


Deflection angle of light for an observer and source at finite distance from a rotating wormhole

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By using a method improved with a generalized optical metric, the deflection of light for an observer and source at finite distance from a lens object in a stationary, axisymmetric and asymptotically flat spacetime has been recently discussed [Ono *et al.* *Phys. Rev. D* **96**, 104037 (2017)]. By using this method, in the weak field approximation, we study the deflection angle of light for an observer and source at finite distance from a rotating Teo wormhole, especially by taking account of the contribution from the geodesic curvature of the light ray in a space associated with the generalized optical metric. Our result of the deflection angle of light is compared with a recent work on the same wormhole but limited within the asymptotic source and observer [Jusufi, Övgün, *Phys. Rev. D* **97**, 024042 (2018)], in which they employ another approach proposed by Werner with using the Nazim's osculating Riemannian construction method via the Randers-Finsler metric. We show that the two different methods give the same result in the asymptotic limit. We obtain also the corrections to the deflection angle due to the finite distance from the rotating wormhole.

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I. INTRODUCTION

Studies on wormholes can be dated back to the celebrated paper by Einstein and Rosen [1], in which they investigated what is a particle in the theory of general relativity, and consequently they noticed a spacetime bridge connecting two distinct spacetime events, called Einstein-Rosen bridges. Decades later, Wheeler argued that such spacetime bridges should be unstable even for a traveling photon [2]. Misner and Wheeler dubbed such a handle of multiply connected spacetime wormholes [3]. Morris *et al.*, nevertheless, discussed traversable wormholes by holding a throat of the wormholes open with hypothetical exotic matter (that must have negative energy in the framework of general relativity) [4]. Later, other types of traversable wormholes were found as allowable solutions to the Einstein equation, especially in a 1989 paper by Matt Visser [5], in which a spacetime tunnel through the wormhole can be constructed where a shortcut path does not pass through a region of such exotic matter. This type of wormhole models are called thin-shell wormholes. See Ref. [6] for comprehensive reviews on wormholes. In the Gauss-Bonnet gravity (an alternative to the theory of general relativity), however, exotic matter is not required for wormholes to exist [7]. The latter wormhole model is based on an idea of modifying the left-hand side (namely, the geometrical side) of the Einstein equation, while the former models are due to some modifications of the right-hand side, especially inclusions of hypothetical exotic matter.

Null and causal structures of such wormhole spacetimes are expected to be very different from those around stellar

objects and even those in black hole spacetimes. Therefore, the light propagation in wormhole spacetimes has attracted a lot of interest. The deflection of light in an Ellis wormhole was first discussed by Chetouani and Clement [8,9]. The gravitational lensing as an observational probe of wormholes was investigated [10–21]. In the weak field approximation, the deflection angle of light was derived in terms of the inverse power of the photon impact parameter, for instance by Dey and Sen [22]. However, Nakajima and Asada showed that this result breaks down at the next-to-leading order, though the leading order term is correct [23]. This problem occurs due to the regularity at the center of wormholes and therefore some methods valid for black holes no longer work for wormholes. On the observational side of wormholes, Takahashi and Asada showed that the Sloan Digital Sky Survey Quasar Lens Search (SQLS) put the upper bound on the cosmic abundance of Ellis wormholes [24].

Most of the work on the wormhole lensing mentioned above is for nonrotating wormholes. Very recently, Jusufi and Övgün [25] discussed the gravitational lensing by rotating Teo wormholes [26], in which they use the Gibbons-Werner approach based on the Gauss-Bonnet theorem [27]. An extension of the Gibbons-Werner approach for calculating the deflection of light for the case of a Kerr black hole was done by Werner [28], in which he used Nazim's method of constructing the osculating Riemannian manifold and computed the Randers-Finsler form of the metric for the Kerr spacetime. To be more precise, Jusufi and Övgün employed Werner's method to calculate the deflection angle of light for the asymptotic

observer and source in the weak field approximation of a rotating Teo wormhole. The condition that the observer and source are located at the null infinity is a requirement for using Werner's method, because the Werner's extension by using the Nazim's osculating Riemannian method needs that two ends of the light ray (corresponding to the observer and source, respectively) are in a Euclidean space. We should note that it is an open issue how to define angles in the Finsler geometry, though angles are well-defined in Euclidean regions of the Finsler geometry.

The main purpose of this paper is to discuss the deflection of light for an observer and source at finite distance from a rotating Teo wormhole as the gravitational lens. For this purpose, we shall use a formulation developed in Ref. [29], which we shall call *generalized optical metric method* henceforth.

The method for investigating the light propagation in a static and spherically symmetric spacetime was reexamined by Gibbons and Werner, who discussed a problem of how to determine a curve on a spatial surface in the optical geometry, where the metric used in the optical geometry was first called the optical metric [27]. The idea of what Gibbons and Werner call the optical geometry may be related with the optical reference geometry that was used to describe inertial forces in general relativity by Abramowicz *et al.* [30], and may be connected also with the idea of the optical 3-geometry that was introduced to discuss thermal Green's functions for black holes by Gibbons and Perry [31]. The optical geometry may be also called the optical reference geometry or Fermat geometry [28]. The merit of the optical metric is that the arc length along the light ray with this metric is directly related with the time associated with the timelike Killing vector, when the spacetime is stationary. Namely, the optical metric describes the Fermat's principle for the light propagation in a manner simpler than other spatial projections of the four-dimensional metric such as the intrinsic metric in the Arnowitt-Deser-Misner (ADM) formulation. The generalized optical method is an improved method for calculating the deflection angle of light especially for the nonasymptotic observer and source with the Weyl-Lewis-Papapetrou metric form of a stationary, axisymmetric and asymptotically flat spacetime (but in the polar coordinates, though it is usually described in the cylindrical coordinates [32–34]), by extending an earlier work on static, spherically symmetric and asymptotically flat spacetimes [35]. The generalized optical metric method has been used for discussions on the light deflection for the case of Kerr black holes [29].

There are the pros and cons in the generalized optical metric method. The merit of this method is that it enables us to calculate the light deflection not only for an asymptotic observer and source but also for nonasymptotic cases. As stated already, Werner's method, which was used by Jusufi and Övgün, is currently limited within the case of an asymptotic observer and source, because the observer and

source are needed to be in a Euclidean space of the Finsler geometry. The price for using the generalized optical metric method is that we have to take account of the geodesic curvature of the light ray in the optical geometry and have to do the path integral of the geodesic curvature. We note that the light ray is not necessarily geodesic in the optical geometry, though the light ray follows the null geodesic in a four-dimensional spacetime [29]. In the present paper, we shall explicitly calculate the geodesic curvature in the optical geometry for rotating Teo wormholes and perform its path integral. A point is that a light ray in Werner's approach is treated as a curve in a space described by the Randers-Finsler type metric, while the generalized optical metric approach discusses a light ray as a curve in a space that is defined by introducing the optical metric. Two spaces in the two methods are different from each other. Therefore, it is important to ask whether both methods give the same deflection angle of light, even if the same limiting case as the asymptotic observer and source is taken. If the deflection angle depended on these calculation methods, it might not be useful for gravitational lensing observations. We shall show that it is not the case. Corrections for the finite distance cases will be also discussed.

In the rest of this paper, the observer is called the receiver (R), in order to avoid a confusion in notations between the observer and the origin of the coordinates (O). This paper is organized as follows. Section II describes a rotating Teo wormhole and its optical metric form. In Sec. III, we perform detailed calculations of the Gaussian curvature and geodesic curvature to obtain the deflection angle of light in the weak field approximation of the rotating Teo wormhole. A comparison with the earlier work [25] is also done. Section IV is devoted to the conclusion. We use the unit of $c = 1$ throughout this paper.

II. GENERALIZED OPTICAL METRIC FOR ROTATING TEO WORMHOLE

A. Rotating Teo wormhole

A general form of a static axially symmetric rotating wormhole was first described by Teo in Ref. [26]. Its spacetime metric reads

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{1 - \frac{b_0}{r}} + r^2 H^2 [d\theta^2 + \sin^2 \theta (d\phi - \omega dt)^2], \quad (1)$$

where the coordinates are $-\infty < t < +\infty$, $b_0 \leq r < +\infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and we denote

$$N = H = 1 + \frac{d(4a \cos \theta)^2}{r}, \quad (2)$$

$$\omega = \frac{2a}{r^3}. \quad (3)$$

The Teo wormhole by Eq. (1) is a rotating generalization of the static Morris-Thorne wormhole. A rigidly rotating wormhole would be a case of $N = H = 1$ and $\omega = \text{const}$. The spacetime of Teo is stationary and axially symmetric and asymptotically flat, and the spatial coordinates r , θ and ϕ coincide asymptotically with the spherical coordinates of a flat space. Here, b_0 denotes the throat radius of the wormhole where two identical asymptotically flat regions are joined together at the throat $r = b_0$. The parameter a is the total angular momentum of the wormhole, and the parameter ω is the angular velocity of the wormhole relative to the asymptotic rest frame, which gives rise to the Lense-Thirring effect in general relativity.

As already noticed by Teo [26], the wormhole metric in Eq. (1) violates the null energy condition. The wormhole (1) has no singularities in the curvature tensor and no event horizon. The Teo wormhole metric is a purely geometrical object in the sense that the metric does not take account of the stress-energy tensor in the Einstein equation. As for the possible matter source of a rotating wormhole, we refer to [36], in which general requirements on the stress-energy tensor were discussed to generate a uniformly rotating wormhole. Here, we are just interested in the geometry of spacetime (1) as being an exact solution of the gravitational field equations.

B. Optical metric

Following Ref. [29], we define the generalized optical metric γ_{ij} ($i, j = 1, 2, 3$) by a relation as

$$dt = \sqrt{\gamma_{ij} dx^i dx^j} + \beta_i dx^i, \quad (4)$$

which is immediately obtained by solving the null condition ($ds^2 = 0$) for dt . Note that γ_{ij} is not the induced metric in the ADM formalism.

For the rotating Teo wormhole by Eq. (1), we find the components of the generalized optical metric as

$$\begin{aligned} \gamma_{ij} dx^i dx^j = & \frac{r^7}{(r-b_0)(r^4-4a^2\sin^2\theta)(16da^2\cos^2\theta+r)^2} dr^2 \\ & + \frac{r^6}{r^4-4a^2\sin^2\theta} d\theta^2 + \frac{r^{10}\sin^2\theta}{(r^4-4a^2\sin^2\theta)^2} d\phi^2. \end{aligned} \quad (5)$$

We obtain the components of β_i as

$$\beta_i dx^i = -\frac{2ar^3\sin^2\theta}{r^4-4a^2\sin^2\theta} d\phi. \quad (6)$$

In the rest of the paper, we focus on the light rays in the equatorial plane, namely $\theta = \pi/2$. Then, the constant d in the metric does not appear.

III. DEFLECTION ANGLE OF LIGHT BY A ROTATING TEO WORMHOLE

A. Deflection angle of light

Let us begin this section with briefly summarizing the generalized optical metric method that enables us to calculate the deflection angle of light for a nonasymptotic receiver (denoted as R) and source (denoted as S) [29].

We define the deflection angle of light as [29]

$$\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}. \quad (7)$$

Here, Ψ_R and Ψ_S are angles between the light ray tangent and the radial direction from the lens object, defined in a covariant manner using the generalized optical metric, at the receiver location and the source, respectively. On the other hand, ϕ_{RS} is the coordinate angle between the receiver and source, where the coordinate angle is associated with the rotational Killing vector in the spacetime. If the space under study is Euclidean, this α becomes the deflection angle of the curve. This is consistent with the thin lens approximation in the standard theory of gravitational lensing.

By using the Gauss-Bonnet theorem [37,38], Eq. (7) can be recast into [29]

$$\alpha = -\iint_{R \square S} K dS + \int_S^R \kappa_g d\ell, \quad (8)$$

where K is defined as the Gaussian curvature at some point on the two-dimensional surface, dS denotes the infinitesimal surface element defined with $\gamma_{ij}^{(2)}$ where $\gamma_{ij}^{(2)}$ denotes the two-dimensional metric in the equatorial plane ($\theta = \pi/2$) and reads $\gamma_{ij}^{(2)} dx^i dx^j = \frac{r^5}{(r-b_0)(r^4-4a^2)} dr^2 + \frac{r^{10}}{(r^4-4a^2)^2} d\phi^2$. $R \square S$ denotes a quadrilateral embedded in a curved space with γ_{ij} , κ_g denotes the geodesic curvature of the light ray in this space and $d\ell$ is an arc length defined with the generalized optical metric (see Fig. 2 in Ref. [29]). It is shown by Asada and Kasai that this $d\ell$ for the light ray is an affine parameter [39]. Note that only the surface integral term appears in the right-hand side of Eq. (8) if $\beta_i = 0$ (see [35]), and the path integral term is proportional to the total angular momentum of the wormhole (as shown in Sec. III C), hence caused by rotational (i.e., Lense-Thirring) effects of the spacetime. We shall make detailed calculations of the rhs of Eq. (8) below.

B. Gaussian curvature

For the equatorial case of a rotating Teo wormhole, the Gaussian curvature in the weak field approximation is calculated as

$$\begin{aligned}
 K &= \frac{R_{r\phi r\phi}}{\det \gamma_{ij}^{(2)}} \\
 &= \frac{1}{\sqrt{\det \gamma_{ij}^{(2)}}} \left[\frac{\partial}{\partial \phi} \left(\frac{\sqrt{\det \gamma_{ij}^{(2)}}}{\gamma_{rr}^{(2)}} \Gamma^{\phi}_{rr} \right) \right. \\
 &\quad \left. - \frac{\partial}{\partial r} \left(\frac{\sqrt{\det \gamma_{ij}^{(2)}}}{\gamma_{rr}^{(2)}} \Gamma^{\phi}_{r\phi} \right) \right] \\
 &= -\frac{b_0}{2r^3} - \frac{56a^2}{r^6} + \mathcal{O}\left(\frac{a^2 b_0}{r^7}, \frac{a^4}{r^{10}}\right), \quad (9)
 \end{aligned}$$

where a and b_0 are bookkeeping parameters in the weak field approximation. As for the first line of Eq. (9), please see e.g., p. 263 in Ref. [40]. We note that the first term in the second line of Eq. (9) does not contribute because $\Gamma^{\phi}_{rr} = 0$. It is not surprising that this Gaussian curvature does not agree with Eq. (26) in Jusufi and Övgün [25], because their Gaussian curvature describes another surface that is associated with the Randers-Finsler metric different from our optical metric.

In order to perform the surface integral of the Gaussian curvature in Eq. (8), we have to determine the boundary of the integration domain. In other words, we need the light ray as a function of $r(\phi)$. For the later convenience, we introduce the inverse of r as $u \equiv r^{-1}$. The orbit equation in this case becomes

$$\begin{aligned}
 \left(\frac{du}{d\phi}\right)^2 &= \frac{1}{b^2} - u^2 - \frac{b_0 u}{b^2} + b_0 u^3 \\
 &\quad - \frac{4au}{b^3} - \frac{4a(b_0 u - b_0^2 u^2)}{b^3} + \mathcal{O}\left(\frac{a^2}{b^6}\right), \quad (10)
 \end{aligned}$$

where b is the impact parameter of the photon. See e.g., Ref. [29] on how to obtain the photon orbit equation in the axisymmetric and stationary spacetime. The orbit equation is iteratively solved as

$$u = \frac{\sin \phi}{b} + \frac{\cos^2 \phi}{2b^2} b_0 - \frac{2}{b^3} a + \mathcal{O}\left(\frac{b_0^2}{b^3}, \frac{ab_0}{b^4}\right). \quad (11)$$

By using Eq. (11) as the iterative solution for the photon orbit, the surface integral of the Gaussian curvature in Eq. (8) is calculated as

$$\begin{aligned}
 - \int \int_{R \square S} K dS &= \int_{\infty}^{r(\phi)} dr \int_{\phi_S}^{\phi_R} d\phi \left(-\frac{b_0}{2r^2} \right) + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\
 &= \frac{b_0}{2} \int_0^{\frac{\sin \phi}{b} + \frac{\cos^2 \phi}{2b^2} b_0 - \frac{2}{b^3} a} du \int_{\phi_S}^{\phi_R} d\phi + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\
 &= \frac{b_0}{2} \int_{\phi_S}^{\phi_R} \left[\frac{\sin \phi}{b} \right] d\phi + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\
 &= \frac{b_0}{2} \left[-\frac{\cos \phi}{b} \right]_{\phi=\phi_S}^{\phi_R} + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\
 &= \frac{b_0}{2b} \left(\sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right), \quad (12)
 \end{aligned}$$

where we used $\sin \phi_R = bu_R + \mathcal{O}(ab^{-2}, b_0 b^{-1})$ and $\sin \phi_S = bu_S + \mathcal{O}(ab^{-2}, b_0 b^{-1})$ by Eq. (11) in the last line.

C. Geodesic curvature

The geodesic curvature provides an important contribution to our calculations of the light deflection, though it is not usually described in standard textbooks on the general relativity. Hence, we follow Ref. [29] to briefly explain the geodesic curvature here. The geodesic curvature can be defined in the vector form as (e.g., [38])

$$\kappa_g \equiv \vec{T}' \cdot (\vec{T} \times \vec{N}), \quad (13)$$

where we assume a parametrized curve with a parameter, \vec{T} is the unit tangent vector for the curve by reparametrizing

the curve using its arc length, \vec{T}' is its derivative with respect to the parameter, and \vec{N} is the unit normal vector for the surface. Equation (13) can be rewritten in the tensor form as

$$\kappa_g = \epsilon_{ijk} N^i a^j e^k, \quad (14)$$

where \vec{T} and \vec{T}' correspond to e^k and a^j , respectively. Here, the Levi-Civita tensor ϵ_{ijk} is defined by $\epsilon_{ijk} \equiv \sqrt{\gamma} \epsilon_{ijk}$, where $\gamma \equiv \det(\gamma_{ij})$, and ϵ_{ijk} is the Levi-Civita symbol ($\epsilon_{123} = 1$). In the present paper, we use γ_{ij} in the above definitions but not g_{ij} . Note that $a^i \neq 0$ in the three-dimensional optical metric by nonvanishing g_{0i} [29], even though the light signal follows a geodesic in the

four-dimensional spacetime. On the other hand, we notice that if we would have a geodesics in the optical metric then $a^i = 0$ and thus $\kappa_g = 0$.

As shown first in Ref. [29], Eq. (14) is rewritten as

$$\kappa_g = -\epsilon^{ijk} N_i \beta_{jk}, \quad (15)$$

where we use $\gamma_{ij} e^i e^j = 1$.

Henceforth, we focus on the equatorial plane ($\theta = \pi/2$). Then, let us denote the unit normal vector as N_p . This vector is normal to the θ - constant surface. Therefore, it satisfies $N_p \propto \nabla_p \theta = \delta_p^\theta$, where ∇_p is the covariant derivative associated with γ_{ij} . Hence, N_p is written in a form as $N_p = N_\theta \delta_p^\theta$. By noting that N_p is a unit vector ($N_p N_q \gamma^{pq} = 1$), we obtain $N_\theta = \pm 1/\sqrt{\gamma^{\theta\theta}}$. Therefore, N_p can be expressed as

$$N_p = \frac{1}{\sqrt{\gamma^{\theta\theta}}} \delta_p^\theta, \quad (16)$$

where we choose the upward direction without loss of generality.

For the equatorial case, one can show

$$\epsilon^{\theta pq} \beta_{q|p} = -\frac{1}{\sqrt{\gamma}} \beta_{\phi,r}, \quad (17)$$

where the comma denotes the partial derivative, we use $\epsilon^{\theta r \phi} = -1/\sqrt{\gamma}$ and we note $\beta_{r,\phi} = 0$ owing to the axisymmetry. By using Eqs. (16) and (17), the geodesic curvature of the light ray with the generalized optical metric becomes [29]

$$\kappa_g = -\sqrt{\frac{1}{\gamma \gamma^{\theta\theta}}} \beta_{\phi,r}. \quad (18)$$

For the Teo wormhole case, this is obtained as

$$\kappa_g = -\frac{2a}{r^3} + \mathcal{O}\left(\frac{a^3}{r^7}, \frac{a^3 b_0}{r^8}\right). \quad (19)$$

We examine the contribution from the geodesic curvature. This contribution is the path integral along the light ray (from the source to the receiver), which is computed as

$$\begin{aligned} \int_S^R \kappa_g d\ell &= \int_R^S \frac{2a}{r^3} d\ell + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\ &= \int_{\pi/2-\phi_R}^{\pi/2-\phi_S} \frac{2a \cos \vartheta}{b^2} d\vartheta + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\ &= \frac{2a}{b^2} \left[\sin\left(\frac{\pi}{2} - \phi_S\right) - \sin\left(\frac{\pi}{2} - \phi_R\right) \right] + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right) \\ &= \frac{2a}{b^2} \left(\sqrt{1 - b^2 u_S^2} + \sqrt{1 - b^2 u_R^2} \right) + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right), \end{aligned} \quad (20)$$

for the retrograde case of the photon orbit. In the last line, we used $\sin \phi_R = bu_R + \mathcal{O}(ab^{-2}, b_0 b^{-1})$ and $\sin \phi_S = bu_S + \mathcal{O}(ab^{-2}, b_0 b^{-1})$ by Eq. (11). The above contribution becomes $4a/b^2$, as $r_R \rightarrow \infty$ and $r_S \rightarrow \infty$. The sign of the right-hand side of Eq. (20) changes, if the photon orbit is prograde.

D. Deflection angle

By combining Eqs. (12) and (20), the deflection angle of light for the prograde case is obtained as

$$\alpha_{\text{prog}} = \frac{b_0}{2b} \left(\sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) - \frac{2a}{b^2} \left(\sqrt{1 - b^2 u_S^2} + \sqrt{1 - b^2 u_R^2} \right) + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right). \quad (21)$$

The deflection angle for the retrograde case is

$$\alpha_{\text{retro}} = \frac{b_0}{2b} \left(\sqrt{1 - b^2 u_R^2} + \sqrt{1 - b^2 u_S^2} \right) + \frac{2a}{b^2} \left(\sqrt{1 - b^2 u_S^2} + \sqrt{1 - b^2 u_R^2} \right) + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right). \quad (22)$$

For both cases, the source and receiver may be located at finite distance from the wormhole. Equations (21) and (22) show that the light deflection is increasing with decreasing impact parameter and increasing throat radius. The light

deflection in the prograde (retrograde) direction is decreasing (increasing) with increasing the angular momentum of the Teo wormhole, because the local inertial frame (in which the light propagates at the light speed c in general

relativity) moves faster (slower), and hence the light signal feels the gravitational pull for shorter (longer) time. Regarding the light propagation around a rotating object, similar physical explanations based on the dragging of the inertial frame were done about the Shapiro time delay by Laguna and Wolsczan [41].

One can see that, in the limit as $r_R \rightarrow \infty$ and $r_S \rightarrow \infty$, Eqs. (21) and (22) become

$$\begin{aligned}\alpha_{\text{prog}} &\rightarrow \frac{b_0}{b} - \frac{4a}{b^2} + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right), \\ \alpha_{\text{retro}} &\rightarrow \frac{b_0}{b} + \frac{4a}{b^2} + \mathcal{O}\left(\frac{b_0^2}{b^2}, \frac{ab_0}{b^3}\right).\end{aligned}\quad (23)$$

They agree with Eqs. (39) and (56) in Jusufi and Övgün [25], in which they are restricted within the asymptotic source and receiver ($r_R \rightarrow \infty$ and $r_S \rightarrow \infty$).

IV. CONCLUSION

In the weak field approximation, we have discussed the deflection angle of light for an observer and source at finite distance from a rotating Teo wormhole. We have shown

that both the Werner's method and the generalized optical metric method give the same deflection angle at the leading order of the weak field approximation, if the receiver and source are at the null infinity. We have also found corrections for the deflection angle due to the finite distance from the wormhole. It is left for the future to study higher order terms in the weak field approximation of a rotating Teo wormhole and to examine also the strong deflection limit.

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