

Conical Morris-Thorne wormholes with a global monopole charge

Kimet Jusufi*

*Physics Department, State University of Tetovo, Ilinden Street nn, 1200, Tetovo, Macedonia
and Institute of Physics, Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University,
Arhimedova 3, 1000 Skopje, Macedonia*



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In this paper, we have established an asymptotically conical Morris-Thorne wormhole solution that is supported by an anisotropic matter fluid and a global monopole charge in the framework of a $1 + 3$ dimensional gravity that is minimally coupled to a triplet of scalar fields ϕ^a , which results from the breaking of a global $O(3)$ symmetry. For the anisotropic matter fluid, we have considered the equation of state (EoS) given by $\mathcal{P}_r = \omega\rho$, with a consequence $\omega < -1$, that implies a so-called phantom energy at the throat of the wormhole, which violates the energy conditions. In addition, we study the weak gravitational lensing effect using the Gauss-Bonnet theorem (GBT) applied to the wormhole optical geometry. We show that the total deflection angle consists of a term given by $4\pi^2\eta^2$, which is independent of the impact parameter b , and an additional term, which depends on the radius of the wormhole throat b_0 as well as the dimensionless constant ζ .

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I. INTRODUCTION

Wormholes are associated with the amazing spacetime topology of the connection of two spacetime geometries that are located in different regions of the Universe or different universes. They are solutions of the Einstein field equations; historically, the first step toward the concept of wormholes was made by Flamm [1] and later a new spin was put forward by Einstein and Rosen [2]. It is interesting to note that Einstein and Rosen proposed a geometric model for elementary particles, such as the electron, in terms of the Einstein-Rosen bridge (ERB). However, this model turns out to be unsuccessful, and moreover, the ERB was shown to be unstable [3–8].

Traversable wormholes (TW) were studied extensively in the past by several authors; notably Ellis [7,8] and Bronnikov [9] studied exact traversable wormhole solutions with a phantom scalar, while a few years later, different wormhole models were discussed by Clement [10], followed by the seminal paper by Morris and Thorne [11]. Afterwards, Visser developed the concept of thin-shell wormholes [12]. Based on physical grounds, it is well-known that all the matter in our Universe obeys certain energy conditions; in this context, as we shall see, the existence of wormholes is problematic. In particular, the geometry of TW requires a special kind of exotic matter concentrated at the wormhole throat (to keep the spacetime region open at the throat). In other words, this kind of matter violates the energy conditions, such as the null

energy condition (NEC) [12]. It is speculated that such a matter can exist in the context of quantum field theory. The second problem is related to the stability of the wormholes. Given the wormhole spacetime geometry, one way to check the stability analyses is the linear perturbation method around the wormhole throat proposed by Visser and Poisson [13]. Wormholes have been studied in the framework of different gravity theories, e.g., the rotating traversable wormhole solution found by Teo [14], spinning wormholes in the scalar-tensor theory [15], wormholes with phantom energy [16], wormholes in gravity's rainbow [17], the traversable Lorentzian wormhole with a cosmological constant [18], wormholes in Einstein-Cartan theory [19], wormholes in Eddington-inspired Born-Infeld gravity [20–22], wormholes with different scalar fields and charged [23–29], wormholes from cosmic strings [30], wormholes by grand unified theories in the early Universe [31], wormholes in $f(R, T)$ gravity [32], and recently, [33–35]. Recently, extensive studies have been conducted by different authors related to the thin-shell wormhole approach [36–45].

Topological defects are interesting objects predicted to exist by particle physics due to the phase transition mechanism in the early Universe [46]. One particular example of topological defects is the global monopole, which is a spherically symmetric object resulting from the self-coupling triplet of scalar fields ϕ^a that undergoes a spontaneous breaking of the global $O(3)$ gauge symmetry down to $U(1)$. The spacetime metric describing the global monopole has been studied in many papers including [47–51]. In this paper, we provide a new Morris-Thorne

*kimet.jusufi@unite.edu.mk

wormhole solution with an anisotropic fluid and a global monopole charge in the 1 + 3 gravity theory minimally coupled to a triplet of scalar fields. The deflection of light by black holes and wormholes has attracted great interest; in this context, the necessary methodology can be found in the papers by Bozza [52–55], Perlick *et al.* [56–59], and Tsukamoto *et al.* [60–65]. For some recent works concerning the strong/weak lensing, see also [66–81]. While for an alternative method to study gravitational lensing via GBT, see Refs. [82–87].

This paper has the following organization. In Sec. II, we deduce the metric for a static and spherically symmetric Morris-Thorne wormhole with a global monopole charge. In Sec. III, we study the weak gravitational lensing applying the Gauss-Bonnet theorem. In Sec. IV, we draw our conclusions.

II. MORRIS-THORNE WORMHOLE WITH A GLOBAL MONOPOLE CHARGE

We start by writing the 3 + 1 dimensional action without a cosmological constant minimally coupled to a scalar field with matter fields in units $c = G = 1$ given by

$$S = \int \left(\frac{\mathfrak{R}}{2\kappa} + \mathcal{L} \right) \sqrt{-g} d^4x + S_m \quad (1)$$

in which $\kappa = 8\pi$. The Lagrangian density describing a self-coupling scalar triplet ϕ^a is given by [47]

$$\mathcal{L} = -\frac{1}{2} \sum_a g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a - \frac{\lambda}{4} (\phi^2 - \eta^2)^2, \quad (2)$$

with $a = 1, 2, 3$, while λ is the self-interaction term, η is the scale of a gauge-symmetry breaking. The field configuration describing a monopole is

$$\phi^a = \frac{\eta f(r) x^a}{r}, \quad (3)$$

in which

$$x^a = \{r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta\}, \quad (4)$$

such that $\sum_a x^a x^a = r^2$. Next, we consider a static and spherically symmetric Morris-Thorne traversable wormhole in the Schwarzschild coordinates given by [11]

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (5)$$

in which $\Phi(r)$ and $b(r)$ are the redshift and shape functions, respectively. In the wormhole geometry, the redshift function $\Phi(r)$ should be finite in order to avoid the formation of an event horizon. Moreover, the shape function $b(r)$ determines the wormhole geometry, with the

following condition $b(r_0) = r_0$, in which r_0 is the radius of the wormhole throat. Consequently, the shape function must satisfy the flaring-out condition [16],

$$\frac{b(r) - rb'(r)}{b^2(r)} > 0, \quad (6)$$

in which $b'(r) = \frac{db}{dr} < 1$ must hold at the throat of the wormhole. The Lagrangian density in terms of f reads

$$\mathcal{L} = -\left(1 - \frac{b(r)}{r}\right) \frac{\eta^2 (f')^2}{2} - \frac{\eta^2 f^2}{r^2} - \frac{\lambda \eta^4}{4} (f^2 - 1)^2. \quad (7)$$

On the other hand, the Euler-Lagrange equation for the field f gives

$$\left(1 - \frac{b}{r}\right) f'' + f' \left[\left(1 - \frac{b(r)}{r}\right) \frac{2}{r} + \frac{1}{2} \left(\frac{b - b'r}{r^2}\right) \right] - f \left[\frac{2}{r^2} + \lambda \eta^2 (f^2 - 1) \right] = 0. \quad (8)$$

The energy momentum tensor from the Lagrangian density (2) is found to be

$$\bar{T}_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^a - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi^a \partial_\sigma \phi^a - \frac{g_{\mu\nu} \lambda}{4} (\phi^a \phi^a - \eta^2)^2. \quad (9)$$

Using the last equation, the energy-momentum components are given as follows:

$$\bar{T}_t^t = -\eta^2 \left[\frac{f^2}{r^2} + \left(1 - \frac{b}{r}\right) \frac{(f')^2}{2} + \frac{\lambda \eta^2}{4} (f^2 - 1)^2 \right], \quad (10)$$

$$\bar{T}_r^r = -\eta^2 \left[\frac{f^2}{r^2} - \left(1 - \frac{b}{r}\right) \frac{(f')^2}{2} + \frac{\lambda \eta^2}{4} (f^2 - 1)^2 \right], \quad (11)$$

$$\bar{T}_\theta^\theta = \bar{T}_\varphi^\varphi = -\eta^2 \left[\left(1 - \frac{b}{r}\right) \frac{(f')^2}{2} + \frac{\lambda \eta^2}{4} (f^2 - 1)^2 \right]. \quad (12)$$

It turns out that Eq. (8) cannot be solved exactly; however, it suffices to set $f(r) \rightarrow 1$ outside the wormhole. Consequently, the energy-momentum components reduces to

$$\bar{T}_t^t = \bar{T}_r^r \simeq -\frac{\eta^2}{r^2}, \quad \bar{T}_\theta^\theta = \bar{T}_\varphi^\varphi \simeq 0. \quad (13)$$

On the other hand, Einstein's field equations (EFE) reads

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \quad (14)$$

where $\mathcal{T}_{\mu\nu}$ is the total energy-momentum tensor which can be written as a sum of the matter fluid part and the matter fields

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu}^{(0)} + \tilde{T}_{\mu\nu}. \quad (15)$$

For the matter fluid, we shall consider an anisotropic fluid with the following energy-momentum tensor components:

$$T^{\mu}_{\nu}{}^{(0)} = (-\rho, \mathcal{P}_r, \mathcal{P}_\theta, \mathcal{P}_\varphi). \quad (16)$$

Einstein tensor components for the generic wormhole metric (5) gives

$$\begin{aligned} G_t^t &= -\frac{b'(r)}{r^2}, \\ G_r^r &= -\frac{b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\Phi'}{r}, \\ G_\theta^\theta &= \left(1 - \frac{b(r)}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)}\Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r}\right], \\ G_\varphi^\varphi &= G_\theta^\theta. \end{aligned} \quad (17)$$

The energy-momentum components yields

$$\begin{aligned} \rho(r) &= \frac{1}{8\pi r^2}[b'(r) - 8\pi\eta^2], \\ \mathcal{P}_r(r) &= \frac{1}{8\pi}\left[2\left(1 - \frac{b(r)}{r}\right)\frac{\Phi'}{r} - \frac{b(r)}{r^3} + \frac{8\pi\eta^2}{r^2}\right], \\ \mathcal{P}(r) &= \frac{1}{8\pi}\left(1 - \frac{b(r)}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)}\Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r}\right], \end{aligned} \quad (18)$$

where $\mathcal{P} = \mathcal{P}_\theta = \mathcal{P}_\varphi$. To simplify the problem, we use the EoS of the form [16–18]

$$\mathcal{P}_r = \omega\rho. \quad (19)$$

In terms of the equation of state, from Eq. (18), it is possible to find the following result:

$$\frac{b(r) - 8\pi\eta^2 r + 8\pi\omega\rho r^3 - 2r(r-b(r))\Phi'(r)}{r^3} = 0. \quad (20)$$

Substituting the energy density relation,

$$\rho(r) = \frac{1}{8\pi r^2}[b'(r) - 8\pi\eta^2], \quad (21)$$

into Eq. (20), we find

$$\frac{b'(r)\omega r + b(r) - 8\pi\eta^2(\omega + 1)r - 2r(r-b(r))\Phi'(r)}{r^3} = 0. \quad (22)$$

In our setup, we shall consider a constant redshift function, namely a wormhole solution with a zero tidal force, i.e., $\Phi' = 0$; therefore, the last equation simplifies to

$$b'(r)\omega r + b(r) - 8\pi\eta^2(\omega + 1)r = 0. \quad (23)$$

Finally, we use the condition $b(r_0) = b_0 = r_0$; thus, by solving the last differential equation, we find the shape function to be

$$b(r) = \left(\frac{r_0}{r}\right)^{1+\omega} r_0(1 - 8\pi\eta^2) + 8\pi\eta^2 r. \quad (24)$$

One can observe that the wormhole solution is not asymptotically flat by checking the following equation (Fig. 1):

$$\lim_{r \rightarrow \infty} \frac{b(r)}{r} \rightarrow \lim_{r \rightarrow \infty} \left[\left(\frac{r_0}{r}\right)^{1+\omega} (1 - 8\pi\eta^2) \right] + 8\pi\eta^2. \quad (25)$$

The first term blows up when $r \rightarrow \infty$, since $\omega < -1$. With the help of the shape function, the wormhole metric reduces to

$$ds^2 = -dt^2 + \frac{dr^2}{(1 - 8\pi\eta^2)\left[1 - \left(\frac{r_0}{r}\right)^{1+\omega}\right]} + r^2 d\Omega^2. \quad (26)$$

Note that the constant factor $\exp(2\Phi) = \text{const}$, is absorbed into the rescaled time coordinate t . To the best of our knowledge, this metric is reported here for the first time. On the other hand, the metric coefficient g_{rr} diverges at the throat $b(r_0) = r_0$; however, this just signals the coordinate singularity. To see this, one can calculate the scalar curvature or the Ricci scalar, which is found to be

$$\mathfrak{R} = \frac{16\pi\eta^2}{r^2} + (1 - 8\pi\eta^2)\frac{2r_0}{\omega r^2}\left(\frac{r_0}{r}\right)^{1+\omega}; \quad (27)$$

from the last equation, we see that the metric is regular at $r = r_0$. Because of the above coordinate singularity, it is convenient to compute the proper radial distance, which should be a finite quantity

$$l = \pm \int_{r_0}^r \frac{dr'}{\sqrt{1 - \frac{b_\pm(r')}{r'}}}. \quad (28)$$

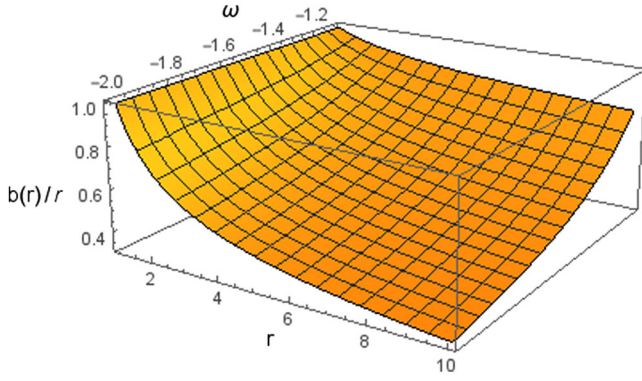


FIG. 1. The figure shows the behavior of the shape function $b(r)/r$ as a function of r and ω , for chosen $b_0 = 1$ and $\eta = 10^{-5}$.

Using Eq. (24), we find

$$l(r) = \pm \frac{\left[rF_1\left(\frac{1}{2}, -\frac{\omega+1}{\omega}, \frac{1}{\omega+1}, \left(\frac{r_0}{r}\right)^{\frac{\omega+1}{\omega}}\right) - \frac{r_0\sqrt{\pi}\Gamma\left(\frac{1}{\omega+1}\right)}{\Gamma\left(\frac{1}{2} - \frac{\omega}{\omega+1}\right)} \right]}{\sqrt{1 - 8\pi\eta^2}} \quad (29)$$

in which \pm stands for the upper and lower part, respectively. Next, we verify whether the null energy condition (NEC) and weak energy condition (WEC) are satisfied at the throat of the wormhole. As we know, WEC is defined by $T_{\mu\nu}U^\mu U^\nu \geq 0$, i.e., $\rho \geq 0$ and $\rho(r) + \mathcal{P}_r(r) \geq 0$, where $T_{\mu\nu}$ is the energy momentum tensor with U^μ being a timelike vector. On the other hand, NEC can be defined by $T_{\mu\nu}k^\mu k^\nu \geq 0$, i.e., $\rho(r) + \mathcal{P}_r(r) \geq 0$, with k^μ being a null vector. In this regard, we have the following energy condition at the throat region:

$$\rho(r_0) = \frac{b'(r_0) - 8\pi\eta^2}{8\pi r_0^2}. \quad (30)$$

Now, using the field equations, one finds the following relations:

$$\rho(r) + \mathcal{P}_r(r) = \frac{1}{8\pi} \left[\frac{b'(r_0)r_0 - b(r_0)}{r_0^3} \right]; \quad (31)$$

considering now the shape function at the throat region, we find

$$(\rho + \mathcal{P}_r)|_{r=r_0} = -\frac{(1 - 8\pi\eta^2)(\omega + 1)}{8\omega\pi r_0^2}. \quad (32)$$

This result verifies that matter configuration violates the energy conditions at the throat ($\rho + \mathcal{P}_r|_{r=r_0} < 0$ (Fig. 2).

Another way to see this is simply by using the flaring-out condition,

$$b'(r_0) = 8\pi\eta^2 - \frac{(1 - 8\pi\eta^2)}{\omega} < 1, \quad (33)$$

which implies $\omega < -1$. This form of exotic matter with $\omega < -1$ is usually known as a phantom energy. Another important quantity is the ‘‘volume integral quantifier,’’ which basically measures the amount of exotic matter needed for the wormhole defined as follows:

$$\mathcal{I}_V = \int (\rho(r) + \mathcal{P}_r(r))dV, \quad (34)$$

with the volume element given by $dV = r^2 \sin\theta dr d\theta d\phi$.

For simplicity, we shall evaluate the volume-integral associated to the phantom energy of our wormhole space-time (26) by assuming an arbitrary small region, e.g., r_0 to a radius situated at ‘‘ a ’’, in which the exotic matter is confined. More specifically, by considering our shape function $b(r)$ given by Eq. (24), for the amount of exotic matter, we find

$$\mathcal{I}_V = \frac{(\omega + 1)(1 - 8\pi\eta^2)}{2\omega} (a - r_0). \quad (35)$$

For an interesting observation when $a \rightarrow r_0$ then it follows

$$\int (\rho + \mathcal{P}_r) \rightarrow 0, \quad (36)$$

and thus, one may interpret that a wormhole can be contracted with arbitrarily small quantities of ANEC violating matter.

As we have already seen from (25), the first term blows up when $r \rightarrow \infty$, since $\omega < -1$. In order to overcome this problem, it is convenient to rewrite the shape function in terms of new dimensionless constants. In particular, following Lobo *et al.* [88], we can consider the following shape function given by

$$\frac{b(r)}{r_0} = a \left[\left(\frac{r}{r_0}\right)^\zeta (1 - 8\pi\eta^2) + 8\pi\eta^2 \left(\frac{r}{r_0}\right) \right] + C, \quad (37)$$

where a , ζ , and C , are dimensionless constants. Without the loss of generality, we choose $a = 1$, then using $b(r_0)/r_0 = 1$, we find $C = 0$. Furthermore, considering a positive energy density implies $\zeta > 0$, while the flaring-out condition imposes an additional constraint at the throat, namely $\zeta < 1$. Moreover, using the equation of state at the throat $\mathcal{P}_r(r_0) = \omega\rho(r_0)$, we find $\zeta\omega = -1$. On the other hand, from Eqs. (21) and (22), we can deduce the following equation:

$$\Phi' = \frac{b(r) - 8\pi\eta^2 r + \omega r(b'(r) - 8\pi\eta^2)}{2r^2(1 - b(r)/r)}. \quad (38)$$

To this end, using the condition $\zeta\omega = -1$ at $r = r_0$, we find that $\Phi = \text{const}$. With this information, we can write our wormhole metric as follows:

$$ds^2 = -dt^2 + \frac{dr^2}{(1 - 8\pi\eta^2)[1 - (\frac{b_0}{r})^{1-\zeta}]} + r^2 d\Omega^2, \quad (39)$$

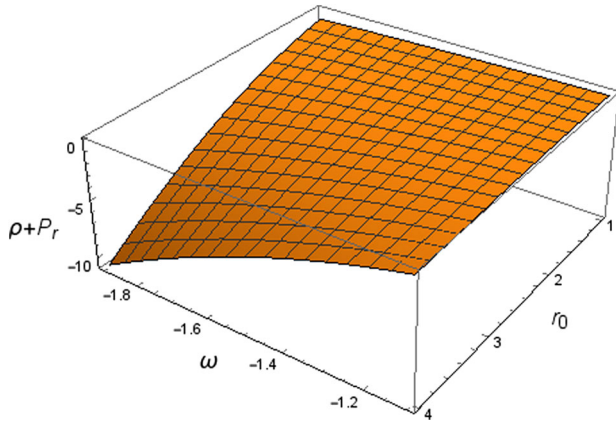


FIG. 2. In this figure, we depict the behavior of $\rho + \mathcal{P}_r$ as a function of r and ω . We have chosen $b_0 = 1$ and $\eta = 10^{-5}$. The energy conditions are violated.

provided that ζ is in the range $0 < \zeta < 1$. Now one can check that

$$\lim_{r \rightarrow \infty} \frac{b(r)}{r} = \lim_{r \rightarrow \infty} \left(\frac{r_0}{r} \right)^{1-\zeta} (1 - 8\pi\eta^2) + 8\pi\eta^2 = 8\pi\eta^2, \quad (40)$$

provided $0 < \zeta < 1$. This equation shows that our wormhole metric (39) is asymptotically conical with a conical deficit angle which is independent of the radial coordinate r . Furthermore, we can construct the embedding diagrams to visualize the conical wormhole by considering an equatorial slice, $\theta = \pi/2$ and a fixed moment of time, $t = \text{const}$. It follows

$$ds^2 = \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\varphi^2. \quad (41)$$

On the other hand, we can embed the metric into three-dimensional Euclidean space written in terms of cylindrical coordinates as follows:

$$ds^2 = dz^2 + dr^2 + r^2 d\varphi^2 = \left[1 + \left(\frac{dz}{dr} \right)^2 \right] dr^2 + r^2 d\varphi^2. \quad (42)$$

From these equations, we can deduce the equation for the embedding surface as follows:

$$\frac{dz}{dr} = \pm \frac{1}{\sqrt{(1 - 8\pi\eta^2) \left[1 - \left(\frac{b_0}{r} \right)^{1-\zeta} \right]}}. \quad (43)$$

Finally, we can evaluate this integral numerically for specific parameter values in order to illustrate the conical wormhole shape given in Fig. 3.

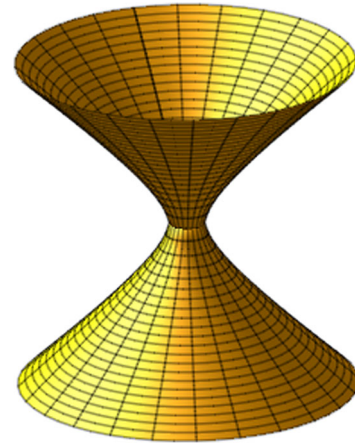


FIG. 3. The embedding diagram of a two-dimensional section along the equatorial plane with $t = \text{const}$ and $\theta = \pi/2$. To visualize this, we plot z vs r sweep through a 2π rotation around the z axis. We chose $b_0 = 1$, $\eta = 0.01$, and $\zeta = 0.5$.

III. GRAVITATIONAL LENSING

We can now proceed to elaborate the gravitational lensing effect in the spacetime of the wormhole metric (39). The wormhole optical metric can be simply found letting $ds^2 = 0$, resulting in

$$dt^2 = \frac{dr^2}{(1 - 8\pi\eta^2) \left[1 - \left(\frac{b_0}{r} \right)^{1-\zeta} \right]} + r^2 d\varphi^2. \quad (44)$$

Consequently, the optical metric can be written in terms of the new coordinates

$$dt^2 = h_{ab} dy^a dy^b = du^2 + \mathcal{H}^2(u) d\varphi^2, \quad (45)$$

in which we have introduced $\mathcal{H} = r$ and

$$du = \frac{dr}{\sqrt{(1 - 8\pi\eta^2) \left[1 - \left(\frac{b_0}{r} \right)^{1-\zeta} \right]}}. \quad (46)$$

It is very important to first compute the Gaussian optical curvature (GOC) \mathcal{K} , which is defined in terms of the following equation [82]:

$$\mathcal{K} = -\frac{1}{\mathcal{H}(u)} \left[\frac{dr}{du} \frac{d}{dr} \left(\frac{dr}{du} \right) \frac{d\mathcal{H}}{dr} + \left(\frac{dr}{du} \right)^2 \frac{d^2 \mathcal{H}}{dr^2} \right]. \quad (47)$$

Applying this to our optical metric, we find

$$\mathcal{K} = -\frac{(1 - 8\pi\eta^2)}{2r^2} \left(\frac{b_0}{r} \right)^{1-\zeta} (1 - \zeta). \quad (48)$$

Obviously, the GOC is affected by the global monopole charge and the state parameter. Note the important negative sign which is implying the divergence of light rays in the wormhole geometry. But, as we are going to see, this is

crucial in evaluating the deflection angle, which is really a result of a global spacetime topology in terms of the Gauss-Bonnet theorem (GBT). Thus, in our setup, we first choose a nonsingular domain, or a region outside the light ray noted as \mathcal{A}_R , with boundaries $\partial\mathcal{A}_R = \gamma_h \cup C_R$. Then, the global GBT in terms of the above construction is formulated as follows:

$$\iint_{\mathcal{A}_R} \mathcal{K} d\sigma + \oint_{\partial\mathcal{A}_R} \kappa dt + \sum_k \psi_k = 2\pi\chi(\mathcal{A}_R). \quad (49)$$

In this equation, κ is usually known as the geodesic curvature (GC) and basically measures the deviation from the geodesics; \mathcal{K} is the GOC; $d\sigma$ is the optical surface element; finally, ψ_k notes the exterior angle at the k^{th} vertex. The domain is chosen to be outside of the light ray, implying the Euler characteristic number to be $\chi(\mathcal{A}_R) = 1$. The GC is defined via

$$\kappa = h(\nabla_{\dot{\gamma}} \dot{\gamma}), \quad (50)$$

where we impose the unit speed condition $h(\dot{\gamma}, \dot{\gamma}) = 1$. For a very large radial coordinate $R \rightarrow \infty$, our two jump angles (at the source \mathcal{S} and observer \mathcal{O}) yields $\psi_{\mathcal{O}} + \psi_{\mathcal{S}} \rightarrow \pi$ [82]. Then the GBT simplifies to

$$\iint_{\mathcal{A}_R} \mathcal{K} d\sigma + \oint_{C_R} \kappa dt \stackrel{R \rightarrow \infty}{=} \iint_{\mathcal{A}_\infty} \mathcal{K} d\sigma + \int_0^{\pi + \hat{\alpha}} d\varphi = \pi. \quad (51)$$

By definition, the GC for γ_h is zero; hence, we are left with a contribution from the curve C_R located at a coordinate distance R from the coordinate system chosen at the wormhole center in the equatorial plane. Hence, we need to compute

$$\kappa(C_R) = |\nabla_{\dot{C}_R} \dot{C}_R|. \quad (52)$$

In components notation, the radial part can be written as

$$(\nabla_{\dot{C}_R} \dot{C}_R)^r = \dot{C}_R^\varphi (\partial_\varphi \dot{C}_R^r) + \Gamma_{\varphi\varphi}^{r(op)} (\dot{C}_R^\varphi)^2. \quad (53)$$

With the help of the unit speed condition and after we compute the Christoffel symbol related to our optical metric in the large coordinate radius R , we are left with

$$\begin{aligned} \lim_{R \rightarrow \infty} \kappa(C_R) &= \lim_{R \rightarrow \infty} |\nabla_{\dot{C}_R} \dot{C}_R|, \\ &\rightarrow \frac{1}{R} \sqrt{1 - 8\pi\eta^2}. \end{aligned} \quad (54)$$

Hence, GC is in fact affected by the monopole charge. To see what this means, we write the optical metric in this limit for a constant R . We find

$$\lim_{R \rightarrow \infty} dt \rightarrow R d\varphi. \quad (55)$$

Putting the last two equation together, we see that $\kappa(C_R) dt = \sqrt{1 - 8\pi\eta^2} d\varphi$. This reflects the conical nature of our wormhole geometry; to put it more simply, our optical metric is not asymptotically Euclidean. Using this result from GBT, we can express the deflection angle as follows:

$$\hat{\alpha} = \pi \left[\frac{1}{\sqrt{1 - 8\pi\eta^2}} - 1 \right] - \frac{1}{\sqrt{1 - 8\pi\eta^2}} \int_0^\pi \int_{\frac{b}{\sin\varphi}}^\infty \mathcal{K} d\sigma. \quad (56)$$

If we used the equation for the light ray, $r(\varphi) = b/\sin\varphi$, in which b is the impact parameter that can be approximated with the closest approach distance from the wormhole in the first order approximation. The surface area is also approximated as

$$d\sigma = \sqrt{h} du d\varphi \simeq \frac{r}{\sqrt{1 - 8\pi\eta^2}}. \quad (57)$$

Finally, the total deflection angle is found to be

$$\hat{\alpha} = 4\pi^2\eta^2 + \left(\frac{b_0}{b}\right)^{1-\zeta} \frac{\sqrt{\pi}\Gamma(1 - \frac{\zeta}{2})}{2\Gamma(\frac{3-\zeta}{2})}. \quad (58)$$

We can recast our wormhole metric (26) in a different form (Fig. 4). In particular, if we introduce the coordinate transformations

$$\mathcal{R} \rightarrow \frac{r}{\sqrt{1 - 8\pi\eta^2}}, \quad (59)$$

and

$$\mathcal{B}_0 \rightarrow \frac{b_0}{\sqrt{1 - 8\pi\eta^2}}. \quad (60)$$

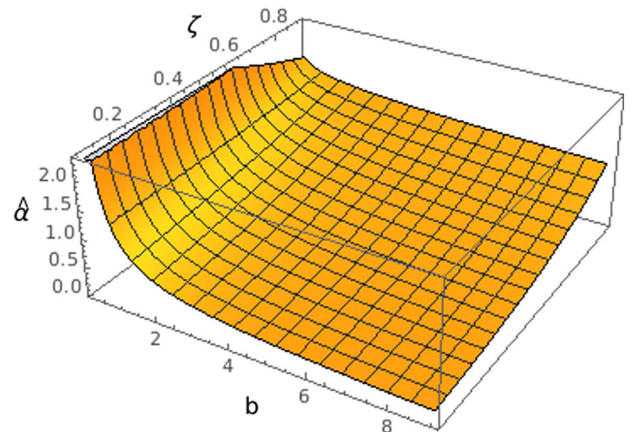


FIG. 4. The figure shows the deflection angle as a function of the impact parameter b and ζ , for chosen $b_0 = 1$ and $\eta = 10^{-5}$.

Taking into the consideration the above transformations, the wormhole metric reduces to

$$ds^2 = -dt^2 + \frac{d\mathcal{R}^2}{1 - \left(\frac{\mathcal{B}_0}{\mathcal{R}}\right)^{1-\zeta}} + (1 - 8\pi\eta^2)\mathcal{R}^2 d\Omega^2. \quad (61)$$

One can show that the deflection angle remains invariant under the coordinate transformations (59)–(60). In a similar fashion, we can apply the following substitutions: $\mathcal{H} = \mathcal{R}\sqrt{1 - 8\pi\eta^2}$ and

$$du = \frac{d\mathcal{R}}{\sqrt{1 - \left(\frac{\mathcal{B}_0}{\mathcal{R}}\right)^{1-\zeta}}}. \quad (62)$$

Then, for the GOP in this case, it is not difficult to find that

$$\mathcal{K} = -\frac{(1-\zeta)}{2\mathcal{R}^2} \left(\frac{\mathcal{B}_0}{\mathcal{R}}\right)^{1-\zeta}. \quad (63)$$

In the limit $R \rightarrow \infty$, GC yields

$$\begin{aligned} \lim_{R \rightarrow \infty} \kappa(C_R) &= \lim_{R \rightarrow \infty} |\nabla_{\dot{C}_R} \dot{C}_R|, \\ &\rightarrow \frac{1}{R}, \end{aligned} \quad (64)$$

but

$$\lim_{R \rightarrow \infty} dt \rightarrow R\sqrt{1 - 8\pi\eta^2} d\varphi. \quad (65)$$

Although GC is independent by η , we see that dt is affected by η . However, we end up with the same result $\kappa(C_R)dt = \sqrt{1 - 8\pi\eta^2} d\varphi$. This time the equation for the light ray can be chosen as $\mathcal{R} = \mathcal{B}/\sin\varphi$, resulting in a similar expression

$$\hat{\alpha} = \pi \left[\frac{1}{\sqrt{1 - 8\pi\eta^2}} - 1 \right] - \frac{1}{\sqrt{1 - 8\pi\eta^2}} \int_0^\pi \int_{\frac{\mathcal{B}}{\sin\varphi}}^\infty \mathcal{K} d\sigma. \quad (66)$$

Solving this integral, we can approximate the solution to be

$$\hat{\alpha} = 4\pi^2\eta^2 + \left(\frac{\mathcal{B}_0}{\mathcal{B}}\right)^{1-\zeta} \frac{\sqrt{\pi}\Gamma(1 - \frac{\zeta}{2})}{2\Gamma(\frac{3-\zeta}{2})}. \quad (67)$$

From the equations of the light rays, we deduce that the impact parameters should be related to

$$\mathcal{B} \rightarrow \frac{b}{\sqrt{1 - 8\pi\eta^2}}, \quad (68)$$

yielding the ratio

$$\frac{\mathcal{B}_0}{\mathcal{B}} \rightarrow \frac{b_0}{b}. \quad (69)$$

Thus, we showed that the final expression for the deflection angle remains invariant under the coordinate transformations (59)–(60). For an important observation, we can compare our result with two special cases. Firstly, we note that the metric (61) reduces to the pointlike global monopole metric by letting $\mathcal{B}_0 = 0$; thus,

$$ds^2 = -dt^2 + d\mathcal{R}^2 + (1 - 8\pi\eta^2)\mathcal{R}^2 d\Omega^2. \quad (70)$$

The deflection angle due to the pointlike global monopole is given by $4\pi^2\eta^2$ (see, e.g., [87]). It is clear that due to the geometric contribution related to the wormhole throat, the light bending is stronger in the wormhole case compared to the pointlike global monopole case.

IV. CONCLUSION

In this paper, we have found an asymptotically conical Morris-Thorne wormhole supported by an anisotropic matter fluid and a triplet of scalar fields ϕ^a minimally coupled to a 1 + 3 dimensional gravity. For the anisotropic fluid, we have used EoS of the form $\mathcal{P}_r = \omega\rho$, resulting with a phantom energy described by the relation $\omega < -1$. Our phantom wormhole solution is characterized by a solid angle deficit due to the global conical geometry that reveals interesting observational effects such as the gravitational lensing. Introducing a new dimensionless constant ζ , we have shown that our wormhole metric is not asymptotically flat, namely $b(r)/r \rightarrow 8\pi\eta^2$ when $r \rightarrow \infty$. We have also studied the deflection of light; more specifically, a detailed analysis using GBT revealed the following result for the deflection angle:

$$\hat{\alpha} = 4\pi^2\eta^2 + \left(\frac{b_0}{b}\right)^{1-\zeta} \frac{\sqrt{\pi}\Gamma(1 - \frac{\zeta}{2})}{2\Gamma(\frac{3-\zeta}{2})}.$$

Clearly, the first term, $4\pi^2\eta^2$, is independent of the impact parameter b , while the second term is a product of a function written in terms of the throat of the wormhole, b_0/b , and the Gamma functions depending on the dimensionless constant ζ . It is worth noting that we have performed our analysis in two different spacetime metrics. In both cases, we find the same result; hence, the deflection angle is form invariant under coordinate transformations. Finally, we pointed out that the gravitational lensing effect is stronger in the wormhole geometry case compared to the pointlike global monopole geometry.

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