Ion traps and the memory effect for periodic gravitational waves

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(Received 6 July 2018; published 23 August 2018)

The Eisenhart lift of a Paul trap used to store ions in molecular physics is a linearly polarized periodic gravitational wave. A modified version of Dehmelt's Penning trap is, in turn, related to circularly polarized periodic gravitational waves, sought in inflationary models. Similar equations also govern the Lagrange points in celestial mechanics. The explanation is provided by anisotropic oscillators.

DOI: 10.1103/PhysRevD.98.044037

I. INTRODUCTION

The memory effect of gravitational waves concerned, originally, the motion of test particles after the passage of a sudden burst of gravitational wave. See [1–11] and references therein for a nonexhaustive list. Later, the meaning of the expression was extended to include also the effect of periodic gravitational waves [12] sought in inflationary models [13,14]. Recent studies [12,15–17] reveal striking similarities with that of storing molecular ions, considered half a century ago [18–22]. In this paper, we argue that this similarity is not a coincidence: Paul traps [18,19] correspond indeed to linearly polarized periodic (LPP) gravitational waves; Dehmelt's Penning trap [20–22] is in turn reminiscent of circularly polarized periodic (CPP) gravitational waves [12], sought in inflationary models [13,14]. A CPP wave is also the "double copy" of Białynicki-Birula's electromagnetic vortex [15,23]. Similar considerations apply to the Lagrange points in the 3-body problem in Celestial Mechanics [24,25].

The similarity between these at first sight far remote physical phenomena, observed on so different scales, is explained mathematically by tracing back to anisotropic oscillators. The motion of a test particle in a CPP GW boils down, in particular, to Hill's equations for a harmonic oscillator in a constant magnetic field.

Time-dependent (or not), anisotropic (or not) oscillators, described by Hill's equations and their particular case studied by Mathieu have indeed a huge literature impossible to cite here. Their general study goes beyond our scope; here, our interest is limited to those cases which have direct relevance for the memory effect for periodic gravitational waves.

Apart of pointing out the far-reaching analogies mentioned above, we argue that applying those well-elaborated tools of ion physics to gravitational waves sheds some new light on the memory effect. To make our paper self-contained, we include some facts which are familiar for specialists of either of the fields, but, perhaps, not for every reader.

II. PAUL TRAPS

The intuitive explanation of the working of Paul's ingenious Ionenkäfig (now called the Paul trap) to capture ions [18,19] has been given by Paul himself in his Nobel Lecture [18]. Let us consider indeed an electric field in the $X^+ - X^-$ plane, given by an anisotropic harmonic electric (quadrupole) potential

$$\Phi = \frac{\Phi_0}{2}((X^+)^2 - (X^-)^2), \qquad \Phi_0 = \text{const.} \tag{2.1}$$

Putting $a = (e/m)\Phi_0$, the equations of motion of a spinless ion with charge e and mass m are

$$\ddot{X}^\pm \pm aX^\pm = 0, \tag{2.2}$$

where the dot $(\dot{.})$ means d/dt with t denoting nonrelativistic time. The opposite signs in (2.2) come from the relative minus sign in (2.1), required by the Laplace condition $\Delta\Phi = 0$ which expresses the fact that there are no sources (charges) inside the trap. For $a > 0$ (say), the

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electric force is thus attractive in the X^+, and repulsive in the X^- coordinate, yielding bounded oscillations in the first, but escaping motion in the second direction. Then Paul proposed stabilizing the position by adding a periodical perturbing electric force, i.e., to consider,[1]

$$\boldsymbol{F} = -e(\Phi_0 - \Gamma_0 \cos \omega t)\begin{pmatrix} X^+ \\ -X^- \end{pmatrix}, \tag{2.3}$$

where Φ_0 and Γ_0 are constants and ω is the frequency of the perturbation. The time dependent inhomogeneous rf voltage changes the sign of the electric force periodically. In a certain range of parameters, this yields stable motions both in the X^+ and X^- directions.

Mathematically, the planar Paul trap is described by the modified equations

$$\ddot{X}^{\pm} = \mp\, (a - 2q \cos \omega t) X^{\pm}, \tag{2.4}$$

where a and $q = e\Gamma_0/2m$ are constants determined by the applied dc and rf voltages, respectively. In Eqs. (2.4), we recognize two uncoupled the Mathieu equations, whose standard form is

$$\frac{d^2\xi}{d\tau^2} + (a - 2q\cos(2\tau))\xi = 0 \tag{2.5}$$

and whose solutions are combinations of the (even/odd) Mathieu cosine/sine functions $C(a,q,\tau)$ and $S(a,q,\tau)$, respectively. Mathieu functions have a rather complicated behavior; in a suitable range of the parameters, the solutions of (2.5) remain bounded, while in another one they are unbounded.

Returning to the Eqs. (2.4), we note that ω, the frequency of the oscillation, does not have (as long as it does not vanish) any influence on what will happen, only on when will it happen. Redefining indeed the time as

$$t \to U = \frac{1}{2}\omega t \Rightarrow \ddot{X}^i \to (\omega^2/4) d^2X^i/dU^2 \equiv (\omega^2/4)X'' \tag{2.6}$$

takes (2.4) into the standard Mathieu form (2.5) with redefined parameters, $d^2X^i/dU^2 \pm (\hat{a} - 2\hat{q}\cos 2U)X^i = 0$, $\hat{a} = (4/\omega^2)a$, $\hat{q} = (4/\omega^2)q$. Thus ω simply sets the time scale. Henceforth we shall use the redefined "time" coordinate U; d/dU will be denoted by prime, $(.)' = d/dU$.

[1]The magnetic field induced by the time-varying electric field is neglected. In his Nobel lecture Paul illustrated his idea by to putting a ball on a rotating saddle surface [18], materially realized in glass; a photo is reproduced in Bialynicki-Birula's lecture [25].

III. PERIODIC GRAVITATIONAL WAVES

Equations similar to (2.4) have been met recently in a rather different context, namely for the memory effect, more precisely, for particle motion in the spacetime of a periodic gravitational wave [12], which is our main interest in this paper, and this is not a coincidence, as we now explain.

A convenient way to study nonrelativistic motion in $(d,1)$ dimensions with coordinates $(\boldsymbol{X}, U)$ is indeed to consider null geodesics in $(d+1,1)$-dimensional "Bargmann" space with coordinates $(\boldsymbol{X}, U, V)$, with the potential $\Phi(\boldsymbol{X}, U)$ entering into the UU component of the metric [26]. In detail, for the planar Paul trap, we have

$$d\boldsymbol{X}^2 + 2dUdV - 2\Phi(\boldsymbol{X}, U)dU^2, \tag{3.1a}$$

$$\Phi(\boldsymbol{X}, U) = \frac{1}{2}(a - 2q\cos 2U)((X^+)^2 - (X^-)^2), \tag{3.1b}$$

whose null geodesics project to nonrelativistic spacetime with coordinates $(X^{\pm}, U)$ precisely following Eqs. (2.4). Let us stress that the anisotropy of the profile follows from the requirement of Ricci-flatness of the metric: $R_{\mu\nu} = 0$ for (3.1a) which implies $\Delta\Phi = 0$. In conclusion, the Bargmann metric of the planar Paul trap is an exact plane gravitational wave.

More generally, an exact plane wave metric in four dimensions can be brought to the form

$$ds^2 = g_{ij}dX^i dX^j + 2dUdV + K_{ij}(U)X^iX^j dU^2, \tag{3.2a}$$

$$K_{ij}(U)X^iX^j = \frac{1}{2}\mathcal{A}_+(U)((X^+)^2 - (X^-)^2) + \mathcal{A}_\times(U)(X^+X^-), \tag{3.2b}$$

where $\mathcal{A}_+$ and $\mathcal{A}_\times$ are the $+$ and $\times$ polarization-state amplitudes [6,27,28]. The geodesic equations,

$$\frac{d^2\boldsymbol{X}}{dU^2} - K(U)\boldsymbol{X} = 0,$$

$$K(U) = (K_{ij}(U)) = \frac{1}{2}\begin{pmatrix} \mathcal{A}_+ & \mathcal{A}_\times \\ \mathcal{A}_\times & -\mathcal{A}_+ \end{pmatrix}, \tag{3.3a}$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}_+}{dU}((X^+)^2 - (X^-)^2) + \mathcal{A}_+\left(X^+\frac{dX^+}{dU} - X^-\frac{dX^-}{dU}\right) + \frac{1}{2}\frac{d\mathcal{A}_\times}{dU}X^+X^- + \mathcal{A}_\times\left(X^-\frac{dX^+}{dU} + X^+\frac{dX^-}{dU}\right) = 0, \tag{3.3b}$$

are decoupled: after solving (3.3a) for the transverse motion, (3.3b) can be integrated.

Equations (3.3a) belong to family of Hill-type equations which describe (possibly time-dependent and/or anisotropic) oscillators. Their general study goes well above our scope here. Having established the fundamental relation we focus henceforth our study to those cases which are

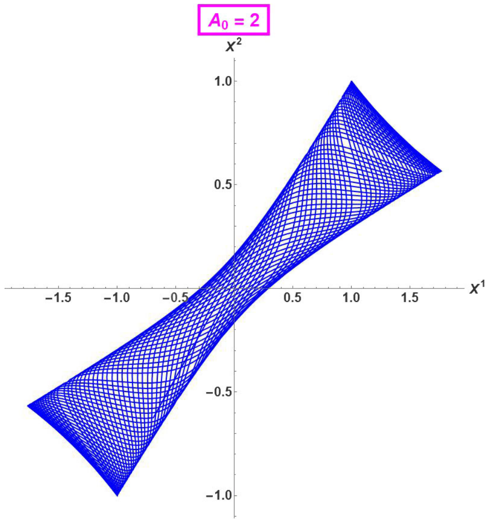


FIG. 1. In a weak linearly polarized periodic (LPP) wave, (3.4), the transverse coordinate $\boldsymbol{X}(U)$ oscillates in a bounded "bow tie"-shaped domain. The initial conditions are $\dot{X}^+(U=0) = \dot{X}^-(U=0) = 0$ (at rest for $U=0$), at initial position $X^+(U=0)=1$, $X^-(U=0)=0$.

directly relevant for us—namely to the motion of test particles initially at rest in a circularly polarized gravitational wave.

Henceforth we focus our attention at the transverse motion.

(i) The Bargmann metric of the Paul trap, (3.1), is a linearly polarized gravitational wave with periodic profile. Its properties for $a=0$, i.e., for the periodic profile

$$\mathcal{A}_+ = A_0 \cos 2U, \qquad \mathcal{A}_\times = 0 \tag{3.4}$$

were studied in [12] (see also e.g., [6]), shown in Fig. 1 above. For particular values of the parameters, one obtains bound motions. The intuitive explanation is precisely that of Paul recalled in Sec. II: in a given "moment" U, one of the oscillators is attractive and the other is repulsive, with strength $\mathcal{A}_+ = A_0 \cos 2U$. However, as "time" goes on, the strength varies, and when the cosine changes sign, the attractive and repulsive sectors are interchanged, as mentioned in Sec. II.

A sufficiently strong wave breaks up the bound motion.

(ii) The general form in Eq. (3.2b) allows, however, for more general profiles, and now we turn to waves with circularly polarized periodic profile (CPP), considered before e.g., in [12],

$$K = (K_{ij}) = \frac{A_0}{2}\begin{pmatrix} \cos 2U & \sin 2U \\ \sin 2U & -\cos 2U \end{pmatrix}$$
$$A_0 = \text{const} > 0. \tag{3.5}$$

The transverse equations of motion, $\boldsymbol{X}'' = K\boldsymbol{X}$, should be supplemented by appropriate initial conditions. In the sandwich case one usually considers particles which are at rest in the before zone. But a periodic wave has no before zone, and here we propose the initial condition,[2]

$$\text{rest at } U=0 \quad \text{i.e.,} \quad \boldsymbol{X}'(0) = 0. \tag{3.6}$$

Then numerical calculations [12] yield Fig. 2: for a sufficiently weak wave, all motions remain confined to a toroidal region; for a strong wave the trajectory becomes instead unbounded: the particle is ejected. Below we show that the problem admits an exact analytic solution. Following a suggestion of Kosinski [29], the first step is to switch to a rotating frame by setting

$$\begin{pmatrix} X^+ \\ X^- \end{pmatrix} = \begin{pmatrix} \cos U & -\sin U \\ \sin U & \cos U \end{pmatrix}\begin{pmatrix} Y^+ \\ Y^- \end{pmatrix}. \tag{3.7}$$

In terms of the new coordinates $Y^\pm$ the harmonic force becomes U-independent—at the price of introducing the cross terms $\mp 2(Y^\mp)'$,[3]

$$\boxed{(Y^\pm)'' \mp 2(Y^\mp)' - \Omega_\pm^2 Y^\pm = 0 \quad \text{where} \quad \Omega_\pm^2 = 1 \pm A_0/2.} \tag{3.8}$$

Our initial condition (3.6) is valid in Brinkmann-coordinates (3.2a); from Eq. (3.7), we infer instead

$$\boldsymbol{Y}'(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\boldsymbol{Y}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\boldsymbol{X}_0, \tag{3.9}$$

i.e., $\boldsymbol{Y}'(0)$ is obtained from $\boldsymbol{Y}(0) = \boldsymbol{X}_0$ by a 90-degree rotation, which corresponds precisely to rotating the coordinate system.

The equations of motion can be conveniently solved by chiral decomposition [32,33]. Equations (3.8) belong indeed to a Hamiltonian system in the plane, whose phase space is thus four-dimensional; it has coordinates $Y^\pm$ and $\Pi^\pm = (Y^\pm)'$. Then the idea is to choose "smart" phase-space coordinates we denote here by Z_+^a, Z_-^b, $a, b = 1, 2$ such that the system decouples onto uncoupled one-dimensional oscillators [32,33]. Searching for real coefficients $\alpha_\pm$ and $\beta_\pm$,

$$\Pi^+ = \alpha_+ Z_+^2 + \alpha_- Z_-^2, \qquad \Pi^- = -\beta_+ Z_+^1 - \beta_- Z_-^1, \tag{3.10a}$$

[2] Ions issued from accelerators and injected into the "Ionenkäfig" require different initial conditions.

[3] In Y-coordinates, U-translational symmetry is restored due to the manifest U-independence of the metric (4.11). Expressed in the original coordinates, the sixth "screw" symmetry [12,15,28,30,31] is recovered.

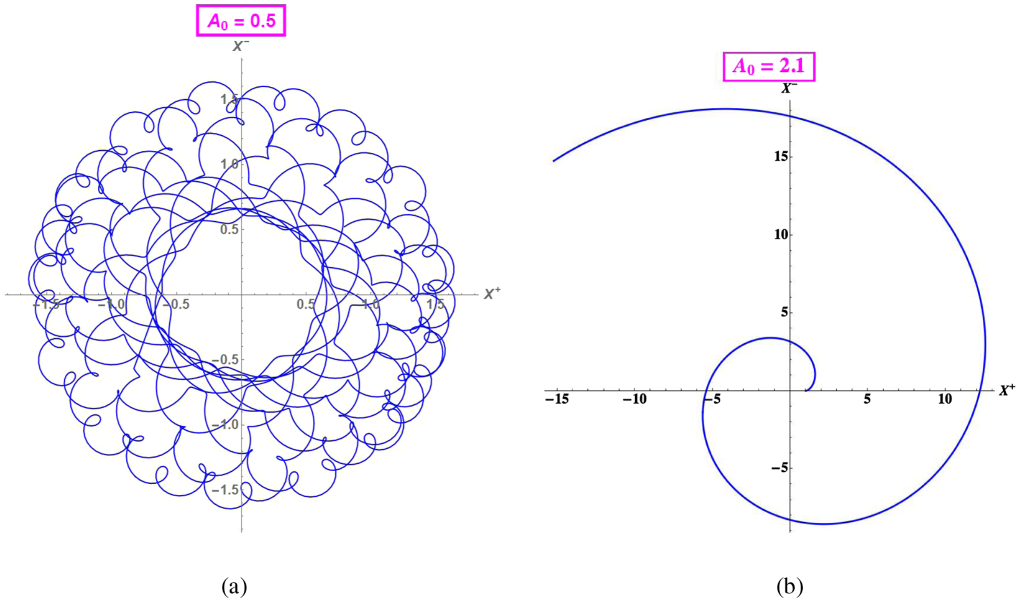


FIG. 2. (a) In a sufficiently weak circularly polarized gravitational wave (3.5) the transverse trajectory of a particle initially at rest remains confined in a toroidal region. (b) For a strong wave the trajectory becomes unbounded. The initial conditions are $\dot{X}^+(U=0) = \dot{X}^-(U=0) = 0$ and $X^+(U=0) = 1$, $X^-(U=0) = 0$.

$$Y^+ = Z^1_+ + Z^1_-, \qquad Y^- = Z^2_+ + Z^2_-, \tag{3.10b}$$

in terms of which both the symplectic form and the Hamiltonian separate, we find that for $\alpha_+ = 1$, $\alpha_- = \Omega_-^2$, $\beta_+ = \Omega_+^2$, $\beta_- = 1$, for example,[4]

$$\sigma = \sigma_+ - \sigma_- = -\frac{A_0}{2}[dZ^1_+ \wedge dZ^2_+ - dZ^1_- \wedge dZ^2_-], \tag{3.11a}$$

$$H = H_+ - H_- = \frac{A_0}{4}[(\Omega_+^2 Z^1_+ Z^1_+ + Z^2_+ Z^2_+) - (Z^1_- Z^1_- + \Omega_-^2 Z^2_- Z^2_-)]. \tag{3.11b}$$

The relative negative signs between the terms reflect here the chiral nature: the two oscillators turn in the opposite direction [34]. The Poisson brackets associated with the symplectic structure (3.11a) are

$$\{Z^1_+, Z^2_+\} = \frac{2}{A_0}, \qquad \{Z^1_-, Z^2_-\} = -\frac{2}{A_0},$$
$$\{Z^1_+, Z^2_-\} = \{Z^2_+, Z^1_-\} = 0. \tag{3.12}$$

Working out the Hamilton equations, we end up with uncoupled oscillator equations,

$$(Z^a_\pm)'' + \Omega_\pm^2 Z^a_+ = 0, \tag{3.13}$$

($a = 1, 2$), whose solutions (when none of the $\Omega_\pm$ vanishes[5]) are

$$Z^1_+ = A\cos(\Omega_+ U) + B\sin(\Omega_+ U), \tag{3.14a}$$

$$Z^2_+ = -\Omega_+(A\sin(\Omega_+ U) - B\cos(\Omega_+ t)), \tag{3.14b}$$

$$Z^1_- = C\cos(\Omega_- U) + D\sin(\Omega_- U), \tag{3.14c}$$

$$Z^2_- = -\frac{1}{\Omega_-}(C\sin(\Omega_- U) - D\cos(\Omega_- U)), \tag{3.14d}$$

where A, B, C, D are constants. Proceeding backwards, we obtain, using (3.10b),

$$Y^+(U) = A\cos(\Omega_+ U) + B\sin(\Omega_+ U) + C\cos(\Omega_- U) + D\sin(\Omega_- U), \tag{3.15a}$$

$$Y^-(U) = -\Omega_+(A\sin(\Omega_+ U) - B\cos(\Omega_+ U)) - \frac{1}{\Omega_-}(C\sin(\Omega_- U) - D(\cos\Omega_- U)). \tag{3.15b}$$

For a weak wave, i.e., whose amplitude is $A_0 < 2$, both frequencies $\Omega_\pm$ in (3.8) are real, implying that the motion, although complicated, remains bounded. A typical trajectory is shown in Figs. 7 and 8 of [12]. However, for a strong wave with amplitude $A_0 > 2$, one (and only one) of the $\Omega_\pm$

[4]Both the symplectic structure and the Hamiltonian are proportional to the wave amplitude, A_0, which drops seemingly out therefore from the equations of motion. It is, however, still hidden in the frequencies $\Omega_\pm$, cf., (3.8).

[5]The choice of the coefficients is not unique; another choice would interchange Ω_+ and Ω_-. When one of the $\Omega_\pm$s vanishes the corresponding motion is free [12,33,34].

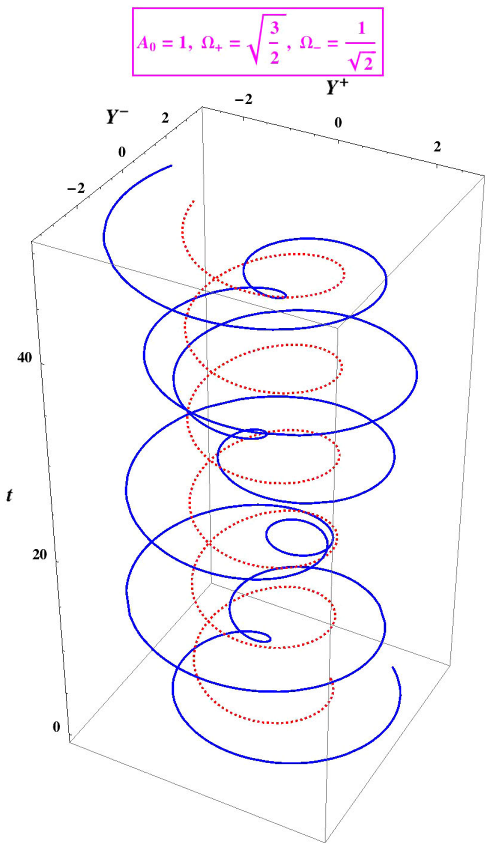


FIG. 3. In the circularly polarized periodic gravitational wave (3.5) the trajectory unfolded into "time" (in heavy blue) winds about the guiding center (dotted in red). If the wave is weak, $A_0 < 2$, the trajectory remains bounded, projecting to the plane consistently with Fig. 2.

becomes imaginary and the corresponding motion is unbounded: a sufficiently strong wave ejects the particle and makes it escape. Between those two regimes i.e., for $A_0 = 2$, one of the Ω's vanishes, and the motion in the corresponding direction is free; we recover Eq. (5.11) of [12], illustrated in Fig. 9 of that paper.

The solutions (3.15) are plotted for $A_0 < 2$ in Fig. 3. The one which has smaller real (or imaginary) frequency can be viewed (somewhat arbitrarily) as a guiding center, around which the one with the larger frequency winds around. For $A_0 = 2$ one of the frequencies vanishes, $\Omega_- = 0$, and the Y-trajectory is an ellipse drifting with constant speed [33,34]. A further rotation (3.7) backward would yield the trajectories $X^{\pm}(U)$.

As noticed by Ilderton [15], Eqs. (3.8) are actually identical to Eqs. (10a)–(10b) of Białynicki-Birula for a charged particle in the field of an electromagnetic vortex [23], and (3.15) above just reproduces his solution # (14)—with some additional insight, though. The relation will be further discussed elsewhere.

So far we studied classical motions only. However, the system could readily be quantized, courtesy of the chiral decomposition [23,32–34]. The Poisson brackets (3.12) are promoted to commutation relations,

$$[Z^i_+, Z^j_+] = \frac{2i\hbar}{A_0}\epsilon^{ij}, \qquad [Z^i_-, Z^j_-] = -\frac{2i\hbar}{A_0}\epsilon^{ij}, \tag{3.16}$$

where we denoted, with a light abuse of notations, the classical and quantum observables by the same symbols. Creation and annihilation operators can now be introduced,

$$a^\dagger = \sqrt{\frac{A_0\Omega_+}{4}}\left(Z^1_+ - \frac{i}{\Omega_+}Z^2_+\right),$$
$$a = \sqrt{\frac{A_0\Omega_+}{4}}\left(Z^1_+ + \frac{i}{\Omega_+}Z^2_+\right), \tag{3.17a}$$

$$b^\dagger = \sqrt{\frac{A_0\Omega_-}{4}}\left(Z^2_- - \frac{i}{\Omega_-}Z^1_-\right),$$
$$b = \sqrt{\frac{A_0\Omega_-}{4}}\left(Z^2_- + \frac{i}{\Omega_-}Z^1_-\right), \tag{3.17b}$$

whose nonvanishing commutators are, by (3.16),

$$[a, a^\dagger] = 1 = [b, b^\dagger]. \tag{3.18}$$

In their terms, the Hamiltonian is

$$H = \left(\Omega_+\left(a^\dagger a + \frac{1}{2}\right) - \Omega_-\left(b^\dagger b + \frac{1}{2}\right)\right). \tag{3.19}$$

The number operators $a^\dagger a$ and $b^\dagger b$ commute and have [$\hbar$-times] integer eigenvalues. The bound-state spectrum is, therefore,

$$E_{n_+,n_-} = \hbar\left[\Omega_+\left(n_+ + \frac{1}{2}\right) - \Omega_-\left(n_- + \frac{1}{2}\right)\right], \quad n_\pm = 0, 1\ldots. \tag{3.20}$$

Let us observe that the spectrum is not bounded from below, consistently with the relative minus sign of H_+ and H_- in the Hamiltonian (3.11b) reflecting the shape of the saddle potential.

IV. STURM-LIOUVILLE PROBLEM AND SWITCHING TO BJR

The key to study the memory effect for gravitational waves is to solve the Sturm-Liouville equation with an auxiliary condition [5,35,36],

$$P''(U) = K(U)P(U), \tag{4.1a}$$

$$P(U)^T P'(U) = (P'(U))^T P(U). \tag{4.1b}$$

This system should be supplemented by initial conditions. Let us recall that in the sandwich case, for which the wave vanishes outside an interval $[U_i, U_f]$, we required in the before zone $U \leq U_i$ the initial conditions

$$P(U) = \mathbb{I} \quad \text{and} \quad P'(U) = 0 \quad \text{for } U \leq U_i. \tag{4.2}$$

Below, we extend our study to the periodic case, which has no before zone. First we note that having solved the SL Eq. (4.1) for the 2×2 matrix $P(U)$,

(1) allows us to switch to Baldwin-Jeffery-Rosen (BJR) coordinates $(\boldsymbol{x}, u, v)$: setting

$$X^i = P^{ij}(u)x^j, \tag{4.3a}$$

$$U = u \tag{4.3b}$$

$$V = v - \frac{1}{4} y^i (G^{ij})'(u) x^j, \quad \text{where } G = P^T P \tag{4.3c}$$

carries the metric (3.2a) with $g_{ij} = \delta_{ij}$ to the BJR form

$$G_{ij}(u) dx^i dx^j + 2 du dv. \tag{4.4}$$

(2) The metric admits a five-parameter isometry [28,30,31,35,37,38]. The system is in particular symmetric with respect to translations and boosts, with associated conserved momenta

$$p_i = G_{ij} \dot{x}^j \tag{4.5a}$$

$$k_i = x_i(u) - H_{ij}(u) p_j \tag{4.5b}$$

where $H(u)$ is the 2×2 matrix $H(u) = \int_{u_0}^{u} G^{-1}(w) dw$ [5,31,38].

(3) Remember that, in the sandwich case, the usual assumption is that the particle is at rest in the before zone, $\boldsymbol{X}'(U) = 0$ for $U < U_i$. Then exporting to BJR by (4.3a),

$$\boldsymbol{x} = P^{-1}\boldsymbol{X} \Rightarrow \boldsymbol{x}'(0) = (-P^{-1}P'(P^{-1})(0)\boldsymbol{X}(0) + (P^{-1})\boldsymbol{X}'(0) = 0,$$

and thus the BJR coordinate also has vanishing initial velocity, $\boldsymbol{x}'(u) = 0$. Consequently the linear momentum vanishes, $\boldsymbol{p} = 0$ by (4.5a); then (4.5b) implies that $\boldsymbol{x}(u) = \boldsymbol{x}_0$ for all u. Returning to Brinkmann coordinates allows us to conclude, using (4.2), that the trajectory is simply

$$\boxed{\boldsymbol{X}(U) = P(U)\boldsymbol{X}_0.} \tag{4.6}$$

Now we extend our theory by replacing the initial conditions (4.2) by requiring that it holds at a chosen initial moment, e.g.,

$$\boxed{P(0) = \mathbb{I}, \quad \text{and} \quad P'(0) = 0.} \tag{4.7}$$

Then (4.6) remains true also in our case: the SL Eqs. (4.1) imply that it satisfies the equations of motion with the initial conditions $\boldsymbol{X}(0) = \boldsymbol{X}_0$ and $\boldsymbol{X}'(0) = 0$. Conversely, following the same argument as in the sandwich case, we observe that inverting (4.3a) shows that $\boldsymbol{X}'(0) = 0$ implies $\boldsymbol{x}'(0) = 0$ and therefore $\boldsymbol{p} = 0$ by (4.5a) from which (4.5b) allows us to infer $\boldsymbol{x}(u) = \boldsymbol{x}_0 = \text{const}$, so that (4.3a) yields once again (4.6).

A. Linearly polarized periodic (LPP) waves

In the linearly polarized case (3.4), MATHEMATICA tells us that Eq. (4.1a) can be solved: Using the shorthands $c_{A_0}(U) \equiv C(0, A_0, U)$ and $s_{A_0}(U) \equiv S(0, A_0, U)$, cf., Sec. II, we get

$$P(U) = \begin{pmatrix} A_{11} c_{A_0}(U) + B_{11} s_{A_0}(U) & A_{12} c_{A_0}(U) + B_{12} s_{A_0}(U) \\ A_{21} c_{-A_0}(U) + B_{21} s_{-A_0}(U) & A_{22} c_{-A_0}(U) + B_{22} s_{-A_0}(U) \end{pmatrix} \tag{4.8}$$

with A_{ij} and B_{ij} constants of integration. Then Eq. (4.1b) yields the compatibility constraints

$$A_{11} B_{12} = A_{12} B_{11} \quad \text{and} \quad A_{22} B_{21} = A_{21} B_{22}. \tag{4.9}$$

Assuming that, e.g., $A_{11} \neq 0$ and $A_{22} \neq 0$ we obtain B_{12} and B_{21}. The solution thus depends on 6 integration constants. Then it follows that from the parity-properties of the Mathieu functions that the initial condition $\boldsymbol{X}'(U) = 0$ in (3.6) can only be satisfied if all B_{ij} vanish (and then the auxiliary conditions (4.9) hold also). Then, consistently with Eq. (IV.3) of [12], the trajectory is given by pure Mathieu cosines with labels $\pm A_0$ and coefficients depending on the initial conditions,

$$X^{\pm}(U) = D^{\pm} c_{\pm A_0}(U), \tag{4.10}$$

where the constants $D^{\pm}$ are determined by the A_{ij} in (4.8) and the initial position $\boldsymbol{X}_0$.

B. Circularly polarized periodic (CPP) waves

Now we turn to the circularly polarized wave (3.5). Switching to a rotating frame by (3.7) allows us to present the metric as

$$ds^2 = d\boldsymbol{Y}^2 + 2dU(dV + A) - 2\Psi dU^2, \tag{4.11a}$$

$$A = -Y^- dY^+ + Y^+ dY^-,$$
$$\Psi = -\frac{1}{2}(\Omega_+^2 (Y^+)^2 + \Omega_-^2 (Y^-)^2). \tag{4.11b}$$

The only nonvanishing component of the Ricci tensor of (4.13) is

$$R_{UU} = -\partial_U(\nabla \cdot \boldsymbol{A}) - \frac{1}{2}B^2 - \Delta\Psi \tag{4.12}$$

where $B = \partial_i A_j - \partial_j A_i$. Ricci-flatness is thus confirmed for (4.11).[6]

This metric is consistent with the Bargmann description of a particle with charge = mass in a combined anisotropic oscillator plus a "magnetic" (alias Coriolis) field. The appearance of the new metric component implies that the potential term Ψ alone does not contain all information. The metric (4.11) has the form of a pp metric sometimes called "gyratonic",[7]

$$d\boldsymbol{Y}^2 + 2dU(dV + A) - 2\Psi dU^2, \tag{4.13}$$

where now $\Psi = -\frac{1}{2}H_{ij}Y^iY^j$ and where the 1-form $A = A_\mu dY^\mu$ is a vector potential. It has a gauge freedom: $A_i \to A_i - \partial_i\Lambda$ can be compensated by the "vertical" coordinate transformation $V \to V + \Lambda(\boldsymbol{Y})$.

Switching to BJR coordinates by replacing $\boldsymbol{X}$ by $\boldsymbol{Y}$ and $\boldsymbol{x}$ by $\boldsymbol{y}$ in (4.3a), the new term in (4.13) becomes

$$\begin{aligned}2dUA_j(Y)dY^j &= 2dud(A_j(Y)Y^j) - 2du\partial_i A_j(Y)dY^iY^j\\ &= 2dud(A_j(Y)Y^j) - 2((P')^T\partial AP)^{(ij)}y^iy^jdu^2\\ &\quad - 2(P^T\partial AP)^{ij}y^jdudy^i,\end{aligned}$$

where we used the shorthand ∂A for the matrix $[\partial A_j/\partial Y_i]$. The first term here can be reabsorbed into the V-change in (4.3c),

$$V = v - \frac{1}{4}y^i(a^{ij})'(u)y^j - 2A_j(Y)P^{jk}y^k.$$

The two other terms modify the Sturm-Liouville equations (4.1)[8]: the auxiliary condition (4.1b) becomes

$$P^TP' - (P')^TP = 2(P^T\partial AP),$$

whose consistency requires $\partial A = -\partial A^T$. Then the SL equation (4.1a) becomes

$$-\frac{1}{2}((P'')^TP + P^TP'') + P^THP = (P')^T\partial AP - P^T\partial AP'.$$

When $\partial_U(\partial A) = 0$, both equations are solved by $P'' = KP + 2\partial AP'$. To sum up, changing our notations to emphasize that the new system concerns the metric obtained after applying the rotational trick, $\boldsymbol{X} \to \boldsymbol{Y}$, $K = (K_{ij}) \to H = (H_{ij})$ and $P \to Q$, Eqs. (4.3) for the Brinkmann ⇔ BJR transcription should be replaced by

$$Y^i = Q^{ij}(u)y^j, \tag{4.14a}$$

$$U = u \tag{4.14b}$$

$$V = v - \frac{1}{4}y^i(G^{ij})'y^j - 2A_jQ^{jk}y^k, \quad G = Q^TQ. \tag{4.14c}$$

Note that Eqs. (4.3a) are formally unchanged while (4.3c) picks up a new term; however, the SL equations to be solved are now rather

$$Q'' = HQ + 2\partial AQ', \tag{4.15a}$$

$$\partial A = -\partial A^T, \tag{4.15b}$$

$$\partial_U(\partial A) = 0. \tag{4.15c}$$

Spelling out our formulas for the circularly polarized periodic wave, from (4.11) we infer that

$$\partial A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix}, \tag{4.16}$$

which are both U-independent. Thus our modified Sturm-Liouville equation becomes

$$\boxed{Q'' + 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}Q' - \begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix}Q = 0} \tag{4.17}$$

which is precisely Eq. (3.8) with the vector $\boldsymbol{Y}$ replaced by the 2×2 matrix Q. Replacing $\boldsymbol{X}$ by $\boldsymbol{Y}$ in (4.6), it follows that

$$\boxed{\boldsymbol{Y}(U) = Q(U)\boldsymbol{Y}_0} \tag{4.18}$$

is a solution of the equations of motion (3.8). Moreover, the initial condition (3.9) is satisfied provided,[9]

$$\boxed{Q(0) = \mathbb{I}, \qquad Q'(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.} \tag{4.19}$$

[6]This is hardly surprising: switching from X to Y is a mere coordinate change.

[7]Considered by Brinkmann back in 1925 [27] and used e.g., in [39]. Here, we deliberately changed our notations, $\Phi \to \Psi$, $K \to H$, to underline the difference with the previous discussion.

[8]Our formulas are valid in four dimensions.

[9]$Q'(0)$ is the matrix of a planar rotation by $\pi/2$.

Our new equation (4.17) has constant coefficients and can be solved analytically. Putting $Q = (Q_{ij})$ (4.17) is mapped indeed into two sets of equations of type (3.8), with the identifications $Q_{11} \leftrightarrow Y^+$, $Q_{21} \leftrightarrow Y^-$, $Q_{12} \leftrightarrow Y^+$, $Q_{22} \leftrightarrow Y^-$: the columns of Q are vectors of the form $(Y^+ Y^-)$ both of which satisfy (3.8). Therefore the general solution is, by (3.15), a combination with eight constants $A_i, \ldots, D_i$, $i = 1, 2$, cf., (4.8),

$$Q_{11}(U) = A_1 \cos(\Omega_+ U) + B_1 \sin(\Omega_+ U) + C_1 \cos(\Omega_- U) + D_1 \sin(\Omega_- U), \tag{4.20a}$$

$$Q_{21}(U) = -\Omega_+ (A_1 \sin(\Omega_+ U) - B_1 \cos(\Omega_+ U)) - \frac{1}{\Omega_-}(C_1 \sin(\Omega_- U) - D_1 \cos(\Omega_- U)), \tag{4.20b}$$

$$Q_{12}(U) = A_2 \cos(\Omega_+ U) + B_2 \sin(\Omega_+ U) + C_2 \cos(\Omega_- U) + D_2 \sin(\Omega_- U), \tag{4.20c}$$

$$Q_{22}(U) = -\Omega_+ (A_2 \sin(\Omega_+ U) - B_2 \cos(\Omega_+ U)) - \frac{1}{\Omega_-}(C_2 \sin(\Omega_- U) - D_2 \cos(\Omega_- U)), \tag{4.20d}$$

The number of constants is halved by the initial condition (4.19) which require,

$$C_1 = 1 - \Omega_+^2 A_1, \qquad D_1 = -\frac{\Omega_+}{\Omega_-} B_1, \qquad C_2 = -\Omega_+^2 A_2, \qquad D_2 = \frac{1}{\Omega_-} - \frac{\Omega_+}{\Omega_-} B_2.$$

Requiring in addition also $Q(0) = \mathbb{I}$ which follows from (4.18) eliminates all constants with the exception of $B_2 = \Omega_+^{-1}$, leaving us with[10]

$$Q = \begin{pmatrix} \cos(\Omega_- U) & \frac{\sin(\Omega_+ U)}{\Omega_+} \\ -\frac{\sin(\Omega_- U)}{\Omega_-} & \cos(\Omega_+ U) \end{pmatrix}. \tag{4.21}$$

V. ION TRAPS IN THREE DIMENSIONS

A. Paul trap in three dimensions

Real traps are three-dimensional: ions are Paul-trapped by a time-dependent quadrupole potential, written, in appropriate units, as [18,19]

$$\Phi = \frac{1}{2}(a + 2q \cos 2U)((X^+)^2 + (X^-)^2 - 2z^2), \tag{5.1}$$

where a and q are parameters and we used again the notation $U = \omega t/2$. (5.1) is clearly an axi-symmetric anisotropic oscillator potential with time-dependent frequencies. The motion of an ion is described therefore by three uncoupled Mathieu equations,

$$(X^\pm)'' + (a + 2q \cos 2U) X^\pm = 0, \tag{5.2a}$$

$$z'' - 2(a + 2q \cos 2U) z = 0. \tag{5.2b}$$

The interaction in the $X^\pm$ plane is attractive, while the one in the z direction is repulsive and has a factor 2. The oscillating term produces bounded motions in an appropriate range of parameters. For details the reader is referred to the literature, e.g., [19]. Some bounded trajectories are shown in Fig. 4.

The three-dimensonal Paul trap can again be lifted to Bargmann space—but one in five dimensions. The recipe is the same as before [26]: the Bargmann metric is (3.2a) but now we have three transverse components; the UU component is $-2\Phi(X^\pm, z, U)$ given in (5.1). The quadratic form is traceless and therefore the metric still satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$: it is a gravitational wave in five dimensions.

B. Penning trap

A similar ion trap was proposed by Dehmelt, who called it the Penning trap [20–22] (and shared the Nobel prize with Paul). It combines an anisotropic but time-independent quadrupole potential with a uniform (constant) magnetic field $\boldsymbol{B} = B\hat{z}$ directed along the z axis,

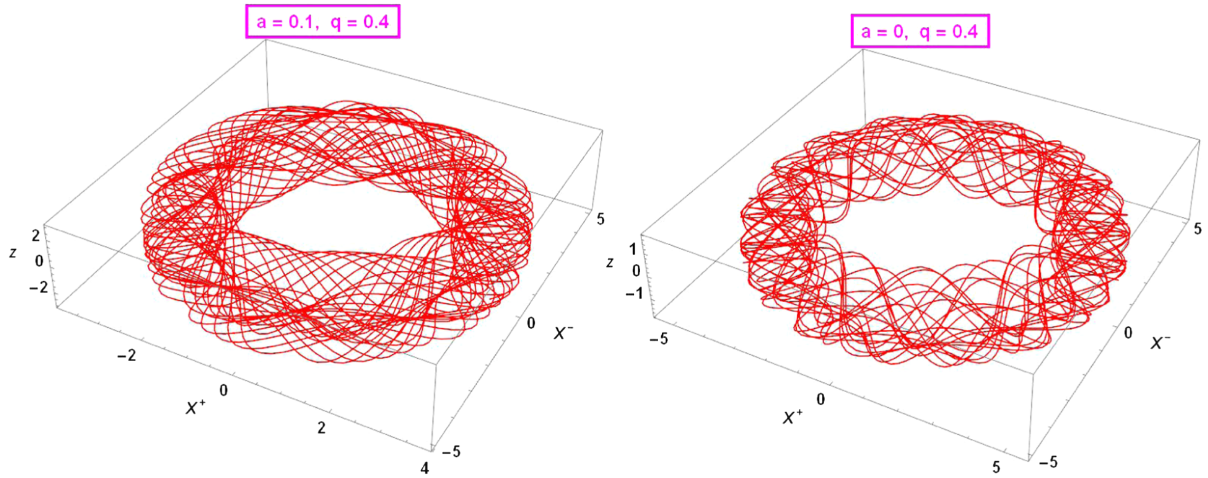
$$\Psi(Y^+, Y^-, z) = -\left(\frac{\omega_z}{2}\right)^2 ((Y^+)^2 + (Y^-)^2 - 2z^2), \tag{5.3a}$$

$$A_+ = -\frac{1}{2} B Y^-, \qquad A_- = \frac{1}{2} B Y^+, \qquad A_z = 0. \tag{5.3b}$$

The Lagrangian

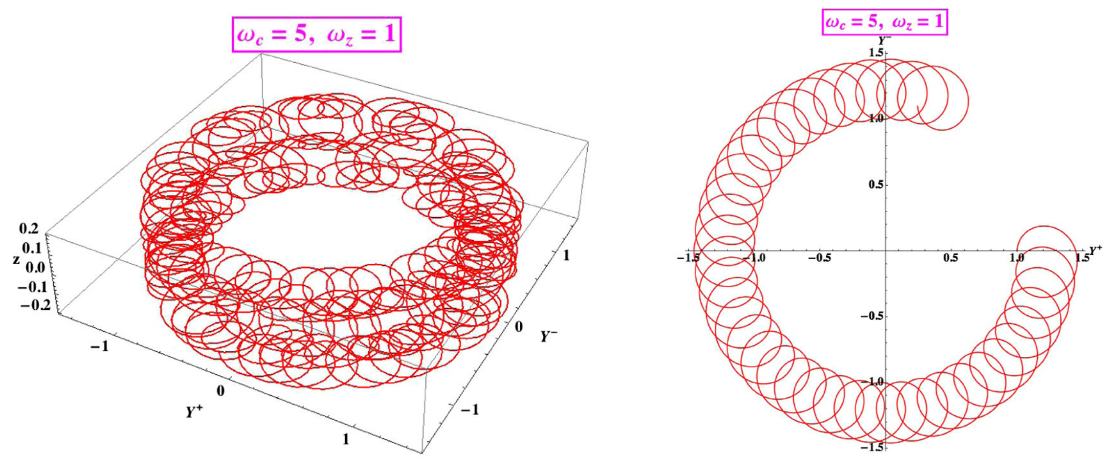
$$L = \frac{1}{2}\dot{\boldsymbol{Y}}^2 + \frac{\omega_c}{2}(\dot{Y}^- Y^+ - \dot{Y}^+ Y^-) + \frac{1}{4}\omega_z^2 (\boldsymbol{Y}^2 - 2z^2), \tag{5.4}$$

[10]The transverse metric in BJR form, $G_{ij}(u) = Q^T(u) Q(u)$ is not illuminating and is therefore omitted.



(a) (b)

FIG. 4. Motion in a three-dimensional Paul trap for (a) $a = 0.1$ (axial symmetry) (b) $a = 0$ (periodic profile). The initial conditions $X^+(0) = 0$, $\dot{X}^+(0) = 1$, $X^-(0) = 5$, $\dot{X}^-(0) = 0$, $z(0) = 0$, $\dot{z}(0) = 1$.



(a) (b)

FIG. 5. Trajectory of a charged particle in a Penning trap (a) in three dimensions. (b) its projection on the $Y^\pm$ plane. The initial conditions are $Y^+(0) = 1.0$, $\dot{Y}^+(0) = 0.0$, $Y^-(0) = 0$, $\dot{Y}^-(0) = 1.0$, $z(0) = 0$, $\dot{z}(0) = 0.2$.

where $\omega_c = B$ in our units is the cyclotron frequency,[11] yields, for a particle of unit charge and mass,

$$\ddot{Y}^\pm \mp \omega_c \dot{Y}^\mp - \frac{1}{2}\omega_z^2 Y^\pm = 0 \tag{5.5a}$$

$$\ddot{z} + \omega_z^2 z = 0, \tag{5.5b}$$

cf., Eqs. (2.5)–(2.7) of Ref. [40].

Let us observe, for further reference, that the upper Eqs. (5.5a) are reminiscent of the circularly polarized periodic (CPP) form (3.8) (as suggested by our notations), while the z-equation is that of a decoupled harmonic oscillator. In terms of the complex coordinate $\Upsilon = Y^+ + iY^-$ Eq. (5.5a) is solved by [40],

$$\Upsilon_\pm(t) = e^{-i\gamma_\pm t}, \qquad \gamma_\pm = \frac{1}{2}\left(\omega_c \pm \sqrt{\omega_c^2 - 2\omega_z^2}\right). \tag{5.6}$$

The constants γ_+ and γ_- here are the modified cyclotron frequency and the magnetron frequency, respectively. Periodic solutions require $\omega_c^2 - 2\omega_z^2 > 0$. Solutions are shown in Fig. 5; bound motions arise when $\omega_z^2 > 0$. In experimentally realistic cases $\omega_c \gg \omega_z$ [21,40]. However, a special case arises when the Penning trap has equal modified-cyclotron and magnetron frequencies,

[11]Once again, the dot means here d/dt where t is nonrelativistic time.

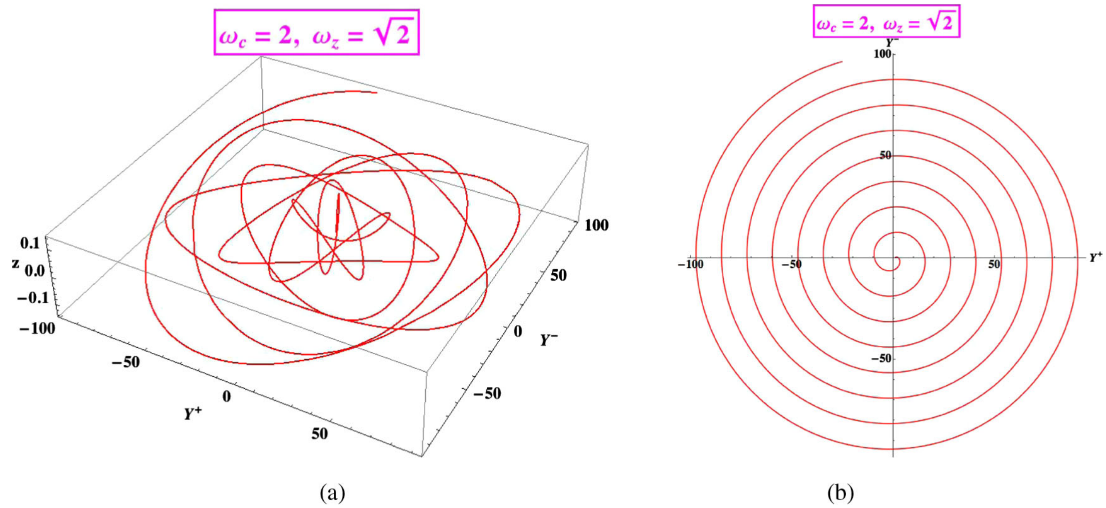


FIG. 6. In the fine-tuned case (5.7) and for initial conditions $Y^+(0) = 1.0$, $\dot{Y}^+(0) = 0.0$, $Y^-(0) = 0$, $\dot{Y}^-(0) = 1.0$, $z(0) = 0$, $\dot{z}(0) = 0.2$, the $2d$ projection (b) of the $3d$ trajectory (a) spirals outward with expanding radius. The z coordinate oscillates with frequency $\omega_z = \omega_c/\sqrt{2}$.

$$\gamma_+ = \gamma_- = \frac{1}{2}\omega_c, \quad \text{i.e. for } \Delta = \left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{1}{2} = 0. \tag{5.7}$$

Then both motions in (5.6) coincide (are purely cyclotronic) whereas the new independent solution spirals outward as shown in Fig. 6, reminiscent of the maximally anisotropic case $\Omega_- = 0$ in Fig. 9 of [12],

$$\Upsilon_0^{(\mathrm{per})}(t) = e^{-i\frac{1}{2}\omega_c t}, \qquad \Upsilon_0^{(\mathrm{esc})}(t) = te^{-i\frac{1}{2}\omega_c t}. \tag{5.8}$$

The toroidal region shrinks to a circle and we get also a new, escaping solution. The general solution of the three-dimensonal system (5.5), a combination of those in (5.6) completed with $z = E\cos(\omega_z t) + F\cos(\omega_z t)$, can be also obtained by chiral decomposition.

For a discussion of the quantum aspects the reader is referred, e.g., [21] to for details. Here we just mention that the spectrum is [21],

$$E_{(n_+,n_-,k)} = \hbar\left[\gamma_+\left(n_+ + \frac{1}{2}\right) - \gamma_-\left(n_- + \frac{1}{2}\right) + \omega_z\left(k + \frac{1}{2}\right)\right], \quad n_\pm, k = 0, 1, \ldots \tag{5.9}$$

as it can also be confirmed by the chiral decomposition method. In the special case (5.7) the bound-state spectrum is that of the z component alone, consistently with (5.8) and Fig. 6.

Now we turn to the GW aspect of three-dimensional traps. As said above, Paul traps correspond to linearly polarized periodic (LPP) waves; now we inquire if the analogy can be extended by relating the Penning trap to CPP waves. We first recall how a nonrelativistic particle in an external electromagnetic field can be described by a five-dimensional Bargmann space [39]. In terms of the coordinates $(\boldsymbol{Y}, z, t, s)$ we have,

$$ds^2 = d\boldsymbol{Y}^2 + dz^2 + 2dt(ds + A_i dY^i) - 2\Psi\, dt^2, \tag{5.10a}$$

whose null geodesics project consistently with (5.5). Note that the metric is not Ricci-flat: the potential (5.3a) is harmonic, $\Delta\Psi = 0$, and therefore $R_{UU} = -\frac{1}{2}B^2 \neq 0$, cf., (4.12). The metric (5.10) is, thus, not vacuum Einstein.

To get further insight, we now eliminate the vector potential in (5.10) by the rotational trick (3.7) [backward] extended to three dimensions,

$$\begin{pmatrix} X^+ \\ X^- \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y^+ \\ Y^- \\ z \end{pmatrix}, \tag{5.11}$$

where ω is a constant. The cross terms $dX^\pm dt$ cancel if $\omega = \omega_c/2$ and we end up with

$$ds^2 = d\boldsymbol{X}^2 + 2dtds - 2\Phi\, dt^2, \tag{5.12a}$$

$$\Phi = \frac{1}{8}(\omega_c^2 - 2\omega_z^2)[(X^+)^2 + (X^-)^2] + \frac{1}{2}\omega_z^2 z^2, \tag{5.12b}$$

which is the Bargmann metric of an axially symmetric [attractive or repulsive, generally anisotropic] oscillator.[12] Therefore, despite the similarity between the upper two Penning Eqs. (5.5a) and the CPP equations (3.8), the Bargmann lift of a Penning trap is not a CPP GW: it is not Ricci-flat (as confirmed again by $\Delta\Phi = \omega_c^2/2 \equiv B^2/2$) and is not brought to the CPP form by the rotational trick.

[12]For the special value (5.7) the oscillator is maximally anisotropic: X-motion is free. Another extreme case would be $\omega_z = 0$ when the z-motion is free. When $\omega_c^2 = 6\omega_z^2$, the X-oscillator (5.12b) is isotropic.

C. Modified Penning trap

Below, we propose instead a modified Penning trap, closer to CPP GWs. We first note a subtle, however important, difference between the two systems: in (5.5a) the $Y^{\pm}$ terms have identical frequencies ω_z, whereas in the CPP case (3.8) the frequencies are different, $\Omega_+^2 \neq \Omega_-^2$, except when $A_0 = 0$—i.e., when there is no wave. Therefore we propose to generalize the scalar Penning potential (5.3a) while keeping the same vector potential (5.3b),

$$\boxed{\Psi \to \tilde{\Psi} = -\left(\frac{\omega_z}{2}\right)^2 \left(\left(1+\frac{A_0}{2}\right)(Y^+)^2 + \left(1-\frac{A_0}{2}\right)(Y^-)^2 - 2z^2\right)} \tag{5.13a}$$

$$A_{\pm} = \mp\frac{1}{2}\omega_c Y^{\mp}, \qquad A_z = A_t = 0, \tag{5.13b}$$

where A_0 is a perturbation parameter. The new term clearly breaks the axial symmetry whenever $A_0 \neq 0$. Spelling out for completeness, the Lagrangian

$$L = \frac{1}{2}(\dot{\mathbf{Y}}^2 + \dot{z}^2) + \frac{\omega_c}{2}(\dot{Y}^- Y^+ - \dot{Y}^+ Y^-) + \frac{\omega_z^2}{4}\left(\left(1+\frac{A_0}{2}\right)(Y^+)^2 + \left(1-\frac{A_0}{2}\right)(Y^-)^2 - 2z^2\right), \tag{5.14}$$

cf., (5.4) yields the equations of motions,

$$\ddot{Y}^{\pm} \mp \omega_c \dot{Y}^{\mp} - \frac{\omega_z^2}{2}\left(1 \pm \frac{A_0}{2}\right)Y^{\pm} = 0, \qquad \ddot{z} + \omega_z^2 z = 0. \tag{5.15}$$

cf., (5.5). Lifting to five-dimensional Bargmann space (Y^+, Y^-, z, t, s), our modification amounts to considering

$$ds^2 = (dY^+)^2 + (dY^-)^2 + (dz)^2 + 2dt(ds + A_i dY^i) - 2\tilde{\Psi}dt^2 \tag{5.16}$$

where the vector potential is still (5.3b). Then applying once again the three-dimensional rotational trick (5.11) allows us to conclude along the same lines as above that choosing $\omega = B/2 \equiv \omega_c/2$ and putting $U = \omega_c t/2$, $V = 2s/\omega_c$, we get

$$ds^2 = (dX^+)^2 + (dX^-)^2 + (dz)^2 + 2dUdV - 2\tilde{\Phi}dU^2, \tag{5.17a}$$

$$\begin{aligned}\tilde{\Phi} = &\left(\frac{1}{2} - \left(\frac{\omega_z}{\omega_c}\right)^2\right)[(X^+)^2 + (X^-)^2] + 2\left(\frac{\omega_z}{\omega_c}\right)^2 z^2 \\ &- \left(\frac{\omega_z}{\omega_c}\right)^2 \frac{A_0}{2}[\cos 2U((X^+)^2 - (X^-)^2) + 2\sin 2U(X^+X^-)],\end{aligned} \tag{5.17b}$$

which is a rather complicated mixture of a time-dependent oscillator with a periodic correction term. However, when

$$\Delta = \left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{1}{2} = 0, \tag{5.18}$$

cf., (5.7), the isotropic part is turned off, leaving us with a CPP GW embedded into five-dimensional Bargmann space,

$$\boxed{\tilde{\Phi}_{\text{spec}} = -\frac{1}{2}\tilde{K}_{ij}X^iX^j = -\frac{A_0}{4}[\cos 2U((X^+)^2 - (X^-)^2) + 2\sin 2U(X^+X^-)] + z^2,} \tag{5.19}$$

which identifies the constant A_0 as the amplitude of the CPP GW in five dimensions, the Bargmann space of the modified Penning trap. For $A_0 = 0$, we recover the maximally anisotropic Penning case $\Phi = (\omega_c^2/4)z^2$, cf., (5.12b). In the special case (5.18), the chiral decomposition of the system (5.14)–(5.15) is found as

$$\begin{aligned}H &= H_+ - H_- + H_z \\ &= \frac{1}{2}\left[\frac{\omega_c^2 A_0}{8}((1 + A_0/2)Z_+^1 Z_+^1 + Z_+^2 Z_+^2) - \frac{\omega_c^2 A_0}{8}((1 - A_0/2)Z_-^2 Z_-^2 + Z_-^1 Z_-^1) + p_z^2 + \frac{\omega_c^2}{2}z^2\right],\end{aligned} \tag{5.20a}$$

$$\sigma = \sigma_+ - \sigma_- + \sigma_z = -\frac{\omega_c A_0}{4}[dZ_+^1 \wedge dZ_+^2 - dZ_-^1 \wedge dZ_-^2] + dp_z \wedge dz. \tag{5.20b}$$

cf., (3.10)–(3.11). The resulting uncoupled equations,

$$\ddot{Z}_\pm^{1,2} + \frac{\omega_c^2}{4}(1 \pm A_0/2)Z_\pm^{1,2} = 0, \qquad \ddot{z} + \frac{\omega_c^2}{2}z = 0, \tag{5.21}$$

are solved at once; in Y-coordinates, we get,

$$\begin{aligned} Y^+ = {} & A\cos\left(\frac{\omega_c}{2}\sqrt{1+\frac{A_0}{2}}t\right) + B\sin\left(\frac{\omega_c}{2}\sqrt{1+\frac{A_0}{2}}t\right) \\ & + C\cos\left(\frac{\omega_c}{2}\sqrt{1-\frac{A_0}{2}}t\right) + D\sin\left(\frac{\omega_c}{2}\sqrt{1-\frac{A_0}{2}}t\right) \end{aligned} \tag{5.22a}$$

$$\begin{aligned} Y^- = {} & \sqrt{1+\frac{A_0}{2}}\left[B\cos\left(\frac{\omega_c}{2}\sqrt{1+\frac{A_0}{2}}t\right) - A\sin\left(\frac{\omega_c}{2}\sqrt{1+\frac{A_0}{2}}t\right)\right] \\ & + \frac{1}{\sqrt{1-\frac{A_0}{2}}}\left[D\cos\left(\frac{\omega_c}{2}\sqrt{1-\frac{A_0}{2}}t\right) - C\sin\left(\frac{\omega_c}{2}\sqrt{1-\frac{A_0}{2}}t\right)\right], \end{aligned} \tag{5.22b}$$

The quantum spectrum can be obtained using creation/annihilation operators,

$$\boxed{E_{n_+,n_-,k} = \hbar\left[\sqrt{1+A_0/2}\left(n_+ + \frac{1}{2}\right) - \sqrt{1-A_0/2}\left(n_- + \frac{1}{2}\right) + \sqrt{2}k + \frac{1}{\sqrt{2}}\right]} \tag{5.23}$$

where $n_\pm = 0, 1\ldots$ are the eigenvalues of the appropriate number operators, and $k = 0, 1$ is that of the z-oscillator.[13] For a weak wave, $A_0 \ll 1$, we have,

$$E_{n_+,n_-,k} \approx \hbar\left[(n_+ - n_-) + \frac{A_0}{4}(1 + (n_+ + n_-)) + \sqrt{2}k + \frac{1}{\sqrt{2}}\right]. \tag{5.24}$$

VI. LAGRANGE POINTS IN CELESTIAL MECHANICS

In [24,25] Białynicki-Birula *et al.* discuss the stability of Lagrange points in the Newtonian 3-body problem using a linearized Hamiltonian [41]. In the co-rotating x–y plane defined by the two main orbiting bodies the Hamiltonian takes the form

$$H_{\rm osc} = \frac{p_x^2 + p_y^2}{2} + \frac{a\omega^2x^2 + b\omega^2y^2}{2} - \omega(xp_y - yp_x), \tag{6.1}$$

where the values of ω and the dimensionless a and b depend on the parameters of the original problem. The authors discuss in particular islands of stability in the space of parameters. The equations of motion arising from (6.1) are,[14]

$$\ddot{x} - 2\omega\dot{y} = \omega^2(1-a)x, \qquad \ddot{y} + 2\omega\dot{x} = \omega^2(1-b)y. \tag{6.2}$$

The values of a and b can be found by comparing with the results in textbooks such as [41]. In this reference units are chosen so that distances, time, and masses are expressed by dimensionless quantities. Distances are measured from the center of mass of the two main rotating bodies, and rescaled by their relative distance. In the co-rotating frame the two rotating bodies lie on the x axis, and the unit of time is chosen so that the angular velocity of rotation of the co-rotating frame is $\omega = 1$. Our "big masses" are labeled so that $M_1 \leq M_2$, which implies that

[13] $\omega_c = 2$ in our units.

[14] The "one-sided" "Hill" case studied in [33] corresponds to $a = -2$ and $b = 1$ and was found unstable.

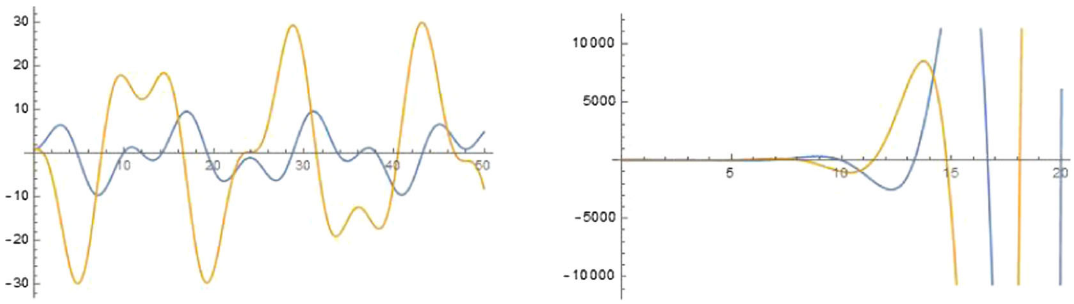


FIG. 7. Motions around a Lagrange point are bounded for $\mu = 0.02 < 1/27$ and unbounded for $\mu = 0.4 > 1/27$.

$$0 < \mu \equiv \frac{M_1}{M_1 + M_2} \leq 1/2. \tag{6.3}$$

Here we are interested in the two Lagrangian points, traditionally denoted by L_4 and L_5. The displacements ξ, η in the x–y plane satisfy the coupled equations:

$$\xi'' - 2\eta' = \frac{3}{4}\xi + \frac{3\sqrt{3}}{4}(1-2\mu)\eta, \tag{6.4a}$$

$$\eta'' + 2\xi' = \frac{9}{4}\eta + \frac{3\sqrt{3}}{4}(1-2\mu)\xi, \tag{6.4b}$$

from which we can read off a scalar potential $V(\xi,\eta) = -\frac{3}{8}\xi^2 - \frac{3\sqrt{3}}{4}(1-2\mu)\xi\eta - \frac{9}{8}\eta^2$, whose Hessian $\partial_i\partial_j V$ has eigenvalues

$$\lambda_{1,2} = \frac{3}{2}\left(-1 \mp \sqrt{1-3\mu+3\mu^2}\right). \tag{6.5}$$

The potential can be diagonalized by a rotation, $\binom{\hat{\xi}}{\hat{y}} = R\binom{\xi}{\eta}$ which brings the equations of motion to the form,

$$\hat{\xi}'' - 2\hat{\eta}' = -\lambda_1\hat{\xi}, \tag{6.6a}$$

$$\hat{\eta}'' + 2\hat{\xi}' = -\lambda_2\hat{\eta}. \tag{6.6b}$$

Then by comparison with (6.2) we get $\omega = 1$, $\lambda_1 = a - 1$, $\lambda_2 = b - 1$. Equation (6.5) implies $a + b = -1$, cf., [24], which allows us to infer that

$$a = -\frac{1}{2} - \frac{3}{2}\sqrt{1-3\mu+3\mu^2},$$
$$b = -\frac{1}{2} + \frac{3}{2}\sqrt{1-3\mu+3\mu^2}, \tag{6.7}$$

A naïve argument for stability for all possible values of μ would be to observe that replacing the $-\lambda_i$ by $\Omega_\pm^2$ Eq. (6.6) go over to our Eqs. (3.8) for which we found stability (trigonometric functions) provided that they are both positive. But $\lambda_i < 0$, $i = 1, 2$ requires only $\mu(1-\mu) > 0$, which is always satisfied in the range $0 < \mu \leq 1/2$, seemingly contradicting the stability condition given in [24],

$$\mu(1-\mu) < 1/27, \tag{6.8}$$

which is however confirmed both numerically (cf., Fig. 7) and as follows. Let us test our equations on simple exponential solutions, $\hat{\xi}(\tau) = e^{\alpha\tau}$. Nontrivial solutions arise only if $\alpha^4 + \alpha^2 + \frac{27}{4}\mu(1-\mu) = 0$ by (6.5). The solution is bounded only if $\alpha^2 > 0$ which requires precisely (6.8).[15]

The apparent contradiction is resolved by noting that the λ_is in (6.5) were tacitly assumed to be real, which requires either $\mu \leq \mu_1 = \frac{3-\sqrt{21}}{6} < 0$ or $\mu \geq \mu_2 = \frac{3+\sqrt{21}}{6} > 1$—and neither of these conditions can be satisfied for μ in the range (6.3).

VII. CONCLUSION

The striking similarities of the apparently far remote topics discussed in this paper have, from the mathematical point of view, a simple explanation: in all cases, the problem boils down to study an *anisotropic oscillator* [32–34]. In the Paul [alias linearly polarized GW] case, the solution is expressed in terms of Mathieu functions [6,12,18]; in the circularly polarized periodic GW case, they involve trigonometric/hyperbolic functions.

Previous investigations of the memory effect focused on sudden bursts of sandwich waves which vanish outside a short "wave zone". It was advocated [42] that their observation would (theoretically) be possible due to the velocity memory effect: in the flat "afterzone" the particles move indeed with constant velocity, as required by … Newton's 1st law [5,11,12].

In this paper we study instead periodic waves sought in inflationary models [13,14]. Such waves have no "before and after zone," and their observation would require a different technique. Here, we argue that, by analogy with ion trapping [18,20], one might study bound motions.

[15] For the Sun-Jupiter system $\mu \approx 10^{-3} \ll 1/27 \approx 0.037037$, consistently with the observed stability of the "Greek/Trojan" minor planets (asteroids). For the Earth-Moon system $\mu \approx 0.012 \Rightarrow \mu(1-\mu) \approx 0.0118 < 1/27$. The discovery of the first Earth-Trojan, 2010 TK7, was announced by NASA in 2011. For the Sun-Earth system $\mu \approx \mu(1-\mu) \approx 3.10^{-6}$–stable. This has its importance for, say, LISA, with GW detectors planned to be sent to the Lagrange points.

The Eisenhart lift of the three-dimensional Paul trap is a linearly polarized periodic gravitational wave in five dimensions. For three-dimensional Penning traps, we find that, despite strong similarity with the equations which govern circularly polarized gravitational waves, their Eisenhart lift is not a CPP wave. However, a slight modification (see Sec. V C) allows for anisotropy and for a special value (5.18) of the frequencies we do get circularly polarized gravitational waves in five dimensions. Such perturbations were actually considered before as due to imperfections; see Eq. (2.71) of Ref. [21].

It is remarkable that molecular physicists, who worked on ion traps decades ago, were, like Molière's Monsieur Jourdain, studying gravitational waves.

While our investigations here are classical, the chiral decomposition, (3.11b), makes it easy to study the quantum problem. Observing, in particular, the bound-state spectrum (5.23) could, theoretically, lead to the detection of such a wave. We mention that ion traps have also been studied recently in connection with (space)time crystals [43].

It is worthwhile to emphasize that these kinds of analogies extend very generally, even for nonperiodic waves: the geodesic deviation equation in a vacuum background always looks like $\ddot{\xi} = R\xi$ in a parallel-propagated frame, where R is an appropriate matrix. But this is an anisotropic oscillator equation, and such equations are ubiquitous in physics. Moreover, time dependence in R can give parametric resonance and similar things, as is well known in other contexts. Geodesic motion in curved spacetimes is, therefore, generically linked to time-dependent anisotropic oscillators. Plane wave spacetimes are special here because (i) it's not awkward to let R have any time dependence you like, and (ii) the geodesic deviation equation is exact even for finite separations.

ACKNOWLEDGMENTS

P. H. is grateful to Piotr Kosinski and Csaba Sükösd for correspondence. The last paragraph of our Conclusion has been copied verbatim from the report of our anonymous referee to whom we express our indebtedness. M. E. and P. H. want to thank the Institute of Modern Physics of the Chinese Academy of Sciences in Lanzhou (China) for hospitality. This work was supported by the Chinese Academy of Sciences President's International Fellowship Initiative (No. 2017PM0045) and by the National Natural Science Foundation of China (Grant No. 11575254). M. C. acknowledges CNPq support from project (303923/2015-6) and a Pesquisador Mineiro Project No. PPM-00630-17.

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