS_2 solution as

$$ds^2 = L^2\left[-\frac{r_0^{2z-3\theta}}{R^{2z}r_{hv}^{-3\theta}L^4}(r-r_0)^2dt^2 + \frac{dr^2}{(r-r_0)^2}\right], \qquad L^2 \equiv \frac{R r_0^{1-\frac{3\theta}{2}} r_{hv}^{\frac{3\theta}{2}}}{(2+z-\theta)(1+z-\theta)},$$
$$\Phi^2 = \left(\frac{r_0}{r_{hv}}\right)^{-\theta}\left(\frac{r_0}{R}\right)^2, \quad e^\Psi = e^{\Psi_0}\left(\frac{r_{hv}r_0}{R^2}\right)^{\sqrt{(2-\theta)(2z-2-\theta)}},$$
$$Q^2 = \frac{(2+z-\theta)}{(z-\theta)}r_0^{2(1+z-\theta)}. \tag{3.22}$$

Let us choose conformal gauge by doing a coordinate transformation,

$$\rho = \frac{R^{z+1}r_0^{1-z}}{(2+z-\theta)(1+z-\theta)}\frac{1}{(r-r_0)}. \tag{3.23}$$

In the conformal gauge, the AdS_2 metric in (3.22) can be written as

$$ds^2 = e^{2\omega}(-dt^2 + d\rho^2) = e^{2\omega}(-dx^+ dx^-), \qquad e^{2\omega} = \frac{L^2}{\rho^2}, \tag{3.24}$$

where the light-cone coordinates are $x^{\pm} = t \pm \rho$ and L is the radius of AdS_2. To see that (3.21) admits the above AdS_2 solution, we compute $\frac{\partial U}{\partial \Phi^2}$ for the above solution, which gives

$$\frac{\partial U}{\partial \Phi^2} = -2\frac{(2+z-\theta)(1+z-\theta)}{R r_0^{1-\frac{3\theta}{2}} r_{hv}^{\frac{3\theta}{2}}} = -\frac{2}{L^2}. \tag{3.25}$$

From (3.20) for $\Psi =$ constant [from (3.22)], we get the Ricci scalar as

$$\mathcal{R} = \frac{\partial U}{\partial \Phi^2} = -\frac{2}{L^2}. \tag{3.26}$$

1. *Perturbations about* AdS_2

As before, we turn on perturbations to fields with background profiles, i.e., to the metric, the dilaton Φ, and the scalar field Ψ,

$$\Phi = \Phi_b + \phi(x^+, x^-), \qquad \omega = \omega_b + \Omega(x^+, x^-), \qquad \Psi = \Psi_b + \sqrt{2z-2-\theta}\,\psi(x^+, x^-), \tag{3.27}$$

where Φ_b, ω_b, and Ψ_b denote the (3.22) background solution. Expanding the action (3.21) (in conformal gauge) about this background gives

$$S = \frac{1}{16\pi G_2}\int d^2x \left(4\Phi^2 \partial_+\partial_-\omega + \Phi^2 \partial_+\Psi\partial_-\Psi - \frac{e^{2\omega}}{2}U(\Phi,\Psi)\right) \equiv S_0 + S_1 + S_2, \tag{3.28}$$

where

$$S_0 = \frac{1}{16\pi G_2}\int d^2x \left(4\Phi_b^2 \partial_+\partial_-\omega_b + \Phi_b^2 \partial_+\Psi_b\partial_-\Psi_b - \frac{e^{2\omega_b}}{2}U(\Phi_b,\Psi_b)\right) \tag{3.29}$$

is the background action and S_1 vanishes by the equations of motion. S_2 is quadratic in perturbations and is given by

$$\begin{aligned} S_2 = \frac{1}{16\pi G_2}\int d^2x \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)}\Bigg[8\phi\partial_+\partial_-\Omega + \frac{16}{(x^+-x^-)^2}\phi\Omega \\ + \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)}\left((2z-2-\theta)\partial_+\psi\partial_-\psi - \frac{4(z-1)}{(x^+-x^-)^2}\psi^2\right) \\ + \frac{1}{(x^+-x^-)^2}\left(-\frac{16L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\phi^2 + \frac{8\theta}{\sqrt{(2-\theta)}}\psi\phi\right)\Bigg]. \end{aligned} \tag{3.30}$$

Varying this action, we get the linearized equations of motion for the perturbations,

$$\begin{aligned} \partial_+\partial_-\phi + \frac{2}{(x^+-x^-)^2}\phi = 0, \\ (2z-2-\theta)\partial_+\partial_-\psi + \frac{1}{(x^+-x^-)^2}\left(4(z-1)\psi - \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\frac{4\theta}{\sqrt{(2-\theta)}}\phi\right) = 0, \\ \partial_+\partial_-\Omega + \frac{1}{(x^+-x^-)^2}\left(2\Omega - \frac{4L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\phi + \frac{\theta}{\sqrt{(2-\theta)}}\psi\right) = 0. \end{aligned} \tag{3.31}$$

These equations are consistent at linear order with the "constraint" equations for the $\pm\pm$ components of the Einstein equation in (3.20): see Appendix, Eq. (A8)–(A10). We see that the equation for ψ is coupled to ϕ as well: defining a new field ζ,

$$\zeta = \psi - \frac{2}{\sqrt{2-\theta}} \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \phi, \qquad (3.32)$$

decouples the equations for ζ and ϕ, which now become

$$\partial_+\partial_-\phi + \frac{2}{(x^+ - x^-)^2}\phi = 0,$$
$$(2z-2-\theta)\partial_+\partial_-\zeta + 2(z-1)\frac{2}{(x^+ - x^-)^2}\zeta = 0,$$
$$\partial_+\partial_-\Omega + \frac{1}{(x^+ - x^-)^2}\left(2\Omega + \frac{2(3\theta-4)}{(2-\theta)} \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\phi + \frac{\theta}{\sqrt{(2-\theta)}}\zeta\right) = 0. \qquad (3.33)$$

In this form, the perturbations ϕ and ζ are equivalent to scalars with positive mass propagating in a perturbed AdS_2 background, with equation of motion $\frac{1}{\sqrt{-g}}\partial_\mu \times (\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0$: in conformal gauge this is $\partial_+\partial_-\phi + \frac{m^2L^2}{(x^+-x^-)^2}\phi = 0$. Let us look at a few special cases here:

(i) For $z = 1$, $\theta = 0$, we have seen that this system reduces to the relativistic brane case studied earlier (2.29), and the Ψ scalar (the nonrelativistic scalar in higher dimensions) can be then seen to decouple from the system: in particular, the terms containing ψ-perturbations vanish in the action (3.30) for quadratic perturbations. This is expected from the fact that the original action for the higher-dimensional nonrelativistic theory reduces to the relativistic brane theory as $z \to 1, \theta \to 0$, as discussed after (3.1). In effect, we have defined the ψ perturbation in (3.27) so that the relativistic brane limit arises smoothly, and the Ψ scalar freezes out. This is also reflected in the linearized equations for perturbations.

(ii) For $\theta = 0$ and $z > 1$, both ϕ and ζ have positive mass term coefficients, and further ζ decouples entirely from the Ω equation. This means that in fact any linear combination of the fields $A\phi + B\zeta$ also in fact has a positive mass term coefficient in its linearized fluctuation equation, as can be seen by taking that linear combination of the two equations $\partial_+\partial_-(A\phi + B\zeta) + \frac{2}{(x^+-x^-)^2}(A\phi + B\zeta) = 0$. The linear fluctuation analysis, thus, suggests that the attractor point is in fact perfectly stable for small fluctuations.

(iii) For $\theta \neq 0$ and $z = 1$, we see that the ζ field is a massless mode and further it does not decouple from the Ω equation. This suggests that the linear stability analysis is insufficient to determine stability of the attractor point. However, in this case, there is a more basic concern: looking back at the higher-dimensional system (3.16), we see that in fact $\frac{\partial^2 V_{\mathrm{eff}}}{\partial\Psi^2} = 0$ in this case [in fact, all derivatives vanish, (A5)], so that the higher-dimensional theory is also not manifestly a stable attractor. Thus, the relevance of the two-dimensional theory is less clear in this case.

(iv) For generic z, θ values satisfying the energy conditions (3.6), (3.7), (3.8), we see that the mass term coefficients for both ϕ and ζ perturbations are positive. Now a generic linear combination of the fields $A\phi + B\zeta$ satisfies

$$\partial_+\partial_-(A\phi + (2z-2-\theta)B\zeta) + \frac{2}{(x^+ - x^-)^2}(A\phi + (2z-2-\theta)B\zeta) = -\frac{2}{(x^+ - x^-)^2}\theta B\zeta. \qquad (3.34)$$

This is akin to a scalar field $A\phi + (2z-2-\theta)B\zeta$ with positive mass, driven by the source field ζ. Since ζ is also a positive mass scalar, small fluctuations do not contain any unstable modes growing in time. Thus, the general perturbation also is stable. To elaborate a bit further, imagine long-wavelength modes of ϕ, ζ which are spatially uniform, i.e., $\phi = \phi(t), \zeta = \zeta(t)$. Now the linearized equations are of the form $\ddot{\phi} + m_\phi^2\phi = 0, \ddot{\zeta} + m_\zeta^2\zeta = 0$, so that these fields are effectively decoupled harmonic oscillators. Then the general field is a driven oscillator, with the driving force itself executing small oscillations: so there are no unstable modes growing in time. It is important to note that the positivity of the mass term coefficients and the stability they imply stems from the energy conditions and asymptotic boundary conditions, which force $z > 1$ and $2z-2-\theta > 0$ for generic z, θ values.

It is worth noting that, for fixed ζ, the relative sizes of the dilaton ϕ and hvLif scalar ψ perturbations are $\frac{\psi}{\phi} \sim \frac{L^2}{r_0^2}(\frac{r_0}{r_{hv}})^{2\theta} \ll \frac{L^2}{r_0^2}$ for $\theta > 0$ since $\frac{r_0}{r_{hv}} \ll 1$.

It is worth comparing this analysis with that for the higher-dimensional theory discussed earlier in (3.14), (3.16): the scalar Ψ has a canonical kinetic term and the equation governing small fluctuations of Ψ about the attractor point acquires a mass term from $\frac{\partial^2 U}{\partial \Psi^2}$, whose positivity dictates the stability of the attractor point. For a theory with two scalars ϕ_1, ϕ_2 with canonical kinetic terms, the stability of the linearized fluctuations can again be studied by studying the second derivative matrix of the potential $U(\phi_1, \phi_2)$ or the Hessian $[\frac{\partial^2 U}{\partial \phi_i \partial \phi_j}]$. Positivity of the Hessian then translates to stability of the attractor extremum. In the present case, however, the effective action is (3.21), and the kinetic terms for Φ, Ψ are not canonical; thus, the naive Hessian analysis to study the stability of $U(\Phi, \Psi)$ about the attractor point is not valid. Instead, we must analyze perturbations about the attractor point, which are governed by the above equations. From these equations, we see that the mass terms for the decoupled fields ζ and ϕ are positive.

In terms of ϕ and ζ, the quadratic action becomes

$$
\begin{aligned}
S_2 = \frac{1}{16\pi G_2} \int d^2x \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \Bigg[& 8\phi \partial_+ \partial_- \Omega + \frac{16}{(x^+ - x^-)^2} \phi \Omega \\
& + \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \left((2z-2-\theta) \partial_+ \zeta \partial_- \zeta - \frac{4(z-1)}{(x^+ - x^-)^2} \zeta^2 \right) \\
& + \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \left(\frac{4(2z-2-\theta)}{(2-\theta)} \partial_+ \phi \partial_- \phi - \frac{16(z+1-2\theta)}{(2-\theta)(x^+ - x^-)^2} \phi^2 \right) \\
& + 2\sqrt{\frac{2z-2-\theta}{2-\theta}} (\partial_+ \zeta \partial_- \phi + \partial_- \zeta \partial_+ \phi) - \frac{8(2z-2-\theta)}{\sqrt{2-\theta}} \frac{\zeta \phi}{(x^+ - x^-)^2} \Bigg]. \qquad (3.35)
\end{aligned}
$$

It can be checked that varying this action leads to the linearized equations written in terms of ϕ, ζ above.

2. The Schwarzian

In this section, we switch to Euclidean time $\tau = it$. From the linearized equations (3.31), we see that the dilaton fluctuation ϕ is decoupled from the metric and scalar fluctuations Ω and ψ, as in the case of the relativistic brane. So solving the equation for ϕ [i.e., the Euclidean form of (3.31)] gives, as before,

$$
\phi = \frac{a + b\tau + c(\tau^2 + \rho^2)}{\rho}, \qquad (3.36)
$$

where a, b, c are independent constants. Substituting ϕ in the equation for ψ in (3.31), we can solve for the scalar perturbation ψ. Using these solutions for ϕ and ψ in the equation for Ω in (3.31), we can solve for the metric perturbation Ω. We see that the AdS$_2$ metric gets corrected at the same order as the dilaton and the scalar field. The Euclidean on-shell (boundary) action obtained by using linearized field equations in (3.35) and changing to Euclidean time $\tau = it$ is

$$
\begin{aligned}
S_2^E = -\frac{1}{16\pi G_2} \int d\tau \sqrt{\gamma} n^\mu \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \Bigg[& 4(\Omega \partial_\mu \phi - \phi \partial_\mu \Omega) \\
& - \frac{(2z-2-\theta)}{\sqrt{2-\theta}} (\phi \partial_\mu \zeta + \zeta \partial_\mu \phi) - \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \frac{2(2z-2-\theta)}{(2-\theta)} \phi \partial_\mu \phi \\
& - \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} (2z-2-\theta) \zeta \partial_\mu \zeta \Bigg]. \qquad (3.37)
\end{aligned}
$$

The discussion of the Gibbons-Hawking term is very similar to that in Sec. II A 2, so we will not be detailed. The Gibbons-Hawking boundary term for the Euclidean form of the bulk action (3.21) is

$$
S_{\mathrm{GH}} = -\frac{1}{8\pi G_2} \int d\tau \sqrt{\gamma} \Phi^2 K, \qquad (3.38)
$$

arising as discussed in the case of the relativistic electric brane earlier. As in Sec. II A 2, we now redefine the dilaton perturbation after rescaling the background value Φ_b out, so that the perturbation satisfies $\frac{\Phi-\Phi_b}{\Phi_b} \equiv \tilde{\phi} \ll 1$. A similar redefinition is appropriate for the hvLif scalar Ψ as well (we have, however, retained the perturbations in (3.27) without this rescaling simply with a view to not cluttering the resulting expressions). Then the perturbation, the background value (3.22) and the entropy (3.12) are

$$\Phi = \Phi_b(1+\tilde{\phi}), \qquad \Phi_b^2 = \left(\frac{r_0}{r_{hv}}\right)^{-\theta}\left(\frac{r_0}{R}\right)^2,$$
$$S_{\rm BH} = \frac{\Phi_b^2 V_2}{4G_4} = \frac{\Phi_b^2}{4G_2}. \tag{3.39}$$

This gives

$$S_{\rm GH}^{(1)} = -\frac{2\Phi_b^2}{8\pi G_2}\int d\tau\sqrt{\gamma}\tilde{\phi}K \to -\frac{\Phi_b^2}{4\pi G_2}\int du\,\phi_r(u)\{\tau(u),u\}. \tag{3.40}$$

In evaluating the last term, we take the boundary of AdS_2 as a slightly deformed curve $(\tau(u),\rho(u))$ parametrized by the boundary coordinate u, as discussed in [2] (reviewed in [11]), and expand the extrinsic curvature using the outward unit normal n^μ to the boundary. Expanding $S_{\rm GH}^{(1)}$ leads to the action above, which contains the Schwarzian derivative $Sch(\tau(u),u) = \{\tau(u),u\} = \frac{\tau'''}{\tau'} - \frac{3}{2}(\frac{\tau''}{\tau'})^2$. The integral above pertains simply to the AdS_2 scale L, into which the various length scales in the nonrelativistic theory have been absorbed. We have also, as before, defined $\tilde{\phi} = \frac{\phi_r(u)}{\epsilon}$ and $\sqrt{\gamma} = \frac{L}{\epsilon}$.

As for the relativistic brane, Sec. II A 2 and (2.39), we note that the coefficient of the Schwarzian effective action is proportional to the entropy (3.12) and (3.39) of the compactified black brane, with V_2 finite. As in Sec. II A 2, this coefficient as the entropy arises after making the reasonable definition of the dilaton perturbation as in (3.39), scaling out the background Φ_b. The entropy now contains only Φ_b, which controls the transverse area. Since the entropy captures the number of microstates of the unperturbed background, this is akin to a central charge.

This is the leading term in the total boundary action $I_{\rm bdy} = S_2^E + S_{\rm GH}$. The remaining terms in the expansion of $S_{\rm GH}$ and S_2^E are all quadratic in perturbations and, hence, are subleading compared to $S_{\rm GH}^{(1)}$ which contains the dilaton perturbation alone at linear order, as for the relativistic brane discussed earlier. This universal behavior is in accord with the general arguments in, e.g., [2].

Thus, overall, expanding in the perturbations $\tilde{\phi}, \Omega, \psi$, we have $I = S^E + S_{\rm GH} = I_0 + I_1 + I_2 + \cdots$, where

$$I_0 = -\frac{\Phi_b^2}{16\pi G_2}\left(\int d^2x\sqrt{g}\mathcal{R} + 2\int_{\rm bndry}\sqrt{\gamma}K\right) \tag{3.41}$$

is the background action [see (3.29)]: here, Ψ_b is constant and it can be checked that $U(\Phi_b,\Psi_b) = 0$. This is a topological term and gives the extremal entropy, very similar to the detailed discussion for the relativistic brane in Sec. II A 2. The linear terms are contained in

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2}\int d^2x\sqrt{g}\tilde{\phi}\left(\mathcal{R} - \frac{\partial U}{\partial\Phi^2} - \frac{1}{2}(\partial\Psi_b)^2\right) - \frac{2\Phi_b^2}{8\pi G_2}\int_{\rm bndry}\sqrt{\gamma}\tilde{\phi}K - \frac{1}{16\pi G_2}\int d^2x\sqrt{g}\left(-\frac{\Phi_b^2}{2}\partial_\mu\Psi_b\partial^\mu\psi - \psi\frac{\partial U}{\partial\Psi}\right). \tag{3.42}$$

On the AdS_2 background with a constant dilaton Φ_b and a constant hvLif scalar field Ψ_b, we get $\frac{\partial U}{\partial\Phi^2}|_{(\Phi_b,\Psi_b)} = -\frac{2}{L^2}$ and the second line in the expression for I_1 above vanishes by the Ψ equation in (3.20). Thus, I_1 reduces to

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2}\int d^2x\sqrt{g}\tilde{\phi}\left(\mathcal{R} + \frac{2}{L^2}\right) - \frac{2\Phi_b^2}{8\pi G_2}\int_{\rm bndry}\sqrt{\gamma}\tilde{\phi}K, \tag{3.43}$$

which is the Jackiw-Teitelboim theory. The fluctuations of the scalar Ψ now propagate on the fixed AdS_2 background at this order. However, we see, as in Sec. II A 2, that there are various subleading terms at quadratic order ((3.37) and from the Gibbons-Hawking term, see (2.42), as well as possible counterterms), containing the perturbations to the dilaton Φ, metric and scalar Ψ, which all mix (at the same order as the metric): the fluctuation spectrum is stable for physically sensible theories satisfying the energy conditions as we have seen. These encode information about the regularization of the AdS_2 theory by the particular higher-dimensional hvLif theory.

3. More general perturbations

In the above analysis, we have restricted ourselves to the dimensional reduction of perturbations to only those components of fields (metric, gauge fields, scalar) which have nontrivial background values in the higher-dimensional theory. More generally, considering the dimensional reduction of perturbations to all the components of all the fields (some of which have trivial background values) gives

$$h_{MN} \to h_{\mu\nu}, \quad h_{\mu i}, \quad h_{ij}; \qquad A_M^{(1,2)} \to A_\mu^{(1,2)}, \quad A_i^{(1,2)}; \qquad \phi \to \phi, \tag{3.44}$$

i.e., tensor, vector, and scalar perturbations in the two-dimensional theory (note that the two-dimensional dilaton is g_{xx}). For instance this includes the shear perturbation h_{xy} in the higher-dimensional theory as well the spatial components of the gauge fields A_{1i}, A_{2i} for $i = x, y$ which reduce respectively to a nonminimally coupled scalar ($h = g^{(4)xx} h_{xy}$) and minimally coupled scalars $A_{1i} = \chi_i^{(1)}$, $A_{2i} = \chi_i^{(2)}$ in the two-dimensional theory. The terms in the full two-dimensional action which govern these perturbations are

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g} \times \left[\cdots - \frac{\Phi^2}{2}(\partial h)^2 - \frac{e^{\lambda_1 \Psi}}{2}(\partial\chi_i^{(1)})^2 - \frac{e^{\lambda_2 \Psi}}{2}(\partial\chi_i^{(2)})^2\right]. \tag{3.45}$$

The terms involving h_{xy} arise from the higher-dimensional Ricci scalar and so contain the overall dilaton factor Φ^2 under reduction to two dimensions. The linearized equations for h_{xy} in the higher-dimensional theory in, e.g., [31] can be dimensionally reduced to two dimensions: at zero momentum, this is consistent with the Kaluza-Klein ansatz for reduction and the action above. Expanding these terms around the background AdS_2, the leading contributions from these terms appear at quadratic order in perturbations

$$S_2 = \frac{1}{16\pi G}\int d^2x\sqrt{-g} \times \left[\cdots - \frac{\Phi_b^2}{2}(\partial h)^2 - \frac{e^{\lambda_1 \Psi_b}}{2}(\partial\chi_i^{(1)})^2 - \frac{e^{\lambda_2 \Psi_b}}{2}(\partial\chi_i^{(2)})^2\right]. \tag{3.46}$$

These are subleading compared to $S_{\mathrm{GH}}^{(1)}$ and, thus, do not contribute to the Schwarzian.

IV. ON A NULL REDUCTION OF THE CHARGED AdS_5 BLACK BRANE

In [29] (see also [32]), it was argued that the null reduction of AdS plane waves, highly boosted limits of uncharged black branes, gives rise to hvLif theories with certain specific z, θ exponents. The lower-dimensional hvLif gauge field and scalar arise as the KK gauge field and scalar under x^+-reduction. One might imagine that considering such a null reduction of the charged relativistic black brane might be interesting along these lines. In this section, we describe an attempt to obtain the charged hvLif black branes here by a null x^+-reduction of the charged relativistic black brane in one higher dimension. Unfortunately this turns out to be close, but not quite on the nose: while the charge electric gauge field upstairs does give rise to an electric field in the lower-dimensional theory, it also leads to an additional background scalar profile. It would be interesting to understand if this can be refined further.

The action for a charged AdS_5 black brane [15] is[4]

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[\mathcal{R} - 2\Lambda - \frac{2\kappa^2}{e^2}\frac{F^2}{4}\right]. \tag{4.1}$$

The charged AdS_5 black brane metric is

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2 + dx_3^2\right), \tag{4.2}$$

$$f(r) = 1 - \left(1 + \frac{r_0^2\mu^2}{\gamma^2}\left(1 - \frac{r^2}{r_0^2}\right)\right)\left(\frac{r}{r_0}\right)^4, \qquad \gamma^2 = \frac{3e^2L^2}{2\kappa^2}, \tag{4.3}$$

where the horizon is at $r = r_0$ and the boundary at $r \to 0$. The gauge field A_t, charge density ρ and temperature are

$$A_t = \mu\left(1 - \left(\frac{r}{r_0}\right)^2\right), \qquad \rho = \frac{2L}{e^2 r_0^2}\mu, \qquad T = \frac{1}{4\pi r_0}\left(4 - 2\frac{r_0^2\mu^2}{\gamma^2}\right). \tag{4.4}$$

Transforming to light-cone coordinates, $x^\pm = \frac{t \pm x_3}{\sqrt{2}}$ and performing a boost $x^\pm \to \lambda^\pm x^\pm$, the metric becomes

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)\left(\frac{\lambda dx^+ + \lambda^{-1}dx^-}{\sqrt{2}}\right)^2 + \left(\frac{\lambda dx^+ - \lambda^{-1}dx^-}{\sqrt{2}}\right)^2 + \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2\right). \tag{4.5}$$

Completing squares in dx^+, we get

$$ds^2 = -\frac{2L^2 r_0^2 f(r)}{\lambda^2 r^6 (1 + \frac{r_0^2\mu^2}{\gamma^2}(1 - \frac{r^2}{r_0^2}))}(dx^-)^2 + \frac{L^2}{r^2}\left(\frac{dr^2}{f(r)} + dx_1^2 + dx_2^2\right) + \frac{L^2\lambda^2 r^2}{2r_0^4}\left(1 + \frac{r_0^2\mu^2}{\gamma^2}\left(1 - \frac{r^2}{r_0^2}\right)\right)(dx^+ + \mathcal{A}_- dx^-)^2, \tag{4.6}$$

$$\mathcal{A}_- = \frac{-1 + \frac{r^4}{2r_0^4}(1 + \frac{r_0^2\mu^2}{\gamma^2}(1 - \frac{r^2}{r_0^2}))}{\frac{\lambda^2 r^4}{2r_0^4}(1 + \frac{r_0^2\mu^2}{\gamma^2}(1 - \frac{r^2}{r_0^2}))}. \tag{4.7}$$

The first line in (4.6) after incorporating the conformal factor from x^+-reduction leads approximately to the

[4] In this section, $r \to 0$ is the boundary.

four-dimensional hvLif metric with $z = 3$, $\theta = 1$, in the vicinity of $r \to 0$ and $r \to r_0$. The KK-gauge field becomes the F_1 gauge field in the lower-dimensional theory: its form becomes that of F_{1rt} only in the vicinity of the horizon $r \to r_0$, giving $A_{1-} \equiv A_{1t} \sim -\frac{1}{(\lambda^2/r_0^4)r^4} + \frac{1}{\lambda^2}$, where we hold $\frac{\lambda^2}{r_0^4}$ fixed which preserves the first term (while the second term dies). This reduction to hvLif is exact if $\mu = 0$, as in [29] for zero temperature (and [31,32] for finite temperature).

Likewise, the $A_t \equiv A_{2t}$ gauge field giving charge becomes in the lower-dimensional theory

$$A_{2+} = \lambda A_{2t}, \qquad A_{2-} = \frac{1}{\lambda} A_{2t} \to A_{2t}^{4d}. \tag{4.8}$$

Scaling the chemical potential as $\mu \to \frac{\mu}{\lambda} =$ fixed, we obtain precisely the gauge field profile for A_{2t}; however, A_{2+} survives as a scalar background in the lower-dimensional theory.

It can also be seen that the relativistic brane action (2.1) gives rise upon x^+ reduction to the hvLif action (3.1), up to the extra scalar arising from A_{2+}. It would be interesting look for refinements of the discussion here, towards decoupling this extra scalar.

V. DISCUSSION

We have studied dilaton-gravity theories in two dimensions obtained by dimensional reduction of certain families of extremal charged hyperscaling violating Lifshitz black branes in Einstein-Maxwell-scalar theories with an extra gauge field in four dimensions. We have argued that the near-horizon AdS_2 backgrounds here can be obtained in equivalent theories of two-dimensional dilaton-gravity with an extra scalar, descending from the higher-dimensional scalar, and an interaction potential with the dilaton. A simple subcase is the relativistic black brane with $z = 1$, $\theta = 0$ (which has no extra scalar), which we have analyzed in detail. Studying linearized fluctuations of the metric, dilaton, and extra scalar about these AdS_2 backgrounds suggests stability of the attractor background generically. This is correlated with the requirements imposed by the energy conditions on these backgrounds. From the study of small fluctuations, we have seen that the leading corrections to AdS_2 arise at linear order in the dilaton perturbation resulting in a Schwarzian derivative effective action from the Gibbons-Hawking term, and Jackiw-Teitelboim theory at leading order. We have also seen that the coefficient of the Schwarzian derivative term, (2.39), (3.40), is proportional to the entropy of the (compactified) extremal black branes after defining the perturbations by scaling out the background values (2.37), (3.39): this being the number of microstates of the unperturbed background is, thus, akin to a central charge. The background entropy arises automatically as a topological term from the compactification. There are, of course, various subleading terms in the action at quadratic order which mix at the same order as the metric: these encode information on the higher-dimensional realization of these AdS_2 backgrounds.

We have explored certain classes of such extremal backgrounds: it would be interesting to understand the space of such AdS_2 theories in a more systematic manner. One might imagine that the parameters in these theories, for instance the dynamical exponents, are reflected in the spectrum of correlation functions, thus distinguishing the specific ultraviolet regularization of the AdS_2 regimes. This requires better understanding of the subleading terms beyond the Schwarzian, which in turn requires a systematic treatment of counterterms and holographic renormalization. We hope to explore these further.

From the point of view of the dual theories, it would seem that the present two-dimensional backgrounds are dual to one-dimensional theories arising from T^2 compactifications of the dual field theories. It would be interesting to understand these better, in part towards possibly exploring parallels with the SYK models [8,9], discussed more recently in, e.g., [2,10,33,34] and related SYK/tensor models (see, e.g., [35]).

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APPENDIX: SOME DETAILS

Relativistic electric black brane: The Einstein equation and the dilaton equation from the action (2.16) are

$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\left(2\Lambda\Phi + \frac{16\pi G_2 V_2 \Phi^3 F_{\mu\nu}F^{\mu\nu}}{4}\right) - \frac{16\pi G_2 V_2 \Phi^3}{2} F_{\mu\rho}F_\nu{}^\rho = 0,$$
$$\mathcal{R} - \frac{\Lambda}{\Phi} - (6\pi G_2) V_2 \Phi F_{\mu\nu}F^{\mu\nu} = 0. \tag{A1}$$

1. Charged hvLif black brane

Effective scalar potential in four-dimensional hvLif black brane and its derivatives: The first and second derivatives of the effective scalar potential in four-dimensional charged hvLif black brane are

$$\frac{\partial V_{\mathrm{eff}}}{\partial \Psi} = -\frac{\gamma(2+z-\theta)(1+z-\theta)e^{\gamma(\Psi-\Psi_0)}}{R^{2-2\theta} r_{hv}^{2\theta}} - \frac{1}{g_{xx}^2}\left(\frac{\lambda_1(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{\lambda_2(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\right), \tag{A2}$$

$$\frac{\partial^2 V_{\mathrm{eff}}}{\partial \Psi^2} = -\frac{\gamma^2(2+z-\theta)(1+z-\theta)e^{\gamma(\Psi-\Psi_0)}}{R^{2-2\theta} r_{hv}^{2\theta}} + \frac{1}{g_{xx}^2}\left(\frac{\lambda_1^2(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{\lambda_2^2(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\right). \tag{A3}$$

Differentiating V_{eff} n times, we get

$$\frac{\partial^n V_{\mathrm{eff}}}{\partial \Psi^n} = -\frac{\gamma^n(2+z-\theta)(1+z-\theta)e^{\gamma(\Psi-\Psi_0)}}{R^{2-2\theta} r_{hv}^{2\theta}} + \frac{(-)^n}{g_{xx}^2}\left(\frac{\lambda_1^n(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{\lambda_2^n(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\right), \tag{A4}$$

which at the extremal point becomes

$$\frac{\partial^n V_{\mathrm{eff}}}{\partial \Psi^n} = \frac{r_0^\theta(2+z-\theta)}{r_{hv}^\theta R^2}\left[\frac{-\theta^n(1+z-\theta)+(-)^n(\theta-4)^n(z-1)}{(2-\theta)^{\frac{n}{2}}(2z-2-\theta)^{\frac{n}{2}}} + \frac{(-)^n(2z-2-\theta)^{\frac{n}{2}}(2-\theta)}{(2-\theta)^{\frac{n}{2}}}\right]. \tag{A5}$$

At $z=1$, $\theta\neq 0$, we see that $\frac{\partial^n V_{\mathrm{eff}}}{\partial \Psi^n} = 0 \forall n$ at the extremal point.

Dimensional reduction to two dimensions: The two-dimensional action obtained by reducing (3.1) on T^2 is (retaining only fields with background profiles)

$$S = \int d^2x\sqrt{-g^{(2)}}\left[\frac{1}{16\pi G_2}\left(\Phi^2\mathcal{R}^{(2)} + 2\partial_\mu\Phi\partial^\mu\Phi - \frac{\Phi^2}{2}\partial_\mu\Psi\partial^\mu\Psi + V\Phi^2 - \frac{\Phi^2}{4}Z_1F_{1\mu\nu}F_1^{\mu\nu}\right) - \frac{V_2\Phi^2}{4}Z_2F_{2\mu\nu}F_2^{\mu\nu}\right], \tag{A6}$$

Equations of motion from two-dimensional action (3.17): The equations of motion obtained by varying the action (3.17) are

$$\begin{aligned}
&g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\left(\frac{\Phi^2}{2}(\partial\Psi)^2 - V\Phi + \frac{\Phi^3}{4}(Z_1(F_1)^2 + 16\pi G_2V_2Z_2(F_2)^2)\right)\\
&\quad - \frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi - \frac{\Phi^3}{2}(Z_1F_{1\mu\rho}F^\rho_{1\nu} + 16\pi G_2V_2Z_2F_{2\mu\rho}F^\rho_{2\nu}) = 0,\\
&\mathcal{R} - \frac{1}{2}(\partial\Psi)^2 + \frac{V}{2\Phi} - \frac{3}{8}\Phi(Z_1(F_1)^2 + 16\pi G_2V_2Z_2(F_2)^2) = 0,\\
&\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Phi^2\partial^\mu\Psi) + \gamma V\Phi - \frac{\Phi^3}{4}(\lambda_1Z_1(F_1)^2 + \lambda_2 16\pi G_2V_2Z_2(F_2)^2) = 0.
\end{aligned} \tag{A7}$$

The equations of motion (3.20) in conformal gauge and in light-cone coordinates are

$$-e^{2\omega}\partial_\pm(e^{-2\omega}\partial_\pm\Phi^2) - \frac{\Phi^2}{2}\partial_\pm\Psi\partial_\pm\Psi = 0,$$
$$\partial_+\partial_-\Phi^2 - \frac{e^{2\omega}}{4}U = 0,$$
$$4\partial_+\partial_-\omega + \partial_+\Psi\partial_-\Psi - \frac{e^{2\omega}}{2}\frac{\partial U}{\partial(\Phi^2)} = 0,$$
$$\partial_+(\Phi^2\partial_-\Psi) + \partial_-(\Phi^2\partial_+\Psi) + \frac{e^{2\omega}}{2}\frac{\partial U}{\partial\Psi} = 0. \quad \text{(A8)}$$

Expanding the constraint equations in the first line of (A8) to linear order in perturbations (3.27) gives

$$\partial_\pm\partial_\pm\phi \pm \frac{2}{(x^+ - x^-)}\partial_\pm\phi = 0, \quad \text{(A9)}$$

the other terms vanishing at linear order. To see that these linearized constraint equations are consistent with the linearized equations (3.31), we differentiate the $++$ constraint equation with respect to x^- to get

$$\partial_+(\partial_+\partial_-\phi) + \frac{2}{(x^+ - x^-)}\partial_+\partial_-\phi + \frac{2}{(x^+ - x^-)^2}\partial_+\phi = 0, \quad \text{(A10)}$$

which is satisfied after using the equation for ϕ in (3.31). Similarly differentiating the $--$ constraint equation with respect to x^+, we can show that the resulting equation is satisfied upon substituting the equation for ϕ in (3.31).

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