Yangian symmetry for the action of planar $\mathcal{N}=4$ super Yang-Mills and $\mathcal{N}=6$ super Chern-Simons theories

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In this article, we establish the notion of classical Yangian symmetry for planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and for related planar gauge theories. After revisiting Yangian invariance for the equations of motion, we describe how the bilocal generators act on the action of the model such that the latter becomes exactly invariant. In particular, we elaborate on the relevance of the planar limit and how to act nonlinearly with bilocal generators on the cyclic action.

I. INTRODUCTION

Integrability of planar $\mathcal{N}=4$ supersymmetric Yang-Mills (sYM) theory, its dual string theory on $\mathrm{AdS}_5 \times S^5$ and several other AdS/CFT dual pairs of models has proven to be extremely powerful feature in computing various quantities at weak, strong, and even intermediate coupling strength; see Ref. [1] for reviews of the subject. However, despite the many successes related to planar integrability, a rigorous and unified notion of this feature is still missing. In particular, such a notion would be highly desirable toward finding a proof for planar integrability and thus toward firmly establishing a solid foundation for the results alluded to above.

In this regard, the world sheet theory of $\mathrm{AdS}_5 \times S^5$ strings at strong 't Hooft coupling is in rather good shape because there exist well-established notions for integrability in such two-dimensional field theories. Most importantly, it was shown (see also the historical accounts by Polchinski [2]) that the structures of integrability can be formulated in terms of a family of flat connections [3]. However, the nonultralocality of the algebra of world sheet currents poses some difficulty, see Ref. [4], and the quantization of these algebraic structures has several unresolved issues.

The exploration of integrable structures in planar $\mathcal{N} = 4$ sYM took a different path: various remarkable features were discovered in the investigation of particular observables. On a case-by-case basis, these discoveries led to the application, refinement, and development of several integrability-based methods to compute these quantities much more efficiently than by ordinary field theory methods. Some of these features and methods directly matched known integrable structures; for example, the spectral problem of local operators at the leading one-loop order was related to an integrable spin chain [5], and the Bethe ansatz could be applied to the computation of the spectrum in various interesting limits.

Integrability of a physics model goes hand in hand with an enhancement of its ordinary symmetry algebra by a large amount of hidden symmetries. A unifying structure of the integrability-related features of planar $\mathcal{N} = 4$ sYM turned out to be the Yangian algebra $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ [6] based on the superconformal algebra $\mathfrak{psu}(2,2|4)$ of the model. The Yangian $\mathrm{Y}[\mathfrak{g}]$ is an infinite-dimensional quantum algebra based on a (finite-dimensional) Lie (super)algebra $\mathfrak{g}$ [7]. Yangian algebras underlie large classes of integrable models where the Lie algebra $\mathfrak{g}$ usually describes the ordinary symmetries of the model. The latter is spanned by the so-called level-zero generators J^A, $A = 1, \ldots, \dim \mathfrak{g}$, and the Yangian algebra enhances it by the level-one generators $\hat{\mathrm{J}}^A$, $A = 1, \ldots, \dim \mathfrak{g}$, which form a similar set as the J^A. Importantly, the level-one generators act in a certain bilocal fashion, and as nonlocal transformations, they are not easily identified as symmetries and are therefore hidden from a standard analysis. Beyond these, the Yangian contains infinitely many other generators at higher levels which will not be of concern in this work.

The Yangian algebra has been observed and applied for various observables, but in most cases, some caveats which prevent the action of the Yangian algebra from

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being considered a proper invariance exist. In the case of the spectral problem of local operators, the spin chain Hamiltonian almost commutes with the Yangian algebra [6]. However, the cyclic boundary conditions represent an obstacle here because the Yangian generators typically map cyclic physical states to noncyclic states which have no proper meaning in the field theory context. Furthermore, at higher loops, some issues related to gauge symmetry which obscure the closure of the algebra have been observed in Ref. [8]. The S-matrix describing the scattering of magnon excitations of the spin chain was shown to have exact Yangian symmetry [9], even at finite coupling strength, but this symmetry is merely based on a $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$ subalgebra of $\mathfrak{psu}(2,2|4)$ or, more precisely, an extension thereof. Also, color-ordered scattering amplitudes of ordinary particles of the four-dimensional gauge theory have been shown to display Yangian invariance [10] which can be understood as an interplay between ordinary and dual conformal symmetry [11]. Unfortunately, also, here an exact and complete invariance is spoiled, this time by the presence of infrared divergences due to the scattering of massless particles. The divergences have been regularized in many different ways in order to arrive at suitable results, see, e.g., Ref. [12]; however, such regularizations obscure invariances and thus make it much less clear in what sense the Yangian algebra can be regarded as a symmetry or how to prove a corresponding statement in general. Nevertheless, Yangian symmetry has proven useful in constructions of scattering amplitudes; see, e.g., Ref. [13]. A similar picture arises from the study of null polygonal Wilson loops which are T-dual to scattering amplitudes [14]. The main difference with respect to scattering amplitudes is that the divergences arising from the cusps between lightlike segments of the polygon are associated to the ultraviolet rather than the infrared regime, and are thus potentially easier to address. Yangian symmetry also plays a role in the construction of many other observables such as correlation functions and form factors of local operators. However, here, the corresponding world sheet topology of the observables is a clear obstruction for Yangian invariance. Yangian invariance without any of the above restrictions has only been observed for expectation values of smooth, nonintersecting Maldacena-Wilson loops with disk topology [15,16] (even though the analysis is limited to one loop so far).

Altogether, these results demonstrate that the Yangian algebra appears to be some kind of symmetry of planar $\mathcal{N} = 4$ sYM, but it is not evident in what sense this statement can be made precise. They also show that the symmetry is restricted to finite objects with the world sheet topology of a disk which are necessarily symmetric under ordinary superconformal symmetry. A relevant object with this set of features is the (unrenormalized, single-trace) action of $\mathcal{N} = 4$ sYM itself. Showing invariance of the

action could in fact be considered as a proof that the theory is symmetric under the Yangian algebra. Unfortunately, it was unclear how to show invariance largely because it was not understood precisely how to act with a Yangian generator on the action and how to make sense of the planar limit and, eventually, of quantum effects.

In Ref. [17], we have laid the foundations for our work to establish the Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$ as a symmetry of planar $\mathcal{N} = 4$ sYM by proposing that it is a symmetry of the equations of motion in a strong sense. In this article, we provide a more detailed account of our construction and present explicitly in what precise way the action is classically invariant under the Yangian algebra. The fact that the model possesses a very nontrivial extended symmetry algebra and at the same time is apparently exactly integrable can hardly be a coincidence given the usual relationship between these two features. It should be evident that Yangian symmetry is a way to formally express the integrability of planar $\mathcal{N} = 4$ sYM. Hence, we may declare integrability of a planar gauge theory model to be the presence of Yangian symmetry (or some other alike symmetry algebra).[1]

To lend further credibility to our claim, we will treat $\mathcal{N} = 6$ supersymmetric Chern-Simons theory, also known as ABJ(M) theory [18], as the second main example of a gauge field theory known to be integrable in the planar limit following Refs. [19]. Here, the Yangian $Y[\mathfrak{osp}(6|4)]$ has been established as a symmetry of color-ordered scattering amplitudes [20]. We will show that the action of this model is indeed invariant under a Yangian symmetry, and thus our proposed definition of integrability matches with the expectation. In order to convince ourselves of the non-triviality of Yangian symmetry—after all, it might in principle be a feature of a broad range of models—we will furthermore consider pure supersymmetric Yang-Mills theories with $\mathcal{N} < 4$ supersymmetries. These sample field theories can be addressed straightforwardly within our framework, and there is no indication that they become integrable in the planar limit. In line with this expectation, we will show that these models do not possess Yangian symmetry in the planar limit.

The present work is structured as follows. In Sec. II, we introduce $\mathcal{N} = 4$ sYM along with its superconformal symmetry, and by doing that, we outline the notation used in this work. We then review the results of our previous paper [17] on this subject in Sec. III with a much more detailed account of our constructions and further explanations and discussions. In Sec. IV, we convert these results to a statement on the invariance of the action which not only

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[1]As there is no formal definition of integrability applicable to this case, it is evidently impossible to prove that Yangian invariance of the action implies integrability (or the converse statement). We can merely argue why it makes sense to identify the two features.

serves a definition of Yangian symmetry of the model but can also be viewed as an exact notion of integrability. In Secs. V and VI, we consider ABJ(M) theory and pure $\mathcal{N} < 4$ sYM theory serving as two sample planar gauge theories for which we can test our proposal. Indeed, we find that Yangian invariance of the action coincides with our expectations on planar integrability of these models. Finally, we conclude in Sec. VII, and we present a list of open issues regarding integrability within the AdS/CFT correspondence that can be addressed with our framework. The Appendices contain a complete account of the superconformal algebra $\mathfrak{psu}(2, 2|4)$ and its nonlinear representation on the fields (Appendix A) as well as the enhancement to the Yangian algebra (Appendix B).

II. $\mathcal{N} = 4$ sYM AND SUPERCONFORMAL SYMMETRY

We start by introducing $\mathcal{N} = 4$ sYM, the corresponding superconformal algebra, and how the algebra acts as a symmetry of our model. In particular, we shall discuss the latter point at length by presenting several different notions of "symmetry." They will later serve as the foundation for our formulation of Yangian symmetry in this model.

A. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

The fields of $\mathcal{N} = 4$ sYM consist of a gauge potential A with associated field strength F, four Dirac fermions Ψ, and six real scalars Φ. All matter fields transform in the adjoint representation of the gauge group which we assume to be $\mathrm{U}(N_c)$. We will thus express all (real) fields as $N_c \times N_c$ (Hermitian) matrices, and sequences of fields correspond to matrix products.

As we will be mainly interested in aspects of the classical field theory model related to $\mathfrak{psu}(2, 2|4)$ superconformal symmetry, it makes sense to use a notation where all vector indices are expressed as bispinors. In our convention, Latin indices $a, b, \ldots = 1, 2, 3, 4$ denote $\mathfrak{su}(4)$ internal spinors, and undotted/dotted Greek indices $\alpha, \beta, \ldots = 1, 2/\dot{\alpha}, \dot{\beta}, \ldots = 1, 2$ correspond to left/right chiral $\mathfrak{sl}(2, \mathbb{C})$ spacetime spinors. The coordinates x are given by a Hermitian 2×2 matrix, and they carry the index structure $x^{\beta\dot{\alpha}}$ with the reality condition $(x^{\alpha\dot{\gamma}})^\dagger = x^{\gamma\dot{\alpha}}$. The corresponding partial derivatives $\partial_{\dot{\alpha}\beta}$ are defined such that $\partial_{\dot{\alpha}\beta}x^{\delta\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}\delta_\beta^\delta$. We shall refrain from implicitly raising or lowering spinor indices. Instead, we will explicitly contract indices with the help of the totally antisymmetric symbols $\varepsilon^{\alpha\gamma}$ and $\varepsilon^{\dot{\alpha}\dot{\gamma}}$ as well as ε^{abcd} (all these symbols equal $+1$ when the upper or the lower indices are in proper order).

In this notation, the fields of $\mathcal{N} = 4$ sYM all carry two spinor indices of various kinds,

$$\Phi^{ac}, \bar{\Phi}_{ac}, \Psi^a_{\ \gamma}, \bar{\Psi}_{\dot{\alpha}c}, A_{\dot{\alpha}\gamma}, F_{\alpha\gamma}, \bar{F}_{\dot{\alpha}\dot{\gamma}}. \quad (2.1)$$

Here, the bar denotes the Hermitian conjugate of the unbarred field.[2] Importantly, the two scalar fields Φ and $\bar{\Phi}$ are related by the reality condition

$$\bar{\Phi}_{ab} = \frac{1}{2}\varepsilon_{abcd}\Phi^{cd} \iff \Phi^{ab} = \frac{1}{2}\varepsilon^{abcd}\bar{\Phi}_{cd}. \quad (2.2)$$

The gauge-covariant derivative ∇ of some matter field Z and of the gauge potential A itself is defined as[3]

$$\nabla_{\dot{\alpha}\beta}Z := \partial_{\dot{\alpha}\beta}Z + i[A_{\dot{\alpha}\beta}, Z],$$
$$\nabla_{\dot{\alpha}\beta}A_{\dot{\gamma}\delta} := \partial_{\dot{\alpha}\beta}A_{\dot{\gamma}\delta} - \partial_{\dot{\gamma}\delta}A_{\dot{\alpha}\beta} + i[A_{\dot{\alpha}\beta}, A_{\dot{\gamma}\delta}]. \quad (2.3)$$

Note that the latter definition of ∇A as the associated field strength F is merely a convenient notational assignment, which will later allow us to write some expressions in a more uniform fashion. In the spinor notation, we can split this field strength into chiral and antichiral components F and $\bar{F}$,

$$\nabla_{\dot{\alpha}\beta}A_{\dot{\gamma}\delta} = \varepsilon_{\dot{\alpha}\dot{\gamma}}F_{\beta\delta} + \varepsilon_{\beta\delta}\bar{F}_{\dot{\alpha}\dot{\gamma}}. \quad (2.4)$$

The Lagrangian is tightly constrained by supersymmetry and possesses two marginal couplings, the Yang-Mills coupling constant g_{YM} as well as the topological angle θ; we shall not be interested in the latter, and the former can be expressed as an overall factor of the Lagrangian, which we will also drop. The Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\varepsilon^{\alpha\epsilon}\varepsilon^{\gamma\kappa}\mathrm{tr}(F_{\alpha\gamma}F_{\epsilon\kappa}) - \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\epsilon}}\varepsilon^{\dot{\gamma}\dot{\kappa}}\mathrm{tr}(\bar{F}_{\dot{\alpha}\dot{\gamma}}\bar{F}_{\dot{\epsilon}\dot{\kappa}})$$
$$+ i\varepsilon^{\dot{\kappa}\dot{\alpha}}\varepsilon^{\beta\gamma}\mathrm{tr}(\bar{\Psi}_{\dot{\kappa}d}\nabla_{\dot{\alpha}\beta}\Psi^d_{\ \gamma}) - \frac{1}{4}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\mathrm{tr}(\nabla_{\dot{\alpha}\beta}\bar{\Phi}_{ef}\nabla_{\dot{\gamma}\delta}\Phi^{ef})$$
$$+ \frac{i}{2}\varepsilon^{\alpha\gamma}\mathrm{tr}(\bar{\Phi}_{ef}\{\Psi^e_{\ \alpha}, \Psi^f_{\ \gamma}\}) + \frac{i}{2}\varepsilon^{\dot{\alpha}\dot{\gamma}}\mathrm{tr}(\Phi^{ef}\{\bar{\Psi}_{\dot{\alpha}e}, \bar{\Psi}_{\dot{\gamma}f}\})$$
$$+ \frac{1}{16}\mathrm{tr}([\bar{\Phi}_{ab}, \bar{\Phi}_{ef}][\Phi^{ab}, \Phi^{ef}]). \quad (2.5)$$

The equations of motion are obtained by varying the action with respect to the fields of the theory. To this end, we introduce the notation $\check{Z}$ for the variational derivative of the action with respect to a generic field Z,[4]

$$\check{Z} := \frac{\delta\mathcal{S}}{\delta Z}. \quad (2.6)$$

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[2]All fields are spinor matrices, and Hermitian conjugation is meant to act on the indices of the fields as well such that their order is reversed. Furthermore, Hermitian conjugation raises or lowers internal $\mathfrak{su}(4)$ indices and exchanges undotted with dotted $\mathfrak{sl}(2, \mathbb{C})$ indices. Effectively, we have $(\Phi^{ac})^\dagger = \bar{\Phi}_{ca} = -\bar{\Phi}_{ac}$, $(\Psi^a_{\ \gamma})^\dagger = \bar{\Psi}_{\dot{\gamma}a}$ and $(F_{\alpha\gamma})^\dagger = \bar{F}_{\dot{\gamma}\dot{\alpha}} = \bar{F}_{\dot{\alpha}\dot{\gamma}}$. In practice, we will not need these relations.

[3]It is useful to remember the commutator of covariant derivatives $[\nabla_{\dot{\alpha}\beta}, \nabla_{\dot{\gamma}\delta}]Z = i[\nabla_{\dot{\alpha}\beta}A_{\dot{\gamma}\delta}, Z]$.

[4]Variational derivatives are defined by $\delta Z(y)/\delta Z(x) = \delta^4(x - y)$. In this case, the delta-function eliminates the integral over spacetime within the action.

More concretely, the set of all variations $\check{Z}$ reads

$$\check{\Phi}_{lk} = \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\nabla_{\dot{\alpha}\beta}\nabla_{\dot{\gamma}\delta}\bar{\Phi}_{kl} - \frac{1}{2}[\bar{\Phi}_{ab},[\Phi^{ab},\bar{\Phi}_{kl}]]$$

$$+ \frac{i}{2}\varepsilon_{klgh}\varepsilon^{\alpha\gamma}\{\Psi^g_\alpha,\Psi^h_\gamma\} + i\varepsilon^{\dot{\alpha}\dot{\gamma}}\{\bar{\Psi}_{\dot{\alpha}k},\bar{\Psi}_{\dot{\gamma}l}\},$$

$$\check{\bar{\Psi}}^{l\dot{k}} = i\varepsilon^{\dot{k}\dot{\alpha}}\varepsilon^{\beta\gamma}\nabla_{\dot{\alpha}\beta}\Psi^l_\gamma - i\varepsilon^{\dot{k}\dot{\alpha}}[\Phi^{le},\bar{\Psi}_{\dot{\alpha}e}],$$

$$\check{\Psi}^\lambda_k = i\varepsilon^{\lambda\beta}\varepsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\dot{\alpha}\beta}\bar{\Psi}_{\dot{\gamma}k} - i\varepsilon^{\lambda\beta}[\bar{\Phi}_{ke},\Psi^e_\beta],$$

$$\check{A}^{\lambda\dot{k}} = \varepsilon^{\lambda\beta}\varepsilon^{\dot{k}\dot{e}}\left(-\varepsilon^{\alpha\gamma}\nabla_{\dot{e}\alpha}F_{\gamma\beta} - \varepsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\dot{\alpha}\beta}\bar{F}_{\dot{\gamma}\dot{e}}\right.$$

$$\left. - \frac{i}{2}[\bar{\Phi}_{ac},\nabla_{\dot{e}\beta}\Phi^{ac}] - \{\Psi^a_\beta,\bar{\Psi}_{\dot{e}a}\}\right). \tag{2.7}$$

With this notation, the equations of motion can be concisely written as $\{\check{Z}=0\}$. Moreover, the Bianchi identity for the field strength relates the derivatives of the chiral components (with the opposite sign with respect to $\check{A}$)

$$\varepsilon^{\alpha\gamma}\nabla_{\dot{e}\alpha}F_{\gamma\beta} = \varepsilon^{\dot{\alpha}\dot{\gamma}}\nabla_{\dot{\alpha}\beta}\bar{F}_{\dot{\gamma}\dot{e}}. \tag{2.8}$$

B. Superconformal algebra

The superconformal algebra $\mathfrak{psu}(2,2|4)$ is spanned by the following generators: the $\mathfrak{sl}(2,\mathbb{C})$ Lorentz generators L^α_γ and $\bar{L}^{\dot{\alpha}}_{\dot{\gamma}}$, the translations $P_{\dot{\alpha}\beta}$, the conformal boosts $K^{\beta\dot{\alpha}}$, the dilatation D, the internal $\mathfrak{su}(4)$ symmetry generators R^a_b, the supersymmetries $Q_{a\beta}$ and $\bar{Q}_{\dot{\alpha}}^b$, and the superconformal boosts $S^{\beta a}$ and $\bar{S}_b^{\dot{\alpha}}$.

In this article, we mainly consider a *gauge-covariant* representation of the superconformal symmetries $J \in \mathfrak{psu}(2,2|4)$ on the fundamental fields $Z \in \{A,\Phi,\Psi,\bar{\Psi}\}$ of our model.[5] The representation of the translation generators P on a generic field Z reads

$$P_{\dot{\alpha}\beta} \cdot Z = i\nabla_{\dot{\alpha}\beta}Z. \tag{2.9}$$

Note that in our representation, shifts are generated by the covariant derivative ∇ rather than by a plain partial derivative ∂.[6] As usual, the gauge potential $Z = A$ is a somewhat special case because it is not gauge covariant in

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[5]Note that the representation on *fields* acts by replacing each fundamental field with a transformed field. It does not act on the *coordinates*, i.e., $J \cdot x = 0$ for all generators J, and correspondingly it does not act on *partial derivatives* ∂ either. Consequently, it does not act on functions of the coordinates which are independent of the fundamental fields, such as, e.g., the parameter fields for gauge transformations (unless the latter are defined in terms of the fundamental fields).

[6]The difference between a plain and a covariant representation amounts to a gauge transformation generated by the gauge field contracted to the Killing vector of the symmetry. However, this requires a full superspace formulation where all components (in particular the ones along the fermionic directions) of the gauge field and Killing vectors exist. Therefore, a formulation in components must remain covariant to some extent.

contradistinction to the matter fields. Notwithstanding, the above rule also explicitly applies to $Z = A$, which is mapped to the (gauge-covariant) field strength ∇A according to the definition (2.3). Lorentz rotations of the fields are generated by the rule

$$L^\beta_\delta \cdot Z = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\delta}Z + \frac{i}{2}\delta^\beta_\delta x^{\kappa\dot{\alpha}}\nabla_{\dot{\alpha}\kappa}Z + (L_{\text{spin}})^\beta_\delta \cdot Z,$$

$$\bar{L}^{\dot{\alpha}}_{\dot{\gamma}} \cdot Z = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\gamma}\beta}Z + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}} x^{\beta\dot{k}}\nabla_{\dot{k}\beta}Z + (\bar{L}_{\text{spin}})^{\dot{\alpha}}_{\dot{\gamma}} \cdot Z, \tag{2.10}$$

where L_{spin} and $\bar{L}_{\text{spin}}$ denote the spin contribution of the operator which acts nontrivially only on the spacetime indices of spinor fields

$$(L_{\text{spin}})^\beta_\delta \cdot \Psi^c_\epsilon = -i\delta^\beta_\epsilon\Psi^c_\delta + \frac{i}{2}\delta^\beta_\delta\Psi^c_\epsilon,$$

$$(\bar{L}_{\text{spin}})^{\dot{\alpha}}_{\dot{\gamma}} \cdot \bar{\Psi}_{\dot{e}d} = -i\delta^{\dot{\alpha}}_{\dot{e}}\bar{\Psi}_{\dot{\gamma}d} + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}}\bar{\Psi}_{\dot{e}d}. \tag{2.11}$$

Note that in particular there is no spin action on the gauge field even though it carries spacetime indices. Likewise, the dilatation generator D is represented on a generic field Z by a universal rule,

$$D \cdot Z = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\beta}Z - i\Delta_Z Z, \tag{2.12}$$

with the coefficients $\Delta_\Phi = 1$, $\Delta_\Psi = \Delta_{\bar{\Psi}} = 3/2$, and $\Delta_A = 0$. Here, the case $Z = A$ is special because $\Delta_A = 0$ does not match the mass dimension 1 of the gauge potential. This curiosity is related to the fact that our representation also involves a gauge transformation, as we shall discuss below.[7] The representation of the supercharges reads

$$Q_{a\beta} \cdot \Phi^{cd} = \delta^c_a\Psi^d_\beta - \delta^d_a\Psi^c_\beta,$$

$$Q_{a\beta} \cdot \bar{\Phi}_{cd} = \varepsilon_{acde}\Psi^e_\beta,$$

$$Q_{a\beta} \cdot A_{\dot{\gamma}\delta} = -i\varepsilon_{\beta\delta}\bar{\Psi}_{\dot{\gamma}a},$$

$$Q_{a\beta} \cdot \Psi^c_\delta = -2\delta^c_a F_{\beta\delta} + i\varepsilon_{\beta\delta}[\Phi^{ce},\bar{\Phi}_{ae}],$$

$$Q_{a\beta} \cdot \bar{\Psi}_{\dot{\gamma}d} = 2i\nabla_{\dot{\gamma}\beta}\bar{\Phi}_{ad}. \tag{2.13}$$

We present the representation of the remaining superconformal generators explicitly in Appendix A.

This representation of the superconformal algebra closes on shell up to field-dependent gauge transformations. For example, it is easy to confirm that

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[7]The assignment is perfectly consistent; for example, one can convince oneself that the commutation relation $[D,P] = -iP$ holds for all fields Z including $Z = A$. To that end, it is important to realize that the second generator will also act on the gauge potentials A hidden within covariant derivatives ∇. Furthermore, for the gauge-covariant field strength, one finds $\Delta_F = 2$ as expected.

$$[P_{\dot\alpha\beta}, P_{\dot\gamma\delta}] \cdot Z = -iG[\nabla_{\dot\alpha\beta} A_{\dot\gamma\delta}] \cdot Z,$$

$$[Q_{a\beta}, P_{\dot\gamma\delta}] \cdot Z = i\varepsilon_{\beta\delta} G[\bar\Psi_{\dot\gamma a}] \cdot Z,$$

$$\{Q_{a\beta}, Q_{c\delta}\} \cdot Z = 2i\varepsilon_{\beta\delta} G[\bar\Phi_{ac}] \cdot Z, \qquad (2.14)$$

where $G[X]$ generates a gauge transformation with the gauge parameter field X as follows:

$$G[X] \cdot Z = [X, Z], \qquad G[X] \cdot A_{\dot\alpha\beta} = i\nabla_{\dot\alpha\beta} X. \qquad (2.15)$$

The appearance of a gauge transformation in the algebra is not surprising given the fact the we use covariant rather than partial derivatives to generate shifts; namely, one can view the difference between ∇ and ∂ as a (field-dependent) gauge transformation $iG[A]$.[8,9]

The advantage of this representation is that covariant quantities remain manifestly covariant. The price to pay, however, is that the relations of the original algebra $\mathfrak{psu}(2,2|4)$ are not manifest—to recover them, the ideal generated by gauge transformations must be quotiented out. The related complications are minor for the superconformal algebra, but they introduce considerable difficulty in considering the algebraic relations for the Yangian algebra, as will be discussed in Ref. [21].

C. Notions of symmetry

In the following, we shall discuss three notions of symmetry of a model: invariance of the action as well as a strong and a weak version of invariance of the equations of motion. They will serve as a starting point for our discussion of Yangian symmetry of planar $\mathcal{N} = 4$ sYM which has several complications that can be avoided by some of the notions. We will consider some generator $J \in \mathfrak{psu}(2,2|4)$ in the context of $\mathcal{N} = 4$ sYM, but the following arguments will be rather general and apply to generic local symmetries of a field theory.

Invariance of the action.—Typically, a symmetry J of a model is understood as an invariance of the action $\mathcal{S}$,

$$J \cdot \mathcal{S} = 0. \qquad (2.16)$$

This statement is strong because it implies important structures and relationships for a field theory. For instance, one can derive conserved currents and charges by means of Noether's theorem. Moreover, in a quantum field theory, the Ward-Takahashi identities imply a large set of relationships between various correlation functions. For the case of $\mathcal{N} = 4$ sYM, invariance of the action (2.5) can be checked directly using the representations of J such as (2.9), (2.12), (2.13).

One can also perform one step of evaluating the symmetry representation on the action and write the above symmetry statement (2.16) in terms of the combination $\check{Z} = \delta\mathcal{S}/\delta Z$ expressing the equation of motion as

$$J \cdot \mathcal{S} = (J \cdot Z^I) \frac{\delta\mathcal{S}}{\delta Z^I} = (J \cdot Z^I)\check{Z}_I = 0. \qquad (2.17)$$

Here, the field index I refers to all dependencies of all fields including the gauge degrees of freedom as well as the full coordinate dependence.[10] As such, it is evident that the statement must hold off shell for it becomes trivial when the equations of motion $\check{Z} = 0$ are imposed. In other words, a symmetry implies that (2.17) holds without making use of the equations of motion.

Strong invariance of the equations of motion.—An analogous statement can be obtained by considering the variation of (2.17) with respect to a generic field Z^K,

$$J \cdot Z^I \frac{\delta^2\mathcal{S}}{\delta Z^I \delta Z^K} + \frac{\delta(J \cdot Z^I)}{\delta Z^K} \frac{\delta\mathcal{S}}{\delta Z^I} = 0. \qquad (2.18)$$

This statement can readily be expressed in a more concise form as

$$J \cdot \check{Z}_K = -\frac{\delta(J \cdot Z^I)}{\delta Z^K} \check{Z}_I. \qquad (2.19)$$

It now has an open field index K, which means that it is localized at some point x of spacetime and that there is one statement for each component of each fundamental field. In the case of a gauge theory, the statement is no longer gauge invariant but rather gauge covariant. Here, the open gauge indices imply an $N_c \times N_c$ matrix of relationships. This statement dictates how the equation of motion on the lhs transforms under the symmetry J. Importantly, the statement is still valid off shell. Arguably, it is as powerful as the invariance of the action (2.16) because the variation toward (2.19) merely discards a constant term from the statement which plays an insignificant role for almost all purposes.

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[8]Incidentally, the relationship $\nabla Z = \partial Z + iG[A] \cdot Z$ not only provides the standard covariant derivative on covariant fields Z, but for $Z = A$, it also yields precisely the non-Abelian field strength as defined in (2.3).

[9]In a full superspace formulation of a supersymmetric gauge theory, one could choose to remove the gauge transformations by subtracting from each generator J a gauge transformation $G[\xi \cdot A]$ sourced by the superspace gauge field contracted to the Killing vector ξ associated to J. In a component formulation, this is not possible due to the lack of superspace coordinates and components.

[10]Due to the implicit integration over all space, one can expect that partial integration is necessary to confirm the statement. Analogously, one should take cyclicity of the trace over the gauge degrees of freedom into account.

We shall therefore denote (2.19) as *strong invariance of the equations of motion*. Again, verification of this statement for $\mathcal{N} = 4$ sYM is straightforward using the concrete equations of motion (2.7) and the representations of J.

Weak invariance of the equations of motion.—We can now apply the equations of motion $\check{Z} = 0$ to the statement (2.19) in order to remove the rhs,

$$ \text{J} \cdot \check{Z}_K \approx 0. \qquad (2.20) $$

Note that, even though the lhs contains $\check{Z}$, it does not vanish automatically because of the variation within J which hits $\check{Z}$ before the equations of motion are imposed. By construction, this statement is only valid on shell, which is expressed by the symbol $\approx$ here and in the following. Both relationships (2.19) and (2.20) state that the variation of the equations of motion is proportional to the equations of motion. In other words, they imply that symmetries map solutions to solutions. Nevertheless, the second version of the statement is clearly weaker because it does not predict the specific linear combinations on the rhs. We therefore call (2.20) *weak invariance of the equations of motion*.

Let us now reason in the opposite direction and ask ourselves the following question: suppose we have a transformation J such that (2.20) holds; to what extent can we consider J a symmetry of the theory? When can we promote this transformation to a symmetry of the action of the theory? The following example shows that (2.20) can hardly be considered a sufficient condition for a symmetry. Weak invariance of the equations of motion is only a necessary condition. We propose that the strong version (2.19) is a sufficient condition for J to be a symmetry of the action.

Example.—Let us now demonstrate that the strong invariance condition (2.19) indeed allows one to differentiate which invariance of the equations of motion stems from a true symmetry of the action. To this end, consider a simple example, the free complex scalar field ϕ defined by the following action:

$$ \mathcal{S} = \int \mathrm{d}x^d \, \bar{\phi} \, \partial^2 \phi. \qquad (2.21) $$

The equations of motion following from (2.21) are the wave equations

$$ \check{\bar{\phi}} = \partial^2 \phi = 0, \qquad \check{\phi} = \partial^2 \bar{\phi} = 0. \qquad (2.22) $$

The above equations of motion are weakly invariant as in (2.20) under a global complex rescaling of the fields

$$ \phi \mapsto e^{\rho+i\theta}\phi, \qquad \bar{\phi} \mapsto e^{\rho-i\theta}\bar{\phi}, \qquad (2.23) $$

with ρ and θ real parameters. However, the above transformation leaves the action invariant (2.16) only for pure rotations $\rho = 0$.

Can we observe the distinction of ρ and θ on the level of equations of motion, using only the strong invariance formula (2.19)? Let us introduce the generators of the infinitesimal form of the above transformations, separating the complex *rotation* R and *scaling* S,

$$ \text{R} \cdot \phi = i\theta\phi, \qquad \text{R} \cdot \bar{\phi} = -i\theta\bar{\phi}, \qquad (2.24) $$

$$ \text{S} \cdot \phi = \rho\phi, \qquad \text{S} \cdot \bar{\phi} = \rho\bar{\phi}. \qquad (2.25) $$

It is then a simple exercise to see that R indeed satisfies (2.19), whereas for S, one finds (2.19) with the opposite sign on the rhs. Hence, as claimed, Eq. (2.19) allows us to verify whether a given symmetry of the equations of motion is also a symmetry of the action.

III. YANGIAN SYMMETRY OF THE EQUATIONS OF MOTION

In this section, we will review and elaborate on our results of Ref. [17] on Yangian symmetry of the equations of motion of $\mathcal{N} = 4$ sYM. The goal of that paper was to establish classical Yangian symmetry of planar $\mathcal{N} = 4$ sYM theory and thus to establish a clear notion of integrability for this model which can later be carried over to quantum field theory.

The Yangian algebra is an extension of $\mathfrak{psu}(2, 2|4)$ superconformal symmetry, and the discussion of Yangian symmetry can follow along same lines as in the previous section. However, there are also several features that distinguish the Yangian algebra and its representation on fields from the more basic superconformal symmetry:

(i) Yangian symmetry alias integrability ought to hold only in the planar limit; nonplanar terms are expected to violate the symmetry. However, there is no evident notion for the planar part of the action; the planar limit is largely related to the composition of Feynman diagrams. How should one realize the planar limit in the context of Yangian symmetry?

(ii) The Yangian is a quantum algebra rather than a Lie algebra. As such, there is a large number of presentations which are equivalent for physical purposes. Furthermore, most of the algebraic relations comprise many terms and thus require a lot of patience and attention to verify.

(iii) Yangian representations are typically nonlocal. Nonlocal symmetries are harder to detect, and they may not be accessible with conventional tools of field theory. In particular, Noether's theorem does not apply, at least not directly, and also the notion of quantum anomalies is obscure. In fact, nonlocality

does not refer to spacetime in our case, but rather to color space. This again calls for unconventional methods and ideas.

(iv) A related issue is cyclicity violation. The action of $\mathcal{N} = 4$ sYM is cyclic due to the trace in color space. Yangian representations respect cyclicity only under very special conditions. We have to formulate these conditions and make sure that they are satisfied.

(v) Symmetry representations in interacting field theories are often nonlinear in the fields.[11] It is largely unknown whether Yangian algebras possess such nonlinear representations.

(vi) A related issue is gauge symmetry. Gauge symmetry inevitably leads to interacting field theories with nonlinear representations. The Yangian representation should be compatible with gauge symmetry such that the unphysical degrees of freedom do not interfere with the symmetry. Moreover, gauge symmetries have to be fixed in the process of quantization such that the gauge field propagator can be defined. The process of gauge fixing can potentially violate Yangian symmetry.

Many of these features and in particular their interplay complicate the investigation of Yangian symmetry. We will therefore start with a thorough discussion of the algebra and more elementary considerations of symmetry before we move on to the two kinds of invariances of the equations of motion. The treatment of the equations of motion will be manifestly gauge covariant, which provides some structural constraints on the novel terms that we shall encounter. Eventually, we will address Yangian invariance of the action in the next section where all of the complications come into play.

A. Algebra and representations

Given a semisimple Lie algebra[12] $\mathfrak{g}$, the associated Yangian algebra $Y[\mathfrak{g}]$ is an infinite-dimensional quantum algebra generated by two sets of generators J^A and $\hat{J}^A$ with $A = 1, \ldots, \dim(\mathfrak{g})$ [22]. The so-called *level-zero* generators J^A generate the universal enveloping algebra $U[\mathfrak{g}]$ of the underlying Lie algebra $\mathfrak{g}$, and they obey the commutation relations

$$[J^A, J^B] = if^{AB}{}_C J^C \qquad (3.1)$$

with $f^{AB}{}_C$ the structure constants of $\mathfrak{g}$. The *level-one* generators $\hat{J}^A$ transform in the adjoint representation of $\mathfrak{g}$,

$$[J^A, \hat{J}^B] = if^{AB}{}_C \hat{J}^C, \qquad (3.2)$$

and they obey the so-called Serre relation at level two,

$$[\hat{J}^A, [\hat{J}^B, J^C]] + \text{cyclic} = f^{AD}{}_E f^{BF}{}_H f^{CG}{}_L f_{\text{DFG}} J^{(E} J^H J^{L)}. \qquad (3.3)$$

Here, adjoint indices are raised and lowered with the Killing form k_{AB}.[13] All the higher-level generators are given as polynomials in the level-zero and level-one generators subject to the above commutation relations.

The Yangian $Y[\mathfrak{g}]$ is actually a Hopf algebra, and as such, it possesses a coalgebra structure $\Delta : Y \to Y \otimes Y$ defined by

$$\Delta \hat{J}^A = \hat{J}^A \otimes 1 + 1 \otimes \hat{J}^A + f^A{}_{BC} J^B \otimes J^C. \qquad (3.4)$$

The coproduct is coassociative and compatible with the algebra product in a certain sense. These features allow one to use the coproduct for the definition of tensor product representations. The $(n-1)$-fold iterated coproduct of J and $\hat{J}$ acting on n tensor factors reads

$$\Delta^{n-1} J^A = \sum_{k=1}^{n} J_k^A, \qquad \Delta^{n-1} \hat{J}^A = \sum_{k=1}^{n} \hat{J}_k^A + f^A{}_{BC} \sum_{1 \le k < l \le n} J_k^B J_l^C. \qquad (3.5)$$

Here, J_k^A indicates the action of the generator J^A on the kth factor or site of the tensor product (with a trivial action on the other sites). The above coproduct of J is how one would conventionally define a tensor product representation for the Lie algebra $\mathfrak{g}$. It acts on all sites in the same fashion, and therefore the ordering of the sites plays no role. The coproduct of a level-one generator $\hat{J}$ consists of a local part $\sum_k \hat{J}_k$ analogous to the level-zero coproduct and of a bilocal contribution $\sum_{j<k} J_j \otimes J_k$. This latter bilocal term does depend on the ordering of the sites of the tensor product; in other words, the coproduct is noncocommutative.

As we have seen, the coproduct makes us deal with several generators of $\mathfrak{g}$ at the same time requiring the use of adjoint indices $A, B, \ldots$ and structure constants f. A convenient abbreviation to avoid indices is Sweedler's notation: here, one denotes by $J^{(1)}$ and $J^{(2)}$ the level-zero generators in the first and second factors, respectively, of

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[11]In the field theory context, nonlinearity of a representation refers to nonlinearity in fields. Importantly, such a representation acts linearly on polynomials of fields, thus displaying linearity as one of its elementary properties.

[12]The discussion and notation will assume an ordinary Lie algebra generated by bosonic operators J. The generalization to superalgebras is straightforward by inserting appropriate signs; we will write out these signs only when we discuss specific generators of the superalgebra $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ and their representations.

[13]The Killing form in the simple Lie superalgebra $\mathfrak{psu}(2, 2|4)$ vanishes, but the algebra nevertheless possesses an invariant bilinear form k'_{AB} that is used to raise and lower adjoint indices.

the coproduct of some level-one generator $\hat{J} = \hat{J}^A$ with an implicit sum over all combinations,[14]

$$\mathrm{J}^{(1)} \otimes \mathrm{J}^{(2)} := f^A{}_{BC} \mathrm{J}^B \otimes \mathrm{J}^C = -\mathrm{J}^{(2)} \otimes \mathrm{J}^{(1)}. \quad (3.6)$$

The above coproduct rule can thus be expressed more concisely as

$$\Delta \hat{J} = \hat{J} \otimes 1 + 1 \otimes \hat{J} + \mathrm{J}^{(1)} \otimes \mathrm{J}^{(2)}. \quad (3.7)$$

Our analysis of Yangian symmetry can furthermore be simplified by the fact that only a single level-one generator along with the level-zero generators is sufficient to generate the whole Yangian algebra; all other level-one generators follow from the adjoint property (3.2), and the higher-level ones follow from the Serre relation (3.3). We are thus free to choose a particular generator for which the resulting expressions simplify as much as possible. Arguably, this is the level-one momentum $\hat{P}$, which is also known as the dual conformal generator. Its coproduct reads (including all appropriate signs due to fermionic terms)[15]

$$\Delta \hat{P}_{\dot{\alpha}\beta} = \hat{P}_{\dot{\alpha}\beta} \otimes 1 + 1 \otimes \hat{P}_{\dot{\alpha}\beta}$$
$$- \mathrm{L}^{\gamma}{}_{\beta} \wedge \mathrm{P}_{\dot{\alpha}\gamma} - \bar{\mathrm{L}}^{\dot{\gamma}}{}_{\dot{\alpha}} \wedge \mathrm{P}_{\dot{\gamma}\beta} - \mathrm{D} \wedge \mathrm{P}_{\dot{\alpha}\beta} - \frac{i}{2} \mathrm{Q}_{c\beta} \wedge \bar{\mathrm{Q}}_{\dot{\alpha}}{}^c, \quad (3.8)$$

where the antisymmetric tensor product $\wedge$ of any two objects X and Y is defined as

$$X \wedge Y := X \otimes Y - (-1)^{|X||Y|} Y \otimes X. \quad (3.9)$$

Note that the action of $\hat{P}$ conveniently only needs the dilatation D next to the super Poincaré generators L, $\bar{\mathrm{L}}$, Q, $\bar{\mathrm{Q}}$, and P. All of their representations are reasonably simple compared to the representations of the superconformal boosts S, $\bar{\mathrm{S}}$, and K. In fact, we will encounter further simplifications due to the choice of $\hat{P}$ later on. Expressions for the coproducts of the level-one generators $\hat{J}$ with $J \in \{\mathrm{Q}, \bar{\mathrm{Q}}, \mathrm{R}\}$ as well as the level-one bonus symmetry $\hat{\mathrm{B}}$ introduced in Ref. [23] can be found in Appendix B.

B. Issues

The major complication in considering Yangian symmetry within an (interacting) field theory is that symmetry representations are often *nonlinear*. By definition, the representation is still a linear map between observables;

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[14]In fact, we exclude the trivial appearance of the level-one generator, so that $\mathrm{J}^{(1)} \otimes \mathrm{J}^{(2)}$ describes a sum of pairs of level-zero generators.

[15]The overall factor of the bilocal terms depends on the coefficient of the invariant form k_{AB}, and it thus varies with conventions.

here, nonlinearity refers to the fact that a single field can be mapped to a product of fields. In a gauge theory, the covariant derivative is a major source of nonlinearity; see, e.g., Eqs. (2.9) and (2.12). Furthermore, the representation of supersymmetry (2.13) contains nonlinear terms which are not due to covariant derivatives.

The issue is that the concept of nonlinear representations is in competition with the definition of tensor product representations via the coproduct: in a linear representation, each field corresponds to a single tensor factor. The representation acts on fields one to one, thus preserving the structure of the tensor product. For nonlinear symmetries, the representation changes the number of fields and consequently the structure of the tensor product; see Ref. [24] for a discussion and construction of some aspects of nonlinear representations. In fact, the representation does not split into subrepresentations with a definite number of fields, but there is only the indecomposable representation on polynomials of the fields. For the local action of level-zero generators, this complication is minor, and one can still view the full representation as the sum of representations on component fields. Conversely, the construction of the bilocal action for the level-one generators is less evident, cf. some comments in Ref. [25]. Here, the action of the first constituent generator $\mathrm{J}^{(1)}$ changes the tensor product on which the second constituent generator $\mathrm{J}^{(2)}$ is supposed to act. This mainly refers to the precise definition of the bounds in the double sum in (3.5) given that n is not well defined anymore. For example, it is conceivable that $\mathrm{J}^{(2)}$ acts on the output of $\mathrm{J}^{(1)}$ corresponding to an overlapping action. This leaves some ambiguity for the precise definition of a nonlinear level-one representation. Unfortunately, this becomes a rather serious issue when taking gauge symmetry into account; for instance, the covariant derivative consists of a partial derivative and a gauge potential, and as such, it relates terms of a different number of constituent fields. In that sense, the nonlinear terms of the Yangian representations must be chosen delicately such that they will not violate gauge symmetry.

We will thus need representations which are analogous to the actions defined in (3.5). In particular, they should reduce to the ordinary coproduct rule (3.5) when restricting to linear terms. To that end, it will make sense to distinguish the local and bilocal contribution in the level-one representation and write

$$\hat{J} = \hat{J}_{\mathrm{loc}} + \hat{J}_{\mathrm{biloc}} \quad \text{where} \quad \hat{J}_{\mathrm{biloc}} := \mathrm{J}^{(1)} \otimes \mathrm{J}^{(2)}. \quad (3.10)$$

Here, the tensor product symbol $\otimes$ should be interpreted such that the first factor acts on a field which is to the left of the field on which the second factor acts. Formally, we define the tensor product acting on a sequence of fields by the rule

$$(\mathrm{J}^{(1)} \otimes \mathrm{J}^{(2)}) \cdot (\cdots Z^I \cdots Z^J \cdots)$$
$$:= \dots + \cdots (\mathrm{J}^{(1)} \cdot Z^I) \cdots (\mathrm{J}^{(2)} \cdot Z^J) \cdots + \dots, \quad (3.11)$$

where the dots on the rhs represent further terms due to pairwise contractions with the omitted fields in the sequence. The local part $\hat{\mathrm{J}}_{\mathrm{loc}}$ will just act on all fields homogeneously. Note that any overlapping contributions that could be part of some alternative definition of a bilocal action can and should be interpreted as local contributions.

The other main complication is that representations of the Yangian algebra tend to be incompatible with the boundary conditions imposed by the physical system at hand; consequently, the Yangian can merely be considered a symmetry of the bulk rather than of the system as a whole. Nevertheless, for planar $\mathcal{N} = 4$ sYM, our aim is to show exact Yangian invariance of the action. The color trace within the action imposes periodic boundary conditions on the above tensor product. Moreover, the trace projects to states which are invariant under cyclic permutations; the physical information within the action is cyclic. However, the representations of the Yangian generically do not respect cyclicity.

To understand the violations of cyclicity, let us consider a cyclic state in a tensor product of n sites. The Yangian algebra acts on this state by the (linear) representation (3.5). To that end, we need to define an ordering of the sites within the tensor product. The adjacency relationship of the sites defines a local ordering, which is, however, globally inconsistent due to the periodic identification. In order to define an ordering, we need to choose a base point where to "cut open" the cycle. Let $\hat{\mathrm{J}}_{(k,n+k-1)}$ denote the action of $\hat{\mathrm{J}}$ on the range of sites from k through $n + k - 1$ (modulo n), i.e., where we cut between sites $k - 1$ and k. We then have that

$$\hat{\mathrm{J}}^A_{(1,n)} - \hat{\mathrm{J}}^A_{(k,n+k-1)} = 2 f^A{}_{BC} \mathrm{J}^B_{(1,k-1)} \mathrm{J}^C_{(k,n)}. \quad (3.12)$$

Here, $\mathrm{J}_{(j,k)}$ denotes the representation of the level-zero generator on sites j through k. In a generic situation, the rhs is not zero, and therefore the resulting state is not cyclic. Likewise, the action of $\hat{\mathrm{J}}$ will typically depend on the particular cyclic representative on which it acts. This means that there is no universal answer to the Yangian representation on cyclic states, and it makes no sense to ask whether or not any such state is Yangian invariant.

Gladly, the above difference can be rewritten as [10]

$$\hat{\mathrm{J}}^A_{(1,n)} - \hat{\mathrm{J}}^A_{(k,n+k-1)} = -i f^A{}_{BC} f^{BC}{}_D \mathrm{J}^D_{(1,k-1)} + 2 f^A{}_{BC} \mathrm{J}^B_{(1,k-1)} \mathrm{J}^C. \quad (3.13)$$

The first term on the rhs contains the combination $f^A{}_{BC} f^{BC}{}_D$ which is proportional to the dual Coxeter number of the underlying Lie algebra $\mathfrak{g}$. For the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4)$, this number is 0, and therefore the term does not contribute. The second term is a product of two level-zero representations. The level-zero

generator J^C acts on the complete state and annihilates it if is invariant. In other words, the level-one representation of a Yangian with vanishing dual Coxeter number respects cyclicity for those states which are invariant under the level-zero symmetry. In principle, this is an ideal starting point for considering Yangian invariance of the action of planar $\mathcal{N} = 4$ sYM because these two prerequisites are satisfied.

Unfortunately, the argument about cyclicity is based on linear representations. For nonlinear representations, one may suspect something equivalent to hold, but it is unclear how to set up the representation precisely and how to show cyclicity in this case. The major challenge is thus to properly define nonlinear representations of level-one generators and to learn how to work with them. In this situation, proving invariance of the action turns out to be a rather difficult task, and we shall at first consider the invariance of the equations of motion in order to gain a better understanding of nonlinear representations.

C. Weak invariance of the equations of motion

We start by considering the weak notion of a symmetry of the equations of motion (2.20): given a generator J and its action on the fields Z, it must leave the equation of motion $\check{Z} = 0$ invariant,

$$\mathrm{J} \cdot \check{Z} \approx 0. \quad (3.14)$$

For the Yangian level-one generators $\hat{\mathrm{J}}$, the nontrivial coproduct poses two problems: first, if we want to act with $\hat{\mathrm{J}}_{\mathrm{biloc}}$ on the equations of motion, we have to fix an ordering prescription for each term that appears; second, we need to specify how $\hat{\mathrm{J}}_{\mathrm{loc}}$ acts on a single field.

Gladly, we can impose a "natural" ordering for the fields within the equations of motion for Yang-Mills theories with a U(N_c) gauge group. All the fields in $\mathcal{N} = 4$ sYM can be treated as $N_c \times N_c$ matrices, and the (noncommutative) matrix product provides the ordering within a monomial of the fields. The equations of motion inherit this matrix structure supposing that all structure constants of the gauge group are written in terms of commutators of the fields. This is also where the large-N_c limit comes into play: the possibility to write arbitrary (adjoint) combinations of the fields in terms of matrix polynomials requires a U(N_c) gauge group with correspondingly large N_c.[16] Note that

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[16] For matrix products of N_c fields or more, there are certain identities of polynomials related to antisymmetrization which eventually introduce some ambiguity for the ordering of fields. Arguably, this ambiguity does not apply to the equations of motion when $N_c > 3$ (at least). Nevertheless, we will later want to establish Yangian symmetry for arbitrary correlators of the fields, in which case an arbitrarily large N_c will be necessary. Of course, there is the option to reverse-engineer the ordering rule such that the level-one generators are directly represented on fields contracted by structure constants (as long as N_c is sufficiently large); however, any such rule will be rather messy and cumbersome.

't Hooft's (double) line notation for adjoint fields, which is prominently used in the discussion of the planar limit, directly depicts our ordering prescription.

For what concerns the single-field action $\hat{J}_{\text{loc}}$, we will fix it by first computing the bilocal action of $\hat{J}_{\text{biloc}} = J^{(1)} \otimes J^{(2)}$ on some equations of motion, which is completely determined by the conformal representation. We shall then require that the local terms eliminate all remaining terms,

$$[\hat{J}_{\text{biloc}} + \hat{J}_{\text{loc}}] \cdot \check{Z} \approx 0. \qquad (3.15)$$

The fact that suitable local terms can be found will be a first test for Yangian symmetry. Subsequently, we will consider the other equations of motion. Yangian symmetry will pass a more elaborate test if the same local terms compensate the remainders from the bilocal action for all equations of motion.

In the following, we will focus on the level-one momentum generator $\hat{J} = \hat{P}$ of which the bilocal action is determined by the coproduct in (3.8). We will first compute the action of this bilocal term on the Dirac equation (2.7)

$$\check{\Psi}^{l\check{k}} = i\varepsilon^{\check{k}\check{\alpha}}\varepsilon^{\beta\gamma}\nabla_{\dot{\alpha}\beta}\Psi^l{}_\gamma - i\varepsilon^{\check{k}\check{\alpha}}[\Phi^{le}, \bar{\Psi}_{\dot{\alpha}e}] \approx 0. \qquad (3.16)$$

We act on the latter with the coproduct (3.8) by means of the tensor product action (3.11) using the superconformal representation (2.9), (2.12), and (2.13), to obtain[17]

$$\hat{P}_{\dot{\alpha}\beta,\text{biloc}} \cdot \check{\Psi}^{d\check{\gamma}} = i\varepsilon^{\check{\gamma}\check{e}}\{\nabla_{\check{e}\beta}\Phi^{df}, \bar{\Psi}_{\dot{\alpha}f}\} + i\varepsilon^{\check{\gamma}\check{e}}\{\Phi^{df}, \nabla_{\dot{\alpha}\beta}\bar{\Psi}_{\check{e}f}\}$$
$$+ i\delta_{\dot{\alpha}}^{\check{\gamma}}\{[\Phi^{df}, \bar{\Phi}_{ef}], \Psi^e{}_\beta\}. \qquad (3.17)$$

This expression does not vanish on shell on its own. However, we can make an ansatz for a single-field action $\hat{P} \cdot Z$ based on all terms with appropriate quantum numbers and symmetries and add their contribution to the overall action of $\hat{P}$. By choosing the coefficients of the ansatz as

$$\hat{P}_{\dot{\alpha}\beta} \cdot \Phi^{cd} := 0,$$
$$\hat{P}_{\dot{\alpha}\beta} \cdot \Psi^c{}_\delta := -\varepsilon_{\beta\delta}\{\Phi^{ce}, \bar{\Psi}_{\dot{\alpha}e}\},$$
$$\hat{P}_{\dot{\alpha}\beta} \cdot \bar{\Psi}_{\check{\gamma}d} := -\varepsilon_{\dot{\alpha}\check{\gamma}}\{\bar{\Phi}_{de}, \Psi^e{}_\gamma\},$$
$$\hat{P}_{\dot{\alpha}\beta} \cdot A_{\check{\gamma}\delta} := \frac{i}{4}\varepsilon_{\dot{\alpha}\check{\gamma}}\varepsilon_{\beta\delta}\{\Phi^{ef}, \bar{\Phi}_{ef}\}, \qquad (3.18)$$

we get that

$$\hat{P}_{\dot{\alpha}\beta} \cdot \check{\Psi}^{d\check{\gamma}} = -\delta_{\dot{\alpha}}^{\check{\gamma}}\varepsilon_{\beta\epsilon}\{\Phi^{df}, \check{\Psi}^\epsilon{}_f\} \approx 0. \qquad (3.19)$$

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[17]Note that all explicit x-dependence originating from the generators L, $\bar{\text{L}}$, and D cancels out exactly. This convenient feature is related to the fact that $\hat{P}$ commutes with P in the Yangian algebra.

This shows that $\hat{P}_{\dot{\alpha}\beta}$ is a weak symmetry of the Dirac equation of $\mathcal{N} = 4$ sYM.

One comment on the structure (3.18) of the single-field action is in order: it may appear unconventional as it is formulated in terms of anticommutators in places where commutators are usually expected. However, this configuration is actually natural considering the action of a level-one Yangian generator $\hat{J}_{\text{biloc}}$ on a commutator,

$$f^A{}_{BC}(J^B \otimes J^C) \cdot [Z^I, Z^J]$$
$$= f^A{}_{BC}(J^B \cdot Z^I\ J^C \cdot Z^J - J^B \cdot Z^J\ J^C \cdot Z^I)$$
$$= f^A{}_{BC}\{J^B \cdot Z^I, J^C \cdot Z^J\}. \qquad (3.20)$$

This means that level-one generators typically map commutators to anticommutators.

With the above single-field action of (3.18), it is now possible to show that *all* the equations of motion of $\mathcal{N} = 4$ sYM are weakly invariant under $\hat{P}$:

$$\hat{P}_{\dot{\alpha}\beta} \cdot \check{\Psi}^{d\check{\gamma}} = -\delta_{\dot{\alpha}}^{\check{\gamma}}\varepsilon_{\beta\epsilon}\{\Phi^{df}, \check{\Psi}^\epsilon{}_f\},$$

$$\hat{P}_{\dot{\alpha}\beta} \cdot \check{\Psi}^\delta{}_c = -\delta_\beta^\delta\varepsilon_{\dot{\alpha}\check{e}}\{\bar{\Phi}_{cf}, \check{\bar{\Psi}}^{f\check{e}}\},$$

$$\hat{P}_{\dot{\alpha}\beta} \cdot \check{\Phi}_{dc} = -i\varepsilon_{\dot{\alpha}\check{k}}\varepsilon_{\beta\lambda}\{\bar{\Phi}_{cd}, \check{A}^{\lambda\check{k}}\} + \frac{5}{2}\varepsilon_{cdef}\varepsilon_{\dot{\alpha}\check{k}}[\Psi^e{}_\beta, \check{\bar{\Psi}}^{f\check{k}}]$$
$$+ \frac{5}{2}\varepsilon_{\beta\lambda}[\bar{\Psi}_{\dot{\alpha}c}, \check{\Psi}^\lambda{}_d] - \frac{5}{2}\varepsilon_{\beta\lambda}[\bar{\Psi}_{\dot{\alpha}d}, \check{\Psi}^\lambda{}_c],$$

$$\hat{P}_{\dot{\alpha}\beta} \cdot \check{A}^{\delta\check{\gamma}} = -\frac{i}{2}\delta_{\dot{\alpha}}^{\check{\gamma}}\delta_\beta^\delta\{\Phi^{ef}, \check{\Phi}_{ef}\}$$
$$- \frac{5i}{2}\delta_{\dot{\alpha}}^{\check{\gamma}}[\Psi^e{}_\beta, \check{\Psi}^\delta{}_e] + \frac{i}{2}\delta_{\dot{\alpha}}^{\check{\gamma}}\delta_\beta^\delta[\Psi^e{}_\lambda, \check{\Psi}^\lambda{}_e]$$
$$- \frac{5i}{2}\delta_\beta^\delta[\bar{\Psi}_{\dot{\alpha}e}, \check{\bar{\Psi}}^{e\check{\gamma}}] + \frac{i}{2}\delta_{\dot{\alpha}}^{\check{\gamma}}\delta_\beta^\delta[\bar{\Psi}_{\check{k}e}, \check{\bar{\Psi}}^{e\check{k}}]. \qquad (3.21)$$

Moreover, we have verified explicitly that they are weakly invariant under the level-one generators $\hat{J}$ with $J \in \{Q, \bar{Q}, R\}$ as well as the level-one bonus symmetry $\hat{B}$ which extends the Yangian of $\mathfrak{psu}(2, 2|4)$. This also fixes the single-field actions of these generators, and we present our results in Appendix B.

Up to some issues with respect to the closure of the Yangian algebra onto gauge transformations, to be discussed in Ref. [21], we conclude that the Yangian of $\mathfrak{psu}(2, 2|4)$ is a weak symmetry of the classical planar equations of motion of $\mathcal{N} = 4$ sYM. As previously discussed, this represents a necessary condition for Yangian symmetry in planar $\mathcal{N} = 4$ sYM. It is reassuring to see that it is met, and we can continue with the construction of a sufficient condition.

D. Strong invariance of the equations of motion

Next, we would like to promote the above results to a strong invariance of the equations of motion. As we argued

in Sec. II C, this would amount to an honest statement of Yangian symmetry in classical planar $\mathcal{N} = 4$ sYM. To that end, we need an expression which predicts the exact form of the lhs of $\mathrm{J} \cdot \check{Z} \approx 0$ as a linear combination of the $\check{Z}$ (which are 0 on shell).

Let us start from the ordinary superconformal symmetry. As we have already shown, the invariance of the action under the generators J of $\mathfrak{psu}(2,2|4)$ implies an off-shell relationship for the equations of motion of (2.19):

$$\mathrm{J} \cdot \check{Z}_K = -\check{Z}_I \frac{\delta(\mathrm{J} \cdot Z^I)}{\delta Z^K}. \tag{3.22}$$

On the rhs of this equality, we have a sum of terms $\check{Z}$ of which the coefficients are completely determined by the superconformal representation on the fields. Note that the sequence of fields for the term on the rhs takes a particular form which is not completely evident from the above expression. It can be inferred from the color structure which provides the correct adjacency information due to contraction of the field indices I and K. In particular, the contraction of indices I implies taking a color trace. Let us explain it by means of an example: suppose $\mathrm{J} \cdot Z^I = Z^L Z^M Z^N$ yields some cubic combination of fields; the term on the rhs would take the form

$$\check{Z}_I \frac{\delta(\mathrm{J} \cdot Z^I)}{\delta Z^K} = \delta_K^L Z^M Z^N \check{Z}_I + \delta_K^M Z^N \check{Z}_I Z^L + \delta_K^N \check{Z}_I Z^L Z^M. \tag{3.23}$$

It is straightforward to generalize this expression to any number of fields.

We can now ask ourselves if we can find an analogous expression for level-one Yangian generators: an off-shell relationship for the action of $\hat{\mathrm{J}}$ on $\check{Z}_K$, the right-hand side of which is a sum of $\check{Z}_I$ with coefficients completely determined by the action of the generators on the fields of the theory. With some inspiration from the expected structures related to level-one symmetry, we find that the answer is positive, and this novel relationship reads

$$\hat{\mathrm{J}} \cdot \check{Z}_K = -\check{Z}_I \frac{\delta(\hat{\mathrm{J}} \cdot Z^I)}{\delta Z^K} + \check{Z}_I \left[\mathrm{J}^{(1)} \wedge \frac{\delta}{\delta Z^K} \right] \cdot (\mathrm{J}^{(2)} \cdot Z^I). \tag{3.24}$$

The antisymmetric tensor product $\wedge$ was defined in (3.9), and it acts in analogy to (3.11). The insertion of the new field $\check{Z}_I$, however, follows different rules based on the color structure associated to the indices I and K in analogy to (3.23). For example, assume that $\mathrm{J}^{(2)} \cdot Z^I = Z^L Z^M Z^N$ again is a cubic combination of fields. Then, the bilocal term in (3.24) expands to

$$\check{Z}_I \left[\mathrm{J}^{(1)} \wedge \frac{\delta}{\delta Z^K} \right] \cdot (\mathrm{J}^{(2)} \cdot Z^I)$$
$$= -\delta_K^L [(\mathrm{J}^{(1)} \cdot Z^M) Z^N \check{Z}_I + Z^M (\mathrm{J}^{(1)} \cdot Z^N) \check{Z}_I]$$
$$+ \delta_K^M [Z^N \check{Z}_I (\mathrm{J}^{(1)} \cdot Z^L) - (\mathrm{J}^{(1)} \cdot Z^N) \check{Z}_I Z^L]$$
$$+ \delta_K^N [\check{Z}_I (\mathrm{J}^{(1)} \cdot Z^L) Z^M + \check{Z}_I Z^L (\mathrm{J}^{(1)} \cdot Z^M)]. \tag{3.25}$$

Note that the color trace implied by the contraction of I is cut open by the operator $\delta/\delta Z^K$, so that $\check{Z}_I$ can appear in the middle of the resulting polynomial. This expression also generalizes to any number of fields, and it can be nonzero only if $\mathrm{J}^{(2)} \cdot Z^I$ consists of (at least) two fields. In other words, the bilocal term in (3.24) only sees the nonlinear contributions of $\mathrm{J}^{(2)}$.

First, we shall show that the above relationship holds for the Dirac equation in $\mathcal{N} = 4$ sYM. The lhs of (3.24) has been computed in (3.19). We need to show that it matches with the combination of terms on the rhs of (3.24). The first term involves a variation of $\hat{\mathrm{P}} \cdot Z^I$ by $\bar{\Psi}$. The only single-field action (3.18) containing a field $\bar{\Psi}$ is the one for $Z^I = \Psi$. Thus, we can set Z^I to Ψ^f_ϵ in the first term. The second term turns out to yield no contribution for a combination of reasons. First of all, $\mathrm{J}^{(2)} \cdot Z^I$ must yield an expression nonlinear in the fields. One option is $\mathrm{J}^{(2)} = \mathrm{Q}, \bar{\mathrm{Q}}$ acting on $Z^I = \Psi, \bar{\Psi}$, which, however, never produces any terms containing $Z^K = \bar{\Psi}$. Another option is $\mathrm{J}^{(2)} = \mathrm{L}, \bar{\mathrm{L}}, \mathrm{D}$, but all nonlinear terms have an explicit x-dependence which will eventually cancel against other terms. It remains to check $\mathrm{J}^{(2)} = \mathrm{P}$. It must act on $Z^I = \bar{\Psi}$ if the result is to contain $Z^K = \bar{\Psi}$. The only other field is $Z^J = A$ (within $\nabla \bar{\Psi}$), which can be acted upon by $\mathrm{J}^{(1)} = \mathrm{L}, \bar{\mathrm{L}}, \mathrm{D}$. However, all these terms are explicitly x-dependent, and as such, they are needed to cancel against other terms. Hence, the second term does not contribute, and we are left with

$$\hat{\mathrm{P}}_{\dot{\alpha}\beta} \cdot \check{\bar{\Psi}}^{d\dot{\gamma}} = -\mathrm{tr} \left[\check{\Psi}^\epsilon_f \frac{\delta(\hat{\mathrm{P}}_{\dot{\alpha}\beta} \cdot \Psi^f_\epsilon)}{\delta \bar{\Psi}_{\dot{\gamma}d}} \right]$$
$$= \varepsilon_{\beta\epsilon} \mathrm{tr} \left[\check{\Psi}^\epsilon_f \frac{\delta \{\Phi^{fg}, \bar{\Psi}_{\dot{\alpha}g}\}}{\delta \bar{\Psi}_{\dot{\gamma}d}} \right]$$
$$= \varepsilon_{\beta\epsilon} \mathrm{tr} \left[\{\check{\Psi}^\epsilon_f, \Phi^{fg}\} \frac{\delta \bar{\Psi}_{\dot{\alpha}g}}{\delta \bar{\Psi}_{\dot{\gamma}d}} \right]$$
$$= \delta_{\dot{\alpha}}^{\dot{\gamma}} \varepsilon_{\beta\epsilon} \{\check{\Psi}^\epsilon_f, \Phi^{fd}\} = \delta_{\dot{\alpha}}^{\dot{\gamma}} \varepsilon_{\epsilon\beta} \{\Phi^{df}, \check{\Psi}^\epsilon_f\}. \tag{3.26}$$

This agrees precisely with (3.19). We have also verified that the relation (3.24) holds exactly for all equations of motion of $\mathcal{N} = 4$ sYM and for the level-one generators $\hat{\mathrm{J}}$ with $\mathrm{J} \in \{\mathrm{P}, \mathrm{Q}, \bar{\mathrm{Q}}, \mathrm{R}, \mathrm{B}\}$; i.e., it correctly reproduces the rhs of all terms in (3.21) and corresponding relations for the other generators.

Next, let us now explain the meaning of the two terms appearing on the rhs in (3.24). The first of them is a direct

counterpart of the one from the level-zero formula (3.22). The other term should be viewed a result of the nontrivial coproduct of the level-one generators (3.7). Interestingly, the action of the two constituent operators $J^{(1)}$ and $J^{(2)}$ is overlapping, as $J^{(1)}$ acts purely on the output of $J^{(2)}$. We will thus call this contribution the overlapping term. Note that the overlapping term is a purely nonlinear effect.[18]

To conclude this discussion, we would like to emphasize that, unlike its level-zero counterpart (3.22), the level-one formula (3.24) for the invariance of the equations of motion has not been derived from first principles. We have merely verified that it holds exactly for all equations of motion of $\mathcal{N} = 4$ sYM and for several level-one generators $\hat{J}$. In other words, we have derived a strong form of invariance of the equations of motion for level-one generators. This invariance amounts to a set of nontrivial off-shell identities which hold in classical planar $\mathcal{N} = 4$ sYM. Such identities are hard to get a hold of, and independently of how we obtained them and of their precise form, they clearly indicate a (hidden) property of classical planar $\mathcal{N} = 4$ sYM which is equivalent to a global symmetry.

IV. YANGIAN INVARIANCE OF THE ACTION

We would now like to address Yangian invariance of the action. Our starting point is the strong invariance of the equations of motion (3.24) which we claimed to be a valid statement of symmetry. In the following, we will rearrange the terms in the relationship such that they take the form of a Yangian invariance of the action

$$\hat{J} \cdot \mathcal{S} = 0. \qquad (4.1)$$

The goal is to find a precise formulation of this statement that holds for planar $\mathcal{N} = 4$ sYM.

A. Notation

In order to perform the rearrangements, we will need a concise notation for the various terms that arise. First of all, we decompose all objects with respect to the number of fields that they contain. The action of $\mathcal{N} = 4$ sYM has quadratic, cubic, and quartic terms, and the superconformal representation has terms which preserve the number of fields as well as terms that increase the number of fields by one unit,[19]

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[18]This is in agreement with the fact the overlapping contribution from two linear operators $J^{(1)}$ and $J^{(2)}$ essentially boils down to their commutator. The latter is equivalent to a local term and could therefore be absorbed into the definition of the local part of the bilocal operator.

[19]We define the expansion coefficients $\mathcal{O}_{[n]}$ of traced polynomials $\mathcal{O}$ of the fields with an explicit symmetry factor of $1/n$. Conversely, there is no symmetry factor for the expansion coefficients $\mathcal{X}_{[n]}$ and $J_{[n]}$ of open polynomials $\mathcal{X}$ and operators J, respectively. Even though these different assignments may be confusing at times, they will avoid many combinatorial factors.

$$\mathcal{S} = \frac{1}{2}\mathcal{S}_{[2]} + \frac{1}{3}\mathcal{S}_{[3]} + \frac{1}{4}\mathcal{S}_{[4]}, \qquad J = J_{[0]} + J_{[1]}. \qquad (4.2)$$

We choose a particular level-one generator which minimizes the nonlinear terms, such as the level-one momentum $\hat{J} = \hat{P}$; see also Appendix A for further calculational simplifications. The relevant local and bilocal terms in $\hat{J} = \hat{J}_{\text{loc}} + \hat{J}_{\text{biloc}}$ then expand as follows[20]:

$$\hat{J}_{\text{loc}} = \hat{J}_{\text{loc},[1]},$$
$$\hat{J}_{\text{biloc}} = J^{(1)}_{[0]} \otimes J^{(2)}_{[0]} + J^{(1)}_{[1]} \otimes J^{(2)}_{[0]} + J^{(1)}_{[0]} \otimes J^{(2)}_{[1]}. \qquad (4.3)$$

Next, we need to write out somewhat more explicitly how the representations act on the individual fields in the action. The action is a polynomial in the fields of which the ordering matters. The contribution $J_{[0]}$ to the representation maps one field to one, and it is natural to denote the action on the field at site k by $J_{[0],k}$.[21] For the higher contributions $J_{[m]}$, $m > 0$, which map one field to $m + 1$ fields, the situation is not as evident; see Ref. [24] for discussions of this matter in a similar context. Here, we make the choice that $J_{[m],k}$ acts on the field at site k and replaces it by an appropriate sequence of $m + 1$ fields. Consequently, the fields at sites $1, \ldots, k - 1$ are mapped to themselves, whereas the fields at sites $k + 1, k + 2, \ldots$ are shifted by m steps to sites $k + m + 1, k + m + 2, \ldots$, e.g.,

$$J_{[1],3}(\overset{1}{Z^I} \overset{2}{Z^J} \overset{3}{Z^K} \overset{4}{Z^L} \overset{5}{Z^M}) := \overset{1}{Z^I} \overset{2}{Z^J} (J_{[1]}\overset{3,4}{Z^K}) \overset{5}{Z^L} \overset{6}{Z^M}. \qquad (4.4)$$

The change of length has a relevant implication on the commutation of two operators J and J'. It is evident that two one-to-one operators acting on two different sites commute. For operators which change the length, their insertion points have to be adjusted as follows:

$$J_{[l],j}J'_{[m],k} = J'_{[m],k+l}J_{[l],j} \quad \text{if } j < k. \qquad (4.5)$$

Cyclicity.—All the terms in the action are gauge invariant by means of a trace. The trace establishes a neighboring relationship between the last and the first site, i.e., periodic boundary conditions. Moreover, the trace is cyclic; i.e., it is invariant under cyclic shifts. In order to deal with periodic boundary conditions, we introduce U as the operator that performs a cyclic shift by one site to the left; e.g., it shifts

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[20]Even though there is in principle a bilocal term which adds a field at both insertion points, this term will not contribute in practice.

[21]To avoid excessive clutter, we may suppress the symbol · when an operator acts on a specific field.

the second site to the first and the first site to the last. Cyclicity of a traced operator $\mathcal{O}$ is then expressed as

$$U\mathcal{O} = \mathcal{O} \quad \text{or expanded} \quad U\mathcal{O}_{[n]} = \mathcal{O}_{[n]}. \quad (4.6)$$

Note that we will also work with polynomials $\mathcal{O}$ in the fields which are periodic but not cyclic. The cyclic shift operator shifts the insertion of an operator J according to the rule

$$J_{[m],k}U\mathcal{O}_{[n]} = \begin{cases} UJ_{[m],k+1}\mathcal{O}_{[n]} & \text{for } k \neq n, \\ U^{m+1}J_{[m],1}\mathcal{O}_{[n]} & \text{for } k = n. \end{cases} \quad (4.7)$$

It it worth emphasizing that a term of the form $U^k J_{[m],1}$ with $k \leq m$ never appears on the rhs of (4.7). Such a term cannot be written with the shift U residing to the right of J; the shift must remain to the left of J. This is because the term has a form which is not covered by $J_{[m],k}$ alone and which is different from (4.4). In this term, the output of $J_{[m]}$ extends past the last site of the polynomial and continues periodically to the first site or beyond. An example analogous to (4.4) could be written as

$$UJ_{[1],1}(\overset{1}{Z^I} \overset{2}{Z^J} \overset{3}{Z^K} \overset{4}{Z^L} \overset{5}{Z^M}) := (\overset{6,1}{J_{[1]}Z^I}) \overset{2}{Z^J} \overset{3}{Z^K} \overset{4}{Z^L} \overset{5}{Z^M}$$
$$= (J_{[1]}Z^I)_2 \overset{2}{Z^J} \overset{3}{Z^K} \overset{4}{Z^L} \overset{5}{Z^M} (J_{[1]}Z^I)_1, \quad (4.8)$$

where $(J_{[m]}Z)_j$ denotes the jth site of the output of $J_{[m]}$ on Z (with an implicit sum over all m-tuples of fields in the polynomial $J_{[m]}Z$ similarly to Sweedler's notation).

B. Level zero

Equipped with this notation, we will first address the superconformal symmetry of the action. This will provide us with some identities that we shall need later when we consider Yangian symmetry.[22] First, we decompose the statement $J \cdot \mathcal{S} = 0$ by the number of fields, and we obtain the following set of four statements:

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[22]We would like to remind the reader of a general issue with respect to homogeneous nonlinear representations acting on periodic objects. On the one hand, the periodic shift operator U classifies periodic objects according to the eigenvalue $e^{2\pi in/L}$ where $n = 0, …, L - 1$. On the other hand, the nonlinear representation changes the length L of the object. Now, the spectra of U for different lengths L are largely distinct; the only eigenvalue which is present for all lengths is 1, corresponding to cyclic objects. Therefore, nonlinear representation can be homogeneous, i.e., commute with the shift operator U, only on the subspace of cyclic objects. In other words, the periodic object on which they act must be cyclic, and the representation must be constructed such that the result is cyclic as well.

$$\frac{1}{2}J_{[0]} \cdot \mathcal{S}_{[2]} = \frac{1}{3}J_{[0]} \cdot \mathcal{S}_{[3]} + \frac{1}{2}J_{[1]} \cdot \mathcal{S}_{[2]}$$
$$= \frac{1}{4}J_{[0]} \cdot \mathcal{S}_{[4]} + \frac{1}{3}J_{[1]} \cdot \mathcal{S}_{[3]}$$
$$= \frac{1}{4}J_{[1]} \cdot \mathcal{S}_{[4]} = 0. \quad (4.9)$$

When making the fields explicit, the first statement reads

$$\frac{1}{2}J_{[0],1}\mathcal{S}_{[2]} + \frac{1}{2}J_{[0],2}\mathcal{S}_{[2]} = 0. \quad (4.10)$$

Using cyclicity of the trace, we may as well write this even more concisely as $J_{[0],1}\mathcal{S}_{[2]} \simeq 0$ where the symbol $\simeq$ denotes equality up to cyclic permutations. The cubic relationship following from superconformal symmetry reads

$$\frac{1}{3}J_{[0],1}\mathcal{S}_{[3]} + \frac{1}{3}J_{[0],2}\mathcal{S}_{[3]} + \frac{1}{3}J_{[0],3}\mathcal{S}_{[3]} + \frac{1}{2}J_{[1],1}\mathcal{S}_{[2]}$$
$$+ \frac{1}{2}J_{[1],2}\mathcal{S}_{[2]} \simeq 0. \quad (4.11)$$

This relationship can be rewritten in two alternative ways using cyclicity. Collecting terms, we arrive at the simpler form $J_{[0],1}\mathcal{S}_{[3]} + J_{[1],1}\mathcal{S}_{[2]} \simeq 0$. However, we can also write the relationship in a manifestly cyclic fashion as

$$\frac{1}{3}J_{[0],1}\mathcal{S}_{[3]} + \frac{1}{3}J_{[0],2}\mathcal{S}_{[3]} + \frac{1}{3}J_{[0],3}\mathcal{S}_{[3]} + \frac{1}{3}J_{[1],2}\mathcal{S}_{[2]}$$
$$+ \frac{1}{3}UJ_{[1],1}\mathcal{S}_{[2]} + \frac{1}{3}J_{[1],1}\mathcal{S}_{[2]} = 0. \quad (4.12)$$

Here, it is necessary to use the cyclic shift operator U for one term to distribute the sequence of fields resulting from $J_{[1]}$ over the last and first sites.

It is also useful to consider the strong invariance of the equations of motion (3.22). For any Z^K specifying the equation of motion, it yields a relationship consisting of a polynomial of fields. In order to handle the identities for all fields Z^K at the same time, we prepend this field to the relationship polynomial and sum over all fields,

$$\mathcal{Y} := Z^K \left[(J \cdot Z^I) \frac{\delta^2 \mathcal{S}}{\delta Z^I \delta Z^K} + \frac{\delta(J \cdot Z^I)}{\delta Z^K} \frac{\delta \mathcal{S}}{\delta Z^I} \right] = 0. \quad (4.13)$$

In this definition, we explicitly do not take the color trace such that the above field Z^K will always reside at site 1 of the polynomial by construction. The expansion $\sum_n \mathcal{Y}_{[n]}$ of the open polynomial $\mathcal{Y}$ in the number n of fields takes the form

$$\mathcal{Y} = \sum_n \mathcal{Y}_{[n]},$$

$$\mathcal{Y}_{[n]} = \sum_{m=0}^{n-2} \sum_{j=2}^{n-m} J_{[m],j} \mathcal{S}_{[n-m]} + \sum_{m=0}^{n-2} \sum_{j=1}^{m+1} U^{j-1} J_{[m],1} \mathcal{S}_{[n-m]}. \tag{4.14}$$

Here, the symmetry factors $1/(n-m)$ for $\mathcal{S}_{[n-m]}$ in (4.2) have been cancelled by varying the component of the (cyclic) action $\mathcal{S}_{[n-m]}$ consisting of $n-m$ fields by Z^{K}.[23] Note that the statements $\frac{1}{2}\mathcal{Y}_{[2]} = 0$ and $\frac{1}{3}\mathcal{Y}_{[3]} = 0$ are precisely the above (4.10) and (4.12), respectively. In fact, the expression $\mathcal{Y}$ is cyclic; by rewriting the first term using (4.7) such that $J_{[m]}$ will always act on site 1, we can make cyclicity manifest,

$$\mathcal{Y}_{[n]} = \sum_{m=0}^{n-2} \sum_{j=1}^{n} U^{j-1} J_{[m],1} \mathcal{S}_{[n-m]}. \tag{4.15}$$

Therefore, we lose no information by identifying terms by cyclic permutations, and we find a more concise statement,

$$\mathcal{Y}_{[n]} \simeq n \sum_{m=0}^{n-2} J_{[m],1} \mathcal{S}_{[n-m]} \simeq 0. \tag{4.16}$$

To wrap this discussion up, we can rewrite the above definition (4.13) as

$$\mathcal{Y} = Z^{K} \frac{\delta(J \cdot \mathcal{S})}{\delta Z^{K}}. \tag{4.17}$$

As the operation $Z^{K}(\delta/\delta Z^{K})$ just counts the number of fields, it is not surprising that the symmetry variation of the action expands precisely to the coefficients $\mathcal{Y}_{[n]}$,

$$J \cdot \mathcal{S} = \sum_n \frac{1}{n} \mathcal{Y}_{[n]}. \tag{4.18}$$

Therefore, the above transformations are a bit of a detour in this case, but they will help us find a corresponding expression $\hat{J} \cdot \mathcal{S}$ for the level-one Yangian generators $\hat{J}$ acting on the action $\mathcal{S}$.

C. Level one

We would now like to construct a suitable level-one Yangian representation on the action, $\hat{J} \cdot \mathcal{S}$, such that invariance amounts to $\hat{J} \cdot \mathcal{S} = 0$. We will approach this construction by expanding in the number of fields,

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[23]We assume that $\delta\mathcal{S}/\delta Z^{K}$ pulls one of the $n-m$ fields and moves the empty spot within the polynomial to site 1. The subsequent variation $\delta/\delta Z^{I}$ pulls another field (but not from the empty site 1) and replaces it by $J \cdot Z^{I}$.

$$\hat{J} \cdot \mathcal{S} = \sum_n \frac{1}{n} (\hat{J} \cdot \mathcal{S})_{[n]}, \tag{4.19}$$

where the prefactors $1/n$ account for cyclic symmetry. We will start with the simplest cases at $n = 2, 3$ and later address arbitrary lengths. Before doing so, we derive some more identities which are needed to transform the expressions.

Commuting constituents.—Consider the commutator

$$H := \frac{1}{2}[J^{(1)}, J^{(2)}], \tag{4.20}$$

which we can expand in fields as usual as $H = H_{[0]} + H_{[1]}$ with

$$H_{[m],k} = \sum_{l=0}^{m} \sum_{j=0}^{l} J^{(1)}_{[m-l],k+j} J^{(2)}_{[l],k}. \tag{4.21}$$

The leading term $H_{[0]}$ contains the combination $f^{A}{}_{BC} f^{BC}{}_{D}$, which is proportional to the dual Coxeter number of the level-zero algebra. The dual Coxeter number for our superconformal algebra $\mathfrak{psu}(2,2|4)$ is zero, and consequently the linear term vanishes, $H_{[0]} = 0$. The nonlinear term $H_{[1]}$ is also zero for $\mathcal{N} = 4$ sYM by explicit computation. One can relate this finding to the G-identity for $\mathcal{N} = 4$ sYM found in Ref. [16]. Altogether, the constituent operators of the bilocal level-one generator commute (when summed over all pairs as implied by Sweedler's notation),

$$H = 0. \tag{4.22}$$

Two fields.—Next, we address level-one Yangian symmetry. The starting point is the strong invariance statement (3.24) of the equations of motion which we have already shown to hold. We will treat it analogously to the strong invariance of the equation of motion at level zero (3.24): we prepend the field that selects the equation of motion to the relationship polynomial and denote the resulting expression by

$$\hat{\mathcal{Y}} := Z^{K} \left[\hat{J} \cdot \check{Z}_{K} + \check{Z}_{I} \frac{\delta(\hat{J} \cdot Z^{I})}{\delta Z^{K}} \right.$$
$$\left. - \check{Z}_{I} \left(J^{(1)} \wedge \frac{\delta}{\delta Z^{K}} \right) \cdot (J^{(2)} \cdot Z^{I}) \right] = 0. \tag{4.23}$$

It can be expanded in fields leading to some additional identities of a similar kind as (4.10), (4.12)

$$\hat{\mathcal{Y}} = \sum_n \hat{\mathcal{Y}}_{[n]}. \tag{4.24}$$

However, the relationship $\hat{\mathcal{Y}}_{[2]} = 0$ at the level of two fields is empty; the first nontrivial relationship is at three fields.

What are the implications of Yangian symmetry at two fields? We use the fact that the combination $\mathcal{Y}_{[2]}$ in (4.15) is exactly zero[24] and apply another generator J to it to obtain a new relationship:

$$J^{(1)}_{[0],1}\mathcal{Y}^{(2)}_{[2]} = J^{(1)}_{[0],1}J^{(2)}_{[0],1}\mathcal{S}_{[2]} + J^{(1)}_{[0],1}J^{(2)}_{[0],2}\mathcal{S}_{[2]} = 0. \quad (4.25)$$

The first term has the form $H_{[0],1}\mathcal{S}_{[2]}$, and therefore it vanishes by (4.22). The second term is the bilocal part of the level-one representation on the quadratic part of the action. Furthermore, there is no linear contribution to the local part of the level-one representation. Altogether, Eq. (4.25) boils down to the statement

$$\hat{J}_{[0]} \cdot \mathcal{S}_{[2]} = (J^{(1)}_{[0]} \otimes J^{(2)}_{[0]}) \cdot \mathcal{S}_{[2]} = 0. \quad (4.26)$$

In other words, the level-one symmetry of the action at quadratic order in the fields is a plain consequence of level-zero symmetry and the vanishing of the dual Coxeter number. This agrees with the fact that the equations of motions are trivially invariant at linear order.

Three fields.—The strong invariance relationship (3.24) at three fields reads $\hat{\mathcal{Y}}_{[3]} = 0$ with

$$\hat{\mathcal{Y}}_{[3]} := \hat{J}_{[1],2}\mathcal{S}_{[2]} + U\hat{J}_{[1],1}\mathcal{S}_{[2]} + \hat{J}_{[1],1}\mathcal{S}_{[2]}$$
$$+ J^{(1)}_{[0],2}J^{(2)}_{[0],3}\mathcal{S}_{[3]} - J^{(2)}_{[0],2}J^{(1)}_{[1],1}\mathcal{S}_{[2]} + UJ^{(2)}_{[0],1}J^{(1)}_{[1],1}\mathcal{S}_{[2]}. \quad (4.27)$$

We would now like to reformulate this combination such that it looks more like the cubic term in the expansion of $\hat{J} \cdot \mathcal{S}$. To that end, let us first address the cyclicity of $\hat{\mathcal{Y}}_{[3]}$. The first three terms are manifestly cyclic; the latter three are not. We therefore look at the violation of cyclicity,

$$(U - 1)\hat{\mathcal{Y}}_{[3]} = -J^{(1)}_{[0],2}\mathcal{Y}^{(2)}_{[3]} - UJ^{(1)}_{[1],1}\mathcal{Y}^{(2)}_{[2]} + H_{[0],2}\mathcal{S}_{[3]}$$
$$+ UH_{[1],1}\mathcal{S}_{[2]}. \quad (4.28)$$

Here, we have made use of the algebraic identities (4.5) and (4.7) and the implicit antisymmetry in Sweedler's notation (3.6). All remaining terms could then be collected in the combinations $\mathcal{Y}$ (4.15) and H (4.21), which are zero as discussed above. Therefore, the relationship $\hat{\mathcal{Y}}_{[3]} = 0$ is

[24]Importantly, the relationship $\mathcal{Y}_{[2]} = 0$ holds without assuming the equation of motion to hold or without applying cyclic permutations.

effectively cyclic; all the noncyclic contributions to the relationship are zero for lesser reasons than Yangian symmetry. It also allows us to compare modulo cyclic identifications ($\simeq$) without losing relevant information,

$$\hat{\mathcal{Y}}_{[3]} \simeq 3\hat{J}_{[1],1}\mathcal{S}_{[2]} + J^{(1)}_{[0],2}J^{(2)}_{[0],3}\mathcal{S}_{[3]} - J^{(2)}_{[0],2}J^{(1)}_{[1],1}\mathcal{S}_{[2]}$$
$$+ J^{(2)}_{[0],1}J^{(1)}_{[1],1}\mathcal{S}_{[2]}. \quad (4.29)$$

Let us now turn to the reformulation. Due to the effective cyclicity of $\hat{\mathcal{Y}}_{[3]}$, it is reasonable to define $(\hat{J} \cdot \mathcal{S})_{[3]}$ to be proportional to it. To figure out the factor of proportionality, we will compare the terms in (4.29) to some canonical terms in $(\hat{J} \cdot \mathcal{S})_{[3]}$. The first term in (4.29) is equivalent to the local part of the level-one representation on $\mathcal{S}_{[2]}$,

$$\hat{J}_{\mathrm{loc},[1]} \cdot \mathcal{S}_{[2]} :\simeq 2\hat{J}_{[1],1}\mathcal{S}_{[2]}, \quad (4.30)$$

while the second term in (4.29) is equivalent to the bilocal part on $\mathcal{S}_{[3]}$,

$$\hat{J}_{\mathrm{biloc},[0]} \cdot \mathcal{S}_{[3]} := J^{(1)}_{[0],1}J^{(2)}_{[0],2}\mathcal{S}_{[3]} + J^{(1)}_{[0],1}J^{(2)}_{[0],3}\mathcal{S}_{[3]} + J^{(1)}_{[0],2}J^{(2)}_{[0],3}\mathcal{S}_{[3]}$$
$$\simeq J^{(1)}_{[0],1}J^{(2)}_{[0],2}\mathcal{S}_{[3]}. \quad (4.31)$$

The remaining two terms in (4.29) have a special form; one may interpret them as a nonlinear bilocal contribution where both legs overlap. Such terms are not provided by the usual coproduct rule for tensor product representations, but our nonlinear representation is not exactly a tensor product, and hence it is conceivable to have them,

$$\hat{J}_{\mathrm{o'lap},[1]} \cdot \mathcal{S}_{[2]} :\simeq \frac{2}{3}J^{(1)}_{[0],2}J^{(2)}_{[1],1}\mathcal{S}_{[2]} - \frac{2}{3}J^{(1)}_{[0],1}J^{(2)}_{[1],1}\mathcal{S}_{[2]}. \quad (4.32)$$

The (insignificant) prefactors in this definition were chosen to agree with a desirable form of expression further below. Finally, we notice that $\hat{\mathcal{Y}}_{[3]}$ does not contain the standard bilocal contribution to the representation on $\mathcal{S}_{[2]}$. However, this term is zero modulo cyclic identifications,

$$J^{(1)}_{[1],1}J^{(2)}_{[0],2}\mathcal{S}_{[2]} + J^{(1)}_{[0],1}J^{(2)}_{[1],2}\mathcal{S}_{[2]} \simeq 0. \quad (4.33)$$

Altogether, we derive an invariance statement for the action at cubic order in the fields,

$$\frac{1}{3}(\hat{J} \cdot \mathcal{S})_{[3]} :\simeq \frac{1}{2}\hat{J}_{\mathrm{loc},[1]} \cdot \mathcal{S}_{[2]} + \frac{1}{3}\hat{J}_{\mathrm{biloc},[0]} \cdot \mathcal{S}_{[3]} + \frac{1}{2}\hat{J}_{\mathrm{o'lap},[1]} \cdot \mathcal{S}_{[2]}$$
$$\simeq \frac{1}{3}\hat{\mathcal{Y}}_{[3]} \simeq 0. \quad (4.34)$$

Note that the prefactors of all terms agree with the symmetry factors $1/m$ of the $\mathcal{S}_{[m]}$ which is acted

upon.[25] We have verified explicitly that (4.34) holds for the level-one momentum $\hat{J} = \hat{P}$.

Nonlinear level-one invariance.—Now, let us turn to a general number of fields n in order to understand the precise structure of the unconventional terms. The combination $\hat{\mathcal{Y}}_{[n]}$ of the strong invariance condition (3.24) can be written in our present notation as

$$
\begin{aligned}
\hat{\mathcal{Y}}_{[n]} = &\sum_{m=0}^{n-2}\sum_{j=2}^{n-m}\hat{J}_{[m],j}\mathcal{S}_{[n-m]} + \sum_{m=0}^{n-2}\sum_{j=1}^{m+1}U^{j-1}\hat{J}_{[m],1}\mathcal{S}_{[n-m]} \\
&+ \sum_{m=0}^{n-2}\sum_{l=0}^{m}\sum_{j=2}^{n-m-1}\sum_{k=j+1}^{n-m}J^{(1)}_{[l],j}J^{(2)}_{[m-l],k}\mathcal{S}_{[n-m]} \\
&+ \sum_{m=0}^{n-2}\sum_{l=0}^{m}\sum_{j=1}^{l}\sum_{k=j+1}^{l+1}(U^{j-1}J^{(1)}_{[m-l],k} - U^{k-1+m-l}J^{(1)}_{[m-l],j}) \\
&\times J^{(2)}_{[l],1}\mathcal{S}_{[n-m]}.
\end{aligned} \tag{4.35}
$$

Using the same identities as at the level of three fields, we find that violations of cyclicity are given by terms containing $\mathcal{Y}$ and H,

$$
(U-1)\hat{\mathcal{Y}}_{[n]} = \sum_{m=0}^{n-2}(1+U^{m+1})[J^{(1)}_{[m],1}\mathcal{Y}^{(2)}_{[n-m]} - H_{[m],1}\mathcal{S}_{[n-m]}]. \tag{4.36}
$$

This means that $\hat{\mathcal{Y}}$ is in fact cyclic provided that the action is invariant under level zero and that the combined generator H defined in (4.20) is zero. We may thus interpret the statement $\hat{\mathcal{Y}} = 0$ as level-one invariance of the action.

To bring this statement somewhat closer to the expected form of $\hat{J} \cdot \mathcal{S}$, we introduce a modified combination $\hat{\mathcal{Y}}'$ with some convenient extra terms which are zero due to $\mathcal{Y} = H = 0$:

$$
\begin{aligned}
\hat{\mathcal{Y}}'_{[n]} :=&\ \hat{\mathcal{Y}}_{[n]} + \sum_{m=0}^{n-2}[J^{(1)}_{[m],1}\mathcal{Y}^{(2)}_{[n-m]} + H_{[m],1}\mathcal{S}_{[n-m]}] \\
=&\ \sum_{m=0}^{n-2}\sum_{j=1}^{n}U^{j-1}\hat{J}_{[m],1}\mathcal{S}_{[n-m]} \\
&+ \sum_{m=0}^{n-2}\sum_{l=0}^{m}\sum_{k=l+2}^{n-m+l}\sum_{j=l+2}^{k}U^{j-1}J^{(1)}_{[m-l],k}J^{(2)}_{[l],1}\mathcal{S}_{[n-m]} \\
&+ \sum_{m=0}^{n-2}\sum_{l=0}^{m}\sum_{k=1}^{l+1}\left[\sum_{j=2}^{m-l+k} - \sum_{j=k+1}^{m+1}\right]U^{j-1}J^{(1)}_{[m-l],k}J^{(2)}_{[l],1}\mathcal{S}_{[n-m]}.
\end{aligned} \tag{4.37}
$$

The terms in the first line of the result represent the local terms of the level-one representation. The terms in the second line are clearly bilocal with two nonoverlapping insertions. The terms in the third line take a similar form as the bilocal terms, but here the insertion of $J^{(1)}$ is within the range of $J^{(2)}$, and thus their actions overlap.[26]

Some comments regarding the bilocal and overlapping terms are in order: even through the prefactors for all terms in (4.37) are 1, those of the nonlinear bilocal terms appear somewhat unnatural; when compared to their linear counterparts, they are off by a factor of $(n-m)/n$. On the one hand, this may be worrisome in view of gauge symmetry because such relative factors could easily upset the composition of covariant derivatives and thus spoil this essential symmetry. On the other hand, such factors can be compensated by the relative length of the objects and thus by the multiplicity of operator insertions. Moreover, the overlapping terms have no analog in the conventional level-one representation based on the coproduct rule.

How can we make sense of this behavior? Clearly, the novel nonlinear representation of level-one symmetry on cyclic polynomials can have new features, which do not need to follow the coproduct rule strictly. What matters is that the above form follows by elementary transformations from the strong invariance relationship (3.24). The latter holds in planar $\mathcal{N} = 4$ sYM for the level-one momentum $\hat{J} = \hat{P}$. It therefore makes sense to view $\hat{\mathcal{Y}}'_{[n]} = 0$ as given by (4.37) as the definition of the nonlinear level-one invariance of the action. Whether or not it has the expected from, it certainly does describe a nontrivial relationship of planar $\mathcal{N} = 4$ sYM. In any case, the structures in (4.37) clearly deserve further theoretical scrutiny.

Cyclic level-one representation.—As the expression (4.37) is effectively cyclic, we may furthermore identify terms related by cyclic permutation. This yields a simpler expression:

$$
\begin{aligned}
\hat{\mathcal{Y}}'_{[n]} \simeq&\ \sum_{m=0}^{n-2}n\hat{J}_{[m],1}\mathcal{S}_{[n-m]} \\
&+ \sum_{m=0}^{n-2}\sum_{l=0}^{m}\sum_{k=1}^{n-m+2l+1}\left(k-l-\frac{1}{2}n+\frac{1}{2}m-1\right) \\
&\times J^{(1)}_{[m-l],k}J^{(2)}_{[l],1}\mathcal{S}_{[n-m]}.
\end{aligned} \tag{4.38}
$$

Note that all bilocal terms could be combined in a uniform expression. Here, the first and the last $l+1$ terms labeled by $k=1,\ldots,l+1$ and $k=n-m+l+1,\ldots,n-m+2l+1$ represent overlapping bilocal terms, whereas the insertions

[25]For the novel overlapping terms, this is a choice which was used to fix the perviously chosen prefactors in its definition (4.32).

[26]Consequently, the combination of both insertions is effectively local; nevertheless, we will still associate them to the bilocal part due to their structural similarity.

do not overlap in the remaining middle range $k = l + 2, ..., n - m + l$.

Since invariance of the action can and should be expressed modulo the cyclicity of the trace, we may write the symmetry statement as

$$\hat{J} \cdot \mathcal{S} \simeq 0 \quad \text{with} \quad \hat{J} \cdot \mathcal{S} :\simeq \sum_n \frac{1}{n} \hat{\mathcal{Y}}'_{[n]}. \quad (4.39)$$

This form is a complete expression for level-one invariance of the action including all standard and nonstandard terms. We have verified explicitly by computer algebra that it holds for planar $\mathcal{N} = 4$ sYM for the level-one generators $\hat{J}$ with $J \in \{P, Q, \bar{Q}, R, B\}$. Unfortunately, the calculations produce hundreds of intermediate terms which are subject to cyclic identifications and integrations by parts. Only the invariance under the level-one bonus symmetry $\hat{B}$ has a reasonably simple structure, and we shall show it explicitly below.

As such, we have not yet fully established that the extended symmetries generate a Yangian algebra; we would need to show that the adjoint property (3.2) as well as the Serre relation (3.3) hold. A complication, to be addressed in Ref. [21], is that the algebra is mixed with gauge transformations of a novel kind. Independently of which algebraic relations the symmetry generators obey, they give rise to novel relations for planar $\mathcal{N} = 4$ sYM.[27]

Level-one bonus symmetry.—In order to show invariance of the action under the level-one bonus symmetry $\hat{B}$ introduced in Ref. [23], see also Ref. [26], we first introduce the additional algebraic structures; see Appendix A and B for further details. The coproduct depends only on the odd level-zero generators

$$\Delta\hat{B} = \hat{B} \otimes 1 + 1 \otimes \hat{B} - \frac{1}{4} S^{ab} \wedge Q_{ba} - \frac{1}{4} \bar{S}_b{}^{\dot{\alpha}} \wedge \bar{Q}_{\dot{\alpha}}{}^b, \quad (4.40)$$

and the single-field action is trivial,

$$\hat{B} \cdot Z = 0. \quad (4.41)$$

Moreover, the representation of the superconformal boosts S and $\bar{S}$ are almost completely given in terms of the supersymmetries $\bar{Q}$ and Q, respectively,

$$S^{ab} \sim ix^{\alpha\dot{\epsilon}}\bar{Q}_{\dot{\epsilon}}{}^b + S'^{ab}, \qquad \bar{S}_a{}^{\dot{\gamma}} \sim -ix^{\beta\dot{\gamma}}Q_{a\beta} + \bar{S}_a'{}^{\dot{\gamma}}. \quad (4.42)$$

Both of the additional operators S' and $\bar{S}'$ act nontrivially only on a single type of field,

$$S'^{ab} \cdot \Psi^c{}_\delta = -2\delta_\delta^\alpha\Phi^{bc}, \qquad \bar{S}_a'{}^{\dot{\gamma}} \cdot \bar{\Psi}_{\dot{\epsilon}d} = 2\delta_{\dot{\epsilon}}^{\dot{\gamma}}\bar{\Phi}_{ad}. \quad (4.43)$$

Based on these relationships, we can in fact show that almost all contributions to the invariance condition are trivially zero. First of all, by combining (4.40) and (4.42), one can observe that the x-dependent terms due to $\bar{S} \wedge \bar{Q}$ and $S \wedge Q$ mutually cancel irrespectively of what they act on. Therefore, the only nonzero contributions to the invariance condition can originate from the x-independent operators S' and $\bar{S}'$ as well as when S and $\bar{S}$ act on a partial derivative which subsequently removes the x-dependence,

$$S^{ab} \cdot (\partial_{\dot{\gamma}\delta}Z) = i\partial_{\dot{\gamma}\delta}(x^{\alpha\dot{\epsilon}}\bar{Q}_{\dot{\epsilon}}{}^b \cdot Z) + \partial_{\dot{\gamma}\delta}(S'^{ab} \cdot Z)$$
$$= ix^{\alpha\dot{\epsilon}}\partial_{\dot{\gamma}\delta}(\bar{Q}_{\dot{\epsilon}}{}^b \cdot Z) + \partial_{\dot{\gamma}\delta}(S'^{ab} \cdot Z) + i\delta_\delta^\alpha\bar{Q}_{\dot{\gamma}}{}^b \cdot Z,$$
$$\bar{S}_a{}^{\dot{\gamma}} \cdot (\partial_{\dot{\epsilon}\delta}Z) = -ix^{\beta\dot{\gamma}}\partial_{\dot{\epsilon}\delta}(Q_{a\beta} \cdot Z) + \partial_{\dot{\epsilon}\delta}(\bar{S}_a'{}^{\dot{\gamma}} \cdot Z) - i\delta_{\dot{\epsilon}}^{\dot{\gamma}}Q_{a\delta} \cdot Z.$$
$$(4.44)$$

First, we consider overlapping terms in the action of $\hat{B}$; overlapping terms require the first level-zero generator to act nonlinearly. However, all nonlinear contributions from the odd generators act on fermions only, and they produce two bosons without derivatives. When acting further on the result, all x-dependent contributions must cancel by the above arguments, and as we have seen, bosons without derivatives cannot generate x-independent terms. Moreover, extra terms cannot be generated from the action of the first generator because S' and $\bar{S}'$ are purely linear and any partial derivatives acting on the original field can be pulled out from the calculation. Hence, there are no overlapping terms. Analogously, the single-field action of $\hat{B}$ is zero.

Let us now consider the various terms in the action: the quadratic terms $\mathcal{S}_{[2]}$ can only contribute via overlapping terms, and therefore they are all trivially invariant on their own. Furthermore, all the quartic terms $\mathcal{S}_{[4]}$ are purely bosonic, and they do not involve partial derivatives. Since the operators S' and $\bar{S}'$ act on fermions only, and the absence of partial derivatives prevents the generation of further x-independent terms, also all terms in $\mathcal{S}_{[4]}$ are invariant on their own. It remains to consider the cubic terms $\mathcal{S}_{[3]}$.

By elementary transformations, we can summarize the (nonlinear) action (4.38) on the cubic terms as

$$\sum_n \hat{\mathcal{Y}}'_{[n]} \simeq J_1^{(1)}J_2^{(2)}\mathcal{S}_{[3]}$$

$$\simeq -\frac{1}{4}(S^{ab})_1(Q_{ba})_2\mathcal{S}_{[3]} - \frac{1}{4}(Q_{ba})_1(S^{ab})_2\mathcal{S}_{[3]}$$

$$- \frac{1}{4}(\bar{S}_b{}^{\dot{\alpha}})_1(\bar{Q}_{\dot{\alpha}}{}^b)_2\mathcal{S}_{[3]} - \frac{1}{4}(\bar{Q}_{\dot{\alpha}}{}^b)_1(\bar{S}_b{}^{\dot{\alpha}})_2\mathcal{S}_{[3]}.$$
$$(4.45)$$

---

[27]By construction, these symmetries form some algebra. If it is not of Yangian kind, it will inevitably be much larger than that. So, the default assumption is that the algebra is as small as possible, and thus of Yangian kind. Nevertheless, it intrinsically interesting to understand the actual symmetry algebra and its relations in detail.

Effectively, only x-independent terms can potentially contribute, and therefore we need to consider only two types of terms from this expression: the generators S and $\bar{\text{S}}$ can act on a derivative ∂Z in such a way that the derivative eliminates the x-dependence from the action of S and $\bar{\text{S}}$. Alternatively, S and $\bar{\text{S}}$ can act on fermions and yield terms via the extra operators S and $\bar{\text{S}}$. Curiously, the cubic terms in the action (2.5) split into two corresponding classes, namely terms with fermions and terms with derivatives,

$$\mathcal{L}_{[3]} = -i\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\varepsilon^{\dot{\epsilon}\dot{\kappa}}\varepsilon^{\zeta\lambda}\text{tr}([A_{\dot{\kappa}\lambda}, \partial_{\dot{\alpha}\beta}A_{\dot{\epsilon}\zeta}]A_{\dot{\gamma}\delta})$$
$$-\frac{i}{2}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\text{tr}([\Phi^{ef}, \partial_{\dot{\alpha}\beta}\bar{\Phi}_{ef}]A_{\dot{\gamma}\delta})$$
$$+\varepsilon^{\dot{\kappa}\dot{\alpha}}\varepsilon^{\beta\gamma}\text{tr}(\{\Psi^d{}_\gamma, \bar{\Psi}_{\dot{\kappa}d}\}A_{\dot{\alpha}\beta})$$
$$+\frac{i}{2}\varepsilon^{\alpha\gamma}\text{tr}(\{\Psi^e{}_\alpha, \Psi^f{}_\gamma\}\bar{\Phi}_{ef})$$
$$+\frac{i}{2}\varepsilon^{\dot{\alpha}\dot{\gamma}}\text{tr}(\{\bar{\Psi}_{\dot{\alpha}e}, \bar{\Psi}_{\dot{\gamma}f}\}\Phi^{ef}). \tag{4.46}$$

Let us first consider the fermionic terms on the second line on which S and $\bar{\text{S}}$ can effectively act only via the extra operators S$'$ and $\bar{\text{S}}'$. By considering the types which can potentially be generated along with the parity-reversing nature of the level-one generators, we find terms of the types

$$\varepsilon^{\alpha\gamma}\text{tr}([\Psi^e{}_\alpha, \Psi^f{}_\gamma]\bar{\Phi}_{ef}), \qquad \varepsilon^{\alpha\gamma}\text{tr}(\{\bar{\Phi}_{ef}, \Phi^{ef}\}F_{\alpha\gamma}),$$
$$\varepsilon^{\dot{\alpha}\dot{\gamma}}\text{tr}([\bar{\Psi}_{\dot{\alpha}e}, \bar{\Psi}_{\dot{\gamma}f}]\Phi^{ef}), \qquad \varepsilon^{\dot{\alpha}\dot{\gamma}}\text{tr}(\{\bar{\Phi}_{ef}, \Phi^{ef}\}\bar{F}_{\dot{\alpha}\dot{\gamma}}),$$
$$\text{tr}(\{\bar{\Phi}_{ef}, \Phi^{de}\}[\bar{\Phi}_{db}, \Phi^{bf}]), \qquad \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\text{tr}(\{A_{\dot{\alpha}\beta}, \bar{\Phi}_{ef}\}[A_{\dot{\gamma}\delta}, \Phi^{ef}]),$$
$$\tag{4.47}$$

as well as

$$\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\text{tr}(\partial_{\dot{\gamma}\delta}\{\bar{\Phi}_{ef}, \Phi^{ef}\}A_{\dot{\alpha}\beta}). \tag{4.48}$$

Now, the former six terms are trivially zero because they all involve simultaneous symmetrization and antisymmetrization of indices. Only the last term is nonzero. It can only be generated from the term $\{\Psi, \bar{\Psi}\}A$ in the action, and the contributions read

$$\frac{1}{12}\varepsilon^{\dot{\kappa}\dot{\alpha}}\varepsilon^{\beta\gamma}\text{tr}(\{S'^{\epsilon f}\Psi^d{}_\gamma, Q_{fe}\bar{\Psi}_{\dot{\kappa}d}\}A_{\dot{\alpha}\beta} + \{\bar{S}'_f{}^{\dot{\epsilon}}\bar{\Psi}_{\dot{\kappa}d}, \bar{Q}_{\dot{\epsilon}}{}^f\Psi^d{}_\gamma\}A_{\dot{\alpha}\beta}). \tag{4.49}$$

However, by explicit computation, the two terms cancel precisely.

By similar arguments as above, the purely bosonic cubic terms of the action can yield terms of the kinds

$$\varepsilon^{\alpha\gamma}\text{tr}([\Psi^e{}_\alpha, \Psi^f{}_\gamma]\bar{\Phi}_{ef}), \qquad \varepsilon^{\dot{\alpha}\dot{\gamma}}\text{tr}([\bar{\Psi}_{\dot{\alpha}e}, \bar{\Psi}_{\dot{\gamma}f}]\Phi^{ef}),$$
$$\varepsilon^{\dot{\kappa}\dot{\alpha}}\varepsilon^{\beta\gamma}\text{tr}([\Psi^d{}_\gamma, \bar{\Psi}_{\dot{\kappa}d}]A_{\dot{\alpha}\beta}). \tag{4.50}$$

As before, the former two terms have incompatible symmetrizations and are zero. Only the last term is nonzero. By acting on the term $[\Phi, \partial\Phi]A$, we find

$$-\frac{i}{24}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\text{tr}([\partial_{\dot{\alpha}\beta}(S^{\kappa g}\bar{\Phi}_{ef}), Q_{g\kappa}\Phi^{ef}]A_{\dot{\gamma}\delta}$$
$$+ [\partial_{\dot{\alpha}\beta}\bar{S}_g{}^\kappa\Phi^{ef}, \bar{Q}_\kappa{}^g\bar{\Phi}_{ef}]A_{\dot{\gamma}\delta}), \tag{4.51}$$

which evaluates to

$$-\frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\text{tr}([\Psi^f{}_\beta, \bar{\Psi}_{\dot{\alpha}f}]A_{\dot{\gamma}\delta}). \tag{4.52}$$

Conversely, the action on the term $[A, \partial A]A$ yields

$$\frac{i}{12}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\varepsilon^{\dot{\epsilon}\dot{\kappa}}\varepsilon^{\zeta\lambda}\text{tr}([\partial_{\dot{\alpha}\beta}S^{\eta g}A_{\dot{\epsilon}\zeta}, Q_{g\eta}A_{\dot{\gamma}\delta}]A_{\dot{\kappa}\lambda}$$
$$+ [\partial_{\dot{\alpha}\beta}\bar{S}_g{}^{\dot{\eta}}A_{\dot{\epsilon}\zeta}, \bar{Q}_{\dot{\eta}}{}^g A_{\dot{\gamma}\delta}]A_{\dot{\kappa}\lambda})$$
$$-\frac{i}{12}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\varepsilon^{\dot{\epsilon}\dot{\kappa}}\varepsilon^{\zeta\lambda}\text{tr}([\partial_{\dot{\epsilon}\zeta}S^{\eta g}A_{\dot{\alpha}\beta}, Q_{g\eta}A_{\dot{\gamma}\delta}]A_{\dot{\kappa}\lambda}$$
$$+ [\partial_{\dot{\epsilon}\zeta}\bar{S}_g{}^{\dot{\eta}}A_{\dot{\alpha}\beta}, \bar{Q}_{\dot{\eta}}{}^g A_{\dot{\gamma}\delta}]A_{\dot{\kappa}\lambda}), \tag{4.53}$$

which amounts to

$$\frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta}\text{tr}([\Psi^f{}_\beta, \bar{\Psi}_{\dot{\alpha}f}]A_{\dot{\gamma}\delta}). \tag{4.54}$$

Therefore, both remaining contributions cancel, and altogether, this proves that the bonus level-one Yangian generator $\hat{\text{B}}$ is a symmetry of planar $\mathcal{N} = 4$ sYM.

V. $\mathcal{N} = 6$ SUPERSYMMETRIC CHERN-SIMONS THEORY

Now that we have established an extended symmetry related to planar integrability of classical $\mathcal{N} = 4$ sYM theory, we should consider further gauge theories where planar integrability has been observed. The main example of a supposedly integrable gauge theory in the planar limit which is substantially different from $\mathcal{N} = 4$ sYM is ABJ(M) theory. This model is a superconformal field theory in three dimensions with an AdS/CFT dual string theory, it appears to be integrable in the planar limit, and its observables display Yangian symmetry.

ABJ(M) theory will serve as a crucial testing ground for our proposal; if our definition of integrability in the form of Yangian symmetry is correct, we should be able to show that its holds also for this theory.

A. Action

ABJ(M) theory is a $\mathcal{N} = 6$ supersymmetric Chern-Simons theory with gauge group $U(M) \times U(N)$. The vector multiplet is coupled to two scalar multiplets (Φ, Ψ) and $(\bar{\Phi}, \bar{\Psi})$, transforming in the $(M, \bar{N})$ and $(\bar{M}, N)$ representations of the gauge group, respectively.

As for $\mathcal{N} = 4$ sYM, we use a spinor-matrix notation $x^{\alpha\beta}$ to denote the coordinates of three-dimensional spacetime where $\alpha, \beta, \ldots = 1, 2$ denote $\mathfrak{sl}(2, \mathbb{R})$ spacetime spinor indices. The coordinate matrices are symmetric $x^{\alpha\beta} = x^{\beta\alpha}$ and real. The corresponding partial derivatives $\partial_{\alpha\beta} = \partial_{\beta\alpha}$ are normalized by the rule $\partial_{\alpha\beta} x^{\gamma\delta} = \delta^\gamma_\alpha \delta^\delta_\beta + \delta^\delta_\alpha \delta^\gamma_\beta$. Furthermore, Latin letters $a, b, \ldots = 1, 2, 3, 4$ denote $\mathfrak{su}(4)$ internal indices.

For simplicity, we introduce two gauge fields $A_{\alpha\beta}$ and $\tilde{A}_{\alpha\beta}$, one for each $U(K)$ component of the gauge group. The fields of the theory can be represented as matrices: the field $A_{\alpha\beta}$ ($\tilde{A}_{\alpha\beta}$) is a Hermitian $M \times M$ ($N \times N$) matrix, and the

fields Φ^a and Ψ_{ab} ($\bar{\Phi}_a$ and $\bar{\Psi}_a{}^b$) are complex $M \times N$ ($N \times M$) matrices; any sequence of fields is to be understood as the appropriate matrix product.

The gauge-covariant derivatives act on a covariant $M \times N$ field Z ($N \times M$ field $\bar{Z}$) as

$$\nabla_{\alpha\beta} Z := \partial_{\alpha\beta} Z + i A_{\alpha\beta} Z - i Z \tilde{A}_{\alpha\beta},$$
$$\nabla_{\alpha\beta} \bar{Z} := \partial_{\alpha\beta} \bar{Z} + i \tilde{A}_{\alpha\beta} \bar{Z} - i \bar{Z} A_{\alpha\beta}, \tag{5.1}$$

and we define the field strengths as for $\mathcal{N} = 4$ sYM,

$$\nabla_{\alpha\beta} A_{\gamma\delta} := \partial_{\alpha\beta} A_{\gamma\delta} - \partial_{\gamma\delta} A_{\alpha\beta} + i[A_{\alpha\beta}, A_{\gamma\delta}],$$
$$\nabla_{\alpha\beta} \tilde{A}_{\gamma\delta} := \partial_{\alpha\beta} \tilde{A}_{\gamma\delta} - \partial_{\gamma\delta} \tilde{A}_{\alpha\beta} + i[\tilde{A}_{\alpha\beta}, \tilde{A}_{\gamma\delta}]. \tag{5.2}$$

The action of the theory is completely fixed by supersymmetry, and its Lagrangian reads

$$\begin{aligned}
\mathcal{L} = \ &\varepsilon^{\beta\gamma} \varepsilon^{\delta\epsilon} \varepsilon^{\kappa\alpha} \mathrm{tr}\left(-\frac{1}{4} A_{\alpha\beta} \partial_{\gamma\delta} A_{\epsilon\kappa} - \frac{i}{6} A_{\alpha\beta} A_{\gamma\delta} A_{\epsilon\kappa} + \frac{1}{4} \tilde{A}_{\alpha\beta} \partial_{\gamma\delta} \tilde{A}_{\epsilon\kappa} + \frac{i}{6} \tilde{A}_{\alpha\beta} \tilde{A}_{\gamma\delta} \tilde{A}_{\epsilon\kappa}\right) \\
&+ i\varepsilon^{\alpha\beta} \varepsilon^{\gamma\delta} \mathrm{tr}\left(\bar{\Psi}_\alpha{}^e \nabla_{\beta\gamma} \Psi_{\delta e} + \frac{1}{2} \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} \nabla_{\alpha\beta} \bar{\Phi}_e \nabla_{\gamma\delta} \Phi^e\right) \\
&+ \varepsilon^{\epsilon\kappa} \mathrm{tr}(i\varepsilon^{abcd} \bar{\Phi}_a \Psi_{\epsilon b} \bar{\Phi}_c \Psi_{\kappa d} - i\varepsilon_{abcd} \Phi^a \bar{\Psi}_\epsilon{}^b \Phi^c \bar{\Psi}_\kappa{}^d) \\
&+ \varepsilon^{\epsilon\kappa} \mathrm{tr}(-i\Phi^a \bar{\Phi}_a \Psi_{\epsilon b} \bar{\Psi}_\kappa{}^b + i\bar{\Phi}_a \Phi^a \bar{\Psi}_\epsilon{}^b \Psi_{\kappa b} + 2i\Phi^a \bar{\Phi}_b \Psi_{\epsilon a} \bar{\Psi}_\kappa{}^b - 2i\bar{\Phi}_a \Phi^b \bar{\Psi}_\epsilon{}^a \Psi_{\kappa b}) \\
&+ \mathrm{tr}\left(\frac{1}{3} \Phi^a \bar{\Phi}_a \Phi^b \bar{\Phi}_b \Phi^c \bar{\Phi}_c + \frac{1}{3} \Phi^a \bar{\Phi}_b \Phi^b \bar{\Phi}_c \Phi^c \bar{\Phi}_a + \frac{4}{3} \Phi^a \bar{\Phi}_b \Phi^c \bar{\Phi}_a \Phi^b \bar{\Phi}_c - 2\Phi^a \bar{\Phi}_b \Phi^c \bar{\Phi}_c \Phi^b \bar{\Phi}_a\right). \tag{5.3}
\end{aligned}$$

B. Symmetries

As already stated, ABJ(M) theory is a superconformal theory; the algebra of superconformal transformations is[28] $\mathfrak{osp}(6|4)$. The action of the Poincaré and dilatation generators reads

$$\mathrm{P}_{\alpha\beta} \cdot Z = i\nabla_{\alpha\beta} Z,$$
$$\mathrm{L}^\alpha{}_\beta \cdot Z = -ix^{\alpha\epsilon} \nabla_{\epsilon\beta} Z + \frac{i}{2} \delta^\alpha_\beta x^{\delta\epsilon} \nabla_{\delta\epsilon} Z + (\mathrm{L}_{\mathrm{spin}})^\alpha{}_\beta \cdot Z,$$
$$\mathrm{D} \cdot Z = -\frac{i}{2} x^{\alpha\beta} \nabla_{\alpha\beta} Z - i\Delta_Z Z, \tag{5.4}$$

where the conformal dimensions are $\Delta_\Phi = 1/2$, $\Delta_\Psi = 1$, $\Delta_A = 0$, and where the spin operator $\mathrm{L}_{\mathrm{spin}}$ acts nontrivially only on the spinor fields $Z_\alpha \in \{\Psi_{\alpha c}, \bar{\Psi}_\alpha{}^c\}$,

$$(\mathrm{L}_{\mathrm{spin}})^\alpha{}_\beta \cdot Z_\gamma = -i\delta^\alpha_\gamma Z_\beta + \frac{i}{2} \delta^\alpha_\beta Z_\gamma. \tag{5.5}$$

The supersymmetry charges transform in the six-dimensional real irreducible representation of the internal algebra $\mathfrak{su}(4)$. They act as

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[28]The maximal even subalgebra is $\mathfrak{sp}(4) \times \mathfrak{so}(6)$, where $\mathfrak{sp}(4) = \mathfrak{so}(2, 3)$ is the conformal algebra in three dimensions, and $\mathfrak{so}(6)$ is the R-symmetry algebra.

$$Q_{\epsilon ab} \cdot \Phi^c = 2\delta^c_{\bar a}\Psi_{\epsilon \bar b},$$

$$Q_{\epsilon ab} \cdot \bar\Phi_c = \varepsilon_{abcd}\bar\Psi_\epsilon{}^d,$$

$$Q_{\epsilon ab} \cdot \Psi_{\kappa c} = i\varepsilon_{abcd}\nabla_{\epsilon\kappa}\Phi^d + \varepsilon_{\epsilon\kappa}\varepsilon_{abcd}(-i\Phi^d\bar\Phi_f\Phi^f + i\Phi^f\bar\Phi_f\Phi^d) - 2i\varepsilon_{\epsilon\kappa}\varepsilon_{abdf}\Phi^d\bar\Phi_c\Phi^f,$$

$$Q_{\epsilon ab} \cdot \bar\Psi_\delta{}^c = 2i\delta^c_{\bar a}\nabla_{\epsilon\delta}\bar\Phi_{\bar b} + 4i\varepsilon_{\epsilon\delta}\bar\Phi_{\bar a}\Phi^c\bar\Phi_{\bar b} + 2i\varepsilon_{\epsilon\delta}\delta^c_{\bar a}\bar\Phi_{\bar b}\Phi^f\bar\Phi_f - 2i\varepsilon_{\epsilon\delta}\delta^c_{\bar a}\bar\Phi_f\Phi^f\bar\Phi_{\bar b},$$

$$Q_{\epsilon ab} \cdot A_{\gamma\delta} = -4i\varepsilon_{\epsilon\bar\gamma}\Psi_{\bar\delta\bar a}\bar\Phi_{\bar b} - 2i\varepsilon_{\epsilon\bar\gamma}\varepsilon_{abfg}\Phi^f\bar\Psi_{\bar\delta}{}^g,$$

$$Q_{\epsilon ab} \cdot \tilde A_{\gamma\delta} = 4i\varepsilon_{\epsilon\bar\gamma}\bar\Phi_{\bar a}\Psi_{\bar\delta b} + 2i\varepsilon_{\epsilon\bar\gamma}\varepsilon_{abfg}\bar\Psi_{\bar\delta}{}^f\Phi^g. \tag{5.6}$$

Here and in the following, we use a shorthand notation $X_{\bar\alpha\bar\gamma} := \frac{1}{2}X_{\alpha\gamma} \pm \frac{1}{2}X_{\gamma\alpha}$ for the symmetrization of indices, where the sign is determined by the manifest symmetries on the defining lhs of the equation.

C. Yangian invariance

In order to check the classical Yangian invariance of ABJ(M) theory, we can proceed the same way we did for $\mathcal{N} = 4$ sYM: we start from the simplest equation of motion, we act on it with the bilocal part of a level-one Yangian generator, we fix the single-field action, and then we check that the relationship of (3.24) is satisfied.

The most convenient choice for the level-one generator of $Y[\mathfrak{osp}(6|4)]$ is again $\hat P_{\alpha\beta}$. The bilocal part of the coproduct reads

$$(\hat P_{\alpha\beta})_{\text{biloc}} = P_{\gamma\bar\alpha} \wedge L^\gamma{}_{\bar\beta} + P_{\alpha\beta} \wedge D + \frac{i}{8}\varepsilon^{cdef}Q_{\alpha cd} \wedge Q_{\beta ef}. \tag{5.7}$$

Furthermore, we choose the simplest of the equations of motion, the Dirac equation $\breve\Psi = 0$ with

$$\breve\Psi^\alpha{}_b = \varepsilon^{\alpha\epsilon}(i\varepsilon^{\gamma\delta}\nabla_{\epsilon\gamma}\Psi_{\delta b} + 2i\varepsilon_{bcdf}\Phi^c\bar\Psi_\epsilon{}^d\Phi^f) + \varepsilon^{\alpha\epsilon}(i\Psi_{\epsilon b}\bar\Phi_c\Phi^c - 2i\Psi_{\epsilon c}\bar\Phi_b\Phi^c - i\Phi^c\bar\Phi_c\Psi_{\epsilon b} + 2i\Phi^c\bar\Phi_b\Psi_{\epsilon c}). \tag{5.8}$$

We find that the Dirac equation is weakly invariant under the above level-one momentum generator with the following unique choice of single-field action:

$$\hat P_{\alpha\beta} \cdot \Phi^a = 0,$$

$$\hat P_{\alpha\beta} \cdot \bar\Phi_a = 0,$$

$$\hat P_{\alpha\beta} \cdot \Psi_{\gamma d} = \varepsilon_{\bar\alpha\gamma}(-\Psi_{\bar\beta d}\bar\Phi_e\Phi^e + 2\Psi_{\bar\beta e}\bar\Phi_d\Phi^e - \Phi^e\bar\Phi_e\Psi_{\bar\beta d} + 2\Phi^e\bar\Phi_d\Psi_{\bar\beta e}),$$

$$\hat P_{\alpha\beta} \cdot \bar\Psi_\gamma{}^d = \varepsilon_{\bar\alpha\gamma}(\bar\Psi_{\bar\beta}{}^d\Phi^e\bar\Phi_e - 2\bar\Psi_{\bar\beta}{}^e\Phi^d\bar\Phi_e + \bar\Phi_e\Phi^e\bar\Psi_{\bar\beta}{}^d - 2\bar\Phi_e\Phi^d\bar\Psi_{\bar\beta}{}^e),$$

$$\hat P_{\alpha\beta} \cdot A_{\gamma\delta} = i\varepsilon_{\bar\alpha\bar\gamma}\nabla_{\bar\beta\bar\delta}(\Phi^e\bar\Phi_e) + \varepsilon_{\bar\alpha\bar\gamma}\varepsilon_{\bar\beta\bar\delta}(-2i\Phi^e\bar\Phi_f\Phi^f\bar\Phi_e + \varepsilon^{\kappa\lambda}\Psi_{\kappa e}\bar\Psi_\lambda{}^e),$$

$$\hat P_{\alpha\beta} \cdot \tilde A_{\gamma\delta} = -i\varepsilon_{\bar\alpha\bar\gamma}\nabla_{\bar\beta\bar\delta}(\bar\Phi_e\Phi^e) + \varepsilon_{\bar\alpha\bar\gamma}\varepsilon_{\bar\beta\bar\delta}(-2i\bar\Phi^e\Phi_f\bar\Phi^f\Phi_e - \varepsilon^{\kappa\lambda}\bar\Psi_\kappa{}^e\Psi_{\lambda e}). \tag{5.9}$$

More concretely, we find that the following equality holds:

$$\hat P_{\alpha\beta} \cdot \breve\Psi^\gamma{}_d = \varepsilon_{\bar\alpha\epsilon}\left(-\frac{1}{2}\delta^\gamma_{\bar\beta}\breve A^{\epsilon\kappa}\Psi_{\kappa d} - \frac{5}{2}\breve A^{\epsilon\gamma}\Psi_{\bar\beta d} + \frac{1}{2}\delta^\gamma_{\bar\beta}\Psi_{\kappa d}\breve A^{\epsilon\kappa} + \frac{5}{2}\Psi_{\bar\beta d}\breve A^{\epsilon\gamma}\right)$$

$$+ \varepsilon_{\bar\alpha\epsilon}\delta^\gamma_{\bar\beta}(2\breve\Psi^\epsilon{}_f\bar\Phi_d\Phi^f - \breve\Psi^\epsilon{}_d\bar\Phi_f\Phi^f + 2\Phi^f\bar\Phi_d\breve\Psi^\epsilon{}_f - \Phi^f\bar\Phi_f\breve\Psi^\epsilon{}_d). \tag{5.10}$$

Here, $\breve A$ and $\breve{\tilde A}$ denote the variations of the action with respect to the gauge fields,

$$\breve A^{\alpha\beta} = \varepsilon^{\bar\alpha\gamma}\varepsilon^{\bar\beta\delta}\left(\frac{1}{2}\varepsilon^{\epsilon\kappa}\nabla_{\gamma\epsilon}A_{\delta\kappa} + i\Phi^e\nabla_{\gamma\delta}\bar\Phi_e - i\nabla_{\gamma\delta}\Phi^e\bar\Phi_e - 2i\Psi_{\gamma e}\bar\Psi_\delta{}^e\right),$$

$$\breve{\tilde A}^{\alpha\beta} = \varepsilon^{\bar\alpha\gamma}\varepsilon^{\bar\beta\delta}\left(-\frac{1}{2}\varepsilon^{\epsilon\kappa}\nabla_{\gamma\epsilon}\tilde A_{\delta\kappa} + i\bar\Phi_e\nabla_{\gamma\delta}\Phi^e - i\nabla_{\gamma\delta}\bar\Phi_e\Phi^e - 2i\bar\Psi_\gamma{}^e\Psi_{\delta e}\right), \tag{5.11}$$

and they vanish on shell as does $\breve\Psi$ so that the Dirac equation is weakly invariant. Moreover, we have verified that the rhs of (5.10) matches exactly the rhs of (3.24), and we have performed this check for all the equations of motion of ABJ(M) theory. Therefore, the equations of motion display Yangian symmetry in a strong sense. Finally, have shown by computer algebra that the ABJ(M) action is invariant in the sense of (4.39). Therefore, classical planar ABJ(M) theory has Yangian symmetry.

VI. PURE $\mathcal{N} < 4$ sYM

In this section, we would like to address a possible concern that our results discussed above, especially the formula (3.24), are a mere result of level-zero (super-conformal) and gauge symmetries of the theories. If that were the case, our work would not provide any criterion for establishing Yangian symmetry of physical models. As we will now demonstrate, this is fortunately not the case and, for formula (3.24) to hold, increased symmetry is indeed required.

To this end, let us consider a pure sYM theory but keep $\mathcal{N}$ arbitrary. The action is again given by

$$
\mathcal{L} = -\frac{1}{2}\varepsilon^{\alpha\epsilon}\varepsilon^{\gamma\kappa}\mathrm{tr}(F_{\alpha\gamma}F_{\epsilon\kappa}) - \frac{1}{2}\varepsilon^{\dot\alpha\dot\epsilon}\varepsilon^{\dot\gamma\dot\kappa}\mathrm{tr}(\bar F_{\dot\alpha\dot\gamma}\bar F_{\dot\epsilon\dot\kappa})
$$
$$
+ i\varepsilon^{\dot\kappa\dot\alpha}\varepsilon^{\beta\gamma}\mathrm{tr}(\bar\Psi_{\dot\kappa d}\nabla_{\dot\alpha\beta}\Psi^d{}_\gamma) - \frac{1}{4}\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\beta\delta}\mathrm{tr}(\nabla_{\dot\alpha\beta}\bar\Phi_{ef}\nabla_{\dot\gamma\delta}\Phi^{ef})
$$
$$
+ \frac{i}{2}\varepsilon^{\alpha\gamma}\mathrm{tr}(\bar\Phi_{ef}\{\Psi^e{}_\alpha,\Psi^f{}_\gamma\}) + \frac{i}{2}\varepsilon^{\dot\alpha\dot\gamma}\mathrm{tr}(\Phi^{ef}\{\bar\Psi_{\dot\alpha e},\bar\Psi_{\dot\gamma f}\})
$$
$$
+ \frac{1}{16}\mathrm{tr}([\bar\Phi_{ab},\Phi^{ef}][\Phi^{ab},\bar\Phi_{ef}]). \tag{6.1}
$$

The action (6.1) looks exactly like (2.5) of $\mathcal{N}=4$ sYM. The only difference is the range of indices $a,b,\dots = 1,\dots,\mathcal{N}$, where $\mathcal{N} \le 4$. As the reality condition (2.2) exists only for $\mathcal{N} = 4$, the scalar potential is written here in a different form, valid also for $\mathcal{N} \ne 4$. Moreover, for $\mathcal{N} = 1$, the scalar fields do not exist at all, which here is implicitly taken care of by the antisymmetry of their indices, $\Phi^{ab} = -\Phi^{ba} = \Phi^{11} = 0$. The symmetry algebra of (6.1) is $\mathfrak{psu}(2,2|\mathcal{N})$, and for any $\mathcal{N} \ge 0$, its Yangian exists. What we want to demonstrate is that $Y[\mathfrak{psu}(2,2|\mathcal{N})]$ is a symmetry of the theory only for the special value $\mathcal{N} = 4$. To that end, it is already enough to show that for other values of $\mathcal{N}$ the weak invariance of the equations of motion does not hold, as its failure will assure that none of the (aforementioned) stronger criteria stand either.

We thus can repeat the computation from Sec. III C keeping $\mathcal{N}$ arbitrary. The action of all the generators carries over from Secs. II and III [see Eqs. (2.13) and (3.18) for supersymmetry and local Yangian action, respectively], of course up to the restricted range of indices mentioned above (e.g., the local action of $\hat P$ vanishes completely for $\mathcal{N} = 1$). For general $\mathcal{N} \ge 1$, we act with $\hat P$ on the Dirac equation of motion,[29]

$$
\check{\bar\Psi}^{l\dot\kappa} = i\varepsilon^{\dot\kappa\dot\alpha}\varepsilon^{\beta\gamma}\nabla_{\dot\alpha\beta}\Psi^l{}_\gamma - i\varepsilon^{\dot\kappa\dot\alpha}[\Phi^{le},\bar\Psi_{\dot\alpha e}]. \tag{6.2}
$$

On shell, we are left with a term of the form

$$
\hat P_{\dot\alpha\beta}\cdot\check{\bar\Psi}^{d\dot\gamma} \approx (4-\mathcal{N})\varepsilon^{\dot\gamma\dot\kappa}\{\bar F_{\dot\alpha\dot\kappa},\Psi^d{}_\beta\}. \tag{6.3}
$$

---

[29]For $\mathcal{N} = 0$, the absence of fermionic fields Ψ requires a more elaborate consideration of the Yang-Mills equations.

Even though we have the principal freedom to adjust the local action of $\hat P$ to the case $\mathcal{N} < 4$, it is easy to see that this cannot remove the residual term; in order to cancel the term, one would need extra contributions of the form $\hat P A \sim \bar F$ or $\hat P \Psi \sim \nabla\Psi$. However, these would merely produce commutators $[\bar F, \Psi]$ rather than the desired anticommutator $\{\bar F, \Psi\}$.

Thus, we see that the value $\mathcal{N} = 4$ is special as it warrants the cancellation of the residual term and the on-shell invariance of the equations of motion of $\mathcal{N} = 4$ sYM. More importantly, we observe that superconformal and gauge symmetries by themselves are not sufficient to warrant our results presented in the preceding sections, and indeed Yangian invariance of a theory is a nontrivial statement.

VII. CONCLUSIONS AND OUTLOOK

In this work, we have elaborated upon the results of our previous paper [17] by spelling out in detail the different kinds of planar invariance conditions for the equations of motion of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and ABJ(M) theory under a Yangian algebra. Subsequently, we have reformulated the strong version of this invariance as an invariance of the action and proved explicitly that this invariance holds indeed. This not only shows that these two classical gauge theory models possess Yangian symmetry in the planar limit, but it may also be viewed as a formal definition for their integrability.

A follow-up work [21] will address Yangian invariance of field correlators at tree level: for a field theory with some set of symmetries, one should expect correlation functions of the fields to obey a corresponding set of Ward-Takahashi identities reflecting these symmetries. An important aspect in the computation of correlators within a gauge theory is gauge fixing. It will be shown how Yangian symmetry can be made compatible with Faddeev-Popov gauge fixing and the additional fermionic symmetry it gives rise to symmetry and how to formulate Slavnov-Taylor identities for Yangian symmetry which properly take into account the effects of gauge fixing. To that end, the algebraic relations of the Yangian need to be studied in detail because they are intertwined with gauge transformations which are henceforth deformed by the gauge fixing procedure. A curious side effect is that this will amount to an extension of the gauge algebra by bilocal and nonlocal gauge symmetries.

Our works lay the foundations for many directions of further study. Most importantly, firm contact should be made with corresponding symmetry structures of the world sheet theory for strings on $AdS_5 \times S^5$. By the AdS/CFT correspondence, such structures should exist, but quantum effects on both sides, potentially including quantum anomalies, may play a significant role for a precise matching; see Ref. [27] for considerations on the string theory side. For example, the algebraic complications due

to nonultralocality of the string theory sigma model [28] may well have a counterpart in the Yangian algebra for gauge theory.

Another obvious question is which other models beyond $\mathcal{N}=4$ sYM and ABJ(M) theory enjoy a Yangian (or related) symmetry in the planar limit. There are several deformations and orbifolds of $\mathcal{N}=4$ sYM which are apparently integrable [29] and which should therefore possess an extended symmetry. This has been shown explicitly for the beta-deformation in Ref. [30]. The simplistic fishnet theory [31] represents a particular contraction limit of an integrable deformation; however, due to the nonvanishing Coxeter number of the level-zero symmetry, some adjustments to our treatment will be inevitable (see Ref. [32]). Furthermore, massive deformations of supersymmetric Chern-Simons theories [33] have an interesting nonconformal supersymmetry algebra [34] on which a Yangian algebra could in principle be established. One may also wonder whether and how the partial integrability of $\mathcal{N}<4$ sYM theories within certain sectors, see Ref. [35], can be formulated in analogy to our framework.

Yangian symmetry could also prove helpful in constructing gauge theory models which are predicted by the AdS/CFT correspondence but remain to be formulated. Such models include two-dimensional gauge theories as duals of the integrable string theories on $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$, a q-deformation of $\mathcal{N}=4$ sYM as a dual of the eta/kappa-deformation of string theory on $AdS_5 \times S^5$ (where the Yangian symmetry would be deformed to a quantum affine group), as well as the infamous six-dimensional $\mathcal{N}=(2,0)$ theory which is intrinsically nonperturbative.

Another direction for research is to firmly derive the applications of integrability to various observables from our framework of Yangian symmetry. For example, the Bethe equations for the spectrum of one-loop anomalous dimensions follow directly from the well-established Bethe ansatz framework for quantum integrable spin chains in connection to (conventional) quantum algebra. More interestingly, the methods developed for the spectrum at higher loops and at finite coupling should have a formal justification in Yangian symmetry of the model (potentially after implementing quantum corrections). Along the same lines, it will be useful to understand the origin of the extended $\mathfrak{psu}(2|2)$ Yangian for the magnon scattering picture including the various master, boost, and secret symmetries. Similarly, symmetries and techniques for other observables such as Wilson loops, scattering amplitudes, correlation functions of local operators, and form factors might be traced back to Yangian symmetry.

Further open questions beyond generalizations and applications include the following. How does one generalize Yangian symmetry to a superspace formulation of $\mathcal{N}=4$ sYM (either in a light-cone superspace [36] or in a full superspace [16])? Does the (somewhat nonlocal)

Yangian symmetry have some associated Noether charges? What is the role of the Casimir operators (central elements) of the Yangian algebra? Can they be applied to a wider class of observables?

Finally, our framework for Yangian symmetry raises several questions of a mathematical nature. First and foremost, does the Yangian algebra actually close; i.e., do the Serre relations hold, and how does one formulate the closure precisely? How does one construct the single-field action of the level-one generators abstractly? Does it follow from some fundamental principles beyond the consistency requirements employed to derive it here?

Last but not least, in what sense is the proposed action of the Yangian generators a representation, in particular when acting on cyclic states such as a the action functional?

ACKNOWLEDGMENTS

APPENDIX A: $\mathcal{N}=4$ SUPERCONFORMAL SYMMETRY

In this Appendix, we summarize the $\mathcal{N}=4$ superconformal algebra $\mathfrak{psu}(2,2|4)$ and its representation on the fields.

Algebra.—The supersymmetry algebra is spanned by the supersymmetry generators $Q_{a\gamma}$ and $\bar{Q}_{\dot{a}}{}^c$ and the momentum generator $P_{\dot{a}\gamma}$. The special conformal generators are given by two fermionic generators S^{ac} and $J[\bar{S}]_a{}^{\dot{\gamma}}$ and the bosonic generator $K^{\alpha\dot{\gamma}}$. Furthermore, the superconformal algebra includes the Lorentz and internal rotation generators $L^\alpha{}_\gamma$, $\bar{L}^{\dot{\alpha}}{}_{\dot{\gamma}}$, and $R^a{}_c$ (whose trace over the indices of which vanishes) and the dilatation generator D. Finally, we will also need gauge transformations to discuss the gauge-covariant representations. These are generated by $G[X]$ where the field X serves as the gauge parameter matrix.

Although we do not explicitly refer to real algebras, a suitable set of reality conditions for $\mathcal{N}=4$ sYM is given by

$$(P_{\dot{a}\gamma})^\dagger = P_{\dot{\gamma}a},$$
$$(K^{\alpha\dot{\gamma}})^\dagger = K^{\gamma\dot{\alpha}},$$
$$(Q_{a\gamma})^\dagger = \bar{Q}_{\dot{\gamma}}{}^a,$$
$$(S^{ac})^\dagger = \bar{S}_c{}^{\dot{\alpha}}, \tag{A1}$$

as well as

$$(L^\alpha{}_\gamma)^\dagger = \bar{L}^{\dot\alpha}{}_{\dot\gamma},$$
$$(R^a{}_c)^\dagger = R^c{}_a,$$
$$D^\dagger = D,$$
$$G[X]^\dagger = G[X^\dagger]. \qquad (A2)$$

The below representations will be unitary with respect to these reality conditions.

In the following, we will list the most relevant algebra relations. The Lorentz and internal algebra relations take the form

$$[L^\alpha{}_\beta, L^\gamma{}_\delta] = i(\delta^\gamma_\beta L^\alpha{}_\delta - \delta^\alpha_\delta L^\gamma{}_\beta) + G[...],$$
$$[\bar{L}^{\dot\alpha}{}_{\dot\beta}, \bar{L}^{\dot\gamma}{}_{\dot\delta}] = i(\delta^{\dot\gamma}_{\dot\beta}\bar{L}^{\dot\alpha}{}_{\dot\delta} - \delta^{\dot\alpha}_{\dot\delta}\bar{L}^{\dot\gamma}{}_{\dot\beta}) + G[...],$$
$$[R^a{}_b, R^c{}_d] = i(\delta^c_b R^a{}_d - \delta^a_d R^c{}_b). \qquad (A3)$$

Here, the Lorentz algebra relations involve gauge transformations $G[...]$, where the omitted gauge parameter is a term of the form $xx\nabla A$ representing the field strength contracted with the Killing spinors of the two rotations. We do not present the long list of algebra relations with the remaining generators, as these can easily be inferred as the transformations of spinor indices compatible with the above relations.

The algebra of the scaling generator D measures the scaling dimension Δ_J of the other generators J,

$$[D, J] = -i\Delta_J J + G[...], \qquad (A4)$$

with the dimensions

$$\Delta_L = \Delta_{\bar{L}} = \Delta_R = \Delta_D = 0,$$
$$\Delta_Q = \Delta_{\bar{Q}} = -\Delta_S = -\Delta_{\bar{S}} = \frac{1}{2},$$
$$\Delta_P = -\Delta_K = 1. \qquad (A5)$$

The remaining purely bosonic conformal algebra relations read

$$[P_{\dot\alpha\beta}, P_{\dot\gamma\delta}] = -iG[\nabla_{\dot\alpha\beta}A_{\dot\gamma\delta}],$$
$$[P_{\dot\alpha\beta}, K^{\gamma\dot\epsilon}] = i\delta^{\dot\epsilon}_{\dot\alpha}L^\gamma{}_\beta + i\delta^\gamma_\beta \bar{L}^{\dot\epsilon}{}_{\dot\alpha} + i\delta^\gamma_\beta\delta^{\dot\epsilon}_{\dot\alpha}D + G[...],$$
$$[K^{\beta\dot\alpha}, K^{\gamma\dot\epsilon}] = G[...]. \qquad (A6)$$

Again, these relations may involve some gauge transformation $G[...]$ in addition to the pure conformal generators.

The nontrivial relations of the fermionic generators read

$$\{Q_{b\alpha}, \bar{Q}_{\dot\gamma}{}^d\} = 2\delta^d_b P_{\dot\gamma\alpha},$$
$$\{S^{\alpha b}, \bar{S}_c{}^{\dot\epsilon}\} = 2\delta^b_c K^{\alpha\dot\epsilon}.$$
$$\{Q_{a\beta}, S^{\gamma d}\} = 2i\delta^\gamma_\beta R^d{}_a - 2i\delta^d_a L^\gamma{}_\beta - i\delta^d_a\delta^\gamma_\beta D,$$
$$\{\bar{Q}_{\dot\alpha}{}^b, \bar{S}_c{}^{\dot\epsilon}\} = 2i\delta^{\dot\epsilon}_{\dot\alpha}R^b{}_c + 2i\delta^b_c\bar{L}^{\dot\epsilon}{}_{\dot\alpha} + i\delta^b_c\delta^{\dot\epsilon}_{\dot\alpha}D, \qquad (A7)$$

while the nontrivial mixed relations read

$$[P_{\dot\alpha\beta}, S^{\gamma e}] = \delta^\gamma_\beta\bar{Q}_{\dot\alpha}{}^e - \varepsilon_{\dot\alpha\dot\kappa}G[x^{\gamma\dot\kappa}\Psi^e{}_\beta],$$
$$[P_{\dot\alpha\beta}, \bar{S}_c{}^{\dot\epsilon}] = -\delta^{\dot\epsilon}_{\dot\alpha}Q_{c\beta} + \varepsilon_{\beta\delta}G[x^{\delta\dot\epsilon}\bar{\Psi}_{\dot\alpha c}],$$
$$[K^{\alpha\dot\gamma}, Q_{de}] = -\delta^\alpha_e\bar{S}_d{}^{\dot\gamma} + G[...],$$
$$[K^{\alpha\dot\gamma}, \bar{Q}_{\dot\epsilon}{}^d] = \delta^{\dot\gamma}_{\dot\epsilon}S^{\alpha d} + G[...]. \qquad (A8)$$

The remaining relations involving the fermionic generators are trivial modulo gauge transformations,

$$[P_{\dot\alpha\beta}, Q_{d\gamma}] = i\varepsilon_{\beta\gamma}G[\bar{\Psi}_{\dot\alpha d}],$$
$$[P_{\dot\alpha\beta}\bar{Q}_{\dot\gamma}{}^d] = i\varepsilon_{\dot\alpha\dot\gamma}G[\Psi^d{}_\beta],$$
$$\{Q_{b\alpha}, Q_{d\gamma}\} = 2i\varepsilon_{\alpha\gamma}G[\bar{\Phi}_{bd}],$$
$$\{\bar{Q}_{\dot\alpha}{}^b, \bar{Q}_{\dot\gamma}{}^d\} = 2i\varepsilon_{\dot\alpha\dot\gamma}G[\Phi^{bd}],$$
$$\{Q_{b\alpha}, \bar{S}_d{}^{\dot\gamma}\} = G[...],$$
$$\{\bar{Q}_{\dot\alpha}{}^b, S^{\gamma d}\} = G[...],$$
$$\{\bar{S}_b{}^{\dot\alpha}, \bar{S}_d{}^{\dot\gamma}\} = G[...],$$
$$\{S_b{}^\alpha, S^{\gamma d}\} = G[...],$$
$$[\bar{S}_b{}^{\dot\alpha}, K^{\delta\dot\gamma}] = G[...],$$
$$[S_b{}^\alpha, K^{\delta\dot\gamma}] = G[...]. \qquad (A9)$$

The relations involving the fermionic generators typically hold only on shell, i.e., modulo the equations of motion.

Representation.—The representation of the supersymmetries on the fields reads

$$Q_{a\beta} \cdot \Phi^{cd} = \delta^c_a\Psi^d{}_\beta - \delta^d_a\Psi^c{}_\beta,$$
$$Q_{a\beta} \cdot \bar{\Phi}_{cd} = \varepsilon_{acde}\Psi^e{}_\beta,$$
$$Q_{a\beta} \cdot A_{\dot\gamma\delta} = -i\varepsilon_{\beta\delta}\bar{\Psi}_{\dot\gamma a},$$
$$Q_{a\beta} \cdot \Psi^c{}_\delta = -2\delta^c_a F_{\beta\delta} + i\varepsilon_{\beta\delta}[\Phi^{ce}, \bar{\Phi}_{ae}],$$
$$Q_{a\beta} \cdot \bar{\Psi}_{\dot\gamma d} = 2i\nabla_{\dot\gamma\beta}\bar{\Phi}_{ad},$$
$$\bar{Q}_{\dot\alpha}{}^b \cdot \bar{\Phi}_{cd} = \delta^b_c\bar{\Psi}_{\dot\alpha d} - \delta^b_d\bar{\Psi}_{\dot\alpha c},$$
$$\bar{Q}_{\dot\alpha}{}^b \cdot \Phi^{cd} = \varepsilon^{bcde}\bar{\Psi}_{\dot\alpha e},$$
$$\bar{Q}_{\dot\alpha}{}^b \cdot A_{\dot\gamma\delta} = -i\varepsilon_{\dot\alpha\dot\gamma}\Psi^b{}_\delta,$$
$$\bar{Q}_{\dot\alpha}{}^b \cdot \bar{\Psi}_{\dot\gamma d} = -2\delta^b_d\bar{F}_{\dot\alpha\dot\gamma} - i\varepsilon_{\dot\alpha\dot\gamma}[\Phi^{be}, \bar{\Phi}_{de}],$$
$$\bar{Q}_{\dot\alpha}{}^b \cdot \Psi^c{}_\delta = 2i\nabla_{\dot\alpha\delta}\Phi^{bc}. \qquad (A10)$$

The standard rules (2.9) and (2.12) for the representation of the momentum and dilatation generators read explicitly

$$P_{\dot{\alpha}\beta} \cdot \Phi^{cd} = i\nabla_{\dot{\alpha}\beta}\Phi^{cd},$$

$$P_{\dot{\alpha}\beta} \cdot \Psi^c{}_\delta = i\nabla_{\dot{\alpha}\beta}\Psi^c{}_\delta,$$

$$P_{\dot{\alpha}\beta} \cdot \bar{\Psi}_{\dot{\gamma}d} = i\nabla_{\dot{\alpha}\beta}\bar{\Psi}_{\dot{\gamma}d},$$

$$P_{\dot{\alpha}\beta} \cdot A_{\dot{\gamma}\delta} = i\nabla_{\dot{\alpha}\beta}A_{\dot{\gamma}\delta},$$

$$D \cdot \Phi^{cd} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\beta}\Phi^{cd} - i\Phi^{cd},$$

$$D \cdot \Psi^c{}_\delta = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\beta}\Psi^c{}_\delta - \frac{3}{2}i\Psi^c{}_\delta,$$

$$D \cdot \bar{\Psi}_{\dot{\gamma}d} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\beta}\bar{\Psi}_{\dot{\gamma}d} - \frac{3}{2}i\bar{\Psi}_{\dot{\gamma}d},$$

$$D \cdot A_{\dot{\gamma}\delta} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\beta}A_{\dot{\gamma}\delta}. \tag{A11}$$

For the Lorentz generators, one finds[30]

$$L^\beta{}_\delta \cdot \Phi^{ce} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\delta}\Phi^{ce} + \frac{i}{2}\delta^\beta_\delta x^{\kappa\dot{\alpha}}\nabla_{\dot{\alpha}\kappa}\Phi^{ce},$$

$$L^\beta{}_\delta \cdot \Psi^c{}_\epsilon = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\delta}\Psi^c{}_\epsilon + \frac{i}{2}\delta^\beta_\delta x^{\kappa\dot{\alpha}}\nabla_{\dot{\alpha}\kappa}\Psi^c{}_\epsilon$$
$$\qquad - i\delta^\beta_\epsilon\Psi^c{}_\delta + \frac{i}{2}\delta^\beta_\delta\Psi^c{}_\epsilon,$$

$$L^\beta{}_\delta \cdot \bar{\Psi}_{\dot{\gamma}e} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\delta}\bar{\Psi}_{\dot{\gamma}e} + \frac{i}{2}\delta^\beta_\delta x^{\kappa\dot{\alpha}}\nabla_{\dot{\alpha}\kappa}\bar{\Psi}_{\dot{\gamma}e},$$

$$L^\beta{}_\delta \cdot A_{\dot{\gamma}e} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\alpha}\delta}A_{\dot{\gamma}e} + \frac{i}{2}\delta^\beta_\delta x^{\kappa\dot{\alpha}}\nabla_{\dot{\alpha}\kappa}A_{\dot{\gamma}e}, \tag{A12}$$

and analogously for the conjugate Lorentz generators,

$$\bar{L}^{\dot{\alpha}}{}_{\dot{\gamma}} \cdot \Phi^{ed} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\gamma}\beta}\Phi^{ed} + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}} x^{\beta\dot{\kappa}}\nabla_{\dot{\kappa}\beta}\Phi^{ed},$$

$$\bar{L}^{\dot{\alpha}}{}_{\dot{\gamma}} \cdot \Psi^e{}_\delta = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\gamma}\beta}\Psi^e{}_\delta + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}} x^{\beta\dot{\kappa}}\nabla_{\dot{\kappa}\beta}\Psi^e{}_\delta,$$

$$\bar{L}^{\dot{\alpha}}{}_{\dot{\gamma}} \cdot \bar{\Psi}_{\dot{\epsilon}d} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\gamma}\beta}\bar{\Psi}_{\dot{\epsilon}d} + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}} x^{\beta\dot{\kappa}}\nabla_{\dot{\kappa}\beta}\bar{\Psi}_{\dot{\epsilon}d}$$
$$\qquad - i\delta^{\dot{\alpha}}_{\dot{\epsilon}}\bar{\Psi}_{\dot{\gamma}d} + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}}\bar{\Psi}_{\dot{\epsilon}d},$$

$$\bar{L}^{\dot{\alpha}}{}_{\dot{\gamma}} \cdot A_{\dot{\epsilon}\delta} = -ix^{\beta\dot{\alpha}}\nabla_{\dot{\gamma}\beta}A_{\dot{\epsilon}\delta} + \frac{i}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}} x^{\beta\dot{\kappa}}\nabla_{\dot{\kappa}\beta}A_{\dot{\epsilon}\delta}, \tag{A13}$$

as well as for the internal rotation generators,

$$R^a{}_b \cdot \Phi^{cd} = i\delta^c_b\Phi^{ad} + i\delta^c_b\Phi^{ad} - \frac{i}{2}\delta^a_b\Phi^{cd},$$

$$R^a{}_b \cdot \Psi^c{}_\delta = i\delta^c_b\Psi^a{}_\delta - \frac{i}{4}\delta^a_b\Psi^c{}_\delta,$$

$$R^a{}_b \cdot \bar{\Psi}_{\dot{\gamma}d} = -i\delta^a_d\bar{\Psi}_{\dot{\gamma}b} - \frac{i}{4}\delta^a_b\bar{\Psi}_{\dot{\gamma}d},$$

$$R^a{}_b \cdot A_{\dot{\gamma}\delta} = 0. \tag{A14}$$

The representation of the special superconformal generators can be summarized as

$$S^{ab} \cdot \Phi^{cd} = ix^{\alpha\dot{\epsilon}}\bar{Q}_{\dot{\epsilon}}{}^b \cdot \Phi^{cd},$$

$$S^{ab} \cdot \Psi^c{}_\delta = ix^{\alpha\dot{\epsilon}}\bar{Q}_{\dot{\epsilon}}{}^b\Psi^c{}_\delta - 2\delta^\alpha_\delta\Phi^{bc},$$

$$S^{ab} \cdot \bar{\Psi}_{\dot{\gamma}d} = ix^{\alpha\dot{\epsilon}}\bar{Q}_{\dot{\epsilon}}{}^b\bar{\Psi}_{\dot{\gamma}d},$$

$$S^{ab} \cdot A_{\dot{\gamma}\delta} = ix^{\alpha\dot{\epsilon}}\bar{Q}_{\dot{\epsilon}}{}^b A_{\dot{\epsilon}\delta},$$

$$\bar{S}_a{}^{\dot{\gamma}} \cdot \Phi^{ed} = -ix^{\beta\dot{\gamma}}Q_{a\beta} \cdot \Phi^{ed},$$

$$\bar{S}_a{}^{\dot{\gamma}} \cdot \Psi^e{}_\delta = -ix^{\beta\dot{\gamma}}Q_{a\beta} \cdot \Psi^e{}_\delta,$$

$$\bar{S}_a{}^{\dot{\gamma}} \cdot \bar{\Psi}_{\dot{\epsilon}d} = -ix^{\beta\dot{\gamma}}Q_{a\beta} \cdot \bar{\Psi}_{\dot{\epsilon}d} + 2\delta^{\dot{\gamma}}_{\dot{\epsilon}}\bar{\Phi}_{ad},$$

$$\bar{S}_a{}^{\dot{\gamma}} \cdot A_{\dot{\epsilon}\delta} = -ix^{\beta\dot{\gamma}}Q_{a\beta} \cdot A_{\dot{\epsilon}\delta}, \tag{A15}$$

whereas the one of the special conformal generators takes the explicit form

$$K^{\alpha\dot{\gamma}} \cdot \Phi^{ed} = ix^{\alpha\dot{\kappa}}x^{\beta\dot{\gamma}}\nabla_{\beta\dot{\kappa}}\Phi^{ed} + ix^{\alpha\dot{\gamma}}\Phi^{ed},$$

$$K^{\alpha\dot{\gamma}} \cdot \Psi^e{}_\delta = ix^{\alpha\dot{\kappa}}x^{\beta\dot{\gamma}}\nabla_{\beta\dot{\kappa}}\Psi^e{}_\delta + ix^{\alpha\dot{\gamma}}\Psi^e{}_\delta + i\delta^\alpha_\delta x^{\kappa\dot{\gamma}}\Psi^e{}_\kappa,$$

$$K^{\alpha\dot{\gamma}} \cdot \bar{\Psi}_{\dot{\epsilon}d} = ix^{\alpha\dot{\kappa}}x^{\beta\dot{\gamma}}\nabla_{\beta\dot{\kappa}}\bar{\Psi}_{\dot{\epsilon}d} + ix^{\alpha\dot{\gamma}}\bar{\Psi}_{\dot{\epsilon}d} + i\delta^{\dot{\gamma}}_{\dot{\epsilon}}x^{\alpha\dot{\kappa}}\bar{\Psi}_{\dot{\kappa}d},$$

$$K^{\alpha\dot{\gamma}} \cdot A_{\dot{\epsilon}\delta} = ix^{\alpha\dot{\kappa}}x^{\beta\dot{\gamma}}\nabla_{\beta\dot{\kappa}}A_{\dot{\gamma}\delta}. \tag{A16}$$

Finally, the representation of a gauge transformation by the field X is defined by

$$G[X] \cdot \Phi^{cd} = [X, \Phi^{cd}],$$

$$G[X] \cdot \Psi^c{}_\delta = [X, \Psi^c{}_\delta],$$

$$G[X] \cdot \bar{\Psi}_{\dot{\gamma}d} = [X, \bar{\Psi}_{\dot{\gamma}d}],$$

$$G[X] \cdot A_{\dot{\gamma}\delta} = i\nabla_{\dot{\gamma}\delta}X. \tag{A17}$$

Extension.—When discussing the level-one momentum generator $\hat{P}$, it turns out useful to superficially extend the level-zero algebra by an operator B. This operator does not represent a symmetry of the action; nor does its action describe a proper representation of some algebra. We simply define its action as

$$B \cdot \Phi^{cd} = 0,$$

$$B \cdot \Psi^c{}_\delta = -\frac{i}{2}\Psi^c{}_\delta,$$

$$B \cdot \bar{\Psi}_{\dot{\gamma}d} = +\frac{i}{2}\bar{\Psi}_{\dot{\gamma}d},$$

$$B \cdot A_{\dot{\gamma}\delta} = 0. \tag{A18}$$

Furthermore, we define two 2×2 matrices of operators L′ and $\bar{L}'$ and a 4×4 matrix of operators R′ as the combinations of Lorentz and internal rotations, scale transformations, and B

$$L'^\beta{}_\delta := L^\beta{}_\delta + \frac{1}{2}\delta^\beta_\delta(D + B),$$

$$\bar{L}'^{\dot{\alpha}}{}_{\dot{\gamma}} := \bar{L}^{\dot{\alpha}}{}_{\dot{\gamma}} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\gamma}}(D - B),$$

$$R'^a{}_c := R^a{}_c + \frac{1}{2}\delta^a_c B. \tag{A19}$$

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[30]In analogy to the curious assignment $\Delta_A = 0$ for scaling transformations, the spacetime indices of the gauge field A are not transformed explicitly.

These operators have a reasonably simpler action on the fields of $\mathcal{N} = 4$ sYM than the corresponding Lorentz generators L,

$$L'^{\beta}_{\ \delta} \cdot \Phi^{ce} = -ix^{\beta\dot\alpha}\nabla_{\dot\alpha\delta}\Phi^{ce} - \frac{i}{2}\delta^{\beta}_{\delta}\Phi^{ce},$$

$$L'^{\beta}_{\ \delta} \cdot \Psi^c_{\ \epsilon} = -ix^{\beta\dot\alpha}\nabla_{\dot\alpha\delta}\Psi^c_{\ \epsilon} - \frac{i}{2}\delta^{\beta}_{\delta}\Psi^c_{\ \epsilon} - i\delta^{\beta}_{\epsilon}\Psi^c_{\ \delta},$$

$$L'^{\beta}_{\ \delta} \cdot \bar\Psi_{\dot\gamma e} = -ix^{\beta\dot\alpha}\nabla_{\dot\alpha\delta}\bar\Psi_{\dot\gamma e} - \frac{i}{2}\delta^{\beta}_{\delta}\bar\Psi_{\dot\gamma e},$$

$$L'^{\beta}_{\ \delta} \cdot A_{\dot\gamma e} = -ix^{\beta\dot\alpha}\nabla_{\dot\alpha\delta}A_{\dot\gamma e}; \tag{A20}$$

the conjugate Lorentz generator $\bar L$,

$$\bar L'^{\dot\alpha}_{\ \dot\gamma} \cdot \Phi^{ed} = -ix^{\beta\dot\alpha}\nabla_{\dot\gamma\beta}\Phi^{ed} - \frac{i}{2}\delta^{\dot\alpha}_{\dot\gamma}\Phi^{ed}$$

$$\bar L'^{\dot\alpha}_{\ \dot\gamma} \cdot \Psi^e_{\ \delta} = -ix^{\beta\dot\alpha}\nabla_{\dot\gamma\beta}\Psi^e_{\ \delta} - \frac{i}{2}\delta^{\dot\alpha}_{\dot\gamma}\Psi^e_{\ \delta},$$

$$\bar L'^{\dot\alpha}_{\ \dot\gamma} \cdot \bar\Psi_{\dot e d} = -ix^{\beta\dot\alpha}\nabla_{\dot\gamma\beta}\bar\Psi_{\dot e d} - \frac{i}{2}\delta^{\dot\alpha}_{\dot\gamma}\bar\Psi_{\dot e d} - i\delta^{\dot\alpha}_{\dot e}\bar\Psi_{\dot\gamma d},$$

$$\bar L'^{\dot\alpha}_{\ \dot\gamma} \cdot A_{\dot e\delta} = -ix^{\beta\dot\alpha}\nabla_{\dot\gamma\delta}A_{\dot e\delta}; \tag{A21}$$

as well as the internal rotation generators R,

$$R'^a_{\ b} \cdot \Phi^{cd} = i\delta^c_b\Phi^{ad} + i\delta^c_b\Phi^{ad} - \frac{i}{2}\delta^a_b\Phi^{cd},$$

$$R'^a_{\ b} \cdot \Psi^c_{\ \delta} = i\delta^c_b\Psi^a_{\ \delta} - \frac{i}{2}\delta^a_b\Psi^c_{\ \delta},$$

$$R'^a_{\ b} \cdot \bar\Psi_{\dot\gamma d} = -i\delta^a_d\bar\Psi_{\dot\gamma b} + \frac{i}{2}\delta^a_b\bar\Psi_{\dot\gamma d},$$

$$R'^a_{\ b} \cdot A_{\dot\gamma\delta} = 0. \tag{A22}$$

Moreover, several terms of the coproduct of level-one generators naturally combine using L′, $\bar L$′, and R′; see Appendix B.

APPENDIX B: LEVEL-ONE GENERATORS

In this Appendix, we will give explicit expressions for the coproducts as well as single-field actions of the level-one Yangian generators

$$\hat P_{\dot\alpha\beta}, \qquad \hat Q_{\alpha\beta}, \qquad \hat{\bar Q}_{\dot\alpha}^{\ b}, \qquad \hat R^a_{\ b}, \quad \text{and} \quad \hat B. \tag{B1}$$

These are the level-one generators which commute with the ordinary momentum P (up to gauge artifacts), and hence their single-field action can be expected to have no explicit dependence on the position x.

Level-one momentum.—The level-one momentum $\hat P$ has the following single-field action:

$$\hat P_{\dot\alpha\beta} \cdot \Phi^{cd} := 0,$$

$$\hat P_{\dot\alpha\beta} \cdot \Psi^c_{\ \delta} := -\epsilon_{\beta\delta}\{\Phi^{ce}, \bar\Psi_{\dot\alpha e}\},$$

$$\hat P_{\dot\alpha\beta} \cdot \bar\Psi_{\dot\gamma d} := -\epsilon_{\dot\alpha\dot\gamma}\{\bar\Phi_{de}, \Psi^e_{\ \gamma}\},$$

$$\hat P_{\dot\alpha\beta} \cdot A_{\dot\gamma\delta} := \frac{i}{4}\epsilon_{\dot\alpha\dot\gamma}\epsilon_{\beta\delta}\{\Phi^{ef}, \bar\Phi_{ef}\}. \tag{B2}$$

The coproduct reads

$$\Delta\hat P_{\dot\alpha\beta} = \hat P_{\dot\alpha\beta} \otimes 1 + 1 \otimes \hat P_{\dot\alpha\beta} + P_{\dot\alpha\gamma} \wedge L^{\gamma}_{\ \beta} + P_{\dot\gamma\beta} \wedge \bar L^{\dot\gamma}_{\ \dot\alpha}$$

$$+ P_{\dot\alpha\beta} \wedge D - \frac{i}{2}Q_{c\beta} \wedge \bar Q_{\dot\alpha}^{\ c}$$

$$= \hat P_{\dot\alpha\beta} \otimes 1 + 1 \otimes \hat P_{\dot\alpha\beta} - L'^{\gamma}_{\ \beta} \wedge P_{\dot\alpha\gamma}$$

$$- \bar L'^{\dot\gamma}_{\ \dot\alpha} \wedge P_{\dot\gamma\beta} - \frac{i}{2}Q_{c\beta} \wedge \bar Q_{\dot\alpha}^{\ c}. \tag{B3}$$

The expression on the second line uses the operators defined in Appendix A which superficially depend on the extra generator B. The benefit of this expression is that in explicit calculations it avoids the generation of several intermediate terms which would be cancelled by other terms.

Level-one supersymmetry.—The level-one supersymmetries $\hat Q$ and $\hat{\bar Q}$ have a nontrivial single-field action only on one of the fermionic fields:

$$\hat Q_{\alpha\beta} \cdot \Psi^c_{\ \delta} = -\frac{1}{2}\delta^c_a\epsilon_{\beta\delta}\{\Phi^{ef}, \bar\Phi_{ef}\},$$

$$\hat{\bar Q}_{\dot\alpha}^{\ b} \cdot \bar\Psi_{\dot\gamma d} = -\frac{1}{2}\delta^b_d\epsilon_{\dot\alpha\dot\gamma}\{\Phi^{ef}, \bar\Phi_{ef}\}. \tag{B4}$$

The single-field action on all the remaining fields Z of the theory is

$$\hat Q_{\alpha\beta} \cdot Z = 0, \qquad \hat{\bar Q}_{\dot\alpha}^{\ b} \cdot Z = 0. \tag{B5}$$

The coproduct for $\hat Q$ is now

$$\Delta\hat Q_{\alpha\beta} = \hat Q_{\alpha\beta} \otimes 1 + 1 \otimes \hat Q_{\alpha\beta} + Q_{\alpha\gamma} \wedge L^{\gamma}_{\ \beta} + \frac{1}{2}Q_{\alpha\beta} \wedge D$$

$$- iP_{\dot\gamma\beta} \wedge \bar S_a^{\ \dot\gamma} - Q_{c\beta} \wedge R^c_{\ a}$$

$$= \hat Q_{\alpha\beta} \otimes 1 + 1 \otimes \hat Q_{\alpha\beta} + Q_{\alpha\gamma} \wedge L'^{\gamma}_{\ \beta} - iP_{\dot\gamma\beta} \wedge \bar S_a^{\ \dot\gamma}$$

$$- Q_{c\beta} \wedge R'^c_{\ a}, \tag{B6}$$

and the corresponding expression for $\hat{\bar Q}$ reads

$$\Delta\hat{\bar Q}_{\dot\alpha}^{\ b} = \hat{\bar Q}_{\dot\alpha}^{\ b} \otimes 1 + 1 \otimes \hat{\bar Q}_{\dot\alpha}^{\ b} + \bar Q_{\dot\gamma}^{\ b} \wedge \bar L^{\dot\gamma}_{\ \dot\alpha} + \frac{1}{2}\bar Q_{\dot\alpha}^{\ b} \wedge D$$

$$+ iP_{\dot\alpha\gamma} \wedge S^{\gamma b} + \bar Q_{\dot\alpha}^{\ c} \wedge R^b_{\ c}$$

$$= \hat{\bar Q}_{\dot\alpha}^{\ b} \otimes 1 + 1 \otimes \hat{\bar Q}_{\dot\alpha}^{\ b} + \bar Q_{\dot\gamma}^{\ b} \wedge \bar L'^{\dot\gamma}_{\ \dot\alpha} + iP_{\dot\alpha\gamma} \wedge S^{\gamma b}$$

$$+ \bar Q_{\dot\alpha}^{\ c} \wedge R'^b_{\ c}. \tag{B7}$$

Level-one internal rotations.—The generator $\hat R^a_{\ b}$ has a trivial single-field action,

$$\hat{R}^a{}_b \cdot Z = 0. \tag{B8}$$

Heuristically, this is easy to understand; the generator $\hat{R}$ carries zero mass dimension, so its action on a field Z should have the same mass dimension as Z, i.e., 1 for bosonic fields or 3/2 for fermionic fields. Furthermore, the action should have no explicit position dependence as argued above. Together, this implies that the result can only be a single field. However, the parity-inverting property of the level-one generators requires having at least two fields in the result.

The following coproduct thus defines this symmetry completely:

$$\Delta \hat{R}^a{}_b = \hat{R}^a{}_b \otimes 1 + 1 \otimes \hat{R}^a{}_b + R^a{}_c \wedge R^c{}_b$$
$$- \frac{1}{2} S^{\gamma a} \wedge Q_{b\gamma} - \frac{1}{2} \bar{S}_b{}^{\dot{\gamma}} \wedge \bar{Q}_{\dot{\gamma}}{}^a + \frac{1}{8} \delta^a_b S^{\gamma d} \wedge Q_{d\gamma}$$
$$+ \frac{1}{8} \delta^a_b \bar{S}_d{}^{\dot{\gamma}} \wedge \bar{Q}_{\dot{\gamma}}{}^d. \tag{B9}$$

Level-one bonus symmetry.—For the same reasons as for $\hat{R}$, the level-one bonus symmetry $\hat{B}$ has a trivial single-field action,

$$\hat{B} \cdot Z = 0. \tag{B10}$$

The coproduct reads

$$\Delta \hat{B} = \hat{B} \otimes 1 + 1 \otimes \hat{B} - \frac{1}{4} \bar{S}_b{}^{\dot{\alpha}} \wedge \bar{Q}_{\dot{\alpha}}{}^b - \frac{1}{4} S^{\alpha b} \wedge Q_{b\alpha}. \tag{B11}$$

In fact, we can now define a modified level-one internal rotation operator,

$$\hat{R}'^a{}_b := \hat{R}^a{}_b + \frac{1}{2} \delta^a_b \hat{B}, \tag{B12}$$

the trace of which is $2\hat{B}$. This generator has a slightly simplified coproduct as compared to $\hat{R}$:

$$\Delta \hat{R}'^a{}_b = \hat{R}'^a{}_b \otimes 1 + 1 \otimes \hat{R}'^a{}_b + R^a{}_c \wedge R^c{}_b$$
$$- \frac{1}{2} S^{\gamma a} \wedge Q_{b\gamma} - \frac{1}{2} \bar{S}_b{}^{\dot{\gamma}} \wedge \bar{Q}_{\dot{\gamma}}{}^a. \tag{B13}$$

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