

Linear stability analysis of a rotating thin-shell wormhole

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We cut and paste two Bañados-Teitelboim-Zanelli (BTZ) spacetimes at a throat by the Darmois-Israel method to construct a rotating wormhole with a thin shell filled with a barotropic fluid. The thin shell at the throat and both sides of the throat corotate. We investigate the linear stability of the thin shell of the rotating wormhole against radial perturbations. We show that the wormhole becomes more and more stable the larger its angular momentum is until the angular momentum reaches a critical value and that the behavior of a condition for stability significantly changes when the angular momentum exceeds the critical value. We find that the overcritical rotating wormhole has the radius of the thin shell, which is stable regardless of the equation of state for the barotropic fluid.

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I. INTRODUCTION

A wormhole is a spacetime structure that connects two regions in our Universe or multiverse. If the wormhole is spherically symmetric, static, and traversable in general relativity, a null energy condition is broken at least at a throat as shown by Morris and Thorne [1]. The first example of wormholes in the Morris-Thorne class was considered in 1973 by Ellis [2] and Bronnikov [3] independently. The Ellis-Bronnikov wormhole spacetime is the solution of Einstein equations and of a wave equation for a ghost scalar field.

The existence of wormholes in nature requires their stability. The instability of the Ellis-Bronnikov wormhole was reported by several authors [4–9], contrary to the conclusion of an earlier work [10]. Torii and Shinkai showed that an all-dimensional Ellis-Bronnikov wormhole is unstable against perturbations with which the throat radius is changed [11]. On the other hand, in 2013, Bronnikov *et al.* found that a wormhole, which is filled with electrically charged dust with a negative energy density, with the same metric as the metric of the massless Ellis-Bronnikov wormhole in four dimensions [12,13], is stable under both spherically symmetric and axial perturbations [14]. Their result shows that stability of wormholes depend on not only metrics but also the matters of the source of the metrics.¹

Teo pioneered a rotating traversable wormhole in four dimensions [16], but Teo did not pay attention to the stability of wormholes as much as Morris and Thorne [1]. Matos and Nuñez [17] claimed that rotating wormholes would have a higher possibility of being stable intuitively, and they considered a rotating wormhole with a ghost scalar field in four dimensions. Dzhunushaliev *et al.* [18] investigated a five-dimensional rotating wormhole with equal angular momenta filled with a ghost scalar field and discussed its stability. They found that the unstable mode of the five-dimensional wormhole disappears when the wormhole rotates fast and that the wormhole has the upper bound of the angular momenta. Their works might show that rotating wormholes have a higher possibility to be stable but the effect of rotation on the stability would depend on the matters supporting the wormholes.

The class of a wormhole which is composed of two parts of spacetimes joined by a thin shell [19–21] at the wormhole throat using Darmois-Israel matching [21–23] was suggested by Visser [24,25]. The linear stability of a thin-shell wormhole against radial perturbations was studied in Refs. [25,26]. The analysis is useful to investigate the stability of wormholes since one can treat it with only a few assumptions with respect to the matters supporting the wormholes. It has been applied to various thin-shell wormholes [27–74]. Recently, the details of stability of thin-shell wormholes have been investigated. Nakao *et al.* considered a collision between a thin shell at a wormhole throat and another dust thin shell that falls into the throat, and they obtained conditions in which the thin-shell wormhole persists after the collision [75]. Akai and Nakao

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¹The stability of a wormhole against decay via gravitational instanton tunneling was discussed by Cox *et al.* [15]. Their work also shows that the stability of wormholes depends on the matters of the source of the metrics.

have investigated the nonlinear stability of a thin-shell wormhole [76].

The treatment of a rotating thin shell is more difficult than a spherically symmetric thin shell because of the low symmetry of the spacetime. The fact, however, was shown that rotating thin shells in three dimensions are more tractable than ones in higher dimensions [77–79]. In Ref. [80], Mazharimousavi and Halilsoy cut and pasted two parts of BTZ spacetimes [81,82] and made a rotating thin-shell wormhole in three dimensions to study the stability of the wormhole. They claimed that one side of a throat counterrotates against the other side of the throat and concluded that counterrotational effects make the wormhole stable. However, it seems to be difficult to understand their conclusion intuitively.

In this paper, we reexamine the stability of the rotating thin-shell wormhole in three dimensions. We cut and paste two rotating BTZ spacetimes [81,82] with the Darmois-Israel matching conditions and construct the rotating thin-shell wormhole. The angular momenta of both sides of the throat have the same absolute value but signs opposite each other. We notice that the opposite signs of the angular momenta mean that the thin shell at the throat and the two sides of the throat must corotate. Then, we investigate the linear stability of the wormhole against radial perturbations.

This paper is organized as follows. In Sec. II, we construct the rotating thin-shell wormhole. In Sec. III, we consider the linear stability of the thin-shell wormhole against a radial perturbation. In Sec. VI, we summarize our results. In this paper, we use the units in which the light speed and $8G$, where G is Newton's constant in three dimensions, are unity as set in Sec. III in Ref. [77].

II. CONSTRUCTION OF ROTATING BTZ WORMHOLE

We construct a rotating thin-shell wormhole by using a cut-and-paste method [21–23]. We consider two BTZ spacetimes [81,82] with a line element given by

$$\begin{aligned} ds_{\pm}^2 &= g_{\mu\nu} dx_{\pm}^{\mu} dx_{\pm}^{\nu} \\ &= -f_{\pm}(r_{\pm}) dt_{\pm}^2 + \frac{dr_{\pm}^2}{f_{\pm}(r_{\pm})} + r_{\pm}^2 \left(d\varphi_{\pm} - \frac{J_{\pm}}{2r_{\pm}^2} dt_{\pm} \right)^2, \end{aligned} \quad (2.1)$$

where $f_{\pm}(r_{\pm})$ is defined by

$$f_{\pm}(r_{\pm}) \equiv -M_{\pm} + \frac{r_{\pm}^2}{l_{\pm}^2} + \frac{J_{\pm}^2}{4r_{\pm}^2}, \quad (2.2)$$

where M_{\pm} , J_{\pm} , and $l_{\pm} \equiv \sqrt{-1/\Lambda_{\pm}}$ are a mass parameter, an angular momentum, and the scale of a curvature related

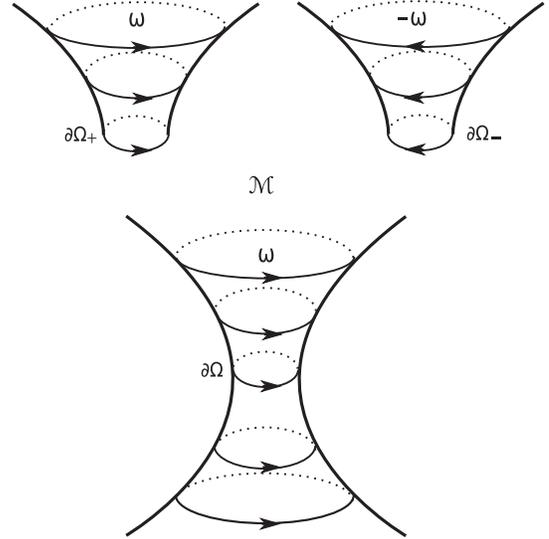


FIG. 1. Construction of a rotating wormhole with a cut-and-paste method. We cut two BTZ spacetimes with angular momenta J_{\pm} and identify the boundaries of the manifolds $\partial\Omega_{\pm}$ that are the timelike hypersurfaces $\partial\Omega \equiv \partial\Omega_{+} = \partial\Omega_{-}$, and we obtain a manifold \mathcal{M} . The second Darmois-Israel junction condition implies $J_{-} = -J_{+}$. Thus, $\omega(a) \equiv \omega_{+}(a) = -\omega_{-}(a)$, where $\omega_{\pm}(r_{\pm})$ is defined as $\omega_{\pm}(r_{\pm}) \equiv -g_{t\varphi_{\pm}}/g_{\varphi\varphi_{\pm}} = J_{\pm}/2r_{\pm}^2$, is obtained.

to a negative cosmological constant $\Lambda_{\pm} < 0$, respectively.² We assume $M \equiv M_{+} = M_{-} > 0$ and $l \equiv l_{+} = l_{-}$ for simplicity. The BTZ spacetime with $lM \geq |J_{\pm}|$ has an event horizon at $r_{\pm} = r_{\pm H}$, where $r_{\pm H}$ is given by

$$r_{\pm H} \equiv l \sqrt{\frac{M}{2} \left(1 + \sqrt{1 - \frac{J_{\pm}^2}{M^2 l^2}} \right)}, \quad (2.3)$$

while the spacetime with $lM < |J_{\pm}|$ has a naked singularity at $r_{\pm} = 0$.

We consider two BTZ spacetimes and remove from them regions

$$\Omega_{\pm} \equiv \{r_{\pm} < a | a > r_b\}, \quad (2.4)$$

where a is a constant and $r_b \equiv r_{\pm H}$ for $lM \geq |J_{\pm}|$ and $r_b \equiv 0$ for $lM < |J_{\pm}|$. The rest of the manifolds has the boundaries that are the timelike hypersurfaces described as

$$\partial\Omega_{\pm} \equiv \{r_{\pm} = a | a > r_b\}. \quad (2.5)$$

We identify the two hypersurfaces

$$\partial\Omega \equiv \partial\Omega_{+} = \partial\Omega_{-}, \quad (2.6)$$

and we obtain a manifold \mathcal{M} with two regions glued by a throat located at $\partial\Omega$ as shown in Fig. 1.

²A wormhole with a negative cosmological constant was considered in Ref. [83].

We permit that the radius of the throat a is a function of time, $a = a(t)$, since we are interested in the stability of the rotating wormhole. By introducing an azimuth coordinate ϕ_{\pm} given by

$$d\phi_{\pm} \equiv d\varphi_{\pm} - \frac{J_{\pm} dt_{\pm}}{2a^2(t_{\pm})}, \quad (2.7)$$

the line element (2.1) in a corotating frame on the timelike hypersurface, i.e., on the throat, is rewritten as

$$ds_{\pm}^2 = -f_{\pm}(r_{\pm})dt_{\pm}^2 + \frac{dr_{\pm}^2}{f_{\pm}(r_{\pm})} + r_{\pm}^2 \left[d\phi_{\pm} + \frac{J_{\pm}}{2} \left(\frac{1}{a^2(t_{\pm})} - \frac{1}{r_{\pm}^2} \right) dt_{\pm} \right]^2. \quad (2.8)$$

We define $[T]$ as

$$[T] \equiv T_+|_{\partial\Omega} - T_-|_{\partial\Omega} \quad (2.9)$$

for any tensorial function T .

From the first junction condition $[h_{ij}] = 0$, the induced metric h_{ij} on the timelike hypersurface with $t_{\pm} = t_{\pm}(\tau)$ and $r_{\pm} = a(t_{\pm}(\tau)) = a(\tau)$ is given by

$$ds_{\pm}^2 = h_{ij} dy^i dy^j = -d\tau^2 + a^2(\tau) d\phi^2, \quad (2.10)$$

where τ is the proper time of an observer on the timelike hypersurface and \dot{t}_{\pm} is obtained as

$$\dot{t}_{\pm} = \frac{\sqrt{\dot{a}^2 + f_{\pm}}}{f_{\pm}}, \quad (2.11)$$

where $f_{\pm} = f_{\pm}(a)$ and where the dot denotes a differentiation with respect to τ . The induced basis vectors $e_{i\pm}^{\mu}$ on the timelike hypersurface are obtained as

$$e_{\tau\pm}^{\mu} \frac{\partial}{\partial x_{\pm}^{\mu}} = \frac{\sqrt{\dot{a}^2 + f_{\pm}}}{f_{\pm}} \frac{\partial}{\partial t_{\pm}} + \dot{a} \frac{\partial}{\partial r_{\pm}} \quad (2.12)$$

and

$$e_{\phi\pm}^{\mu} \frac{\partial}{\partial x_{\pm}^{\mu}} = \frac{\partial}{\partial \phi_{\pm}}. \quad (2.13)$$

The 3-velocity of the observer on the hypersurface is given by $u_{\pm}^{\mu} = e_{\tau\pm}^{\mu}$. The vectors $n_{\mu\pm}$ normal to the hypersurface are given by

$$n_{\mu\pm} dx_{\pm}^{\mu} = \pm \left(-\dot{a} dt_{\pm} + \frac{\sqrt{\dot{a}^2 + f_{\pm}}}{f_{\pm}} dr_{\pm} \right). \quad (2.14)$$

The second junction condition is given by

$$[K_j^i] = 0, \quad (2.15)$$

where K_{ij} is an extrinsic curvature of the timelike hypersurface defined by

$$K_{ij} \equiv e_i^{\mu} e_j^{\nu} \nabla_{\nu} n_{\mu}. \quad (2.16)$$

The extrinsic curvature is obtained as

$$K_{\tau\pm}^{\tau} = \pm \frac{2\dot{a} + f'_{\pm}}{2\sqrt{\dot{a}^2 + f_{\pm}}}, \quad (2.17)$$

$$K_{\phi\pm}^{\phi} = \pm \frac{\sqrt{\dot{a}^2 + f_{\pm}}}{a}, \quad (2.18)$$

$$K_{\phi\pm}^{\tau} = \mp \frac{J_{\pm}}{2a}, \quad (2.19)$$

where $f'_{\pm} = f'_{\pm}(a)$ is defined by

$$f'_{\pm} \equiv \left. \frac{df_{\pm}}{dr_{\pm}} \right|_{r_{\pm}=a} \quad (2.20)$$

and the trace of the extrinsic curvature is obtained as

$$K_{\pm} \equiv K_{i\pm}^i = \pm \frac{2\dot{a} + f'_{\pm}}{2\sqrt{\dot{a}^2 + f_{\pm}}} \pm \frac{\sqrt{\dot{a}^2 + f_{\pm}}}{a}. \quad (2.21)$$

The (τ, ϕ) component of the second junction condition

$$[K_{\phi}^{\tau}] = -\frac{J_+ + J_-}{2a} = 0 \quad (2.22)$$

implies $J \equiv J_+ = -J_-$. Therefore, we obtain

$$f \equiv f_+ = f_- = -M + \frac{a^2}{l^2} + \frac{J^2}{4a^2} \quad (2.23)$$

$$f' \equiv f'_+ = f'_- = \frac{2a}{l^2} - \frac{J^2}{2a^3} \quad (2.24)$$

on the timelike hypersurface. The other components of $[K_j^i]$ given by

$$[K_{\tau}^{\tau}] = \frac{2\dot{a} + f'}{\sqrt{\dot{a}^2 + f}} \quad (2.25)$$

and

$$[K_{\phi}^{\phi}] = \frac{2\sqrt{\dot{a}^2 + f}}{a} \quad (2.26)$$

violate the second junction condition. Thus, the rotating wormhole spacetime is singular at the throat, and a thin shell is needed there. The thin shell at the timelike hypersurface follows the Einstein equations

$$\pi S_j^i = -[K_j^i] + [K] \delta_j^i, \quad (2.27)$$

where S_j^i is the surface stress-energy tensor of the thin shell filled with a perfect fluid described by

$$S_j^i = (\sigma + p)U^i U_j + p\delta_j^i, \quad (2.28)$$

where σ , p , and U_i are the surface energy density, the surface pressure, and the 2-velocity of the thin shell, respectively. Here, the 2-velocity of the thin shell U_i is given by $U_i dy^i \equiv u_{\mu\pm} e_{i\pm}^\mu dy^i = -d\tau$, and $[K]$ is given by $[K] = [K_\tau^\tau] + [K_\phi^\phi]$. The (τ, τ) and (ϕ, ϕ) components of the Einstein equation (2.27) give

$$\sigma = -S_\tau^\tau = -\frac{2\sqrt{\dot{a}^2 + f}}{\pi a} \quad (2.29)$$

and

$$p = S_\phi^\phi = \frac{2\ddot{a} + f'}{\pi\sqrt{\dot{a}^2 + f}}, \quad (2.30)$$

respectively. From Eqs. (2.29) and (2.30), the equation of energy conversation is obtained as

$$\frac{d(2\pi a\sigma)}{d\tau} + p \frac{d(2\pi a)}{d\tau} = 0. \quad (2.31)$$

III. STABILITY OF THE ROTATING WORMHOLE

The (τ, τ) and (ϕ, ϕ) components of the Einstein equation (2.27) are rewritten as

$$\dot{a}^2 + f - \frac{\pi^2 a^2 \sigma^2}{4} = 0 \quad (3.1)$$

and

$$\dot{\sigma} = -(\sigma + p) \frac{\dot{a}}{a}. \quad (3.2)$$

We assume that the thin shell is filled with a barotropic fluid with the surface pressure given by $p = p(\sigma)$. The surface pressure $p = p(\sigma)$ and Eq. (3.2) imply that the surface energy density can be written in the form $\sigma = \sigma(a)$. Thus, the motion of the shell is described by

$$\dot{a}^2 + V(a) = 0, \quad (3.3)$$

where $V(a)$ is the effective potential defined as

$$V(a) \equiv f(a) - \frac{\pi^2 a^2 \sigma^2(a)}{4}. \quad (3.4)$$

The first and second derivatives of $V(a)$ with respect to a are given by

$$V'(a) = f'(a) + \frac{\pi^2 a \sigma p}{2} \quad (3.5)$$

and

$$V''(a) = f''(a) - \frac{\pi^2}{2} [p^2 + (\sigma + p)\sigma\beta^2], \quad (3.6)$$

respectively, where β^2 is defined as

$$\beta^2(\sigma) \equiv \frac{\partial p}{\partial \sigma}, \quad (3.7)$$

the prime is a differentiation with respect to a , and $f''(a)$ is obtained as

$$f''(a) = \frac{2}{l^2} + \frac{3J^2}{2a^4}. \quad (3.8)$$

We consider a situation in which the rotating shell does not move in the radial direction. The shell is characterized by constants a_0 ,

$$\sigma_0 = -\frac{2\sqrt{f_0}}{\pi a_0} \quad (3.9)$$

and

$$p_0 = \frac{f'_0}{\pi\sqrt{f_0}}, \quad (3.10)$$

where f_0 and f'_0 are defined as $f_0 \equiv f(a_0)$ and $f'_0 \equiv f'(a_0)$, respectively. By introducing $x \equiv a/(l\sqrt{M})$ and $x_0 \equiv a_0/(l\sqrt{M})$, the effective potential $V(x)$ can be expanded in the power of $x - x_0$ as

$$V(x) = \frac{1}{2} \frac{d^2 V}{dx^2} \Big|_{x=x_0} (x - x_0)^2 + O((x - x_0)^3) \quad (3.11)$$

since $V(x_0)$ and $dV/dx|_{x=x_0}$ vanish. The thin shell is stable against linearized fluctuations in the radial direction for $d^2 V/dx^2|_{x=x_0} > 0$, while it is unstable for $d^2 V/dx^2|_{x=x_0} < 0$. Here, $d^2 V/dx^2|_{x=x_0}$ is obtained as

$$\frac{d^2 V}{dx^2} \Big|_{x=x_0} = \frac{1}{Mx_0^4} \left[\frac{-8x_0^6 + 12j^2 x_0^4 - 6j^2 x_0^2 + j^4}{4x_0^4 - 4x_0^2 + j^2} + (2x_0^2 - j^2)\beta_0^2 \right], \quad (3.12)$$

where j and β_0 are defined as $j \equiv J/(lM)$ and $\beta_0 \equiv \beta(\sigma_0)$, respectively. The thin shell is stable if the condition

$$\frac{-8x_0^6 + 12j^2 x_0^4 - 6j^2 x_0^2 + j^4}{4x_0^4 - 4x_0^2 + j^2} + (2x_0^2 - j^2)\beta_0^2 > 0 \quad (3.13)$$

is satisfied. The behaviors of inequality (3.13) with an overcritical angular momentum $|j| > 1$ are different from the ones with a critical and a subcritical angular momentum $|j| \leq 1$. We classify the rotating wormhole into three cases below.

A. Subcritical rotating case $|j| < 1$

In the subcritical rotating case $|j| < 1$, the factor $2x_0^2 - j^2$ is positive since x_0 is defined in the region $x_0 > x_H$, where $x_H \equiv r_H/(l\sqrt{M}) = \sqrt{(1 + \sqrt{1 - j^2})}/2$. In this case,

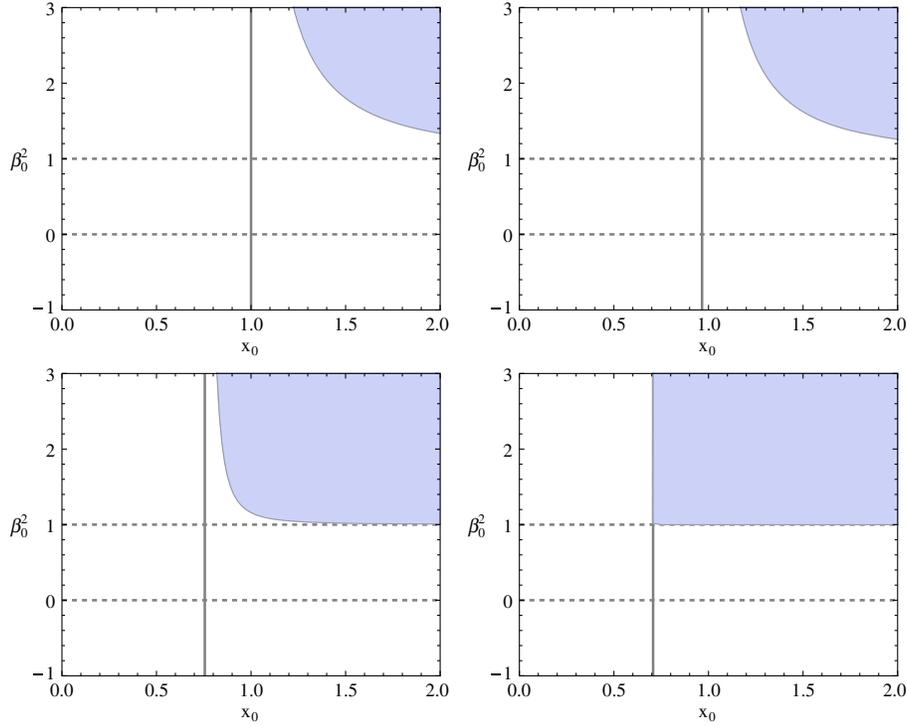


FIG. 2. Stability of the rotating wormhole for $j = 0$ (left top), $|j| = 0.5$ (right top), $|j| = 0.99$ (left bottom), and $|j| = 1$ (right bottom). Shaded regions indicate stability. Solid lines denote the radius of the event horizon for each j . One finds stable regions increase as $|j|$ increases.

the stable condition (3.13) is rewritten as $\beta_0^2 > A_0$, where $A_0 = A_0(x_0)$ is defined as

$$A_0(x_0) \equiv \frac{8x_0^6 - 12j^2x_0^4 + 6j^2x_0^2 - j^4}{(2x_0^2 - j^2)(4x_0^4 - 4x_0^2 + j^2)}. \quad (3.14)$$

A_0 monotonically decreases as x_0 increases for $x_0 > x_H$ and $A_0 \rightarrow 1$ in a limit $x_0 \rightarrow \infty$. Thus, the wormhole with $\beta_0^2 < 1$ is unstable. A_0 diverges $+\infty$ in a limit $x_0 \rightarrow x_H + 0$. This means that the rotating wormhole with any β_0^2 becomes unstable in the near-horizon limit $x_0 \rightarrow x_H + 0$. Figure 2 shows stable and unstable regions. The more rapidly the wormhole rotates, the larger stable region is on the $x_0\beta_0^2$ plane.

B. Critical rotating case $|j| = 1$

In the critical rotating case $|j| = 1$, inequality (3.13) becomes $\beta_0^2 > 1$ and $x_0 > x_H = \sqrt{1/2}$. The wormhole with $\beta_0^2 < 1$ is unstable. The stable and unstable regions are shown in Fig. 2.

C. Overcritical rotating case $|j| > 1$

In the overcritical rotating case $|j| > 1$, the stability condition (3.13) is given by $\beta_0^2 < A_0$ ($\beta_0^2 > A_0$) for $0 < x_0 < |j|/\sqrt{2}$ ($|j|/\sqrt{2} < x_0$), and A_0 approaches ∞ ($-\infty$) in

a limit $x_0 \rightarrow |j|/\sqrt{2} - 0$ ($|j|/\sqrt{2} + 0$). A_0 monotonically increases as x_0 increases in both regions $0 < x_0 < |j|/\sqrt{2}$ and $|j|/\sqrt{2} < x_0$. Note $A_0 \rightarrow 1$ in both the limits $x_0 \rightarrow \infty$ and $x_0 \rightarrow +0$. Figure 3 shows stable and unstable regions. We notice that the rotating wormhole with any β_0^2 , i.e., any barotropic fluid, is stable when the throat is at $x_0 = |j|/\sqrt{2}$. Roughly speaking, the rapidly rotating wormhole is stable if the throat is near $x_0 = |j|/\sqrt{2}$ on the $x_0\beta_0^2$ plane. We also notice that a stable region with $0 < \beta_0^2 < 1$ exists on the $x_0\beta_0^2$ plane.

IV. DISCUSSION AND CONCLUSION

We have cut and pasted two BTZ spacetimes, and we have constructed a rotating thin-shell wormhole by a Darmois-Israel method. The second Darmois-Israel junction condition imposes that two BTZ spacetimes should have an equal absolute value of angular momenta but signs opposite each other. We have shown that the wormhole composed of a thin shell and both sides of a throat corotate as shown in Fig. 1, while Ref. [80] concluded that a side of the throat counterrotates against the other side.

We have investigated the linear stability of the wormhole against a radial perturbation. We have concentrated on the thin shell filled with a barotropic fluid. The wormhole has a critical angular momentum $|j| = 1$, and we find the

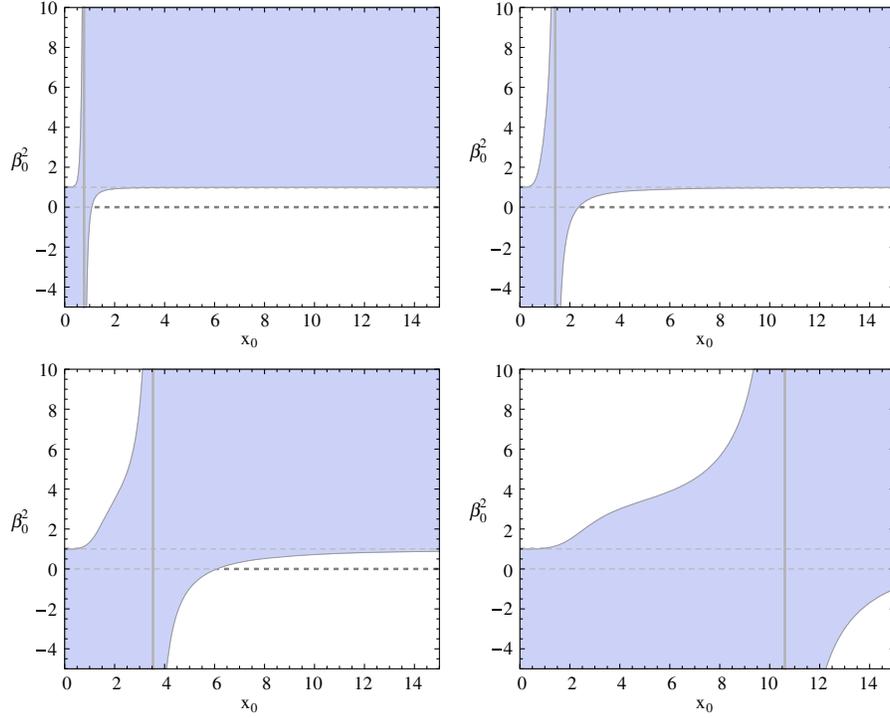


FIG. 3. Stability regions for the rotating wormhole with $|j| = 1.1$ (left top), $|j| = 2$ (right top), $|j| = 5$ (left bottom), and $|j| = 15$ (right bottom). Shaded regions indicate stability. Solid lines represent $x_0 = |j|/\sqrt{2}$.

different behaviors of the stability on the $x_0\beta_0^2$ plane in the following two cases.

In the subcritical and critical rotating cases $|j| < 1$, the more rapidly the wormhole rotates, the larger stable region is on a $x_0\beta_0^2$ plane as shown Fig. 2. In these cases, the wormhole with $\beta_0^2 < 1$ is unstable. In a limit at which the throat approaches the event horizon $x_0 \rightarrow x_H + 0$, the surface pressure p (3.10) diverges, and the wormhole with any β_0^2 is unstable on the $x_0\beta_0^2$ plane.

In the overcritical rotating case $|j| > 1$, we notice that the wormhole with the throat at a radius $x_0 = |j|/\sqrt{2}$ is stable without dependence on β_0^2 , i.e., the equation of state for the barotropic fluid,³ and that the stable region of the wormhole with $0 < \beta_0^2 < 1$ exists on the $x_0\beta_0^2$ plane. In this case, one cannot say that the wormhole becomes more stable the faster wormhole rotates.

We have considered the rotating wormhole which is composed of two parts of two BTZ spacetimes just because we can treat it easily. The simplicity would help us in the further development of the stability of rotating thin-shell wormholes such as the collision of thin shells [75] and nonlinear stability [76]. We hope that this article will stimulate further work in this direction.

³We notice that a charged thin-shell wormhole is also stable with any β_0^2 when some conditions are satisfied [28].

In the rest of this section, we comment on rotating wormholes. In Refs. [84,85], a rotating wormhole solution with a ghost scalar field in four dimensions was obtained, and the maximum of the angular momentum of the rotating wormhole was found. The metric of the maximal rotating wormhole is correspondent with the one of a maximal rotating Kerr black hole. A rotating wormhole solution with an equal angular momenta filled with a ghost scalar field in five dimensions also has a maximal angular momentum [18]. The metric of the maximal rotating wormhole is correspondent with the one of a maximal rotating Myers-Perry black hole [86,87] in five dimensions. These facts do not mean that any wormholes cannot rotate rapidly since the upper bound of the angular momentum would depend on matters that support wormholes.

It was shown that the center-of-mass energy of two falling particles can be very large if the particles collide at the throat of a very highly rotating wormhole [88,89]. This implies that very highly rotating wormholes would be unstable against particle collisions.

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- [1] M. S. Morris and K. S. Thorne, *Am. J. Phys.* **56**, 395 (1988).
- [2] H. G. Ellis, *J. Math. Phys. (N.Y.)* **14**, 104 (1973).
- [3] K. A. Bronnikov, *Acta Phys. Pol. B* **4**, 251 (1973).
- [4] H.-a. Shinkai and S. A. Hayward, *Phys. Rev. D* **66**, 044005 (2002).
- [5] J. A. Gonzalez, F. S. Guzman, and O. Sarbach, *Classical Quantum Gravity* **26**, 015010 (2009).
- [6] J. A. Gonzalez, F. S. Guzman, and O. Sarbach, *Classical Quantum Gravity* **26**, 015011 (2009).
- [7] A. Doroshkevich, J. Hansen, I. Novikov, and A. Shatskiy, *Int. J. Mod. Phys. D* **18**, 1665 (2009).
- [8] K. A. Bronnikov, J. C. Fabris, and A. Zhidenko, *Eur. Phys. J. C* **71**, 1791 (2011).
- [9] K. A. Bronnikov, R. A. Konoplya, and A. Zhidenko, *Phys. Rev. D* **86**, 024028 (2012).
- [10] C. Armendáriz-Picón, *Phys. Rev. D* **65**, 104010 (2002).
- [11] T. Torii and H.-a. Shinkai, *Phys. Rev. D* **88**, 064027 (2013).
- [12] A. Shatskiy, I. D. Novikov, and N. S. Kardashev, *Phys. Usp.* **51**, 457 (2008).
- [13] I. Novikov and A. Shatskiy, *J. Exp. Theor. Phys.* **114**, 801 (2012).
- [14] K. A. Bronnikov, L. N. Lipatova, I. D. Novikov, and A. A. Shatskiy, *Gravitation Cosmol.* **19**, 269 (2013).
- [15] P. H. Cox, B. C. Harms, and S. Hou, *Phys. Rev. D* **93**, 044014 (2016).
- [16] E. Teo, *Phys. Rev. D* **58**, 024014 (1998).
- [17] T. Matos and D. Nunez, *Classical Quantum Gravity* **23**, 4485 (2006).
- [18] V. Dzhunushaliev, V. Folomeev, B. Kleihaus, J. Kunz, and E. Radu, *Phys. Rev. D* **88**, 124028 (2013).
- [19] C. Lanczos, *Phys. Zeits.* **23**, 539 (1922).
- [20] C. Lanczos, *Ann. Phys. (Berlin)* **379**, 518 (1924).
- [21] W. Israel, *Nuovo Cimento B* **44**, 1 (1966); **48**, 463(E) (1967).
- [22] G. Darmois, *Les Équations de la Gravitation Einsteinienne*, Mémorial des Sciences Mathématiques (Gauthier-Villars, Paris, 1927), Vol. XXV.
- [23] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, England, 2004).
- [24] M. Visser, *Phys. Rev. D* **39**, 3182 (1989).
- [25] M. Visser, *Nucl. Phys.* **B328**, 203 (1989).
- [26] E. Poisson and M. Visser, *Phys. Rev. D* **52**, 7318 (1995).
- [27] M. Ishak and K. Lake, *Phys. Rev. D* **65**, 044011 (2002).
- [28] E. F. Eiroa and G. E. Romero, *Gen. Relativ. Gravit.* **36**, 651 (2004).
- [29] F. S. N. Lobo and P. Crawford, *Classical Quantum Gravity* **21**, 391 (2004).
- [30] F. S. N. Lobo and P. Crawford, *Classical Quantum Gravity* **22**, 4869 (2005).
- [31] E. F. Eiroa and C. Simeone, *Phys. Rev. D* **76**, 024021 (2007).
- [32] E. F. Eiroa, *Phys. Rev. D* **78**, 024018 (2008).
- [33] J. P. S. Lemos and F. S. N. Lobo, *Phys. Rev. D* **78**, 044030 (2008).
- [34] S. H. Mazharimousavi, M. Halilsoy, and Z. Amirabi, *Phys. Rev. D* **81**, 104002 (2010).
- [35] P. K. F. Kuhfittig, *Acta Phys. Pol. B* **41**, 2017 (2010).
- [36] G. A. S. Dias and J. P. S. Lemos, *Phys. Rev. D* **82**, 084023 (2010).
- [37] M. La Camera, *Mod. Phys. Lett. A* **26**, 857 (2011).
- [38] Z. Amirabi, M. Halilsoy, and S. Habib Mazharimousavi, *Phys. Rev. D* **88**, 124023 (2013).
- [39] X. Yue and S. Gao, *Phys. Lett. A* **375**, 2193 (2011).
- [40] F. Rahaman, A. Banerjee, and I. Radinschi, *Int. J. Theor. Phys.* **51**, 1680 (2012).
- [41] P. K. F. Kuhfittig, *Adv. High Energy Phys.* **2012**, 462493 (2012).
- [42] A. Banerjee, F. Rahaman, S. Chattopadhyay, and S. Banerjee, *Int. J. Theor. Phys.* **52**, 3188 (2013).
- [43] I. Bochicchio and E. Laserra, *Int. J. Theor. Phys.* **52**, 3721 (2013).
- [44] M. Sharif and M. Azam, *J. Cosmol. Asropart. Phys.* **04** (2013) 023.
- [45] M. Sharif and M. Azam, *J. Phys. Soc. Jpn.* **81**, 124006 (2012).
- [46] M. Sharif and M. Azam, *Eur. Phys. J. C* **73**, 2407 (2013).
- [47] A. Banerjee, *Int. J. Theor. Phys.* **52**, 2943 (2013).
- [48] V. Varela, *Phys. Rev. D* **92**, 044002 (2015).
- [49] S. H. Mazharimousavi, M. Halilsoy, and Z. Amirabi, *Phys. Rev. D* **89**, 084003 (2014).
- [50] C. Bejarano, E. F. Eiroa, and C. Simeone, *Eur. Phys. J. C* **74**, 3015 (2014).
- [51] M. Sharif and S. Mumtaz, *Astrophys. Space Sci.* **352**, 729 (2014).
- [52] P. K. F. Kuhfittig, *Fund. J. Mod. Phys.* **7**, 111 (2014).
- [53] T. Kokubu and T. Harada, *Classical Quantum Gravity* **32**, 205001 (2015).
- [54] P. Bhar and A. Banerjee, *Int. J. Mod. Phys. D* **24**, 1550034 (2015).
- [55] A. Eid, *New Astron.* **39**, 72 (2015).
- [56] T. Kokubu, H. Maeda, and T. Harada, *Classical Quantum Gravity* **32**, 235021 (2015).
- [57] P. K. F. Kuhfittig, *Indian J. Phys.* **90**, 837 (2016).
- [58] M. Sharif and S. Mumtaz, *Can. J. Phys.* **94**, 158 (2016).
- [59] A. Eid, *Eur. Phys. J. Plus* **131**, 23 (2016).
- [60] A. Övgün, *Eur. Phys. J. Plus* **131**, 389 (2016).
- [61] J. B. Dent, D. A. Easson, T. W. Kephart, and S. C. White, *Int. J. Mod. Phys. D* **26**, 1750117 (2017).
- [62] Z. Amirabi, *Eur. Phys. J. Plus* **131**, 270 (2016).
- [63] M. Sharif and F. Javed, *Gen. Relativ. Gravit.* **48**, 158 (2016).
- [64] A. Övgün and K. Jusufi, *Adv. High Energy Phys.* **2017**, 1215254 (2017).
- [65] A. Eid, *New Astron.* **53**, 6 (2017).
- [66] A. Banerjee, K. Jusufi, and S. Bahamonde, *Gravitation Cosmol.* **24**, 71 (2018).
- [67] S. Habib Mazharimousavi, Z. Amirabi, and M. Halilsoy, *Mod. Phys. Lett. A* **32**, 1750064 (2017).
- [68] A. Eid, *Indian J. Phys.* **91**, 1451 (2017).
- [69] E. Rubin de Celis, C. Tomasini, and C. Simeone, *Int. J. Mod. Phys. D* **26**, 1750171 (2017).
- [70] M. G. Richarte, I. G. Salako, J. P. Morais Graca, H. Moradpour, and A. Övgün, *Phys. Rev. D* **96**, 084022 (2017).
- [71] E. F. Eiroa and G. Figueroa Aguirre, *Eur. Phys. J. C* **78**, 54 (2018).
- [72] S. D. Forghani, S. Habib Mazharimousavi, and M. Halilsoy, *Eur. Phys. J. C* **78**, 469 (2018).
- [73] Z. Amirabi, M. Halilsoy, and S. H. Mazharimousavi, *Mod. Phys. Lett. A* **33**, 1850049 (2018).

- [74] A. Eid, *Indian J. Phys.* **92**, 1065 (2018).
- [75] K.-i. Nakao, T. Uno, and S. Kinoshita, *Phys. Rev. D* **88**, 044036 (2013).
- [76] Y. Akai and K.-i. Nakao, *Phys. Rev. D* **96**, 024033 (2017).
- [77] R. B. Mann, J. J. Oh, and M. I. Park, *Phys. Rev. D* **79**, 064005 (2009).
- [78] J. Crisostomo and R. Olea, *Phys. Rev. D* **69**, 104023 (2004).
- [79] A. Övgün, [arXiv:1610.08118](https://arxiv.org/abs/1610.08118).
- [80] S. H. Mazharimousavi and M. Halilsoy, *Eur. Phys. J. C* **74**, 3073 (2014).
- [81] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [82] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993); **88**, 069902(E) (2013).
- [83] E. Ayon-Beato, F. Canfora, and J. Zanelli, *Phys. Lett. B* **752**, 201 (2016).
- [84] B. Kleihaus and J. Kunz, *Phys. Rev. D* **90**, 121503 (2014).
- [85] X. Y. Chew, B. Kleihaus, and J. Kunz, *Phys. Rev. D* **94**, 104031 (2016).
- [86] R. C. Myers and M. J. Perry, *Ann. Phys. (N.Y.)* **172**, 304 (1986).
- [87] R. C. Myers, Myers-Perry black holes, in *Black Holes in Higher Dimensions*, edited by G. Horowitz (Cambridge University Press, Cambridge, England, 2012), p. 101.
- [88] N. Tsukamoto and C. Bambi, *Phys. Rev. D* **91**, 084013 (2015).
- [89] O. B. Zaslavskii, *Mod. Phys. Lett. A* **31**, 1650029 (2016).