


A harmonic solution for two-dimensional adjoint QCD

Uwe Trittmann

Department of Physics, Otterbein University, Westerville, Ohio 43081, USA

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Two-dimensional QCD with adjoint fermions has many attractive features, yet its single-particle content remains largely unknown. To lay the foundation for a crucially improved approximation of the theory's spectrum, we developed a method to find the basis of eigenstates using the symmetry structure of the asymptotic theory where pair production is disallowed. This method produces complete sets of multidimensional harmonic functions for the massless and the massive theory. Previously only part of such a basis was known. The method presented here should be applicable to other theories and has the promise of factoring out the long-range Coulomb-type part of interactions. The role of pair production and implications for the bosonized theory in the case of adjoint QCD₂ are discussed.

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I. INTRODUCTION

Two-dimensional adjoint quantum chromodynamics, QCD_{2A}, is a non-Abelian Yang-Mills theory coupled to fermions in the adjoint representation, and based on the Lagrangian

$$\mathcal{L} = \text{Tr} \left[-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi} \gamma_\mu D^\mu \Psi \right], \quad (1)$$

where $\Psi = 2^{-1/4} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$, with ψ and χ being $N \times N$ matrices. The field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, and the covariant derivative is defined as $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. Throughout the paper, light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ are used, where x^+ plays the role of a time. We will work in the light-cone gauge, $A^+ = 0$, effectively omitting fermionic zero modes.

The theory is discussed extensively in the literature [1–6]. Why is QCD_{2A} interesting? For starters, it is the simplest gauge theory apart from 't Hooft's model [7] that exhibits confinement. Additionally, QCD_{2A} shows several features of higher-dimensional gauge theories. Its adjoint degrees of freedom (d.o.f.) mimic the *transverse* gluons which are dynamical obviously only in two or more spatial dimensions. Also, QCD_{2A} exhibits a phase transition from confinement to screening in the limit of vanishing mass m of the adjoint fermions. This mass furthermore parameterizes the supersymmetric behavior the theory exhibits at

$m^2 = g^2 N$. The theory has an exponentially increasing density of bound states according to [1,3]. Since the density behaves like $\rho(M) \sim e^{M/T_H}$ for large bound state masses M , one expects a deconfinement transition at the Hagedorn temperature T_H . Also, this transition depends crucially on the fermion mass. In particular, the Hagedorn temperature goes to zero in the limit $m \rightarrow 0$, driven by the vanishing string tension at the deconfinement transition. There are several unresolved issues, for instance the existence of Regge-trajectories of single-particle (bound) states. Few single-particle states have been identified to date [3,4,8], which has led to speculations that there is no Regge trajectory in massless QCD_{2A} [6]. This is understandable in a screening theory in which long strings should fall apart, but leaves one to explain what happens to the many *single-trace* states. It seems that most of them have to be interpreted as *multiparticle* states [3,8]. But since some of them can be projected out by bosonization [9] and the rest is close to threshold in a formulation of the theory with a *finite* discretization parameter, this must be considered an unsettled question. On the other hand, the theory is the simplest available¹ exhibiting these interesting features, so the hope is to learn something about the mechanisms underlying more realistic gauge theories. Indeed, there recently has been a renewed interest in three- and four-dimensional versions of the theory [10,11].

While the ultimate goal is a solution of the full theory, our aim here is much more modest. As a foundation for future work, we construct in Sec. II a complete set of eigenstates for the *asymptotic* theory based on symmetries which are broken in the full theory, thereby expanding work begun in [12]. The basis is a set of linear combinations of multidimensional harmonic functions subject to

¹It is a low-dimensional, manifestly finite theory.

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the symmetry constraints furnished by the light-cone Hamiltonian derived from Eq. (1). Since the asymptotic theory has disjoint sectors with different parton numbers, the basis states can be organized by parton number r , which results in $r - 1$ excitation numbers responsible for the multidimensionality of the solution. Technically, the basis will be appropriate only in the high excitation number limit. Empirically, we find that even the lowest states are well represented, probably due to their large separation in (bound state) mass. In the full theory, particle pair production couples the disjoint parton sectors, and eigenfunctions of the full theory will be linear combinations of basis states of different parton numbers. Thus, the harmonic basis states can be used to discuss the role of particle creation and annihilation, as envisioned in [1]. This is done in Sec. III B. As applications of our method we briefly describe a program to solve the full theory with numerical methods in Sec. III A, and point out implications for the bosonized version of the theory in Sec. III C. Finally, we discuss the results and general applicability of the method in Sec. IV.

II. CONSTRUCTING A HARMONIC BASIS

A. Introductory remarks

Starting from the QCD_{2A} Lagrangian, Eq. (1), the dynamics of a system of adjoint fermions interacting via a nondynamical gluon field in two dimensions can be described by a light-cone momentum operator P^+ and Hamiltonian operator P^- . The two operators are expressed in terms of fermionic operators subject to the anticommutation relation

$$\{b_{ij}(k^+), b_{lk}^\dagger(p^+)\} = \delta(k^+ - p^+) \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right). \quad (2)$$

To wit

$$P^+ = \int_0^\infty dk k b_{ij}^\dagger(k) b_{ij}(k), \quad (3)$$

$$\begin{aligned} P^- = & \frac{m^2}{2} \int_0^\infty \frac{dk}{k} b_{ij}^\dagger(k) b_{ij}(k) + \frac{g^2 N}{\pi} \int_0^\infty \frac{dk}{k} C(k) b_{ij}^\dagger(k) b_{ij}(k) \\ & + \frac{g^2}{2\pi} \int_0^\infty dk_1 dk_2 dk_3 dk_4 \left\{ B(k_i) \delta(k_1 + k_2 + k_3 - k_4) \right. \\ & \times (b_{kj}^\dagger(k_4) b_{kl}(k_1) b_{li}(k_2) b_{ij}(k_3) - b_{kj}^\dagger(k_1) b_{jl}^\dagger(k_2) b_{li}^\dagger(k_3) b_{ki}(k_4)) \\ & + A(k_i) \delta(k_1 + k_2 - k_3 - k_4) b_{kj}^\dagger(k_3) b_{ji}^\dagger(k_4) b_{kl}(k_1) b_{li}(k_2) \\ & \left. + \frac{1}{2} D(k_i) \delta(k_1 + k_2 - k_3 - k_4) b_{ij}^\dagger(k_3) b_{kl}^\dagger(k_4) b_{li}(k_1) b_{kj}(k_2) \right\}, \quad (4) \end{aligned}$$

with

$$A(k_i) = \frac{1}{(k_4 - k_2)^2} - \frac{1}{(k_1 + k_2)^2}, \quad (5)$$

$$B(k_i) = \frac{1}{(k_2 + k_3)^2} - \frac{1}{(k_1 + k_2)^2}, \quad (6)$$

$$C(k) = \int_0^k dp \frac{k}{(p - k)^2}, \quad (7)$$

$$D(k_i) = \frac{1}{(k_1 - k_4)^2} - \frac{1}{(k_2 - k_4)^2}, \quad (8)$$

where the trace-splitting term $D(k_i)$ can be omitted at large N_c , and the trace-joining term is proportional to $B(k_i)$. The structure of the Hamiltonian P^- displayed in Eq. (4) is

$$P^- = P_m^- + P_{\text{ren}}^- + P_{PC,s}^- + P_{PC,ns}^- + P_{PV}^- + P_{\text{finite } N}^-. \quad (9)$$

The mass term P_m^- is dropped in the massless theory, but the renormalization operator P_{ren}^- needs to be included. Parton-number violating terms, P_{PV}^- , couple blocks of different parton number. Parton-number conserving interactions P_{PC}^- are block diagonal, and may include singular(s) or non-singular(ns) functions of the parton momenta. The finite N term $P_{\text{finite } N}^-$ is proportional to $D(k_i)$. For details see [2,3,12].

The problem is cast into an eigenvalue equation

$$2P^+ P^- |\Psi\rangle \equiv H_{\text{LC}} |\Psi\rangle = M^2 |\Phi\rangle. \quad (10)$$

Namely, the light-cone Hamiltonian H_{LC} acts on an eigenket $|\Psi\rangle$ yielding the mass (squared) of a bound state as the eigenvalue. The eigenkets are in general linear combinations of states of definite parton(fermion) number r

$$|\Phi_r\rangle = \int_0^1 dx_1 \left(\prod_{i=2}^{r-1} \int_{x_1}^{1-(r-1)x_1 - \sum_{j=2}^{i-1} x_j} dx_i \right) \frac{\phi_r(x_1, x_2, \dots, x_r)}{N_c^{r/2}} \text{Tr}[b(-x_1) \cdots b(-x_r)] |0\rangle. \quad (11)$$

The wave functions ϕ_r distribute momentum between the partons. Note that there are only $r - 1$ integrations since total momentum can be set to unity. The integration is over the effective Hilbert space, which looks complicated due to elimination of redundant operator combinations like $\text{Tr}[b(x)b(y)b(z)] = \text{Tr}[b(y)b(z)b(x)]$. The explicit shape of the integration domain is displayed to emphasize that for $r > 3$ the wave function cannot be reconstructed² in the region around the middle of the naive Hilbert space $[0, 1]^{r-1}$.

In the asymptotic limit, where only highly excited states are considered, both parton-violation and mass terms can be neglected. As a consequence the asymptotic theory splits into decoupled sectors with fixed parton numbers subject to 't Hooft-like equations

$$\frac{M^2}{g^2 N} \phi_r(x_1, \dots, x_r) = - \sum_{i=1}^r (-1)^{(r+1)(i+1)} \int_{-\infty}^{\infty} \frac{\phi_r(y, x_i + x_{i+1} - y, x_{i+2}, \dots, x_{i+r-1})}{(x_i - y)^2} dy. \quad (12)$$

The total momentum is set to unity, and thus the momentum fractions x_i add up to one, $\sum_i x_i = 1$. Clearly, the number of partons r is even (odd) for bosonic (fermionic) states. In [1,12], additionally the approximation

$$\int_0^1 \frac{dy}{(x-y)^2} \phi(y) \approx \int_{-\infty}^{\infty} \frac{dy}{(x-y)^2} \phi(y) \quad (13)$$

was used. Mathematically this is helpful because the solutions of the eigenvalue problem then are harmonic functions, see Eq. (15). The approximation makes sense physically, because for the highly excited states the integral is dominated by the interval around $x = y$ which is associated with the long-range Coulomb-type force.

In [12], we showed that the integral equation (12) can be solved algebraically using the ansatz

$$|n_1, n_2, \dots, n_{r-1}\rangle \doteq \prod_j^{r-1} e^{i\pi n_j x_j} = \phi_r(x_1, x_2, \dots, x_r), \quad (14)$$

where $x_r = 1 - \sum_j^{r-1} x_j$. The ansatz is motivated by its simplest ($r = 2$) version, which solves the 't Hooft equation of *fundamental* QCD₂ [7]

$$\frac{M^2}{g^2 N} e^{i\pi n x} = - \int_{-\infty}^{\infty} \frac{dy}{(x-y)^2} e^{i\pi n y} = \pi |n| e^{i\pi n x}, \quad (15)$$

where the excitation number n is integer. We thus use the single-particle states of a Hamiltonian appropriate for the problem to construct a Fock basis, inspired by [13].

Since the integral equation (12) is more involved than the 't Hooft equation, we have to symmetrize the ansatz (14) to comply with the constraints inherent in the Hamiltonian (10). Namely, the solutions of the adjoint 't Hooft problem have to be (pseudo)cyclic,

$$\phi_r(x_1, x_2, \dots, x_r) = (-1)^{r+1} \phi_r(x_2, x_3, \dots, x_r, x_1), \quad (16)$$

since the fermions are real. To implement this constraint we introduce the cyclic permutation operator

$$\mathcal{C}: (x_1, x_2, \dots, x_r) \rightarrow (x_2, x_3, \dots, x_r, x_1).$$

Since the Hamiltonian is unchanged by a color index reversal of its operators, the string or trace of fermionic operators in the states (11) can be reversed at will. The solutions can therefore be organized into sectors of definite parity under the orientation symmetry,

$$\mathcal{T}: b_{ij} \rightarrow b_{ji}. \quad (17)$$

The two symmetry operators act on the ansatz, Eq. (14), as follows.

$$\begin{aligned} \mathcal{C}: |n_1, n_2, \dots, n_{r-1}\rangle &\rightarrow (-1)^{n_{r-1}} | -n_{r-1}, n_1 - n_{r-1}, n_2 - n_{r-1}, \dots, n_{r-2} - n_{r-1} \rangle, \\ \mathcal{T}: |n_1, n_2, \dots, n_{r-1}\rangle &\rightarrow (-1)^{n_1} | -n_1, n_{r-1} - n_1, n_{r-2} - n_1, \dots, n_2 - n_1 \rangle. \end{aligned} \quad (18)$$

In [12], we constructed eigenfunctions in the two- and three-parton sectors ($r = 2, 3$) for both the massive and the massless theory, with bound-state masses

$$M^2 = g^2 N \pi \left(|n_1| + |n_{r-1}| + \sum_{k=1}^{r-2} |n_k - n_{k+1}| \right). \quad (19)$$

²Meaning we cannot determine the wave function in this region from symmetries and its values close to the domain boundaries. For $r = 2$ we can: knowing $\phi_2(x)$ in $[0, \frac{1}{2}]$ and $\phi_2(x) = -\phi_2(1-x)$ is obviously enough.

The crucial ingredient of the method is the symmetrization due to \mathcal{C} ,

$$|n_1, n_2, \dots, n_{r-1}\rangle_{\text{sym}} \equiv \frac{1}{\sqrt{r}} \sum_{k=1}^r (-1)^{(r-1)(k-1)} \mathcal{C}^{k-1} |n_1, n_2, \dots, n_{r-1}\rangle, \quad (20)$$

where $\mathcal{C}^0 = 1$, because only the symmetrized states reflect the (pseudo)cyclic structure of the Hamiltonian.

While the $r = 2, 3$ wave functions reproduce known (DLCQ) solutions of the theory remarkably well down to the lowest states, the ansatz fails to work for $r > 3$ in general. In [12], we were able to express a subset of the $r = 4$ solutions as a linear combination of the symmetrized four-parton states. These solutions coincide with the ones derived earlier [1]. To introduce notation and show the complexity of the problem, we display a five-parton state symmetrized under \mathcal{C} and \mathcal{T} , i.e., of definite C and T quantum numbers.³ Note that the state consists of $4r = 20$ *statelets*, characterized by an $(r-1)$ -tuple of (ordered) excitation numbers n_i [here: (n, m, l, k)]

$$\begin{aligned} |\phi_5, n, m, l, k; \bar{M}^2 = |n| + |n-m| + |m-l| + |l-k| + |k|\rangle_T \\ = |n, m, l, k\rangle + (-1)^k |k, n-k, m-k, l-k\rangle + (-1)^l |k-l, -l, n-l, m-l\rangle \\ + (-1)^m |l-m, k-m, -m, n-m\rangle + (-1)^n |m-n, l-n, k-n, -n\rangle \\ + T[(-1)^n |n, k-n, l-n, m-n\rangle + |k, l, m, n\rangle + (-1)^k |l-k, m-k, n-k, -k\rangle \\ + (-1)^l |m-l, n-l, -l, k-l\rangle + (-1)^m |n-m, -m, k-m, l-m\rangle]. \end{aligned} \quad (21)$$

Typically, these states are paired with a partner state of negative excitation numbers to create a real wave function, i.e., a sine or cosine. It is convenient to do so with an additional symmetry in mind, which we'll introduce in the next section.

B. Exhaustive symmetrization

As is well known, the eigenfunctions of the massive theory have to vanish when one parton momentum is zero to guarantee Hermiticity of the Hamiltonian [7]

$$\phi_n(0, x_2, \dots, x_n) = 0. \quad (22)$$

This is sometimes called a boundary condition because it behaves as one, which is misleading—after all, we are trying to solve an integral and not a differential equation. In the massive two-parton sector (or the simpler fundamental theory QCD_{2f} [7]), this leads to sine eigenfunctions, and to cosine eigenfunctions in the massless theory. Clearly, the general verdict is that the Hilbert space of the theory splits into two disjoint sets of functions: one even and the other one odd. It seems that the consequences of this straightforward observation have not been fully realized. This is not surprising, since the answer to the simple question—under which transformation or symmetry the eigenfunctions are odd or even—is trivial for few partons, and fairly complicated for many. Indeed, the symmetry in question is

³While T is an ordinary quantum number reflecting a symmetry of the Hamiltonian, C (or rather the set of C_i) is fixed by the constraint that the state has an (anti)cyclic wave function, Eq. (16), as required by the structure of the Hamiltonian as a sum over permutations of the parton momenta.

manifest only in the parton sectors with $r \geq 4$, because it is redundant with \mathcal{C} and \mathcal{T} otherwise. To get a handle on it, note that while physically it is true that we need to manage the behavior of the wave functions at the boundaries of the domain of integration, mathematically *boundary conditions* are not the right tool. Rather, we have to implement *symmetries* that will result in the desired behavior of the wave functions at the boundary—just like the \mathcal{C} symmetrization assures that the state so constructed is an eigenstate of the Hamiltonian. We alluded to such a symmetry in [12], but the general method is more involved.

For massive fermions, we need the wave functions to vanish on the hyperplanes characterized by $x_i = 0$ for at least one x_i . Since our wave function ansatz (14) is modular, we can simply add for every term of the form $e^{i\pi \sum_j n_j x_j}$ another term with opposite sign that is the same for $x_i = 0$ but different⁴ for $x_i \neq 0$. This idea can be realized by introducing $r-1$ symmetry operators \mathcal{S}_i which we might call lower-dimensional inversions, because they invert all but one of the excitation numbers

$$\begin{aligned} \mathcal{S}_i: |n_1, n_2, \dots, n_i, \dots, n_{r-1}\rangle \\ \rightarrow | -n_1, -n_2, \dots, n_i - n_{i+1} - n_{i-1}(1 - \delta_{1i}), \dots, -n_{r-1}\rangle. \end{aligned} \quad (23)$$

The replacement of the i th excitation number is such that the mass (squared) of the state remains invariant, see Eq. (19). This symmetry is hidden in the three-parton sector $r = 3$ because the low-dimensional inversion can be

⁴Because the wave function would otherwise be identically zero, of course.

TABLE I. Characteristics of the lowest states in the first few parton sectors of the asymptotic theory including their quantum numbers TIS . The sectors are labeled with a subscript indicating behavior under \mathcal{T} (\pm) and superscripts signifying massless (0) and massive fermions ($^\mu$). For an explanation of the discrepancy between the T quantum number and the \mathcal{T} sector, see Appendix B.

r	TIS	Sector	Excitation numbers of lowest states	Masses ($g^2 N \pi$)
2	$-+$	$ o\rangle_+^0$	(1),(3),(5),(7)	1,3,5,7
	$--$	$ e\rangle_+^\mu$	(2),(4),(6),(8)	2,4,6,8
3	$++$	$ ee\rangle_-^0$	(0,0),(2,2),(4,2),(4,0),(6,2)	0,4,8,8,12
	$--$	$ ee\rangle_+^0$	(4,2),(6,2),(8,2),(8,4)	8,12,16,16
	$+-$	$ ee\rangle_-^\mu$	(2,0),(4,0),(6,2),(6,0)	4,8,12,12
	$-+$	$ ee\rangle_+^\mu$	(6,2),(8,2),(10,4),(10,2)	12,16,20,20
4	$+++$	$ ooo\rangle_+^0$	(1,2,3), (3,-2,-1), (3,-2,-3), (5,-2,-1)	6,10,12,14
	$+++$	$ ooo\rangle_-^0$	(1,0,1), (1,-2,-1), (3,0,1), (2,-2,-1)	4,6,8,10
	$+++$	$ eee\rangle_+^\mu$	(4,-2,0), (6,-2,0), (6,10,10), (8,10,6), (8,10,10)	12,16,20,20,20
	$++-$	$ eee\rangle_-^\mu$	(4,0,2), (4,-2,0), (6,4,6), (6,-2,0)	12,12,16,16
5	$++++$	$ eeee\rangle_+^0$	(0,0,0,0),(2,2,2,2),(2,4,4,4),(4,4,4,2),(4,4,4,4)	0,4,8,8,8
	$----$	$ eeee\rangle_-^0$	(4,4,4,2),(4,6,6,6),(4,6,4,2),(6,6,4,2),(4,8,8,8)	8,12,12,12,16
	$++-$	$ eeee\rangle_+^\mu$	(4,6,6,4),(6,8,8,6),(6,10,10,6),(8,10,10,6)	12,16,20,20
	$---+$	$ eeee\rangle_-^\mu$	(8,10,10,6),(8,12,10,6),(8,14,12,8)	20,24,28
6	$++++$	$ oooo\rangle_+^0$	(1,2,3,2,1),(1,2,3,4,3),(5,4,3,2,1),(3,4,5,4,3)	6,8,10,10
	$++--$	$ oooo\rangle_-^0$	(...)	
	$++-$	$ eeee\rangle_-^\mu$		
	$---+$	$ eeee\rangle_+^\mu$		

expressed in terms of the other symmetry operations⁵ ($\mathcal{S} = \mathcal{T}\mathcal{C}^2$, see Eq. (26) of [12]), so that $r = 4$ is the lowest parton sector in which the full symmetry unfolds.

In Sec. IID, we will see that additional \mathcal{S} operators creep in, so let's call the set of lower-dimensional inversions $\mathcal{E} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{N(r)}\}$. For a given parton number r , the maximal number $N(r)$ of independent \mathcal{S} operators can be determined from a group theoretical argument. In practice, the argument boils down to combinatorics: the number of operators of the permutation group of order r with inversion is $2r!$. We derive the full symmetry group \mathcal{G} in Sec. IID and use only its generic properties here. The group choice is intuitive as we will see, since in the r parton sector, we are in essence permuting r objects. That the objects (the partons) carry different momenta is irrelevant here. We are thus symmetrizing the wave functions under the set of $N(r)$ \mathcal{S} -symmetries. Later we will see that there is only one common multiplicative Z_2 quantum number S . Tentatively, we write

$$|r\rangle_{\mathcal{S}\text{sym}} = \left(1 + \sum_i^{N(r)} \mathcal{S}_i \mathcal{S}_{i=1}\right) |r\rangle_{\text{sym}},$$

where $|r\rangle_{\text{sym}}$ is the ansatz state $|n_1, n_2, \dots, n_{r-1}\rangle$ symmetrized under the cyclic group $\langle \mathcal{C} \rangle$, Eq. (20).

To include all symmetries in our approach, it is convenient to make the inversion of excitation numbers explicit with the operator

$$\mathcal{I}: |n_1, n_2, \dots, n_{r-1}\rangle \rightarrow |-n_1, -n_2, \dots, -n_{r-1}\rangle. \quad (24)$$

⁵At $r = 2$ the situation is fully degenerate with $\mathcal{C} = \mathcal{T} = \mathcal{S}$ up to signs.

Clearly, \mathcal{I} is a Z_2 operator, and states even and odd under \mathcal{I} simply represent cosine and sine wave functions, respectively. We can then write down an orthonormal set of basis states in all sectors of the theory, characterized by their Z_2 quantum numbers (T, I, S) under the symmetry transformations \mathcal{T} , \mathcal{I} , and \mathcal{S} , and their excitation numbers n_i collected in $|r\rangle$

$$|r\rangle_{\text{FullSym}} \equiv \mathcal{G}|r\rangle = \left\{ \frac{(1+TT)(1+II)}{\sqrt{2r!}\mathcal{N}} \times \sum_{k=0}^{r-1} (-)^{(r-1)k} \mathcal{C}^k \left(1 + \sum_i^{N(r)} \mathcal{S}_i \mathcal{S}_{i=1}\right) |r\rangle \right\}, \quad (25)$$

where \mathcal{N} is the volume of the Hilbert space in the r parton sector,

$$\mathcal{N} = \int_0^1 dx_1 \left(\prod_{i=2}^{r-1} \int_{x_1}^{1-(r-1)x_1 - \sum_{j=2}^{i-1} x_j} dx_i \right) = \frac{1}{r!}, \quad (26)$$

cf. Appendix of Ref. [12]. The excitation numbers can, in general, be even or odd integers. Since the states, Eq. (25), are by construction eigenstates of the light-cone Hamiltonian with masses given by Eq. (19), they furnish, in principle, a full solution⁶ of asymptotic QCD_{2A}—keeping in mind that we made the approximation, Eq. (13).

⁶This construction should also settle the issue raised by 't Hooft in the footnote of his seminal paper [7]: since the “boundary conditions” are really symmetrizations, they must hold order by order in N_c .

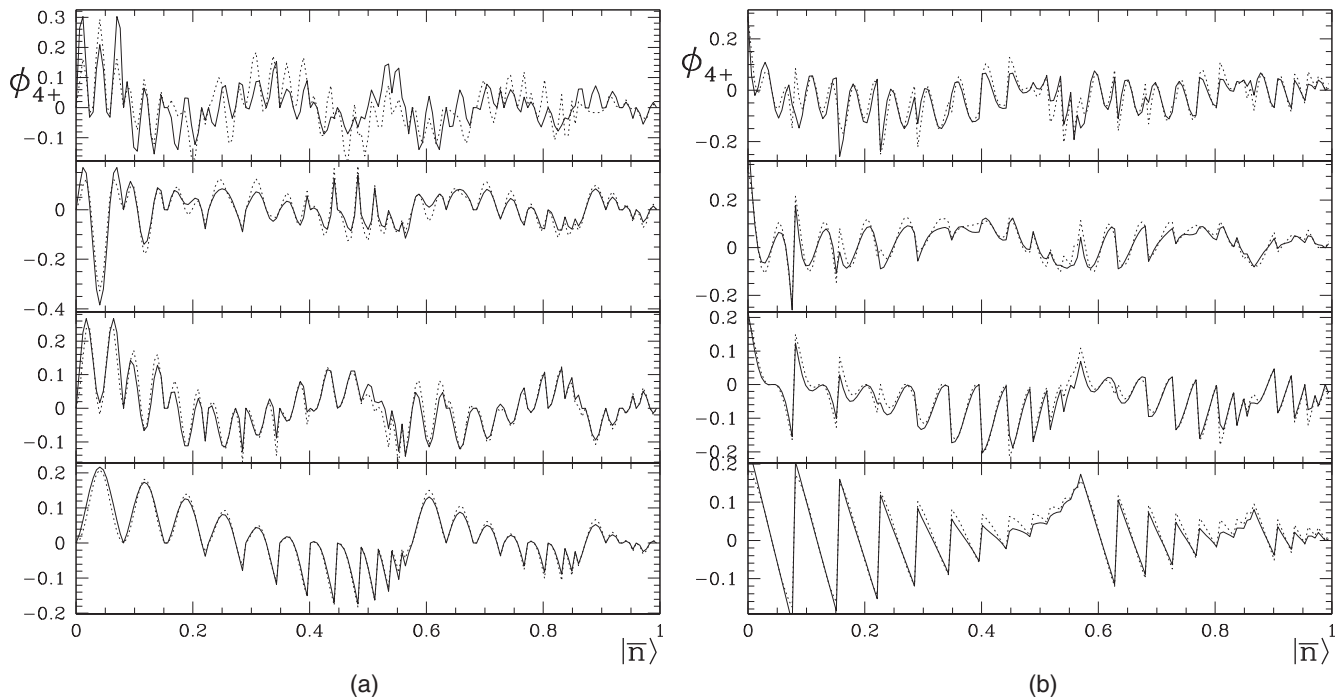


FIG. 1. Lowest DLCQ (solid lines, $K = 32$) and asymptotic eigenfunctions (dashed lines) in the *massless* four-parton sectors, i.e., $|oeo\rangle$ states with $\mu \equiv m^2/g^2N = 0$, plotted as a function of the normalized basis state number \bar{n} . (a) $T+$, hence $I - S-$. (b) $T-$, hence $I + S+$. We used a rather low harmonic resolution K to increase readability, since we do not need to worry about convergence here.

C. Classification of states

In practice, most sectors prescribed by Eq. (25) are empty. As can be gleaned from the symmetry operations, Eqs. (18), (23), and (24), only two sets of quantum numbers lead to viable states. Namely, only states with all even excitation numbers, and states with alternating odd and even numbers give rise to *bona fide* states. In the latter case, the first and last excitation numbers must be odd, so that these states only exist in the even parton sectors. Even so, most of these excitation number combinations are not viable due to cancellations of terms. For example, *bona fide* states are only found in the $(T\pm, I\mp)$ four-parton sectors and in the $(T\pm, I\pm)$ five-parton sectors, see Table I. In the odd parton sectors, these four sectors (for both $S\pm$) are indeed the four sectors necessary to describe the two \mathcal{T} -sectors of the massive and the massless theory, respectively. For an even number of partons, the addition of the mixed excitation number states ($|\text{odd, even, odd}\rangle \equiv |oeo\rangle$ for $r = 4$) does, of course, not lead to more sectors. Rather, two even excitation number sectors do not yield *bona fide* states. For an explanation, see Appendix A. For instance, in the four parton sector, only $(T\pm, I\mp, S\pm)$ in the even excitation number sector and $(T\pm, I\mp, S\mp)$ with mixed even and odd n_i are viable; the former are the two sectors of the massive theory, and the latter represent the two massless sectors. This is a straightforward generalization from the earlier findings [12] that in the two-parton sectors cosines with odd excitation number represent the

massless and sines with even excitation number the massive theory, whereas all excitation numbers in the three-parton sector are even.

To check our musings, we make contact with known results. Our solution (25) reproduces the results of Ref. [1] Eq. (4.13) and classifies them as $|eee; T - I + S-\rangle$ states. The wave functions (4.13) of [1] exhibit only two (not three) excitation numbers, because they represent a subset of the full set of wave functions. We thus find that there are more states than anticipated, and that the counting of states is more involved; it does not seem to lend itself to a string-motivated parametrization. Note that the solution (4.13) in [1] is much more compact and looks differently (double sines versus triple cosines), but this is a superficial disagreement and the price one has to pay for generality: 12 terms⁷ of (4.13) in [1] vs. 48 terms in Eq. (25) at $r = 4$. Next we check how accurate our algebraic solution is by comparing to a numerical (DLCQ) calculation, see Figs. 1 and 2. Some masses are degenerate at $r = 4$ as opposed to $r < 4$, and the agreement is not as good as in the three-parton sector [12], due to considerable mixing of states of equal mass. Recall that the algebraic eigensolutions were derived by making the approximation Eq. (13); using the correct limits of the integral evidently induces interactions between the algebraic basis states. As can be gleaned from Figs. 1 and 2, the discrepancy between algebraic and

⁷Obviously, a sine has *two* exponential terms.

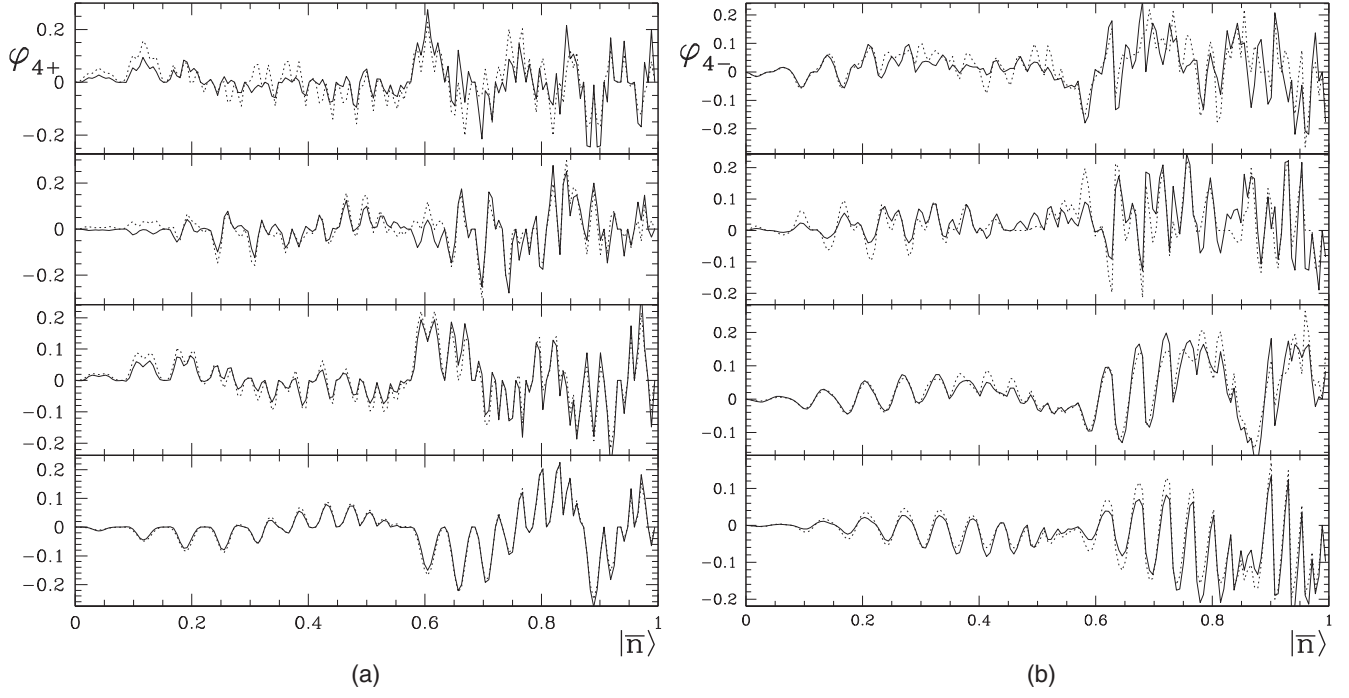


FIG. 2. Lowest eigenfunctions of the massive theory in the four-parton sectors. Solid lines represent numerical results as emulated in DLCQ by letting $\mu = 4$, dashed lines the algebraic eigenfunctions. Note that for $\mu = 1$, the theory is supersymmetric. (a) \mathcal{T} even eigenfunctions. (b) \mathcal{T} odd eigenfunctions.

numerical solutions is noticeable even in the massless sector, where the masses are *not* degenerate. This heralds the worsening of our approximation at large parton number r . Namely, the effective volume of the Hilbert space decreases as $1/r!$, see Eq. (26). As a consequence, the integral looks less and less like $(\int_{-\infty}^{\infty} dx)^{r-1}$, although the core motivation for the approximation (that the region around the singularity is the most important) remains valid.

What do the exact wave functions tell us? First off, the massless and massive sectors show significant differences.

The massive sector is more straightforward, perhaps due to the more stringent constraint of wave functions vanishing on the $x_i = 0$ hyperplanes. Even the lowest states exhibit degenerate masses in the massive, but not the massless theory. Also, the bound-state masses are much lower in the massless sector. This is more surprising than it sounds, since we are omitting the mass term in the asymptotic limit. The difference in bound-state mass is thus generated by symmetry alone. The lowest four states are in the massive theory ($\mu \neq 0$)

$$\begin{aligned}
 |1\rangle_{+-+}^{\mu \neq 0} &= |4, -2, 0\rangle_{12}, & |1\rangle_{-+-}^{\mu \neq 0} &= |4, 0, 2\rangle_{12}, \\
 |2\rangle_{+++}^{\mu \neq 0} &= |6, -2, 0\rangle_{16}, & |2\rangle_{-+-}^{\mu \neq 0} &= |4, -2, 0\rangle_{12}, \\
 |3\rangle_{+++}^{\mu \neq 0} &= \frac{1}{\sqrt{2}}(|6, 10, 10\rangle_{20} + |8, 10, 6\rangle_{20}), & |3\rangle_{-+-}^{\mu \neq 0} &= |6, 4, 6\rangle_{16}, \\
 |4\rangle_{+++}^{\mu \neq 0} &= |8, 10, 10\rangle_{20}, & |4\rangle_{-+-}^{\mu \neq 0} &= |6, -2, 0\rangle_{16}.
 \end{aligned} \tag{27}$$

In the massless theory, they look like

$$\begin{aligned}
 |1\rangle_{+--}^{\mu=0} &= |1, 2, 3\rangle_6, & |1\rangle_{-++}^{\mu=0} &= |1, 0, 1\rangle_4, \\
 |2\rangle_{+--}^{\mu=0} &= |3, -2, -1\rangle_{10}, & |2\rangle_{-++}^{\mu=0} &= |1, -2, -1\rangle_6, \\
 |3\rangle_{+--}^{\mu=0} &= |3, -2, -3\rangle_{12}, & |3\rangle_{-++}^{\mu=0} &= |3, 0, 1\rangle_8, \\
 |4\rangle_{+--}^{\mu=0} &= |5, -2, -1\rangle_{14}, & |4\rangle_{-++}^{\mu=0} &= |3, -2, -1\rangle_{10},
 \end{aligned} \tag{28}$$

where the indices represent the TIS quantum numbers on the right-hand states, and the mass (squared in units $g^2 N\pi$) on the left-hand states. Note that the excitation numbers n_i are not unique. For instance, at $r = 4$ each state has up to $2r! = 48$ different tuples. One way of classifying the state is to pick one statelet's numbers to represent the entire state, e.g., by choosing the lowest positive number for n_i followed by the smallest absolutes $|n_i|$ as in Eq. (28).

D. Some group theory

The solutions (25) are right cosets of the subgroup \mathcal{B} of transformations associated with the full domain of integration (the ‘‘bulk’’). Namely, \mathcal{B} is the direct product of inversions \mathcal{I} , reorientations \mathcal{T} and cyclic permutations \mathcal{C}

$$\mathcal{B} = \{1, \mathcal{C}, \mathcal{C}^2, \dots, \mathcal{C}^{r-1}, \mathcal{T}, \mathcal{TC}, \dots, \mathcal{TC}^{r-1}, \mathcal{I}, \mathcal{IC}, \dots, \mathcal{ITC}^{r-1}\}. \quad (29)$$

We can construct a partition of the full group \mathcal{G} of symmetry transformations of QCD_{2A} by acting on all elements of \mathcal{B} with the lower-dimensional inversions \mathcal{S} concerning symmetrization on the hyperplanes forming the boundary of the integration domain collected in the set⁸ \mathcal{E} alluded to in Sec. II B

$$\mathcal{E} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{1/2(r-1)!-1}\}. \quad (30)$$

In general, we find $N(r) = \frac{1}{2}(r-1)! - 1$ independent lower-dimensional inversions. We work with the right cosets here, because we want to make explicit the symmetrization of the state under the cyclic subgroup $\langle \mathcal{C} \rangle$, as required by the structure of the Hamiltonian, as implicitly defined in Eq. (12). Since the order of \mathcal{B} is $|\mathcal{B}| = 2_{\mathcal{T}} \times 2_{\mathcal{I}} \times r_{\mathcal{C}}$ and its operators act on all elements of \mathcal{E} plus the identity, a fully symmetrized state contains $2r!$ ‘‘statelets’’ owing to the $2r!$ independent automorphisms that can be formed on a set of r objects (here: momentum fractions) if inversions are allowed.

The general theme is hard to prove, so we simply checked with a computer algorithm that the order of \mathcal{G} is indeed $|\mathcal{G}| = 2r!$ —in particular, that it is *finite*. This is expected because permutations form subgroups of the symmetric group. From a physical point of view, permutations are reorderings which leave the bound-state masses invariant, and thus symmetries of the Hamiltonian.

To check the order of \mathcal{G} we proceeded as follows. By acting on all $4r$ statelets $\mathcal{B}_i|r\rangle$ with the first operator $\mathcal{S} \equiv \mathcal{S}_1$ we produced the left coset $\mathcal{S}_1\mathcal{B}$. To keep the

⁸We refer to it as the exhaustive set \mathcal{E} , because its elements are exhausting the ‘‘symmetrization space.’’ It is not a group, because it is not closed under composition of its members which may produce elements of \mathcal{B} .

TABLE II. The \mathcal{E} part of the group multiplication table in the five-parton sector organized as left coset elements. Since there is a small number of primary operators, the table has been transposed, in the sense that the column-heading operators act on the row-leading operators, e.g., $\mathcal{S}_1\mathcal{S}_2 = \mathcal{S}_6$. It is obvious that \mathcal{E} is not a subgroup of \mathcal{G} ; multiplying \mathcal{S} operators generates \mathcal{B} operators.

	\mathcal{S}_1	\mathcal{S}_2	\mathcal{S}_3	\mathcal{S}_4	\mathcal{S}_5
\mathcal{S}_1	id	$\mathcal{S}_{11}\mathcal{IT}$	\mathcal{S}_7	$\mathcal{S}_{11}\mathcal{C}^3$	$\mathcal{S}_7\mathcal{IT}$
\mathcal{S}_2	\mathcal{S}_6	id	\mathcal{S}_{10}	$\mathcal{S}_6\mathcal{ITC}^3$	$\mathcal{S}_{10}\mathcal{ITC}$
\mathcal{S}_3	\mathcal{S}_7	\mathcal{S}_8	id	$\mathcal{S}_7\mathcal{ITC}^4$	$\mathcal{S}_8\mathcal{ITC}$
\mathcal{S}_4	$\mathcal{S}_{11}\mathcal{C}^3$	$\mathcal{S}_6\mathcal{ITC}^3$	$\mathcal{S}_{11}\mathcal{ITC}$	id	$\mathcal{S}_6\mathcal{C}^4$
\mathcal{S}_5	$\mathcal{S}_8\mathcal{C}^4$	$\mathcal{S}_{10}\mathcal{ITC}$	$\mathcal{S}_8\mathcal{ITC}$	$\mathcal{S}_{10}\mathcal{C}$	id
\mathcal{S}_6	\mathcal{S}_2	$\mathcal{S}_4\mathcal{ITC}^3$	$\mathcal{S}_9\mathcal{ITC}^2$	$\mathcal{S}_2\mathcal{ITC}^3$	$\mathcal{S}_4\mathcal{C}$
\mathcal{S}_7	\mathcal{S}_3	\mathcal{S}_9	\mathcal{S}_1	$\mathcal{S}_3\mathcal{ITC}^4$	$\mathcal{S}_1\mathcal{IT}$
\mathcal{S}_8	$\mathcal{S}_5\mathcal{C}$	\mathcal{S}_3	$\mathcal{S}_5\mathcal{ITC}$	$\mathcal{S}_9\mathcal{ITC}^4$	$\mathcal{S}_3\mathcal{ITC}$
\mathcal{S}_9	$\mathcal{S}_{10}\mathcal{ITC}^2$	\mathcal{S}_7	$\mathcal{S}_6\mathcal{ITC}^2$	$\mathcal{S}_8\mathcal{ITC}^4$	\mathcal{S}_{11}
\mathcal{S}_{10}	$\mathcal{S}_9\mathcal{ITC}^2$	$\mathcal{S}_5\mathcal{ITC}$	\mathcal{S}_2	$\mathcal{S}_5\mathcal{C}^4$	$\mathcal{S}_2\mathcal{ITC}$
\mathcal{S}_{11}	$\mathcal{S}_4\mathcal{C}^2$	$\mathcal{S}_1\mathcal{IT}$	$\mathcal{S}_4\mathcal{ITC}$	$\mathcal{S}_1\mathcal{C}^2$	\mathcal{S}_9

symmetry of the Hamiltonian manifest, we need however the right coset $\mathcal{B}\mathcal{S}_1$. Recall that terms of odd \mathcal{C} -parity must be negative in the even r sectors to ensure the alternating signs of the cyclic permutations in the Hamiltonian Eq. (12). Since in general the cosets are not *normal*, $\mathcal{S}_1\mathcal{B} \neq \mathcal{B}\mathcal{S}_1$, we do not off-hand know which \mathcal{C} -parity the members of a left coset of \mathcal{B} might have. In general, both the left and the right coset will therefore contain new operators, which we'll have to include in our growing set of operators to move towards *group closure*. We must then act with these new operators on \mathcal{B} as well as on the existing right cosets, which in turn will yield new operators, right and left cosets. This process continues until no new operators arise—if the group is finite. It is thus a nontrivial test of our hypothesis. We find that the algorithm always converges on the expected number of operators or statelets, namely $2r! = 240, 1440, 10080, 80640$ for $r = 5, 6, 7, 8$ and beyond.

To construct an actual eigenfunction, we need to solve one more problem. It is the assignment of S parity to the individual statelets, in the sense that some of them will carry an odd power of S quantum number, and some an even power. Recall that $S^2 = 1$ since \mathcal{S} is a Z_2 operation. It is convenient to introduce the notion of a *primary operator*. Of the $\frac{1}{2}(r-1)! - 1$ \mathcal{S} operators in the r parton sector, naively only $r-1$ (associated with the $r-1$ excitation numbers) seem necessary to ensure that the wave function is vanishing (maximal) on the hyperplanes $x_i = 0$ in the massive (massless) theory, respectively. We call those $r-1$ operators together with an additional operator⁹ \mathcal{S}_r the

⁹While not intuitive, the last operator is necessary to complete the map $\mathcal{S}_i \times \langle \mathcal{C} \rangle \rightarrow \langle \mathcal{C} \rangle \mathcal{S}_i$, since the cyclic subgroup $\langle \mathcal{C} \rangle$ is of order r , not $r-1$.

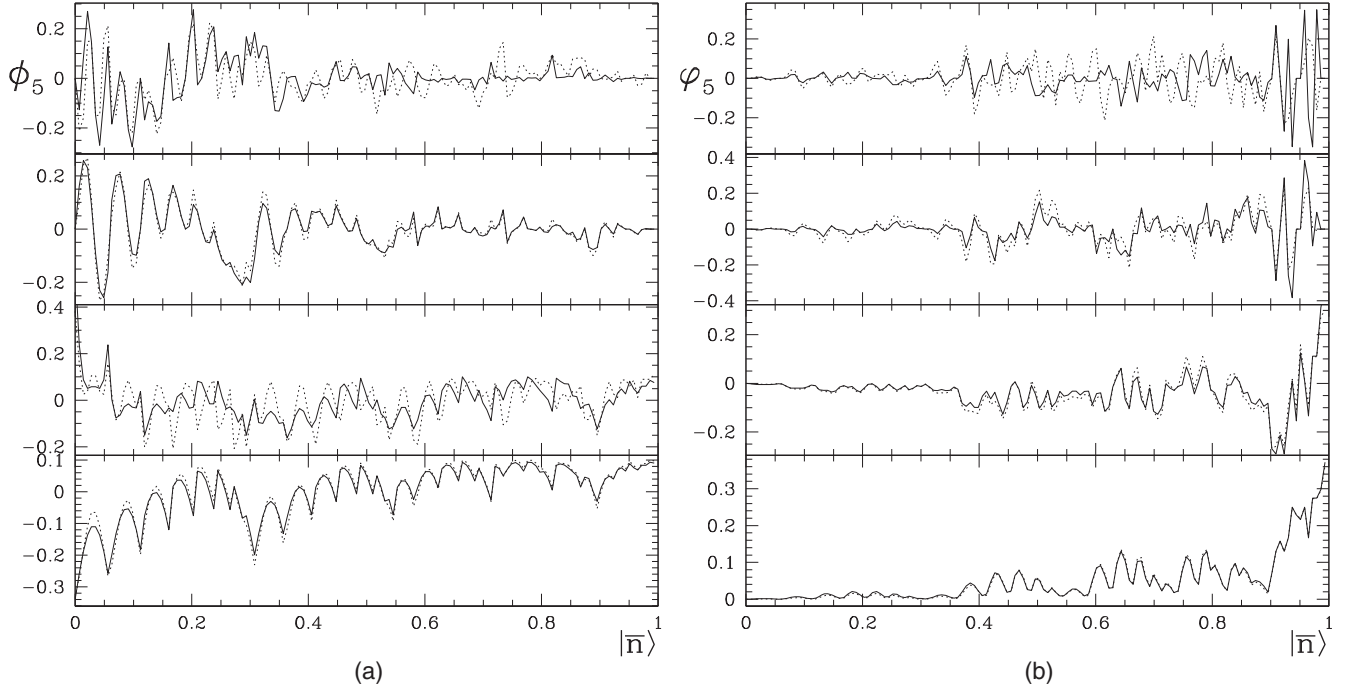


FIG. 3. Lowest eigenfunctions in all five-parton sectors. Solid lines represent numerical (DLCQ) results, dashed lines the algebraic eigenfunctions. (a) Massless theory ($\mu = 0$). (b) Massive theory ($\mu = 4$). The lowest two eigenfunctions are \mathcal{T} even, the upper two \mathcal{T} odd.

primary operators. They form a conjugacy class under $\langle \mathcal{C} \rangle$ and have similar properties, since¹⁰

$$\mathcal{S}_i = \mathcal{C}^{i-1} \mathcal{S} \mathcal{C}^{r-i+1}. \quad (31)$$

It is not hard to show that the primary operators fulfill the following pseudocommutation relations

$$\mathcal{C}^j \mathcal{S}_i = \mathcal{S}_{\text{mod}'_r(i+j)} \mathcal{C}^j \quad \text{and} \quad (\mathcal{T} \mathcal{C}^j) \mathcal{S}_i = \mathcal{S}_{\text{mod}'_r(r-i+j+1)} (\mathcal{T} \mathcal{C}^j), \quad (32)$$

where mod'_r is congruence modulo r shifted by one, so that $\mathcal{S}_0 = \mathcal{S}_r$ and $\mathcal{S}_{r+1} = \mathcal{S}_1$. In other words, the \mathcal{S} operators commute with the members of the subgroup \mathcal{B} under loss of their identity: they transmute into a different primary operator. But then we are done! These relations allow us to compute all other operators in the symmetry group \mathcal{G} . In fact, we must keep adding operators until the $2r!$ slots in the full symmetry group are *exhausted*. The product of two primary operators can be evaluated as follows due to the associativity group axiom¹¹

$$\begin{aligned} \mathcal{S}_k (\mathcal{C}^j \mathcal{S}_i) &= (\mathcal{S}_k \mathcal{C}^j) \mathcal{S}_i = \mathcal{C}^j (\mathcal{S}_{\text{mod}'_r(k-j)} \mathcal{S}_i) \\ &= (\mathcal{S}_k \mathcal{S}_{\text{mod}'_r(i+j)}) \mathcal{C}^j \quad \text{for } 1 \leq i, j, k \leq r. \end{aligned} \quad (33)$$

¹⁰This relation is not unique, for instance $\mathcal{S}_{r-1} = (\mathcal{T} \mathcal{C}) \mathcal{S} (\mathcal{T} \mathcal{C})$ also.

¹¹The \mathcal{T} version of this relation is $\mathcal{S}_k (\mathcal{T} \mathcal{C}^j \mathcal{S}_i) = \mathcal{T} \mathcal{C}^j (\mathcal{S}_{\text{mod}'_r(r-k+j+1)} \mathcal{S}_i) = (\mathcal{S}_k \mathcal{S}_{\text{mod}'_r(r-i+j+1)}) \mathcal{T} \mathcal{C}^j$.

The second half is the commutation relation for the *secondary operator* $\mathcal{S}_{\text{mod}'_r(k-j)} \mathcal{S}_i$. As an example, consider $r = 5$. First one shows with Eqs. (31)–(33) that the operator compositions $\mathcal{S}_1 \mathcal{S}_2$, $\mathcal{S}_1 \mathcal{S}_3$, and $\mathcal{S}_2 \mathcal{S}_3$ cannot be written in terms of powers of \mathcal{C} , \mathcal{T} , \mathcal{I} , or \mathcal{S}_i ($i = 1, \dots, 5$). In this sense, these three combinations are “new” operators, and relabeled \mathcal{S}_6 , \mathcal{S}_7 , and \mathcal{S}_8 , respectively. One then iterates by acting with existing operators on these operators, and puts any further “new” operators into \mathcal{G} . Often though, an equivalent operator combination will already exist in \mathcal{G} . For example, when acting with \mathcal{S}_2 on the right-coset statelet $\mathcal{C} \mathcal{S}_3 |r = 5\rangle$ one finds using Eq. (33)

$$\mathcal{S}_2 (\mathcal{C} \mathcal{S}_3 |r = 5\rangle) = \mathcal{C} \mathcal{S}_1 \mathcal{S}_3 |r = 5\rangle = \mathcal{C} \mathcal{S}_7 |r = 5\rangle,$$

and no new operator needs to be added. Finding new operators thus essentially reduces to the *word problem* of abstract algebra. Proceeding along this way, a group multiplication table can be computed. We display part of it for $r = 5$ in Table II. As a corollary we note that these identities show that the Hamiltonian symmetrization constraint (fixed signs under $\langle \mathcal{C} \rangle$) is intact. Namely, the $\mathcal{T}^i \mathcal{C}^j$ component of a right coset of \mathcal{B} of any \mathcal{S}_k (whether primary or not) is mapped onto the $\mathcal{T}^i \mathcal{C}^j$ component of a right coset of some other operator $\mathcal{S}_{k'}$.

So what are the relative signs of the individual terms? The signs *within* the \mathcal{B} -blocks ($4r$ statelets of the right cosets $\mathcal{B} \mathcal{S}_i$) are fixed by the Hamiltonian and the \mathcal{T} and \mathcal{I} quantum numbers. At $r = 4$ the assignment of the \mathcal{S} quantum number is simple if counter-intuitive: both the \mathcal{S}_1 and the \mathcal{S}_2 block

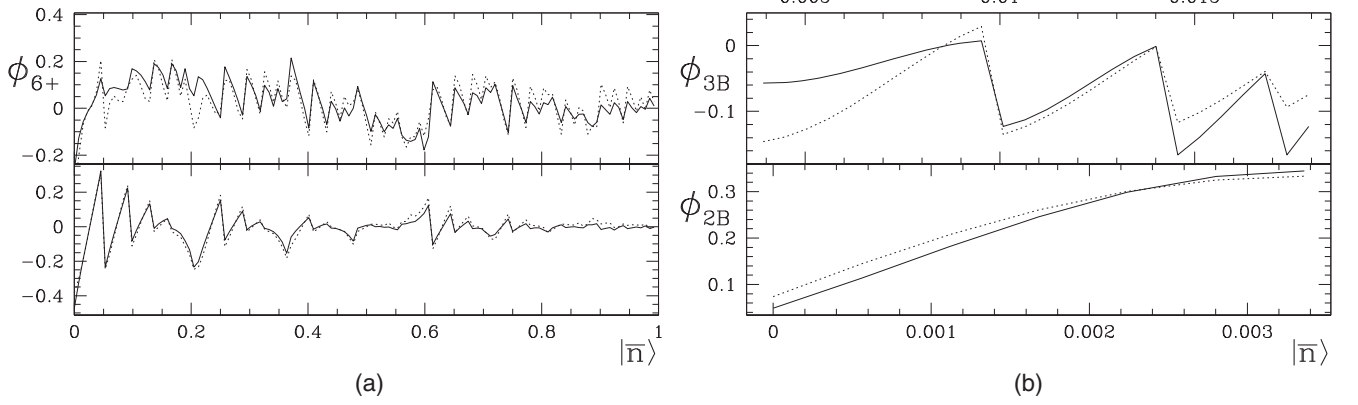


FIG. 4. (a) Massless six-parton \mathcal{T} even eigenfunctions of asymptotic QCD_{2A} at $K = 20$. (b) The two- and three-parton parts of the lowest \mathcal{T} even eigenfunction of the *bosonized theory* at $K_B = 14$ which is equivalent to $K = 28$ in the fermionic theory. In both graphs, DLCQ eigenfunctions are plotted with solid lines and “eLCQ” or model wave functions with dashed lines.

carry an additional S sign (32 statelets); only the identity block’s eight statelets do not acquire a sign. This seems unbalanced, but works due to sign-cancellations unique to the four-parton sector. At $r = 5$ the orders of group and subgroups are such that we get an even split into two “half-groups.” Namely, the one quarternary and four secondary operators plus the identity are S even and the primary operators plus a peculiar secondary operator¹² S_9 are odd and carry an S sign. This makes $r! = 120$ statelets with an S sign, and 120 without. For an algorithm that is more universally applicable, see Appendix A.

This completes our construction of the most general solution in all parton sectors. As a cross-check we construct the eigenfunctions in the five-parton sector and compare them to numerical solutions in Fig. 3. Figure 4(a) shows the comparison for some six-parton eigenfunctions. We used the rather low harmonic resolutions $K = 23$ and $K = 20$, because otherwise the plots get too “crowded”; agreement is just as good at higher K of course. As we glean from Fig. 3, the agreement is near-perfect for the lowest \mathcal{T} even state in the massive theory, good in the massive theory in general, and fair in the massless theory. This might be due to the fact that the density of states is much higher in the massless theory. Apparently, the condition that the wave function vanish for zero parton momenta is quite restrictive. Given that the number of statelets rises to 1440 at $r = 6$ the agreement between algebraic eigenfunctions and numerical solution seems surprisingly good in Fig. 4(a).

It is indeed quite remarkable how perfectly the features of the theory are represented by the properties of the harmonic basis. It is tempting to speculate that this approach—which one might call “exhaustively-symmetrized light-cone quantization (eLCQ)” —applies to other theories, at least in two dimensions where light-cone coordinates are the natural

language. Hopefully, the treatment of the present toy model is a harbinger of wider applicability.

III. APPLICATIONS

A. Using “eLCQ”: Some suggestions

Since we have found a complete set of basis states for the asymptotic theory in Sec. II, we can systematically approximate the full theory in a basis-function approach. This should be pretty straightforward, but implementing the algorithm in the higher parton-sectors is too tedious to be presented here, so we only sketch the general idea. Namely, we can use the set of coupled integral equations derived in [3] from the Hamiltonian Eq. (4) for the full wave functions f_r in the r -parton sectors. The full equation couples sectors of different parton number; f_r and $f_{r\pm 2}$ appear in the equation.

We expand the full eigenfunctions $f_r(x_1, x_2, \dots, x_r)$ into a complete set of asymptotic eigenfunctions $\phi_{r,\vec{n}}$

$$f_r(x_1, x_2, \dots, x_r) = \sum_{\vec{n}} c_{r,\vec{n}} \phi_{r,\vec{n}}(x_1, x_2, \dots, x_r),$$

where \vec{n} represents a tuple of $r - 1$ excitation numbers. Formally, the integral equation looks like

$$\begin{aligned} M^2 f_r &= M^2 \sum_{\vec{n}} c_{r,\vec{n}} \phi_{r,\vec{n}}(x_1, x_2, \dots, x_r) \\ &= 2P^+ P^- \sum_{\vec{n}} c_{r,\vec{n}} \phi_{r,\vec{n}}(x_1, x_2, \dots, x_r). \end{aligned}$$

If we project onto the asymptotic eigenfunctions characterized by (s, \vec{m}) , we get an equation for the associated coefficient

$$\begin{aligned} M^2 \int d^r x f_{s,\vec{m}}^*(\vec{x}) f_r(\vec{x}) &= M^2 \sum_{r,\vec{n}} \int d^r x c_{r,\vec{n}} f_{s,\vec{m}}^* f_{r,\vec{n}} \\ &= M^2 \sum_{r,\vec{n}} c_{r,\vec{n}} \delta_{s,r} \delta_{\vec{n},\vec{m}} = M^2 c_{s,\vec{m}}, \end{aligned}$$

¹²It commutes with the \mathcal{B} subgroup, i.e., $\mathcal{B}S_9 = S_9\mathcal{B}$, which does not imply commutation of individual subgroup elements \mathcal{B}_i with S_9 : $\mathcal{B}_i S_9 \neq S_9 \mathcal{B}_i$ in general.

$$-_{+} \langle \bar{n} | P_{PV}^- | n, m, l \rangle_{+--} \sim \int \int \frac{\sin \pi(n'x + m'y)}{(x+y)^2} dx dy,$$

where the integers n' and m' will depend on all excitation numbers \bar{n} , n , m , l of the two and four-parton states. This integral can be expressed in terms of cosine and sine integrals ($Ci(x)$, $Si(x)$), and is not divergent. There will be several terms, and cancellations are possible. Thus, for specific states, or in certain sectors¹⁶ the matrix element may vanish.

This in turn raises the question whether it is systematically possible to find linear combinations of the asymptotic eigenstates for which the annihilation matrix elements are zero. These would then be states with a definite parton number. In other words, we would have succeeded in summing all pair creation effects, effectively “renormalizing” the theory, i.e., formulating it in terms of effective d.o.f. This program is beyond the scope of the present note, but it seems to be viable—judging from previous work. Namely, in [2,3], it was shown that many of the bound states are very pure in parton number. An open question is whether this behavior is generic or due to the finite discretization used in [2,3].

C. Implications for the bosonized theory

QCD_{2A} can be bosonized by rewriting the Hamiltonian in terms of current operators $J(-p) \sim \int dq b(q)b(p-q)$ subject to a Kac-Moody algebra, see e.g., Refs. [8,14]. Bosonization is in essence a basis transformation, so the eigenfunctions will change while the eigenvalues, i.e., the bound state masses, remain invariant. Straightforward bosonization generates a nonorthonormal basis, with states consisting of color traces of adjoint *current operators* acting on a vacuum state.¹⁷ Numerical approaches [8] produce orthonormal solutions; i.e., the coordinate vectors associated with a chosen basis (whether orthonormal or not) are mutually orthogonal and of unit length. In this sense, a basis of symmetrized harmonics as produced by our “eLCQ” method will be orthonormal.

To get a handle on the bosonized theory, current-number changing operators can be omitted at first [14], and one arrives at an integral equation for the bosonized eigenfunctions ϕ_{rB} ,

$$\begin{aligned} & \frac{M^2}{g^2 N} \phi_{rB}(x_1, \dots, x_r) \\ &= - \sum_{i=1}^r \int_{-\infty}^{\infty} \frac{\phi_{rB}(y, x_i + x_{i+1} - y, x_{i+2}, \dots, x_{i+r-1})}{(x_i - y)^2} dy, \end{aligned} \quad (34)$$

¹⁶Of course, the \mathcal{T} sector is fixed, but the result can be different in the massive and massless sectors of the theory.

¹⁷Either the traditional vacuum state or an “adjoint vacuum,” represented by a fermionic operator of zero momentum acting on the vacuum.

where we have used the subscript B to stress that the partons in the bosonized theory are currents and not fermions. The equation is virtually identical with Eq. (12) save for the nonalternating signs. These signs stemming from anticommuting fermionic operators are clearly absent in the bosonized theory. We should therefore be able to describe the bosonized wave functions with our “eLCQ” ansatz, Eq. (25). We cannot hope for a perfect match, because we cannot decouple sectors of different current number. A consistent asymptotic theory does not exist due to the nature of the Kac-Moody algebra of the current operators. There is an associated problem. Namely, the full theory contains uninteresting nontrivial multiparticle states which interact with the single-particle states of interest at any finite resolution in discretized versions of the theory [4,12]. Indeed, only a few single-particle states have hitherto been identified as such. Furthermore, bosonization only works for massless fermions, so there is no massive sector. Hence, our comparison of numerical and algebraic wave functions will be rather limited. Of course, in the bosonized theory, there is a bosonic and also a fermionic sector. In the latter, the basis states are different owing to the elimination of cyclic symmetry due to the existence of a unique fermionic operator (the “adjoint vacuum”). Here, we will focus on the bosonic sector of the bosonized theory. This is a real test of our ansatz, since the symmetrization clearly cannot be the same as in the theory with fermions. The “eLCQ” ansatz (25) for the two-current wave function ϕ_{2B} is

$$\phi_{2B} = e^{inx} + (-1)^n e^{-inx}$$

(compare to Eq. (B1) which has a minus sign between the terms). Therefore, we “predict” that the massless states with odd n will be sines in the bosonized theory, not cosines as in the fermionic theory. This is consistent with Ref. [14], and also the numerical bosonized eigenfunctions in Fig. 4(b) are consistent with our ansatz.¹⁸ Note that numerical approaches such as DLCQ use different bases for the fermionic and bosonized theories. The fermionic theory is approximated using antiperiodic boundary conditions (odd half-integer momentum fractions), whereas the bosonized theory uses periodic boundary conditions (integer momentum fractions). Of course, regardless of the basis used, they should be decent approximations to the algebraic wave function.

In sum, we find that the “eLCQ” ansatz is compatible with the known solutions of the theory in a significantly different representation. This hints at a wider applicability of our method. What seems crucial is that the structure of the Hamiltonian be of the form

¹⁸We used $\phi_{2B} = \sin(\pi x_1)$ and $\phi_{3B} = -\frac{1}{20} [\cos(\pi x_1) + \cos(\pi x_2) + \cos(\pi x_3)]$ to describe the generic features of the two wave functions. Note that the cosines in the higher parton sectors are consistent with our earlier finding that wave functions of same T have opposite I at subsequent r .

$$P^- \sim \int \frac{dp}{p^2} J_{ij}(-p) J_{ji}(p).$$

In other words, the “eLCQ” ansatz is poised to solve the long-range Coulomb-type part of a strongly interacting system. This is akin to the DLCQ ansatz which was shown in Ref. [13] to decouple the center of mass motion of a system from its more interesting physics.

IV. CONCLUSION AND DISCUSSION

The goal of this paper was to find a complete set of eigenfunctions of QCD_{2A} . We succeeded in constructing a basis of the asymptotic theory without pair-production, consisting of multidimensional harmonic functions. In order to completely and exhaustively symmetrize the wave functions in the r parton sector, a *group* of operators of order $2r!$ is necessary and sufficient. This finite group of abstract symmetries is defined by the relations between the generators Eqs. (31)–(33). Paradoxically, this rather complicated arithmetic is based on the simple observation that “boundary conditions” on the wave functions have to be implemented as symmetries in a Hamiltonian approach (which leads to an *integral* equation for the eigenfunctions). Our finding that the eigenfunctions can be constructed largely algebraically with group theory arguments is corroborated by comparison with numerical solutions of the theory.

At first glance, our method of exhaustively symmetrizing a system quantized on the light-cone (“eLCQ”) applies to a specific family of systems, namely theories with adjoint d.o.f. in one spatial dimension with a Coulomb-type long-range interaction. It seems likely, though, that the method presented here is more widely applicable due to its generality. In particular, exhaustive symmetrization deals with momentum as an abstract entity; i.e., it deals with it regardless of its Lorentz structure. There might well be requirements due to the Poincaré group, but this is not the point here. Rather, the fact that we have r momenta requires us to symmetrize on purely abstract grounds, rather as a preconditioning of the wave functions akin to the Slater determinant implementing the antisymmetrization requirement of the Pauli principle.

We presented evidence that “eLCQ” is a useful method at least for theories exhibiting similar integral equations as QCD_{2A} —like its bosonized version as discussed in Sec. III C. Other theories to which “eLCQ” can be straightforwardly applied include two-dimensional Yang-Mills theory coupled to adjoint scalars [2] and a theory with adjoint Dirac fermions tackled in [15]. Also, many of the supersymmetric models with adjoint particles (for instance [16]) could be re-evaluated with the present method.

We did not have the space to fully exploit the fact that the asymptotic spectrum of QCD_{2A} is now completely mapped out, so there are several opportunities for future projects. For instance, the interaction of asymptotic states and

the formation of multiparticle states can be studied. In Sec. III B, we laid down an initial plan how to proceed. And while the exponential rise of terms in the eigenfunctions presents some difficulty, one might be able to improve the precision of numerical solutions enough by using “eLCQ” basis functions to positively identify the single-particle content of the theory. Accidental symmetries described in Sec. III A will help in this regard. Also, the structure of the asymptotic solutions may help to disentangle salient features of this and other theories. For instance, the “topological sector” of the theory, introduced in [4] to explain the appearance of fermion-fermion multiparticle states in the *fermionic* sectors, might be an artifact of finite group theory. As long as parton number is finite, there is a clear separation of group properties according to their (finite) order. Some combination rules for multiparticle states in terms of T quantum numbers found in [4] might just be the result of such “artificial” group theoretical relations.

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APPENDIX A: DERIVATION OF THE GROUP MULTIPLICATION TABLE

To fully understand why there is only one quantum number S associated with the large set \mathcal{E} of S operators, we work out the details here. We start with the simplest case $r = 4$. We need to show that the fully symmetrized wave function $\mathcal{G}|r\rangle$ is an eigenstate to all three S operators associated with its three excitations numbers (n, m, l) . Note that *de facto* there are only *two* independent S operators at $r = 4$, which we also have to explain, along with the fact that two thirds—not half—of a state’s statelets carry an S sign.

First, we observe that the $4r$ members of the subgroup \mathcal{B} naturally split into four subsets, namely the cosets of the cyclic subgroup $\langle \mathcal{C} \rangle$ under \mathcal{T} , \mathcal{I} , \mathcal{T} , and \mathcal{IT} . But \mathcal{I} is in the center of \mathcal{G} , and therefore can be largely ignored. The cyclic structure of the Hamiltonian means that there is a conserved quantity which we might call \mathcal{TC} -parity. Odd and even powers of the cyclic group carry different signs in general. Therefore, an even(odd) power has to be mapped under any S operation onto an even(odd) power of \mathcal{C} . When we act with \mathcal{T} , then the combined power has to be the same modulo two, e.g., $\mathcal{C}^{\text{mod}_2 j} \leftrightarrow \mathcal{TC}^{\text{mod}_2(j-1)}$. There is a twist. Namely, for states with mixed even and odd excitation

numbers like $|oeo\rangle$, there is an additional sign, which shifts the relative power under \mathcal{T} , so we have $\mathcal{C}^{\text{mod}_2 j} \leftrightarrow \mathcal{T}\mathcal{C}^{\text{mod}_2 j}$. So the subgroup \mathcal{B} has two subsets even and odd under $\mathcal{T}\mathcal{C}$ -parity, \mathcal{B}^e and \mathcal{B}^o . We call their members (i.e., combinations like $\mathcal{T}^i \mathcal{T}^j \mathcal{C}^e$) \mathcal{Z}_j^e and \mathcal{Z}_j^o . At $r=4$ we have the pseudocommutation relations¹⁹

$$\mathcal{S}_i \mathcal{Z}_j^e = \mathcal{Z}_j^e \mathcal{S}_i, \quad (\text{identity preserving}) \quad (\text{A1})$$

$$\mathcal{S}_i \mathcal{Z}_j^e = \mathcal{Z}_j^e \mathcal{S}_{\text{mod}_2(i+1)}. \quad (\text{identity swapping}) \quad (\text{A2})$$

Note that

$$\mathcal{S}_i \mathcal{B}^e = \mathcal{B}^e \mathcal{S}_i \quad \text{and} \quad \mathcal{S}_i \mathcal{B}^o = \mathcal{B}^o \mathcal{S}_{\text{mod}_2(i+1)},$$

i.e., while the individual operators do not pseudocommute, they remain in the same subset. At $r=4$, the following additional identities hold,

$$\begin{aligned} \mathcal{S}_1 \mathcal{S}_2 &= \mathcal{T} \mathcal{Z}_k^o \mathcal{S}_1 = \mathcal{T} \mathcal{S}_2 \mathcal{Z}_k^o, \\ \mathcal{S}_2 \mathcal{S}_1 &= \mathcal{T} \mathcal{Z}_k^o \mathcal{S}_2 = \mathcal{T} \mathcal{S}_1 \mathcal{Z}_k^o, \end{aligned} \quad (\text{A3})$$

for some k , so that $\mathcal{Z}_k^o \in \mathcal{B}^o$. In fact, $\mathcal{Z}_k^o = \mathcal{T} \mathcal{C}^2$. In other words, the product of the two \mathcal{S} -operators can be reduced to the leading one, but an odd \mathcal{Z} operator appears as well as an additional \mathcal{T} operator which switches the \mathcal{T} sign. We are now ready to calculate the action of \mathcal{S}_i on the totally symmetrized state. There are two versions of that state. If focusing on \mathcal{S} symmetry properties, it is natural to write the state in terms of *left* coset statelets $\mathcal{S}\mathcal{B}|r\rangle$

$$\begin{aligned} \mathcal{G}|r=4\rangle &= (\mathcal{B}^e + \mathcal{B}^o)|r\rangle + \mathcal{S}\mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)|r\rangle \\ &\quad + \mathcal{S}\mathcal{S}_2(\mathcal{B}^e + \mathcal{B}^o)|r\rangle. \end{aligned}$$

On the other hand, we have to preserve cyclical properties due to the structure of the Hamiltonian, in which case we should use *right* coset statelets $\mathcal{B}\mathcal{S}|r\rangle$ ²⁰

$$\begin{aligned} \mathcal{G}|r=4\rangle &= (\mathcal{B}^e + \mathcal{B}^o)|r\rangle + \mathcal{S}(\mathcal{B}^e + \mathcal{B}^o)\mathcal{S}_1|r\rangle \\ &\quad + \mathcal{S}(\mathcal{B}^e + \mathcal{B}^o)\mathcal{S}_2|r\rangle. \end{aligned}$$

Both requirements lead to the same constraint on the quantum numbers. In the latter case, we have

$$\begin{aligned} \mathcal{S}_1 \mathcal{G}|r=4; eee\rangle &= \mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + \mathcal{S}\mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)\mathcal{S}_1|r\rangle + \mathcal{S}\mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)\mathcal{S}_2|r\rangle \\ &= (\mathcal{B}^e \mathcal{S}_1 + \mathcal{B}^o \mathcal{S}_2)|r\rangle + \mathcal{S}(\mathcal{B}^e + \mathcal{B}^o \mathcal{S}_2 \mathcal{S}_1)|r\rangle + \mathcal{S}(\mathcal{B}^e \mathcal{S}_1 \mathcal{S}_2 + \mathcal{B}^o)|r\rangle \\ &= \mathcal{S}(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + (\mathcal{B}^e + \mathcal{S}\mathcal{T}\mathcal{B}^o)\mathcal{S}_1|r\rangle + (\mathcal{S}\mathcal{T}\mathcal{B}^e + \mathcal{B}^o)\mathcal{S}_1|r\rangle \\ &= \mathcal{S}\mathcal{G}|r=4; eee\rangle \quad (\text{if } \mathcal{S}\mathcal{T} = 1), \end{aligned} \quad (\text{A4})$$

where we used Eq. (A3) in the last step. The calculation goes analogously for \mathcal{S}_2 . We thus proved that a symmetrized state is an eigenstate of \mathcal{S}_1 (and \mathcal{S}_2) in the $|eee; \mathcal{T} \pm \mathcal{S} \pm\rangle$ sector of the theory. In the mixed sector, the \mathcal{C}^j and $\mathcal{T}\mathcal{C}^{j-1}$ terms get an extra sign for odd j due to $(-1)^n = (-1)^l = -1$. This effectively reverses \mathcal{T} , therefore the viable sectors are $|oeo; \mathcal{T} \mp \mathcal{S} \pm\rangle$.

If we organize in terms of the left cosets we come to the same conclusion. To wit

¹⁹In full detail, they read

$$\begin{aligned} \mathcal{S}_1 \mathcal{C} &= \mathcal{I}\mathcal{T}\mathcal{S}_2 & \mathcal{S}_1 \mathcal{C}^2 &= \mathcal{I}\mathcal{T}\mathcal{C}\mathcal{S}_1 & \mathcal{S}_1 \mathcal{C}^3 &= \mathcal{C}^3 \mathcal{S}_2 \\ \mathcal{S}_1 \mathcal{T} &= \mathcal{I}\mathcal{C}\mathcal{S}_2 & \mathcal{S}_1 \mathcal{T}\mathcal{C} &= \mathcal{I}\mathcal{C}^2 \mathcal{S}_1 & \mathcal{S}_1 \mathcal{T}\mathcal{C}^2 &= \mathcal{T}\mathcal{C}^2 \mathcal{S}_2 & \mathcal{S}_1 \mathcal{T}\mathcal{C}^3 &= \mathcal{T}\mathcal{C}^3 \mathcal{S}_1 \\ \mathcal{S}_1 \mathcal{I} &= \mathcal{I}\mathcal{S}_1 & \mathcal{S}_1 \mathcal{I}\mathcal{C} &= \mathcal{T}\mathcal{S}_2 & \mathcal{S}_1 \mathcal{I}\mathcal{C}^2 &= \mathcal{T}\mathcal{C}\mathcal{S}_1 & \mathcal{S}_1 \mathcal{I}\mathcal{C}^3 &= \mathcal{I}\mathcal{C}^3 \mathcal{S}_2 \\ \mathcal{S}_1 \mathcal{I}\mathcal{T} &= \mathcal{C}\mathcal{S}_2 & \mathcal{S}_1 \mathcal{I}\mathcal{T}\mathcal{C} &= \mathcal{C}^2 \mathcal{S}_1 & \mathcal{S}_1 \mathcal{I}\mathcal{T}\mathcal{C}^2 &= \mathcal{I}\mathcal{T}\mathcal{C}^2 \mathcal{S}_2 & \mathcal{S}_1 \mathcal{I}\mathcal{T}\mathcal{C}^3 &= \mathcal{I}\mathcal{T}\mathcal{C}^3 \mathcal{S}_1 \\ \mathcal{S}_2 \mathcal{C} &= \mathcal{C}\mathcal{S}_1 & \mathcal{S}_2 \mathcal{C}^2 &= \mathcal{I}\mathcal{T}\mathcal{C}^3 \mathcal{S}_2 & \mathcal{S}_2 \mathcal{C}^3 &= \mathcal{I}\mathcal{T}\mathcal{S}_1 \\ \mathcal{S}_2 \mathcal{T} &= \mathcal{I}\mathcal{C}^3 \mathcal{S}_1 & \mathcal{S}_2 \mathcal{T}\mathcal{C} &= \mathcal{T}\mathcal{C}\mathcal{S}_2 & \mathcal{S}_2 \mathcal{T}\mathcal{C}^2 &= \mathcal{T}\mathcal{C}^2 \mathcal{S}_1 & \mathcal{S}_2 \mathcal{T}\mathcal{C}^3 &= \mathcal{I}\mathcal{C}^2 \mathcal{S}_2 \\ \mathcal{S}_2 \mathcal{I} &= \mathcal{I}\mathcal{S}_2 & \mathcal{S}_2 \mathcal{I}\mathcal{C} &= \mathcal{I}\mathcal{C}\mathcal{S}_1 & \mathcal{S}_2 \mathcal{I}\mathcal{C}^2 &= \mathcal{T}\mathcal{C}^3 \mathcal{S}_2 & \mathcal{S}_2 \mathcal{I}\mathcal{C}^3 &= \mathcal{T}\mathcal{S}_1 \\ \mathcal{S}_2 \mathcal{I}\mathcal{T} &= \mathcal{C}^3 \mathcal{S}_1 & \mathcal{S}_2 \mathcal{I}\mathcal{T}\mathcal{C} &= \mathcal{I}\mathcal{T}\mathcal{C}\mathcal{S}_2 & \mathcal{S}_2 \mathcal{I}\mathcal{T}\mathcal{C}^2 &= \mathcal{I}\mathcal{T}\mathcal{C}^2 \mathcal{S}_1 & \mathcal{S}_2 \mathcal{I}\mathcal{T}\mathcal{C}^3 &= \mathcal{C}^2 \mathcal{S}_2. \end{aligned}$$

²⁰If this seems inconsistent, recall that the fully symmetrized state is symmetrized both in \mathcal{B} and in \mathcal{S} . Hence, though parts of the state will appear to be symmetrized with respect to only one symmetry, the state as a whole has to be symmetric with respect to both operations. In general, this is only possible for certain combinations of quantum numbers. This is the reason why many sectors of the theory do not give rise to proper states.

$$\begin{aligned}
\mathcal{S}_1 \mathcal{G}|r = 4; eee\rangle &= \mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + S(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + S\mathcal{S}_1\mathcal{S}_2(\mathcal{B}^e + \mathcal{B}^o)|r\rangle \\
&= \mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + S(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + ST\mathcal{S}_2\mathcal{Z}^o(\mathcal{B}^e + \mathcal{B}^o)|r\rangle \\
&= S[(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + S\mathcal{S}_1(\mathcal{B}^e + \mathcal{B}^o)|r\rangle + T\mathcal{S}_2(\mathcal{B}^e + \mathcal{B}^o)|r\rangle] \\
&= S\mathcal{G}|r = 4; eee\rangle \quad (\text{if } S = T).
\end{aligned} \tag{A5}$$

The I quantum number is necessarily the opposite of T . The reason is that in the four-parton sector the pseudocommutation relations are such that $\mathcal{T}\mathcal{I}$ -parity $(-1)^{t+i}$ is conserved: commuting a (primary) \mathcal{S} -operator with a \mathcal{B} operator will yield a different \mathcal{B} operator with equal $t+i$ (modulo two). This essentially links the identity and \mathcal{T} statelets of \mathcal{S}_1 to the \mathcal{IT} and \mathcal{I} statelets of \mathcal{S}_2 , respectively. Therefore T and I have to be different,²¹ because otherwise the wave function is identically zero.

The higher parton sectors, $r > 4$, are harder to analyze because the pseudocommutation relations are more complicated. In particular, there is no identity preserving relation akin to Eq. (A1). Rather, we have to use Eq. (32). Here \mathcal{I} is decoupled, and I -parity thus independently conserved. So we have to come up with an ‘‘operator calculus’’ that allows us to evaluate the S sign of a combination of operators. As it turns out, $r = 5$ is not the most general case, but tractable and yielding clues for the generic case $r > 5$, so we discuss it here.

The exhaustively symmetrized state in terms of *left* coset elements can formally be written as

$$\mathcal{G}|r\rangle = \mathcal{B}|r\rangle + \sum_{j=1}^{N(r)} \mathcal{S}_j \mathcal{S}_j \mathcal{B}|r\rangle = \left(1 + \sum_{j=1}^{N(r)} \mathcal{S}_j \mathcal{S}_j\right) \mathcal{B}|r\rangle,$$

where the r primary operators \mathcal{S}_j carry an S sign, so $\mathcal{S}_j = S$ for $j \leq r$. The nonprimary operators \mathcal{S}_j carry an unknown sign $\mathcal{S}_j = \pm S$. As before, $N(r) = \frac{1}{2}(r-1)! - 1 = 11, 59, 359, \dots$. The action of a primary operator \mathcal{S}_k on an exhaustively symmetrized state is then

$$\begin{aligned}
\mathcal{S}_k \mathcal{G}|r\rangle &= \left(\mathcal{S}_k + S + \sum_{\substack{j=1 \\ j \neq k}}^{N(r)} \mathcal{S}_j \mathcal{S}_k \mathcal{S}_j\right) \mathcal{B}|r\rangle \\
&= S \left(1 + S\mathcal{S}_k + S \sum_{\substack{j=1 \\ j \neq k}}^{N(r)} \mathcal{S}_j \mathcal{S}_k \mathcal{S}_j\right) \mathcal{B}|r\rangle,
\end{aligned}$$

where we have used $\mathcal{S}_k = S$ for primary operators. Due to the group axioms, the secondary operators $\mathcal{S}_k \mathcal{S}_j$ can be expressed in terms of a single \mathcal{S} operator followed by operators of the \mathcal{B} subgroup, to wit

²¹In fact, this yields exactly two sectors, which makes sense in the even r sectors, since here we have two excitation number sectors (even and mixed) for the required four (two massive and massless) sectors. In the odd r sectors, there is only one condition due to the fact that there is no sign due to relative \mathcal{C} operators; at odd r all cyclic permutations enter with a positive sign.

$$\mathcal{S}_k \mathcal{S}_j = \mathcal{S}_l \mathcal{I}^i \mathcal{T}^t \mathcal{C}^c,$$

where the index l and the powers i, t, c can be looked up in a table like Table II. But to reproduce the exhaustively symmetrized state $\mathcal{G}|r\rangle$, we need to have

$$SS_j \mathcal{I}^i \mathcal{T}^t \mathcal{C}^c = \mathcal{S}_l, \tag{A6}$$

since the \mathcal{B} operators can be absorbed into $\mathcal{B}|r\rangle$, whence only the quantum numbers I, T, C re-emerge. This is the condition we can use to identify the S signs of the nonprimary operators.

As an example, consider $r = 5$, where we can formally set $C = 1$, since cyclic permutations at odd r do not carry signs. Acting with \mathcal{S}_1 , say, on $\mathcal{G}|5\rangle$ yields the operator indices and powers displayed in the \mathcal{S}_1 column of Table II. Since $\mathcal{I}^i \mathcal{T}^t = 1$ for all of them, Eq. (A6) simplifies to $SS_j = \mathcal{S}_l$, so that $\mathcal{S}_6 = \mathcal{S}_7 = \mathcal{S}_8 = \mathcal{S}_{10} = \mathcal{S}_{11} = S^2$ and $\mathcal{S}_9 = S$. Note the interplay between \mathcal{S}_9 and \mathcal{S}_{10} : since $\mathcal{S}_1 \mathcal{S}_9 \propto \mathcal{S}_{10}$ and $\mathcal{S}_1 \mathcal{S}_{10} \propto \mathcal{S}_9$, one operator gets an S sign, and the other does not. Which one it is has actually to be decided by acting with another primary operator on the exhaustively symmetrized state or by writing the state in terms of *right* coset statelets.

The general algorithm is then as follows. Initially, we produce an exhaustive list of operators by acting with primary operators on primary operators until no new operators arise. This leads to a group multiplication table, such as Table II for $r = 5$ which can be used to write down consistent expressions for primary operators acting on the generic state (with yet undetermined S signs) in terms of left or right coset statelets. The condition that the emerging state has to be an eigenstate of the primary operators determines the signs, and may lead to a further condition on I as a function of T . Using the sign condition Eq. (A6) we start with the r known signs \mathcal{S}_j of the primary operators to get another r signs \mathcal{S}_l , and iterate the process. The only time this goes bad is when both \mathcal{S}_j and \mathcal{S}_l are unknown, as in the case of \mathcal{S}_9 and \mathcal{S}_{10} at $r = 5$. In this case we go on to the next \mathcal{S}_k . Another constraint arises from the different powers of I and T in the group multiplication table. It will lead to a condition which determines one of these quantum numbers in terms of the other, e.g., $I = T$ at $r = 5$. At even r there will be another condition linking (the value of) S and T , leading to two viable sectors per (even or mixed) excitation number sector, which is what we need to produce exactly four sectors of bound states at every r .

APPENDIX B: SECTORS, SIGNS, AND SINES

The assignment of symmetry sectors can be confusing due to several signs and quantum numbers involved. Let's straighten things out by considering the simplest case, $r = 2$. Here, we have $\phi_2(x_1, x_2) \doteq |n\rangle$, so we think of the wave function ϕ_2 depending on two momentum fractions x_k , as being represented by an abstract vector $|n\rangle$ labeled by one excitation number. The orientation symmetry \mathcal{T} of the Hamiltonian acts on states only via their fermionic operators $b_{ij}(-x_k)$. By flipping color indices, \mathcal{T} effectively reverses the order of the operators and thus the order of momentum fractions. To compensate, also the wave function must be rewritten with the last momentum fraction now being first. Wave functions might be even or odd under this reversal, and the trace of operators might or might not acquire a sign under \mathcal{T} . To assign a state to a sector, we have to take into account both behaviors. For example, a two-parton state looks like²²

$$|\Phi_2\rangle = \frac{1}{N_c} \int_0^{\frac{1}{2}} dx \phi_2(x, 1-x) \text{Tr}[b(-x)b(1-x)]|0\rangle.$$

Since

$$\begin{aligned} \mathcal{T} b_{ij}(-x) b_{ji}(1-x) &= b_{ji}(-x) b_{ij}(1-x) \\ &= \text{Tr}[b(-x)b(1-x)] \end{aligned}$$

²²The integral only runs to $\frac{1}{2}$ ($\frac{1}{r}$ in general) since operator products with flipped momentum fractions are redundant under the trace. The relative sign between the two Fock states $\text{Tr}[b(-x)b(1-x)]|0\rangle = -\text{Tr}[b(1-x)b(x)]|0\rangle$ will give rise to a wave function odd under argument reversal here, but not in general.

is \mathcal{T} even without the need to reverse the arguments, this state belongs to the \mathcal{T} even sector, even though the wave function ϕ_2 is odd under reversal of its arguments,²³ cf. Eq. (16). Technically, the wave function is represented as

$$\begin{aligned} |\phi_2\rangle &= |n\rangle + \mathcal{T}\mathcal{T}|n\rangle + I(\mathcal{T}|n\rangle) + \mathcal{T}I\mathcal{T}|n\rangle \\ &= |n\rangle + T(-1)^n|-n\rangle + I|-n\rangle + IT(-1)^n|n\rangle, \end{aligned}$$

and also as²⁴

$$|\phi_2\rangle = |n\rangle + (-1)^{r+1}\mathcal{C}|n\rangle = |n\rangle - (-1)^n|-n\rangle, \quad (\text{B1})$$

where we recalled that $\mathcal{T}|n\rangle = \mathcal{C}|n\rangle = (-1)^n|-n\rangle$. So for even n we need $T = I$ and the vanishing of the wave function at $x = 0$ is guaranteed by the latter equation which produces a sine wave function with $I = -1$. Clearly, the even n states constitute the massive sector. If n is odd, on the other hand, we get cosines with $I = 1$. In both cases, $T = -1$ even though the states are \mathcal{T} even, $\mathcal{T}|\phi_2\rangle = |\phi_2\rangle$. In sum, its quantum number T does not directly give away the \mathcal{T} sector of a state,²⁵ but a negative (positive) I quantum number always results in a wave function made of sines(cosines).

²³Of course, we can also flip the b operators and the wave-function arguments, with the same result.

²⁴We are deliberately sloppy here with normalization, which will be taken care of by the Hilbert space integral, Eq. (26).

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