Entanglement entropy in lattice theories with Abelian gauge groups

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We revisit the issue of the geometrical separability of the Hilbert space of physical states on lattice Abelian theories in the context of entanglement entropy. We discuss the conditions under which vectors in the Hilbert space, as well as the gauge-invariant algebra, admit a tensor product decomposition with a geometrical interpretation. With the exception of pure gauge lattices with periodic boundary conditions which contain topological degrees of freedom, we show that the Hilbert space is geometrically separable.

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I. INTRODUCTION

Entanglement entropy in quantum field theories has received increased interest in the past few decades. It has been shown that in many cases it satisfies an area law, prompting questions of how it might be related to the Bekenstein-Hawking black hole entropy, which also satisfies an area law. Unfortunately, entanglement entropy is also UV divergent, thus requiring the use of a cutoff. Lattice field theory is naturally equipped with such a cutoff, making it a good fit for performing entanglement entropy calculations. Except that, while matter fields on the lattice are perfectly localized on lattice vertices, gauge fields are represented by links connecting vertices, which, by definition, have a spatial extent. To further complicate matters, physically measurable quantities must satisfy gauge invariance, and defining a physical entanglement entropy requires the use of such gauge-invariant objects, whose structure is even more complex. In particular, it is believed that degrees of freedom (d.o.f.) in the physical Hilbert space of gauge theories cannot be divided into geometric bipartitions without sacrificing gauge invariance. In this paper we show precisely how this can be done in the case of lattice gauge theories with Abelian groups, while focusing on the particularly simple group Z_2 for clarity.

In Sec. II, we give a technical overview of some of the relevant literature and show the various issues surrounding the geometric separability of the Hilbert space of Abelian lattice gauge theories. Specifically, we show that two inseparability proofs lead to severe consequences that extend beyond gauge theories and can be equally applied to scenarios that are otherwise thought to be geometrically separable. We also show that there is no unique choice for what we call d.o.f. and that the value of the entanglement entropy can depend on that choice.

Section III shows how choices of d.o.f. can be related using dualities and that imposing geometric symmetries on the d.o.f. such that they can be interpreted as d.o.f. in a field theory can narrow down the number of choices. Symmetry arguments can then be applied to various scenarios to obtain minimally constrained gauge-invariant choices of d.o.f.

We proceed with a detailed analysis of the physical Hilbert space in $2+1$ dimensional Z_2 gauge theories with free boundary conditions in Sec. IV and their algebra in Sec. V and show specific examples for a minimal, two-plaquette lattice in Sec. VI. We also show how gauge-invariant density matrices and partial traces can be implemented on the physical Hilbert space, leading to a gauge-invariant entanglement entropy. The work in Secs. IV and V is subsequently used as a basis for the analysis of other lattice configurations. Section VII analyzes $2+1$ dimensional lattices with periodic boundary conditions, which exhibit both a global constraint and topological d.o.f. We show that maintaining lattice symmetries requires the inclusion of the global constraint in the calculations of entanglement entropy. On the other hand, topological d.o.f. cannot be factored in a pure gauge theory. Lattices in $3+1$ dimensions are studied Sec. VIII. They are characterized by the existence of local constraints of a geometrical nature which, again, must also be taken into account if lattice symmetries are to be preserved. Lattice theories that couple gauge fields to bulk matter fields are discussed in Sec. IX. Coupling to matter fields simplifies the Hilbert space since states can be expressed as tensor products of independent electric states on the links and independent matter states at the vertices. Edge charges are addressed in Sec. X as a combination of pure gauge theory in the bulk and matter-coupled theory on the edges.

We conclude with some remarks on extensions to other Abelian gauge groups and the limitations of this analysis.

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II. REVIEW OF RELEVANT LITERATURE

A survey of the literature [1–6] indicates that the commonly held belief is that the physical Hilbert space in pure Abelian gauge lattices is not geometrically separable. In an early paper on the topic by Buividovich *et al.* [1], it is stated that the physical Hilbert space in pure Z_2 gauge lattices in $2+1$ dimensions does not admit a geometrical separation by assigning complementary sets of gauge links to regions. It is then concluded that the physical Hilbert space is not geometrically separable at all, and that defining an entanglement entropy requires embedding the Hilbert space in an extended space. The embedding procedure then results in a contribution to the entanglement entropy that is given by a Shannon term of the probability distribution of d.o.f. on the boundary, which is proportional to the area of the boundary. This is taken as validation of the procedure, since it leads to an entanglement entropy that has an area law for any gauge theory, thus qualitatively matching the area law of black hole entropy [7].

We illustrate briefly why the inseparability proof in [1] is problematic. The proof goes as follows:

> *Assume a Hilbert space $\tilde{\mathcal{H}}$ and a strict subspace of $\tilde{\mathcal{H}}$ generated by the projection operator $\mathcal{P}_{\mathcal{C}}$ of a constraint $\mathcal{C}$, $\mathcal{H}_0 \subset \tilde{\mathcal{H}}$, $\mathcal{H}_0 = \mathcal{P}_{\mathcal{C}}\tilde{\mathcal{H}}$, and take a decomposition $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$ with $\tilde{\mathcal{H}}_{A,B}$ partially supporting the constraint $\mathcal{C}$. That is, there exist vectors $\tilde{\Psi}^c_A \in \tilde{\mathcal{H}}_A$, $\tilde{\Psi}^c_B \in \tilde{\mathcal{H}}_B$ such that $\mathcal{P}_{\mathcal{C}}\tilde{\Psi}^c_{A,B} \neq \tilde{\Psi}^c_{A,B}$ and there exist orthogonal vectors $\tilde{\Psi}^0_A \in \tilde{\mathcal{H}}_A$, $\tilde{\Psi}^0_B \in \tilde{\mathcal{H}}_B$, $\tilde{\Psi}^0_{A,B} \cdot \tilde{\Psi}^c_{A,B} = 0$ such that $\mathcal{P}_{\mathcal{C}}\tilde{\Psi}^0_{A,B} = \tilde{\Psi}^0_{A,B}$. Now, assume that there exists a decomposition $\mathcal{H}_0 = \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\mathcal{H}_A \subseteq \tilde{\mathcal{H}}_A$ and $\mathcal{H}_B \subseteq \tilde{\mathcal{H}}_B$. It follows that any vector in $\mathcal{H}_A$ or $\mathcal{H}_B$ can be written as a vector in $\tilde{\mathcal{H}}_A$ or $\tilde{\mathcal{H}}_B$, respectively. Consider the states $\Psi^0 = \Psi^0_A \otimes \Psi^0_B = \tilde{\Psi}^0_A \otimes \tilde{\Psi}^0_B$ and $\Psi^1 = \Psi^1_A \otimes \Psi^1_B = \tilde{\Psi}^c_A \otimes \tilde{\Psi}^1_B$ with $\mathcal{P}_{\mathcal{C}}\tilde{\Psi}^0_A = \tilde{\Psi}^0_A$, $\mathcal{P}_{\mathcal{C}}\tilde{\Psi}^0_B = \tilde{\Psi}^0_B$, $\mathcal{P}_{\mathcal{C}}\tilde{\Psi}^c_A \neq \tilde{\Psi}^c_A$, $\tilde{\Psi}^0_B \neq \tilde{\Psi}^1_B$. Then, the $\mathcal{C}$-invariant subspace $\mathcal{H}_0$ must also contain the vector $\Psi_2 = \Psi^1_A \otimes \Psi^0_B = \tilde{\Psi}^c_A \otimes \tilde{\Psi}^0_B$. However, $\mathcal{P}_{\mathcal{C}}\Psi_2 = \mathcal{P}_{\mathcal{C}}\tilde{\Psi}^c_A \otimes \mathcal{P}_{\mathcal{C}}\tilde{\Psi}^0_B = \mathcal{P}_{\mathcal{C}}\tilde{\Psi}^c_A \otimes \tilde{\Psi}^0_B \neq \Psi_2$. In other words, Ψ_2 does not satisfy the constraint therefore we arrive at a contradiction.*

A simpler but somewhat inaccurate illustration of the problem is shown in Fig. 1.

The assumptions of the existence of the vectors satisfying the various relations above can be satisfied in lattice gauge theory. For example, $\Psi^0_{A,B}$ can be states with no electric excitations while Ψ^c_A can be an open part of a closed electric string. There remains one assumption, $\mathcal{H}_0 = \mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{H}_A \subseteq \tilde{\mathcal{H}}_A$, $\mathcal{H}_B \subseteq \tilde{\mathcal{H}}_B$, which must be false. It must be noted, however, that this is not the same assumption as $\mathcal{H}_0 = \mathcal{H}_A \otimes \mathcal{H}_B$. Should this distinction not be made, one might be led to believe that no Hilbert space $\mathcal{H}$ is separable, since one can always find some $\mathcal{H}^* \supset \mathcal{H}$ and some constraint $\mathcal{C}^*$ with $\mathcal{P}_{\mathcal{C}^*}\mathcal{H}^* = \mathcal{H}$ such that the conditions in the proof are satisfied. There is, perhaps, some truth to this idea in that constraints can make the notions of d.o.f. and locality ambiguous. One could write $\tilde{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_\perp$, where $\mathcal{H}_\perp$ is the space of all vectors orthogonal to the constrained space $\mathcal{H}_\perp = \ker \mathcal{P}_{\mathcal{C}}$, and then attempt to factorize $\mathcal{H}_{\perp=} = \mathcal{H}_{\perp A} \otimes \mathcal{H}_{\perp B}$ such that $\tilde{\mathcal{H}}_{A,B} = \mathcal{H}_{A,B} \otimes \mathcal{H}_{\perp A,B}$. This requires a meaningful assignment of the d.o.f. in $\mathcal{H}_{\perp A,B}$, which is not always possible: in the example in Fig. 1, $\mathcal{H}_\perp$ is one-dimensional. This is likely the essence of the dilemma. Even when both the physical $\mathcal{H}_0$ and unphysical $\tilde{\mathcal{H}}$ spaces can be factored, a decomposition of the form $\tilde{\mathcal{H}} = (\mathcal{H}_A \otimes \mathcal{H}_{\perp A}) \otimes (\mathcal{H}_B \otimes \mathcal{H}_{\perp B})$ with $\dim(\mathcal{H}_{\perp A}) = \dim(\mathcal{H}_{\perp B})$ or $\dim(\mathcal{H}_{\perp A})/\dim(\mathcal{H}_A) = \dim(\mathcal{H}_{\perp B})/\dim(\mathcal{H}_B)$ may not exist.

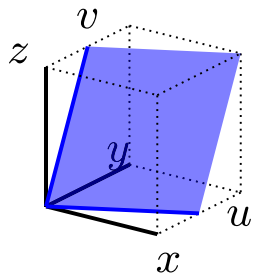


FIG. 1. Separability of nontrivial subspaces. The full Hilbert space $\tilde{\mathcal{H}}$ is a 3-dimensional Cartesian space. The constrained space $\mathcal{H}_0$ (shaded) is a 2-dimensional space. It is impossible to express both $\hat{u}$ and $\hat{v}$ as vectors in any simple bipartition of $\{\hat{x}, \hat{y}, \hat{z}\}$.

Casini *et al.* [2] expand on the work in [1] by looking at the problem from an algebraic perspective. They conclude that expressing a constrained Hilbert space as a product space depends on the method in which one associates a region of space with an algebra of operators. In principle, a factorization of a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is associated with a factorizable algebra $\mathcal{A} = \mathcal{A}_A \otimes \mathcal{A}_B$ such that $O_{A,B}\Psi_{B,A} = \Psi_{B,A}, O_{A,B} \in \mathcal{A}_{A,B}, \Psi_{A,B} \in \mathcal{H}_{A,B}$. That is, operators $O_{A,B}$ have the form $O_A = \tilde{O}_A \otimes \mathbb{1}_B$ and $O_B = \mathbb{1}_A \otimes \tilde{O}_B$, where $\mathbb{1}_{A,B}$ act trivially on their respective subspaces. A necessary and sufficient condition for the above factorization to exist is $\mathcal{A}_A \cap (\mathcal{A}_A)' = \mathbb{1}$, where $(\mathcal{A}_A)'$ is the set of all operators in $\mathcal{A}$ that commute with *all* operators in $\mathcal{A}_A$. The intersection $\mathcal{A}_A \cap (\mathcal{A}_A)'$ is the "center" of the algebra $\mathcal{A}_A$. It is then claimed that lattice gauge theories do not admit local algebras with trivial center and therefore no factorization. The reasoning is:

> *Consider, again, the earlier Hilbert space $\tilde{\mathcal{H}}$ and a constrained subspace $\mathcal{P}_{\mathcal{C}}\tilde{\mathcal{H}} = \mathcal{H}_0$. There exist operators T^c that act trivially on $\mathcal{H}_0$ but not on $\tilde{\mathcal{H}}$. That is, $\exists\, \tilde{\Psi} \in \tilde{\mathcal{H}}$ such that $T^c\tilde{\Psi} \neq \tilde{\Psi}$ and $\forall\, \Psi \in \mathcal{H}_0, T^c\Psi = \Psi$.*

There is an algebra $\mathcal{A}$ associated with $\mathcal{H}_0$. Assume that there exists a factorization of $\mathcal{A} = \mathcal{A}_A \otimes \mathcal{A}_B$ and that there exists a $T^c = AB$ with $A \in \mathcal{A}_A$, $B \in \mathcal{A}_B$, and $B \neq \mathbb{1}_B$. The operator B commutes with all operators in $\mathcal{A}_A$ by the factorization assumption. Then, since $T^c = \mathbb{1}_{AB}$ on $\mathcal{H}_0$, we can write $A^\dagger = A^\dagger \mathbb{1}_{AB} = A^\dagger T^c = A^\dagger AB = B$. Given $A^\dagger \in \mathcal{A}_A$ then also $B \in \mathcal{A}_A$. Hence $B \in \mathcal{A}_A \cap (\mathcal{A}_A)' \neq \mathbb{1}$ and the subalgebra $\mathcal{A}_A$ is not a factor, which contradicts the factorization assumption.

Perhaps the bigger problem would be that we are led to the conclusion that, in a bosonic theory, either the canonical conjugate to $A^\dagger$ is not in $\mathcal{A}_A$ or that it commutes with $A^\dagger$. That is unless $A^\dagger \Psi_A = \Psi_A$, $\forall\, \Psi_A \in \mathcal{H}_A$, in which case $A^\dagger$ is a trivial operator on $\mathcal{H}_A$ and it has no conjugate. The contradictions disappear if, from $T^c = \mathbb{1} = AB$, one concludes that $A^\dagger \equiv B$ and that $T^c = (\mathbb{1}_A \otimes B^\dagger)(\mathbb{1}_A \otimes B)$ on the constrained subspace $\mathcal{H}_0$.

Imposing the requirement of a separable algebra, while sufficient, may not be necessary. Consider a two-spin system with the constraint $\sigma_z^1 \sigma_z^2 = 1$. In other words, the Hilbert space is restricted to wave functions that satisfy $\sigma_z^1 \sigma_z^2 |\Psi\rangle = |\Psi\rangle$. By applying the constraint equation to a general state, we find that $|\Psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\downarrow\downarrow\rangle$. This implies that we are not free to individually manipulate the spins, and the algebra associated with this Hilbert space is $\mathcal{A} = \{\mathbb{1}, \sigma_z^1 = \sigma_z^2, \sigma_x^1 \sigma_x^2, [\sigma_z^1, \sigma_x^1 \sigma_x^2]\}$. Nonetheless, this can represent a legitimate Bell-type experiment, where two spatially separated observers can measure spin correlations and the following entanglement entropy:

$$S_A = -\alpha^2 \log \alpha^2 - \beta^2 \log \beta^2. \tag{1}$$

This setup can be seen as either an entangled state or a single spin, depending on whether σ_z^1 and σ_z^2 are interpreted as distinct operators that measure the same quantity or distinct labels on the same operator. The distinction, however, is not algebraic. Instead, it hinges on whether independent physical measurements can be performed using measurement devices that have a clear spatial separation. In a field theory where the separation is near the scale cutoff, the notion of a clear separation disappears. Furthermore, this is precisely a Hilbert space that exhibits a gaugelike symmetry. To see this, we switch to the transverse basis using $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$ and rewrite the state in the new basis:

$$|\Psi\rangle = \frac{\alpha+\beta}{2}(|{+}{+}\rangle + |{-}{-}\rangle) - \frac{\alpha-\beta}{2}(|{+}{-}\rangle + |{-}{+}\rangle), \tag{2}$$

which is a state that is invariant under global transformations $|\pm\rangle \to |\mp\rangle$.

It can be noted that the algebra $\mathcal{A}$ is already invariant under this particular symmetry, so no additional information on the algebra can come from imposing invariance under this symmetry. We could simply stop here noting that the value of S_A in Eq. (1) is precisely the classical Shannon entropy term found in [1,6] and that this choice corresponds to a separable constrained space $\tilde{\mathcal{H}}$. However, as Casini *et al.* note in [2], S_A would not be the same when gauge fixing is involved. We can remove the redundancy in Eq. (2) by selecting a particular point in the orbit to get:

$$\begin{aligned}|\Psi\rangle &= \frac{\alpha+\beta}{2}|{+}{+}\rangle - \frac{\alpha-\beta}{2}|{+}{-}\rangle \\ &= |+\rangle \otimes \left(\frac{\alpha+\beta}{2}|+\rangle - \frac{\alpha-\beta}{2}|-\rangle\right),\end{aligned} \tag{3}$$

which is a separable state with $S_A = 0$ and corresponds to an entanglement entropy calculated on the unconstrained space $\mathcal{H}_0$. This is consistent with the idea that our Hilbert space contains a single d.o.f.

We could, therefore, adopt the view that a d.o.f. is the smallest entity that can be both measured and manipulated independently [8] while respecting required symmetries. Unfortunately, this too can fail to result in an unambiguous entanglement entropy when multiple choices of basis have equally good geometrical interpretations. Consider the case of four spin d.o.f., one global constraint of the form $\prod_i \sigma_z^i = 1$, and the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\uparrow_2\uparrow_3\uparrow_4\rangle + |\downarrow_1\uparrow_2\downarrow_3\uparrow_4\rangle). \tag{4}$$

The space can be divided by assigning the first two spins to a region and the other two to its complement. If we consider spin 1 as redundant, the unconstrained state reads:

$$|\Psi'\rangle = \frac{1}{\sqrt{2}}|\uparrow_2\rangle \otimes (|\uparrow_3\uparrow_4\rangle + |\downarrow_3\uparrow_4\rangle). \tag{5}$$

Being a separable state, the entanglement entropy is zero. However, choosing spin 4 as redundant, we get:

$$|\Psi''\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\uparrow_2\rangle \otimes |\uparrow_3\rangle + |\downarrow_1\uparrow_2\rangle \otimes |\downarrow_3\rangle), \tag{6}$$

and the entanglement entropy is now log 2. In one dimension, there is a simple solution to the problem which involves a change of basis to eigenstates of products of neighboring σ_z operators. This solution preserves homogeneity of the d.o.f. In two dimensions such a solution does not exist. The resulting ambiguity of S_A is endemic to spaces with global constraints in all but a few cases and stems from the lack of a unique way of meaningfully assigning coordinates to the unconstrained d.o.f. that preserve various qualities that one would expect from a field theory. As we will show in Sec. IV, certain theories admit duals with unconstrained d.o.f. that can be

interpreted as local field theories while others may not (Sec. VIII).

We end this introduction by mentioning a concern introduced by Donnelly in [6]: edge states in gauge theories [9]. This is based on earlier work by Witten [10] and Lowenstein and Swieca [11]. Lattice theories with edge states are special in that they are part of a class of theories that do not fully preserve the homogeneity of d.o.f. from the outset.

III. DEGREES OF FREEDOM AND DUALITIES

We have seen that entanglement can depend on the precise definition of what a d.o.f. is and that there is generally no unique choice of d.o.f. The ambiguity is not necessarily specific to gauge theories, but to spaces with constraints. We will attempt to address two questions. One is whether there exist gauge-invariant d.o.f. that can be deemed as defining a discretized field theory, and the second is whether there exists a set of assumptions that can lead to an unambiguous entanglement entropy when constraints are present.

In general, choices of d.o.f. are related by dualities. Given a field theory defined on a discrete lattice with discrete Abelian group valued d.o.f., there exist dualities that preserve the type and number of unconstrained d.o.f. A way of constructing such dualities consists of finding a basis in the Hilbert space of the theory, a set of distinct operators $\hat{O}_i$, one for each d.o.f., that are simultaneously diagonalized by the basis states, and then selecting some set of operators from the group generated by $\hat{O}_i$ under multiplication to define the new d.o.f. For a theory with constraints, we ask the question of whether we can find suitable dualities that have no constrained d.o.f.

Consider a simple example: a 2-d quantum Ising lattice with d.o.f. on the vertices and a global constraint. The Hilbert space has $N_{\text{d.o.f.}} = [(L_x + a)(L_y + a)/a^2 - 1]\ Z_2$ d.o.f., where a is the lattice spatial dimension. The $+a$ part can be seen as a matter of convention. One can enlarge the lattice by $a/2$ on all sides ($L_i' = L_i + a$) and consider the d.o.f. to be associated with the centers of plaquettes of the enlarged lattice and then write $N_{\text{d.o.f.}} = N_x' N_y' - 1$, with $N_i' = L_i'/a$. We can also write $N_{\text{d.o.f.}} = (N_x + 1)(N_y + 1) - 1 = N_x N_y + (N_x + N_y)$. This suggests that our theory could have a dual with $N_x N_y$ bulk d.o.f. and $N_x + N_y$ edge d.o.f. To see that this duality exists, we follow the geometry suggested by the d.o.f. decomposition and associate with the center of every plaquette a d.o.f. defined by the following operator identities:

$$\tilde{\sigma}^z_{\square}(x + 1/2, y + 1/2) = \sigma^z(x, y)\sigma^z(x + 1, y)\sigma^z(x, y + 1) \times \sigma^z(x + 1, y + 1), \tag{7}$$

where we switched to lattice units $a = 1$. Similarly for edge d.o.f.:

$$\tilde{\sigma}^z_{E_x}(x + 1/2) = \sigma^z(x, 0)\sigma^z(x + 1, 0)$$
$$\tilde{\sigma}^z_{E_y}(y + 1/2) = \sigma^z(0, y)\sigma^z(0, y + 1). \tag{8}$$

The algebra generated by all $\tilde{\sigma}^z_{\square}$ and $\tilde{\sigma}^z_{E_d}$ will contain all products $\sigma^z(x_1, y_1)\sigma^z(x_2, y_2)$, which are all simultaneously invariant under a global spin flip. The dual theory does not contain the global constraint, and it may seem that we should prefer the dual in calculations where geometric ambiguities in the choice of basis are relevant. The problem with the dual, however, is that the standard nearest-neighbor terms in the Hamiltonian are nonlocal. To see this, consider the term $\sigma^z(x_0, y_0)\sigma^z(x_0, y_0 + 1)$. In terms of dual d.o.f., it takes the form:

$$\sigma^z(x_0, y_0)\sigma^z(x_0, y_0 + 1) = \tilde{\sigma}^z_{E_y}(y_0 + 1/2) \prod_{x=0}^{x_0 - 1} \tilde{\sigma}^z_{\square}(x + 1/2, y_0 + 1/2). \tag{9}$$

Imposing a locality condition on the action would exclude the above duality from consideration. We note, however, that the dual has the exact same physics content and the same algebra as the initial theory. The choice of one or the other is a matter of preference, and this preference must be informed by other considerations. It may then be desirable to consider other geometric symmetries that should be satisfied by a duality, such as homogeneity of the d.o.f., (discrete) isotropy, parity transformations, boundary conditions, as well as the preservation of the lattice spacing unit. For example, in two dimensions, only square lattices satisfy the 4-fold isotropy condition. This restricts the number of unconstrained d.o.f. that can be represented on a square lattice with open boundary conditions (see Fig. 2):

$$\begin{aligned} N^d_{\text{d.o.f.}} &= (C + 4P)N^d_{\text{plaq}} + VN^d_{\text{vert}} + LN^d_{\text{links}} + EN^d_{\text{edge links}} \\ &= (C + 4P + 2L + V)N^d_x N^d_y \\ &\quad + (L + 2E + V)(N^d_x + N^d_y) + V, \end{aligned} \tag{10}$$

with $C, P, V, L, E \in \mathbb{N}$ being the number of d.o.f. associated with, respectively, the center of plaquettes, the off-center of

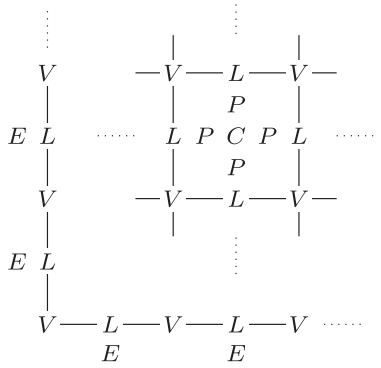


FIG. 2. Possible arrangements of d.o.f. on a square lattice.

plaquettes, vertices, links, and edge links (and/or edge vertices). One can then check numerically that for a 2-d Ising model with a global symmetry there is no satisfactory duality by finding the intersection of sets of solutions $\{(C,P,V,L,E)\}$ to the equation $N^d_{\text{d.o.f.}} = N_{\text{d.o.f.}} = N^0_x N^0_y - 1$ for various N^0_x and N^0_y which are sizes in the original model. In particular, one can verify that $N_{\text{d.o.f.}} = 2\times 2 - 1 = 3$ has no solution except the trivial $C = 3$ solution which places all d.o.f. at the center of one plaquette. The existence of solutions to the above equation does not necessarily imply that a suitable duality exists, but the absence of solutions implies the nonexistence of an unconstrained duality satisfying the geometric symmetries.

It follows that eliminating constraints can, in certain cases, lead to theories that do not have a suitable interpretation as local field theories. In such cases it would seem that we must either accept the constraints as physical or give up locality or other symmetries.

IV. GAUGE-INVARIANT STATES IN PURE Z_2 GAUGE LATTICES

We proceed with an analysis of the physical Hilbert space in Z_2 pure gauge lattices in $2+1$ spatial dimensions with free boundary conditions and a temporal gauge, and consider a time slice with all links in the spatial dimensions. This is the basic setup used in [1] and it is a natural choice for a Hamiltonian lattice theory [12,13]. When working with a Wilsonian theory, one must also consider plaquettes with a time component which correspond to the electric components of the electromagnetic tensor. The two are related, in the continuum limit, by $E_i^2 = F_{i0}^2 \approx (1/a^4)(1-\text{Re}U_{i0})$. The inclusion of electric plaquettes in the Wilsonian theory will be discussed as a particular case of a three-dimensional time slice in Sec. VIII. For the Hamiltonian version of the theory, throughout the paper, we will assume the following Hamiltonian:

$$H = \sum_{x,\mu} L_\mu(x) - \lambda \sum_{x,\mu,\nu} U_\mu(x) U_\nu(x) U_\mu(x+\hat{\nu}) U_\nu(x+\hat{\mu}), \tag{11}$$

where the first sum is taken over all the links and the second is taken over all the plaquettes. This corresponds to the following Wilson action:

$$S = -\lambda \sum_{x,\mu,\nu} U_\mu(x) U_\nu(x) U_\mu(x+\hat{\nu}) U_\nu(x+\hat{\mu}). \tag{12}$$

Consider a lattice with Z_2 links and the standard Z_2 algebra of operators acting on the links:

$$U|u\rangle = u|u\rangle \tag{13}$$

$$L|u\rangle = |-u\rangle, \tag{14}$$

with $u \in \{+1,-1\}$ (or, for consistency with spin systems, $u \in \{\uparrow,\downarrow\}$), and the commutation relations:

$$[U_\mu(x), U_\nu(y)] = 0 \tag{15}$$

$$[L_\mu(x), L_\nu(y)] = 0 \tag{16}$$

$$[U_\mu(x), L_\nu(y)] = 0, \qquad x \neq y \vee \mu \neq \nu \tag{17}$$

$$[U_\mu(x), L_\mu(x)] \neq 0. \tag{18}$$

Gauge transformations are operators parametrized by group elements associated with each vertex which transform links as follows:

$$u_\mu(x) \to g(x) u_\mu(x) g^\dagger(x+\hat{\mu}), \tag{19}$$

where $g(x)$ and $g(x+\hat{\mu})$ represent the vertices associated with the endpoints of link $u_\mu(x)$. In the Z_2 case, $u_\mu(x)$ is flipped if exactly one of $g(x)$ and $g(x+\hat{\mu})$ are -1.

An arbitrary gauge transformation acts on an arbitrary link polynomial as follows:

$$\begin{aligned} &u_{\mu_1}(x_1) u_{\mu_2}(x_2) \ldots u_{\mu_n}(x_n) \\ &\quad \to g(x_1) u_{\mu_1}(x_1) g^\dagger(x_1+\hat{\mu_1}) g(x_2) u_{\mu_2}(x_2) g^\dagger(x_2+\hat{\mu_2}) \ldots \\ &\qquad \times g(x_n) u_{\mu_n}(x_n) g^\dagger(x_n+\hat{\mu_n}). \end{aligned} \tag{20}$$

This polynomial is gauge invariant only if all of the gauge terms cancel out, which can only happen for closed paths (Wilson loops), products of closed paths (for Abelian groups), or a constant. The smallest Wilson loop is the one that goes around a single plaquette.

This implies that we can use Wilson loop functionals to construct functionals $\Psi[U_\mu(x)] = \sum c_k W_k[U_\mu(x)]$ that result in gauge-invariant states:

$$\begin{aligned} |\Psi\rangle &= \sum_{u_\mu(x)=\pm 1} \Psi[U_\mu(x)] \bigotimes_{\mu,x} |u_\mu(x)\rangle \\ &= \sum_{u_\mu(x)=\pm 1} \sum_k c_k W_k[U_\mu(x)] \bigotimes_{\mu,x} |u_\mu(x)\rangle, \end{aligned} \tag{21}$$

where $W_k[U_\mu(x)]$ are any subset of the Wilson loop polynomials, including the identity. These states are gauge invariant because they assign the same coefficients to all microstates $\bigotimes_{\mu,x}|u_\mu(x)\rangle$ related by gauge transformations.

For a single plaquette, the gauge transformations can be seen explicitly in Fig. 3. Since there are $2^4 = 16$ total link states and eight distinct gauge transformations, there are exactly two physical states per plaquette. A gauge transformation on a Z_2 lattice will always flip an even number (including none) of links in any given plaquette. Consequently, a convenient basis for physical states is obtained by dividing the kinematic (link) states into states

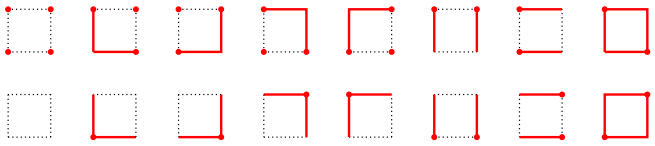


FIG. 3. Gauge transformations on a Z_2 plaquette. The gauge transformations happen at the marked vertices. The solid links are links that are affected by the gauge transformation. Transformations in the same column are equivalent.

with even/odd number of up-type links per plaquette. These states are eigenstates of plaquette operators:

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x). \quad (22)$$

In two dimensions, we can drop the tensor indices and use $U_\square(x) \equiv U_{12}(x)$. The states can be constructed using products of the operators

$$U_\square^\uparrow(x) \equiv \frac{1}{\sqrt{2}}(1 + U_\square(x)) \quad (23)$$

$$U_\square^\downarrow(x) \equiv \frac{1}{\sqrt{2}}(1 - U_\square(x)), \quad (24)$$

by applying them to the weak coupling ground state:

$$|0\rangle = C \sum_{u_\mu(x)=\pm 1} \bigotimes_{\mu,x} |u_\mu(x)\rangle, \quad (25)$$

where C is a normalization constant. Specifically, for a single plaquette, we have that

$$|\uparrow_\square\rangle = U_\square^\uparrow|0\rangle = \frac{1}{2\sqrt{2}}[|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle + \cdots + |\downarrow\downarrow\downarrow\downarrow\rangle](\text{even \# of}\uparrow) \quad (26)$$

$$|\downarrow_\square\rangle = U_\square^\downarrow|0\rangle = \frac{1}{2\sqrt{2}}[|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + \cdots + |\downarrow\downarrow\downarrow\uparrow\rangle](\text{odd \# of}\uparrow). \quad (27)$$

As expected,

$$U_\square|\uparrow_\square\rangle = U_\square U_\square^\uparrow|0\rangle = U_\square\frac{1}{\sqrt{2}}(1+U_\square)|0\rangle = \frac{1}{\sqrt{2}}(U_\square+1)|0\rangle = |\uparrow_\square\rangle \quad (28)$$

$$U_\square|\downarrow_\square\rangle = U_\square U_\square^\downarrow|0\rangle = U_\square\frac{1}{\sqrt{2}}(1-U_\square)|0\rangle = \frac{1}{\sqrt{2}}(U_\square-1)|0\rangle = -|\downarrow_\square\rangle, \quad (29)$$

since $U_\square^2 = 1$ for Z_2.

A physical state on a full lattice can be expressed as a linear combination of basis states which are products of operators $U_\square^\pm$ for each plaquette:

$$|e_\lambda\rangle = \prod_x U_\square^{\lambda_x}(x)|0\rangle \quad (30)$$

$$|\Psi\rangle = \sum_\lambda c_\lambda|e_\lambda\rangle, \quad (31)$$

where $\boldsymbol{\lambda} = (\lambda_x|\lambda_x \in \{\downarrow,\uparrow\})$ are M-dimensional vectors that index the basis vectors of the physical states, M is the number of plaquettes, and c_λ are coefficients satisfying $\sum c_\lambda^2 = 1$. The basis states are orthonormal:

$$\langle e_\lambda|e_{\lambda'}\rangle = \langle 0|\prod_x U_\square^{\lambda_x}(x)U_\square^{\lambda'_x}(x)|0\rangle = \prod_x \delta_{\lambda_x,\lambda'_x}, \quad (32)$$

and therefore we can write:

$$c_\lambda = \langle e_\lambda|\Psi\rangle. \quad (33)$$

The orthogonality is apparent for two reasons. First, any product of the form $U_\square^\uparrow(x)U_\square^\downarrow(x) = (1 - U_\square^2(x))/2$ is zero, since $U_\square^2(x) = 1$. Therefore, λ_x must equal λ'_x for all x in order to get a nonzero result. Second, if all $\lambda_x = \lambda'_x$, then the left-hand side of (32) reduces to:

$$\langle 0|\prod_x(1 \pm U_\square(x))|0\rangle = \langle 0|0\rangle + \langle 0|\sum f(U_\square(x))|0\rangle, \quad (34)$$

where $f(U_\square(x))$ are various terms that contain at least one link operator. Such terms vanish, since they are antisymmetric with respect to $|0\rangle$. In order to obtain a reduced density matrix, we can divide the set of plaquettes into two regions, A and $\bar{A}$, and write $\boldsymbol{\lambda} = (\chi_x, \bar{\chi}_{\bar{x}}'|x \in A, \bar{x} \in \bar{A}) = \boldsymbol{\chi} \oplus \bar{\boldsymbol{\chi}}$, such that $\lambda_x = \chi_x$ if $x \in A$ and $\lambda_{\bar{x}} = \bar{\chi}_{\bar{x}}$ if $\bar{x} \in \bar{A}$. We can then write

$$|\Psi\rangle = \sum_{\chi\bar{\chi}} c_{\chi\otimes\bar{\chi}}|e_{\chi\otimes\bar{\chi}}\rangle \equiv \sum_{\chi\bar{\chi}} c_{\chi\bar{\chi}}|e_{\chi\bar{\chi}}\rangle. \quad (35)$$

The density matrix is then:

$$\rho[\boldsymbol{\lambda};\boldsymbol{\lambda}'] = \rho[\boldsymbol{\chi},\bar{\boldsymbol{\chi}};\boldsymbol{\chi}',\bar{\boldsymbol{\chi}}'] = c_{\chi\bar{\chi}}c_{\chi'\bar{\chi}'}. \quad (36)$$

Consequently, the resulting reduced density matrix, ρ_A, is:

$$\rho_A[\boldsymbol{\chi};\boldsymbol{\chi}'] = \sum_{\bar{\chi}} c_{\chi\bar{\chi}}c_{\chi'\bar{\chi}}. \quad (37)$$

The entanglement entropy [7] is then:

$$S_A = \mathrm{tr}\rho_A \ln \rho_A. \quad (38)$$

The above representation of states is gauge invariant and the density matrix ρ is written explicitly in terms of vectors in the gauge-invariant subspace. It follows that S_A is gauge invariant.

One can also consider the transverse (or "electric") basis, which is the basis in which link operators, L, are diagonal:

$$L|l\rangle = l|l\rangle. \tag{39}$$

Specifically, in terms of link basis vectors:

$$L|+\rangle = L|\uparrow\rangle + L|\downarrow\rangle = |\downarrow\rangle + |\uparrow\rangle = |+\rangle \tag{40}$$

$$L|-\rangle = L|\uparrow\rangle - L|\downarrow\rangle = |\downarrow\rangle - |\uparrow\rangle = -|-\rangle. \tag{41}$$

This basis is particularly useful due to its convenient and suggestive diagrammatic representation, which we will employ later. A gauge transformation at a lattice vertex v is a product of link operators connected to that vertex and acting on a four link state as follows:

$$\begin{aligned} &L_{v,v+\hat{x}}L_{v,v+\hat{y}}L_{v-\hat{x},v}L_{v-\hat{y},v}|l_{v,v+\hat{x}}l_{v,v+\hat{y}}l_{v-\hat{x},v}l_{v-\hat{y},v}\rangle \\ &\quad = l_{v,v+\hat{x}}l_{v,v+\hat{y}}l_{v-\hat{x},v}l_{v-\hat{y},v}|l_{v,v+\hat{x}}l_{v,v+\hat{y}}l_{v-\hat{x},v}l_{v-\hat{y},v}\rangle. \end{aligned} \tag{42}$$

Gauge invariant states must, therefore, satisfy $l_{v,v+\hat{x}}l_{v,v+\hat{y}}l_{v-\hat{x},v}l_{v-\hat{y},v} = 1$. This can be interpreted as a conservation law that ensures that an even number of $|-\rangle$ links are connected to every vertex. It can be seen that the resulting gauge-invariant states take the form of linear combinations of closed loops of links in the $|-\rangle$ state. States can be manipulated using plaquette operators starting from $|0\rangle$, which, in the electric basis, is equal to $\otimes\,|+\rangle$:

$$|e_{\boldsymbol{p}}\rangle = \prod_x (U_\square(x))^{p_x}|0\rangle \tag{43}$$

$$|\Psi\rangle = \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}|e_{\boldsymbol{p}}\rangle, \tag{44}$$

where $\boldsymbol{p} = (p_x|p_x \in \{0,1\})$. Unsurprisingly, we recover the same structure as before [see Eq. (30)] and, consequently, the same dimensionality for the physical Hilbert space.

From a physical standpoint, the Hilbert space discussed in this section can either be seen as the space of closed electric loops or the space of magnetic fluxes going through plaquettes. From the magnetic perspective, it would seem natural that fluxes through extended areas are equal to the sum of individual fluxes through the elementary geometric constructs covered by that area.

V. THE ALGEBRA OF GAUGE-INVARIANT OPERATORS IN 2-D Z_2 GAUGE LATTICES

There exists a duality between 2-d gauge theories and spin chains [14], which was identified initially by Frank Wegner in [15]. We will summarize the relevant parts here.

As seen previously, Wilson loops are gauge invariant. An algebra is generated by the plaquette operators, $U_\square(x)$. This completes the algebra of gauge-invariant operators that can be generated exclusively from link variables $U_\mu(x)$. The remaining gauge-invariant operators are derived from the link-flip operators $L_\mu(x)$. From the commutation relations in Eq. (18), we see that all $L_\mu(x)$ commute with each other. Since gauge transformations are a subalgebra of the algebra generated by link-flip operators, it follows that all $L_\mu(x)$ commute with gauge transformations and are, therefore, gauge invariant. Consequently, the full algebra of gauge-invariant operators is generated by all $U_\square(x)$ and all $L_\mu(x)$. Returning to the duality, the correspondence is:

Gauge theory	Spin chain	Comments
Plaquette	Spin	d.o.f.
$U_\square(x)$	$\sigma_z(x)$	…
$L_\mu(x+\hat{\mu}_\perp)$	$\sigma_x(x)\sigma_x(x+\hat{\mu}_\perp)$	$L_\mu(x)$ not at the edge, $\hat{\mu}\cdot\hat{\mu}_\perp = 0$
$L_\mu^{\text{edge}}(x)$	$\sigma_x(x)$	$L_\mu^{\text{edge}}(x)$ at the edge of the lattice

There is no immediately obvious equivalent between $L_\mu(x)$ operators and the bulk $\sigma_x(x)$ operators. They can be constructed by observing that:

$$\begin{aligned} \sigma_x(x+\hat{\mu}_\perp) &= \sigma_x(x)\sigma_x(x)\sigma_x(x+\hat{\mu}_\perp) \\ &= L_\mu^{\text{edge}}(x)L_\mu(x+\hat{\mu}_\perp), \end{aligned} \tag{45}$$

where $\hat{\mu}\cdot\hat{\mu}_\perp = 0$. This can be generalized for arbitrary plaquettes (see Fig. 4):

$$\begin{aligned} L_\square(x) &\equiv \sigma_x(x) \\ &= L_{\mu_0}^{\text{edge}}(x_0)L_{\mu_1}(x_0+\hat{\mu_0}_\perp) \\ &\quad \times L_{\mu_2}(x_0+\hat{\mu_0}_\perp+\hat{\mu_1}_\perp)\cdots L_{\mu_n}(x). \end{aligned} \tag{46}$$

Given a region A, the algebra generated by the operators $U_\square(x)$, $L_\square(x)$ for $x \in A$ is a factor (an algebra with a trivial center).

Furthermore, the operators $U_\square(x)$ and $L_\square(x)$ are invariant under maximal tree gauge fixing. This type of gauge fixing involves setting a certain set of links to a fixed value and only considering the remaining ones dynamic. This can be done as long as the fixed links do not form any loops (see, e.g., [16]; also Fig. 5). The fixed links are typically set

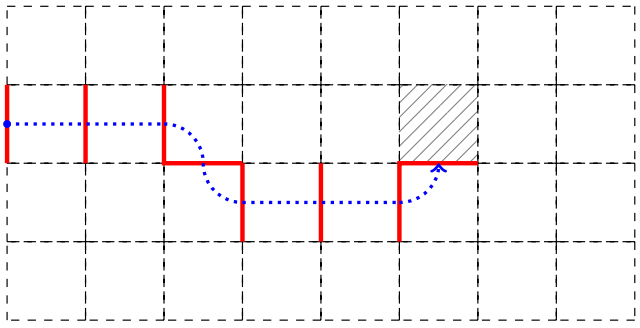


FIG. 4. Construction of one of the possible plaquette-flip operators. The operator acts on the shaded plaquette and is composed of $L_\mu(x)$ operators acting on links that are shown in thick red, lines.

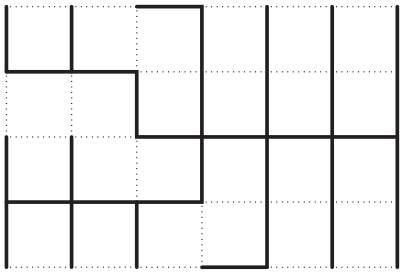


FIG. 5. Example of maximal tree gauge fixing in a simple gauge lattice. The thick links are fixed.

to the state $|\uparrow\rangle = |+\rangle + |-\rangle$ and the corresponding $L_\mu(x)$ operator is removed from the algebra, since its inclusion would be at odds with the link being fixed. Because no loops of fixed links are allowed, we are guaranteed to have at least one dynamic edge link, and we are also guaranteed that a path such as the one shown in Fig. 4 exists between a dynamic edge link and every plaquette. This, in turn, implies that all $L_\square(x)$ operators will exist in the algebra, but no particular $L_\mu(x)$ operator is guaranteed to be there.

We can also check what happens to the Hamiltonian from Eq. (11):

$$H = \sum_{x,\mu} L_\mu(x) - \lambda \sum_x U_\square(x)$$
$$= \sum_{\text{n.n.}} L_\square(x) L_\square(y) + \sum_{\text{edge}} L_\square(x) - \lambda \sum_x U_\square(x), \quad (47)$$

where the sums over x are to be understood as sums over all links or all plaquettes, respectively. The dual Hamiltonian is local, gauge invariant, and well defined for all choices of maximal tree gauge fixing conditions. With the exception of the edge terms, the dual is a quantum transverse Ising model. This dual represents the solution with $C = 1$ in Eq. (10). The duality also underlines the problem with the inseparability proof in [1]. If the observable theory consisted of an unconstrained spin network with Hilbert space $\mathcal{H}_0$, we would have no problem constructing a geometrical bipartition of the spin d.o.f. However, we can also construct a "reverse" Wegner dual gauge theory with a Hilbert space $\tilde{\mathcal{H}}$. On $\tilde{\mathcal{H}}$ there would be no bipartition of links supporting states in $\mathcal{H}_0$, but we should not use that to conclude that $\mathcal{H}_0$ is geometrically inseparable.

The idea floated previously of links being removed from the algebra under maximal tree gauge fixing deserves some more attention. If we adopt a temporal gauge and also gauge-fix a particular time slice t_0 using a maximal tree, we are generally prevented from also fixing links in any subsequent time slice [12]. The Hamiltonian will necessarily contain all link operators in the kinetic term. The link operators are then objects that relate link states at t_0 with link states at other times, and a gauge fixing at t_0 remains associated with the absence of the ability to modify the state of certain links at t_0. Without a temporal gauge, one is free to use the exact same maximal tree of fixed links at all time slices. We can, therefore, completely remove the terms involving nondynamical links from the Hamiltonian. This issue is entirely hidden in the dual Hamiltonian in Eq. (47).

An alternative treatment to the entanglement entropy of Abelian lattice gauge theory based on the duality to spin systems can be found in [17].

VI. THE TWO-PLAQUETTE LATTICE

The simplest two-dimensional pure gauge lattice setup is a two-plaquette Z_2 lattice (Fig. 6). It will be used to illustrate some of the issues presented in the previous section.

The basic gauge-invariant operators are:

$$U^L_\square = U_1 U_2 U_3 U_7 \quad (48)$$

$$U^R_\square = U_4 U_5 U_6 U_7 \quad (49)$$

$$L_i, i \in \{1, \ldots, 7\}. \quad (50)$$

The induced constraints are:

$$L_1 = L_2 = L_3 \equiv L^L_\square \quad (51)$$

$$L_4 = L_5 = L_6 \equiv L^R_\square \quad (52)$$

$$L_7 = L_2 L_5 = L^L_\square L^R_\square. \quad (53)$$

The remaining gauge-invariant operators can be obtained from $U^{L,R}_\square$ and $L^{L,R}_\square$. In particular, we can define:

$$U^{\uparrow\{L,R\}}_\square = \frac{1}{\sqrt{2}}(1 + U^{\{L,R\}}_\square) \quad (54)$$

$$U^{\downarrow\{L,R\}}_\square = \frac{1}{\sqrt{2}}(1 - U^{\{L,R\}}_\square), \quad (55)$$

which can be used to construct the magnetic basis:

$$|\downarrow_\square \downarrow_\square\rangle = U^{\downarrow L}_\square U^{\downarrow R}_\square |0\rangle \quad (56)$$

$$|\downarrow_\square \uparrow_\square\rangle = U^{\downarrow L}_\square U^{\uparrow R}_\square |0\rangle \quad (57)$$

$$|\uparrow_\square \downarrow_\square\rangle = U^{\uparrow L}_\square U^{\downarrow R}_\square |0\rangle \quad (58)$$

$$|\uparrow_\square \uparrow_\square\rangle = U^{\uparrow L}_\square U^{\uparrow R}_\square |0\rangle. \quad (59)$$

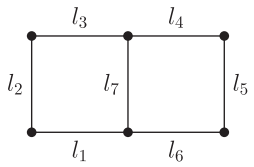


FIG. 6. A simple two-plaquette lattice.

If we switch to the electric basis, we can express the tensor product structure in a less abstract fashion through diagrams in which links in the $|-\rangle$ state are emphasized. The single plaquette states are $|+\rangle_\square = |\square\rangle$, $|-\rangle_\square = |\blacksquare\rangle$, while the two-plaquette states are $|++\rangle_\square = |\square\square\rangle$, $|-+\rangle_\square = |\blacksquare\square\rangle$, $|+-\rangle_\square = |\square\blacksquare\rangle$, $|--\rangle_\square = |\blacksquare\blacksquare\rangle$. It is probably noteworthy that $|\blacksquare\blacksquare\rangle = |\blacksquare\rangle \otimes |\blacksquare\rangle \neq |\square\rangle \otimes |\square\rangle$. This is because the only gauge-invariant operators that can change electric states are operators in the algebra of Wilson loops. If we start with $|\square\square\rangle = |\square\rangle \otimes |\square\rangle$, then $|\blacksquare\blacksquare\rangle = U^L_\square U^R_\square |\square\square\rangle = U^L_\square U^R_\square |\square\rangle \otimes |\square\rangle = |\blacksquare\rangle \otimes |\blacksquare\rangle$.

As pointed out in [2], certain subalgebras are not factors. In particular, if we chose to divide links into regions and considered the algebras of gauge-invariant operators that can be constructed from the operators acting on links in each region, we would encounter algebras such as $\mathcal{A}_1 = \{\mathbb{1}, L_1, L_2, L_3\}$ which satisfies $\mathcal{A}_1 \cap (\mathcal{A}_1)' = \mathcal{A}_1 \neq \mathbb{1}$. Alternatively the algebra of links including the entire left loop would be $\mathcal{A}_2 = \{\mathbb{1}, L_1, L_2, L_3, L_7, U^L_\square\}$, for which we would find that $\{L_4, L_5, L_6\} \subset (\mathcal{A}_2)''$, so $\mathcal{A}_2$ also generates the electric operators in the right loop and $(\mathcal{A}_2)' \cap (\mathcal{A}_2)'' \neq \mathbb{1}$. Neither of the examples are specific to gauge theories. In a two spin system, $\mathcal{A}_1 = \{\mathbb{1}, \sigma^1_x\}$ also satisfies $\mathcal{A}_1 \cap (\mathcal{A}_1)' = \mathcal{A}_1$, whereas, for $\mathcal{A}_2 = \{\mathbb{1}, \sigma^1_x, \sigma^1_x\sigma^2_x, \sigma^1_z\}$, we would necessarily find that $\sigma^2_x \in (\mathcal{A}_2)' \cap (\mathcal{A}_2)''$.

It may be interesting to compare the entanglement entropy in the magnetic, electric, and kinematic spaces. We can simplify the lattice further by doing a partial gauge fixing (see Fig. 7). There is a single gauge transformation remaining, which, in the magnetic basis, flips all the remaining free links. Analyzing an arbitrary state can be somewhat unpalatable, so we stick to states of the form $|\Psi\rangle = \alpha|\downarrow_\square\downarrow_\square\rangle + \beta|\uparrow_\square\uparrow_\square\rangle$, with $\alpha^2 + \beta^2 = 1$. The entanglement entropy is:

$$S^{\text{mag}}_A = -\alpha^2 \log \alpha^2 - \beta^2 \log \beta^2, \tag{60}$$

and it can vary between zero (for $\alpha = 0$ or $\beta = 0$) and log 2 (for $\alpha = \beta = 0.5$). The kinematic state is:

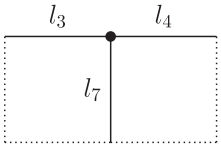


FIG. 7. Partial gauge fixing on the simple lattice. The dotted links are set to 1.

$$|\Psi\rangle = \frac{\alpha}{\sqrt{2}}[|\uparrow_3\downarrow_7\uparrow_4\rangle + |\downarrow_3\uparrow_7\downarrow_4\rangle] + \frac{\beta}{\sqrt{2}}[|\uparrow_3\uparrow_7\uparrow_4\rangle + |\downarrow_3\downarrow_7\downarrow_4\rangle]. \tag{61}$$

The reduced density matrix obtained by tracing over links 7 and 4 is:

$$\rho_{7,4} = \frac{\alpha^2}{2}|\uparrow_3\rangle\langle\uparrow_3| + \frac{\alpha^2}{2}|\downarrow_3\rangle\langle\downarrow_3| + \frac{\beta^2}{2}|\uparrow_3\rangle\langle\uparrow_3| + \frac{\beta^2}{2}|\downarrow_3\rangle\langle\downarrow_3| \tag{62}$$

$$= \frac{1}{2}|\uparrow_3\rangle\langle\uparrow_3| + \frac{1}{2}|\downarrow_3\rangle\langle\downarrow_3|. \tag{63}$$

The resulting entanglement entropy is now $S^{\text{kin}}_A = \log 2$, independent of α and β. This value is not gauge invariant since fixing, e.g., link 7 to $|\uparrow_7\rangle$ yields $|\Psi\rangle = \alpha|\downarrow_3\uparrow_7\downarrow_4\rangle + \beta|\uparrow_3\uparrow_7\uparrow_4\rangle$, and we recover the value of S^{mag}_A.

In the electric basis, after the gauge fixing employed above, the basis vectors are $|++\rangle_\square = |\square\square\rangle = |+_3\,+_7\,+_4\rangle$, $|-+\rangle_\square = |\blacksquare\square\rangle = |-_3\,-_7\,+_4\rangle$, $|+-\rangle_\square = |\square\blacksquare\rangle = |+_3\,-_7\,-_4\rangle$, $|--\rangle = |\blacksquare\blacksquare\rangle = |-_3\,+_7\,-_4\rangle$. The state $|\Psi\rangle$ is:

$$|\Psi\rangle = \frac{\alpha+\beta}{2}[|+_3+_7+_4\rangle + |-_3+_7-_4\rangle] - \frac{(\alpha-\beta)}{2}[|+_3-_7-_4\rangle + |-_3-_7+_4\rangle]. \tag{64}$$

The electric vectors correspond to states with electric fluxes conserved at the shared lattice site. The state expressed in the electric basis has the same form as the kinematic state and one can conclude that $S^{\text{elec}}_A = \log 2$. In the electric picture, one would attribute the entanglement entropy to the constraints on the electric d.o.f. Once again, if we were to fully fix the gauge and remove one of the links l_i from the state, the entanglement entropy of the projected state would take the same value as S^{mag}_A. The exact choice of i is not relevant for the state in this example.

VII. PERIODIC BOUNDARY CONDITIONS

The introduction of periodic boundary conditions in the spatial direction is associated with two new phenomena. One is:

$$\prod_x U_\square(x) = 1, \tag{65}$$

since every link operator $U_\mu(x)$ appears exactly twice in the product. The second is the addition of two new independent topological d.o.f., associated with loops that wind around the two spatial dimensions.

The global constraint in Eq. (65) is a manifestation of the magnetic Gauss's law, which holds that the net magnetic flux through a closed surface is zero. Once again, we can consider the possibility of an unconstrained duality. In the spirit of Eq. (10), for a 2-d lattice with periodic boundary conditions:

$$N^d_{\text{d.o.f.}} = (C + 4P)N^d_{\text{plaq}} + VN^d_{\text{vert}} + LN^d_{\text{links}}. \quad (66)$$

We can notice that in a lattice with periodic boundary conditions in both x and y directions, we have $N^d_{\text{links}} = 2N^d_{\text{plaq}} = 2N^d_{\text{vert}}$. We can then write:

$$N^d_{\text{d.o.f.}} = (C + 4P + V + 2L)N^d_{\text{plaq}}. \quad (67)$$

To see that there is no homogeneous and 4-fold isotropic unconstrained dual lattice with periodic boundary conditions that can support a suitable duality, we can start with a lattice with dimensions $N_x = 2^p$, $N_y = 2^q$. Without the global constraint, there would be precisely one physical d.o.f. per plaquette. However, the presence of the global constraint reduces the number of physical d.o.f. by one, leading to $N_{\text{d.o.f.}} = N_x N_y - 1 = 2^{p+q} - 1$. $N_{\text{d.o.f.}}$. is a Mersenne number, and some p and q would lead to $N_{\text{d.o.f.}}$. being prime. The only solutions to $N_{\text{d.o.f.}} = N^d_{\text{d.o.f.}}$ in Eq. (67) are $N^d_{\text{plaq}} = 1$ and $N^d_{\text{plaq}} = N^d_x N^d_y = N_{\text{d.o.f.}}$. The first solution corresponds to a zero-dimensional geometry, while the second implies either that $N^d_x = 1$ or that $N^d_y = 1$, which are one-dimensional geometries.

The global constraint cannot, therefore, be eliminated without severely spoiling the geometry. However, using the plaquette basis, we can still work in a gauge-invariant space. As before [see Eq. (30)], we express a state as a linear combination of basis vectors for some set X of independent plaquettes:

$$|e_{\boldsymbol{\lambda}}\rangle = \prod_{x\in X} U^{\lambda_x}_{\square}(x)|0\rangle. \quad (68)$$

We can subsequently extract the coefficients of a gauge-invariant state on the constrained space using products of $U_\square$ operators over all plaquettes:

$$\langle e_{\boldsymbol{\lambda},\lambda_0}|e_{\boldsymbol{\lambda}'}\rangle = \langle 0|U^{\lambda_0}_{\square}(x_0)\prod_{x\in X} U^{\lambda_x}_{\square}(x)\prod_{x'\in X} U^{\lambda_{x'}}_{\square}(x')|0\rangle, \quad (69)$$

where x_0 represents the remaining plaquette ($x_0 \notin X$). Since $U^{\lambda_x}_{\square}(x)$ satisfy $U^{\lambda_x}_{\square}(x)U^{\lambda'_x}_{\square}(x) = 2\delta_{\lambda_x,\lambda'_x}U^{\lambda_x}_{\square}(x)$ and we have:

$$\begin{aligned}\langle e'_{\boldsymbol{\lambda}',\lambda'_0}|e_{\boldsymbol{\lambda}}\rangle &= \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}\langle 0|U^{\lambda'_0}_{\square}(x_0)\prod_{x\in X} U^{\lambda_x}_{\square}(x)|0\rangle \\ &= \frac{1}{2}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}\langle 0|1 + \prod_x p(\lambda'_x)U_\square(x) + \cdots|0\rangle \\ &= \frac{1}{2}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}\delta\left(1 + \prod_x p(\lambda'_x)\right), \end{aligned} \quad (70)$$

where $p(\uparrow) = 1$ and $p(\downarrow) = -1$, and the ellipsis stands for terms that are antisymmetric with respect to $|0\rangle$. When the constraint is satisfied, there is an even number of plaquettes in the $|\downarrow\rangle$ state and the product over $p(\lambda_x)$ is equal to 1 leading to $\langle e'_{\boldsymbol{\lambda}',\boldsymbol{\xi}}|e_{\boldsymbol{\lambda}}\rangle = 1$. Conversely, when the constraint is not satisfied, the product over $p(\lambda_x)$ is -1 and the dot product vanishes. The constrained space density matrix corresponding to some state $|\Psi\rangle$ is then:

$$\rho[\boldsymbol{\lambda},\lambda_0;\boldsymbol{\lambda}',\lambda'_0] = \langle\Psi|e_{\boldsymbol{\lambda}',\lambda'_0}\rangle\langle e_{\boldsymbol{\lambda},\lambda_0}|\Psi\rangle. \quad (71)$$

One can then divide d.o.f. using $\boldsymbol{\lambda}\oplus\lambda_0 = \boldsymbol{\chi}\oplus\bar{\boldsymbol{\chi}}$ and proceed as in Eqs. (35)–(37).

With no edge links in the original Hamiltonian [see Eq. (11)], the plaquette basis Hamiltonian would take the form:

$$\begin{aligned} H &= \sum_{x,\mu} L_\mu(x) - \lambda\sum_x U_\square(x) \\ &= \sum_{\text{n.n.}} L_\square(x)L_\square(y) - \lambda\sum_x U_\square(x), \end{aligned} \quad (72)$$

which is the transverse Ising model. In the $\lambda \to 0$ limit, there is no magnetic term, and the entanglement entropy is $S_A = \log 2$ and topological. In the $\lambda \to \infty$ limit, the plaquettes are polarized and $S_A = 0$.

We now return to the topological d.o.f. The topology induced by imposing periodic boundary conditions on the two-dimensional lattice is that of a torus. The two non-plaquette d.o.f. are related to the magnetic flux through the inside and center of the torus [see Fig. 8(a)]. There is no preferential choice for the generating algebra of the topological d.o.f. and the different choices can be related using plaquette operators. However, given such a choice, there exist two classes of bipartitions based on whether exactly one region contains both topological loops [Fig. 8(d)] or not [Figs. 8(b) and 8(c)]. In the former case, at least one of the topological loops cannot be expressed as a nontrivial tensor product. To the extent that one is comfortable with the idea of separating what appears to be an atomic d.o.f., the problem can be alleviated by enlarging the Hilbert space (see, e.g., [6]) or, as will be shown in Sec. IX, by coupling to matter fields.

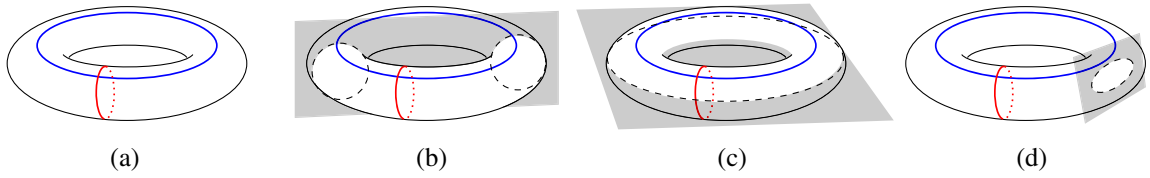


FIG. 8. Topological d.o.f. on a torus (a) and various bipartition choices (b), (c), (d). Bipartition boundaries are shown as dashed lines.

One encounters a similar situation when imposing periodic boundary conditions in a single direction leading to a cylindrical topology. The global constraint is not present, but a topological d.o.f. remains. It is associated with the magnetic flux through the cylinder.

VIII. 3 + 1 DIMENSIONS

The three-dimensional case is characterized by the existence of a local magnetic Gauss constraint of a similar nature as the one in Eq. (65):

$$\prod_C U_{\square\mu\nu}(x) = 1, \tag{73}$$

where C represents the faces of an elementary cube. The constraint is not a gauge constraint since it cannot be eliminated by gauge fixing. For example, a maximal tree gauge fixing on a single cube would result in five remaining dynamic links as shown in Fig. 9, and Eq. (73) would still hold.

To look at the space of unconstrained d.o.f. we can find a subset of plaquettes such that no closed surfaces are formed, a procedure reminiscent of the maximal tree gauge fixing procedure. One such subset is shown in Fig. 10. A count of the number of d.o.f. yields $N_{\text{d.o.f.}} = 2N_xN_yN_z + N_xN_y + N_yN_z + N_zN_x$, where N_i is the number of plaquettes in direction i. Consequently, a dual with unconstrained d.o.f. would necessarily be both anisotropic and would have nonlocal terms in the action. The density matrix can be written in terms of the constrained but gauge-invariant d.o.f. as it was done in the previous section. The Hamiltonian takes the following form:

$$H = \sum_{x,\mu} \prod_{i \in ST(x,\mu)} L^i_{\square} - \lambda \sum_x U_{\square}(x), \tag{74}$$

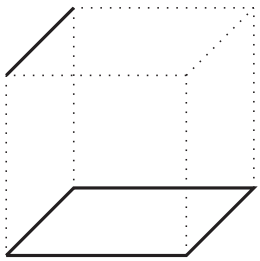


FIG. 9. Maximal tree gauge fixing on an elementary cube. The dotted links are fixed.

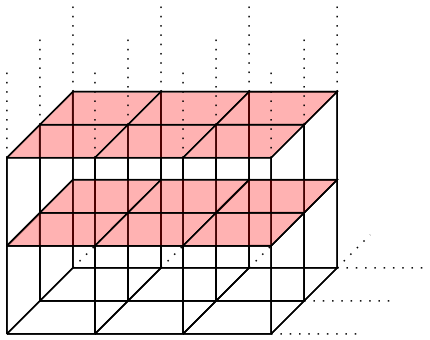


FIG. 10. Example of unconstrained Hilbert space in three-dimensional time slices. The d.o.f. on the red/shaded plaquettes can be expressed in terms of other d.o.f.

where $ST(x,\mu)$ (the "staple") is the set of all plaquettes that contain the link $L_\mu(x)$.

As it was mentioned earlier, in a Wilsonian $2+1$ dimensional theory, a time slice would include plaquettes with a time component, making it a restricted form of a three dimensional space in which no closed surfaces would exist, corresponding to only the bottom layer in Fig. 10. The resulting physical Hilbert space would remain unconstrained.

IX. COUPLING TO MATTER FIELDS

Matter fields in lattice theories are associated with d.o.f. that live on the vertices of the lattice. They transform under a gauge transformation as:

$$\phi(x) \to g(x)\phi(x). \tag{75}$$

This implies a new set of gauge-invariant quantities:

$$\phi^\dagger(x)\phi(x) \to \phi^\dagger(x)g^\dagger(x)g(x)\phi(x) = \phi^\dagger(x)\phi(x). \tag{76}$$

Additionally, since end points of gauge links transform in the same way as matter fields, we can also identify products of the following form as gauge invariant:

$$\begin{aligned}&\phi^\dagger(x_1)u_{\mu_1}(x_1)\ldots u_{\mu_{n-1}}(x_{n-1})\phi(x_n)\\ &\to \phi^\dagger(x_1)g^\dagger(x_1)g(x_1)u_{\mu_1}(x_1)g^\dagger(x_1+\hat{\mu}_1)\ldots\\ &\quad\times g(x_{n-2})u_{\mu_{n-1}}(x_{n-1})g^\dagger(x_{n-1}+\hat{\mu}_{n-1})g(x_n)\phi(x_n),\end{aligned} \tag{77}$$

provided that $x_i + \hat{\mu}_i = x_{i+1}$. In other words, Wilson lines multiplied with matter fields at the ends are gauge-invariant quantities. We can employ a slight redefinition of the fields in order to separate d.o.f. into gauge-dependent and gauge-independent quantities:

$$\phi(x) = |\phi(x)|v(x) = \tilde{\phi}(x)v(x), \tag{78}$$

with $v(x)$ being gauge group valued. The fields $\tilde{\phi}(x)$ now transform trivially under a gauge transformation. Using the $v(x)$ fields, which inherit the transformation properties of $\phi(x)$, we can introduce gauge-invariant d.o.f. associated with individual links:

$$\tilde{u}_\mu(x) = v^\dagger(x)u_\mu(x)v(x+\hat{\mu}). \tag{79}$$

When a product of $\tilde{u}_\mu(x)$ variables is taken over a closed loop, the $v(x)$ fields cancel and we obtain:

$$\prod_{(x,\mu)\in C} \tilde{u}_\mu(x) = \prod_{(x,\mu)\in C} u_\mu(x). \tag{80}$$

The operators that are diagonalized by the vectors $|v(x)\rangle$ and $|u(x)\rangle$ generate part of the gauge-invariant algebra through products of the following form:

$$\tilde{U}_\mu(x) = V^\dagger(x)U_\mu(x)V(x+\hat{\mu}), \tag{81}$$

where $V(x)|v(x)\rangle = v(x)|v(x)\rangle$ and $U_\mu(x)|u_\mu(x)\rangle = u_\mu(x)|u_\mu(x)\rangle$. The remaining gauge-invariant operators are $K(x)$ and $L_\mu(x)$, which are conjugates to $V(x)$ and $U_\mu(x)$, respectively. Together with $\tilde{U}_\mu(x)$ and the algebra of the fields $\tilde{\Phi}(x)$, they generate the full gauge-invariant algebra.

As we did in the case of pure gauge theories, we can explicitly see the form of the states in the electric basis by applying products of $\tilde{U}_\mu(x)$ operators to the vacuum $|0\rangle = \sum_{v(x),u_\mu(x)} \bigotimes_x |v(x)\rangle \bigotimes_{x,\mu} |u_\mu(x)\rangle$. Specifically, for a Z_2 gauge group, the gauge-invariant states can be visualized as the set of all superpositions of nonoverlapping arbitrary length strings. The constraint corresponding to gauge invariance dictates that each vertex must be in the $|-\rangle$ state *iff* there is an odd number of $|-\rangle$ links connected to it. This can be seen by looking at how a gauge transformation acts on the state of a vertex and the links connected to it in the electric basis:

$$\begin{aligned} G|k(x_0)\rangle \otimes \bigotimes_{x,\mu} |l_\mu(x)\rangle &= K(x_0)\prod_{x,\mu} L_\mu(x)|k(x_0)\rangle \otimes \bigotimes_{x,\mu} |l_\mu(x)\rangle \\ &= k(x_0)\prod_{x,\mu} l_\mu(x)|k(x_0)\rangle \otimes \bigotimes_{x,\mu} |l_\mu(x)\rangle, \end{aligned} \tag{82}$$

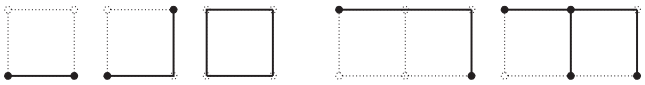


FIG. 11. Example electric states on simple matter and gauge lattices with a Z_2 gauge group. The dotted links/vertices are in the $|+\rangle$ state, while the thick/filled links/vertices are in the $|-\rangle$ state.

where $|k\rangle$ and $|l_i\rangle$ are eigenstates of K and L_i with eigenvalues k and l_i, respectively, and x, μ represent links that have one endpoint at x_0. The requirement that the gauge transformation be the identity for physical states implies that $k\prod_i l_i = 1$ or $k = \prod_i l_i$. Some example gauge-invariant electric states in small two dimensional lattices are shown in Fig. 11.

The unconstrained gauge part of the algebra, $\mathcal{A}_g$, is generated by the $L_\mu(x)$ and $\tilde{U}_\mu(x)$ operators which act on the Hilbert space of gauge invariant link states $\mathcal{H}_g$. To get the full unconstrained gauge-invariant algebra, one adds the algebra of the fields $\tilde{\Phi}$, $\mathcal{A}_{\tilde{\Phi}}$. The choice of d.o.f. on the links and vertices corresponds to $L = 1$, $V = 1$ in Eq. (10). The commutation relations of the operators in $\mathcal{A}_g$ can be inferred from the commutation relations of the non gauge-invariant operators:

$$\begin{aligned} &[L_\mu(x), L_\nu(y)] = 0 \\ &[\tilde{U}_\mu(x), \tilde{U}_\nu(y)] = 0 \\ &[L_\mu(x), \tilde{U}_\nu(y)] = V^\dagger(y)[L_\mu(x), U_\nu(y)]V(y+\hat{\nu}). \end{aligned} \tag{83}$$

The last commutator is zero if $[L_\mu(x), U_\nu(y)] = 0$, which is true if $x \neq y$ or $\mu \neq \nu$. As in the pure gauge case, the electric basis on the full lattice can be built using $\tilde{U}_\mu(x)$ operators and the ground state satisfying $L_\mu(x)|0\rangle = |0\rangle$, $\forall\, x, \mu$:

$$|e_{\boldsymbol{\lambda}}\rangle = \prod_{x,\mu} \tilde{U}_\mu^{\lambda_{x,\mu}}(x)|0\rangle, \tag{84}$$

where $\lambda_{x,\mu} \in \{0, 1\}$. A basis for the full gauge-invariant Hilbert space of the theory would then be formed by tensor products of vectors $|e_{\boldsymbol{\lambda}}\rangle$ and basis vectors in the Hilbert space of the fields $\tilde{\Phi}$, allowing us to write $\mathcal{H}_0 = \mathcal{H}_g \otimes \mathcal{H}_{\tilde{\Phi}} = (\mathcal{H}_{g,A} \otimes \mathcal{H}_{\tilde{\Phi},A}) \otimes (\mathcal{H}_{g,B} \otimes \mathcal{H}_{\tilde{\Phi},B})$.

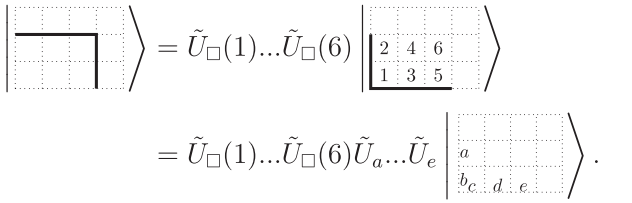
X. SURFACE CHARGES

The case of a theory with surface charges is a special case of coupling to matter fields where matter fields are only defined on the boundaries of a lattice. We allow for both dynamic and nondynamic surface charges, but restrict ourselves to Z_2 gauges and two dimensions for simplicity. We can employ the basis used in the previous section where we decouple the gauge portion from the matter fields. In the electric basis, the gauge-invariant states take the form of loops and strings that open on boundaries. We are concerned with whether strings can be expressed as tensor

products of vectors in bipartitions of plaquettes. We use the diagrammatic representation of states since it provides a more clear picture.

There are two nontrivial situations: bipartitions in which both regions share some of the lattice boundary and bipartitions in which one region is entirely in the bulk of the lattice. In the first case, we seek a tensor product for strings that cross the boundary between regions once, while in the second, we are concerned with open strings that cross the bulk region. The two cases are not exhaustive but illustrate the Hilbert space factorization where it is less obvious.

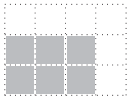
Before proceeding, we note that all open strings along curves C_b with links in the bulk can be created by acting on the vacuum with products of gauge-invariant link operators on the edge $\tilde{U}_i$, $i \in C_e$ to create a closed curve with C_b and then acting on the result with all plaquette operators in the surface enclosed by the curve $C_e \cup C_b$. For example:



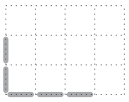
$$= \tilde{U}_\square(1)...\tilde{U}_\square(6)\tilde{U}_a...\tilde{U}_e \left|\; \right\rangle . \tag{85}$$

For more clarity, we can express operators in a diagrammatic form:

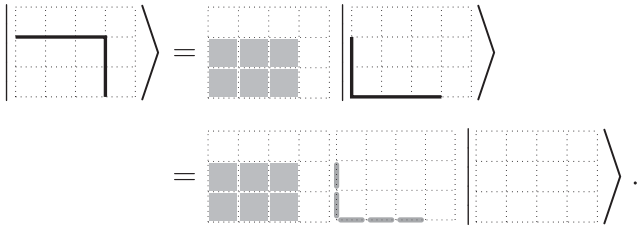
$$\tilde{U}_\square(1)...\tilde{U}_\square(6) = \tag{86}$$



$$\tilde{U}_a...\tilde{U}_e = \tag{87}$$



leading to

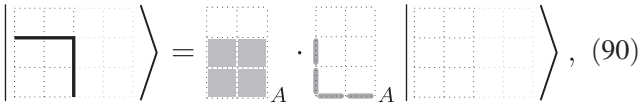


$$= \tag{88}$$

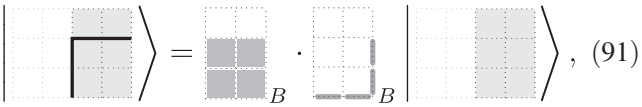
The space of bulk strings exhibits a global symmetry due to the fact that one can close bulk strings using edge strings in two ways. This symmetry is associated with the constraint $\prod_{i\in C_{\text{edge}}}\tilde{U}_i = \prod_x \tilde{U}_\square(x)$, where C_{edge} is the set of links on the edge of the lattice. The constraint can be rewritten as $\prod_{i\in C_e^1}\tilde{U}_i\prod_{x\in V}\tilde{U}_\square(x) = \prod_{i\in C_e^2}\tilde{U}_i\prod_{x\in \bar{V}}\tilde{U}_\square(x)$, with $C_e^1 \cup C_e^2 = C_{\text{edge}}$. Symmetries prevent us from removing this constraint. The tensor product structure arises naturally from the bulk plaquette algebra and the edge gauge-invariant link algebra resulting in identities of the following form:

(89)

with

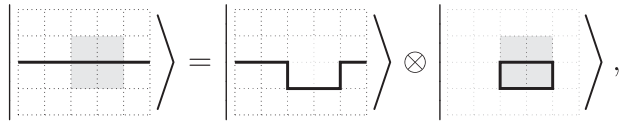


(90)



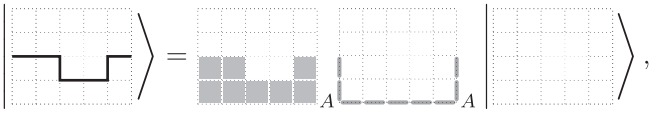
(91)

and

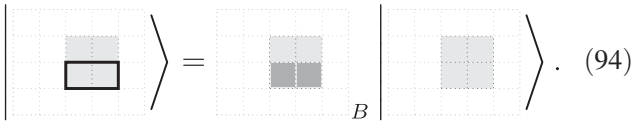


(92)

with



(93)



(94)

XI. CONCLUDING REMARKS

The use of the Z_2 group throughout the paper was motivated by its simplicity and the fact that it enables a useful diagrammatic notation for states. We expect the extension to other Abelian groups to be relatively

straightforward. In a $U(1)$ theory, in the link basis $|e^{i\phi}\rangle$ states take the form of periodic functions, whereas electric states are discrete (see e.g., [18]) and represented by the Fourier modes of the link basis states. The gauge-invariant algebra is generated by loop operators $U_{\square}(x), U^{*}_{\square}(x)$, electric operators $L = -i\partial/\partial_{\phi}$ with electric eigenstates $|q\rangle = \int e^{iq\phi}|e^{i\phi}\rangle$ such that $L|q\rangle = q|q\rangle$, and link rotation operators $L^{\theta}|e^{i\phi}\rangle = |e^{i(\phi+\theta)}\rangle$ such that $L^{\theta}|q\rangle = e^{i\theta q}|q\rangle$. The requirement of gauge invariance at a vertex in the electric basis then reads:

$$\begin{aligned}|q_1 q_2 q_3 q_4\rangle &= G|q_1 q_2 q_3 q_4\rangle \\ &= L_1^{\theta} L_2^{\theta} L_3^{-\theta} L_4^{-\theta}|q_1 q_2 q_3 q_4\rangle \\ &= e^{i\theta(q_1+q_2-q_3-q_4)}|q_1 q_2 q_3 q_4\rangle, \end{aligned} \tag{95}$$

which must be satisfied for all θ, implying $q_1 + q_2 - q_3 - q_4 = 0$. In other words, the electric fluxes are conserved at vertices. Similar to the Z_2 theory, such states can be created by acting on the vacuum with no electric fluxes with operators $U_{\square}(x)$ which raise the electric flux around a plaquette by one and $U^{*}_{\square}(x)$ which lower it.

The case of $3+1$ dimensional theories with periodic boundary conditions is absent. It is a straightforward extension of Secs. VII and VIII. Similarly, the Wilsonian $3+1$ dimensional theory was left out.

The analysis performed in this paper is only valid for discretized spaces. As shown in [19], the entanglement entropy is UV divergent. Furthermore, as Witten argues in [20], Hilbert spaces supported on geometries dense in some connected space may not be separable precisely because the UV divergence of the entanglement entropy is a universal feature not tied to a particular state.

ACKNOWLEDGMENTS

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