Dynamics of compact binary systems in scalar-tensor theories: Equations of motion to the third post-Newtonian order

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Scalar-tensor theories are one of the most natural and well-constrained alternative theories of gravity, while still allowing for significant deviations from general relativity. We present the equations of motion of nonspinning compact binary systems at the third post-Newtonian (PN) order in massless scalar-tensor theories. We adapt the Fokker action of point particles in harmonic coordinates in general relativity to the specificities of scalar-tensor theories. We use dimensional regularization to treat both the infrared and ultraviolet divergences, and we consistently include the tail effects that contribute by a nonlocal term to the dynamics. This work is crucial in order to later compute the scalar gravitational waveform and the energy flux at 2PN order.

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I. INTRODUCTION

The observations by the LIGO-Virgo collaboration of gravitational waves emitted by coalescing compact binary systems have opened a new era in gravitational wave astronomy [1–5]. In the upcoming year, we expect to see many of these events, both in the advanced earth-based interferometric detectors, and in the space-based antenna LISA. The gravitational wave observations will allow us to not only measure the astrophysical properties of these systems, but also to challenge general relativity (GR) in the strong-field and highly dynamical regime of gravity.

The detection and parameter estimation of gravitational wave events require a bank of highly accurate templates for the gravitational waveforms. For the inspiral part of the coalescence of compact binary systems, the post-Newtonian formalism is well-suited to describe the evolution of the system [6]: it consists of an expansion in the small parameter $\varepsilon \equiv v/c \sim (Gm/rc^2)^{1/2}$. The current state of the art in GR concerning the dynamics is the 4PN order[1] [7–14]. The energy flux is known up to 3.5PN order beyond the quadrupole formula [15–18], with the 4.5PN coefficient also being known [19]; while the dominant modes of the gravitational waveform are known up to 3.5PN order [20–22]. The complete waveform is obtained by connecting the PN result with numerical relativity waveforms. At present, this is done using either a direct matching (IMR models) [23] or some resummation techniques (EOB waveforms) [24].

In order to test general relativity, one also has to model waveforms in alternative theories of gravity. Existing tests are performed using either theory-independent or theory-dependent methods. In this paper, we focus on a particular class of theories, namely massless scalar-tensor (ST) theories, which are among the most popular and well-studied theories. They date back to more than sixty years ago, when they were introduced by Jordan, Fierz, Brans and Dicke. See [25,26] for historical reviews of these theories and [27] for current constraints on the parameters. One of the motivations for studying these theories is to explain the accelerated expansion of the universe, as $f(R)$-theories, in which the action is expressed as a function of the Ricci scalar, can be expressed as a scalar-tensor theory [28].

Previous works in order to obtain the waveform at 2PN order have been performed during the last five years. The equations of motion are known at 2.5PN order [29], while the tensor gravitational waveform is known at 2PN order [30]. However, the scalar waveform is only known at 1.5PN and the energy flux at 1PN order, as they respectively start at -0.5PN and -1PN order with respect to the leading GR order [31]. All these ST results were obtained using the direct integration of relaxed Einstein equations (DIRE) method developed by Will, Wiseman and Pati [32–34]. The "Effective One-Body formalism" (EOB) has also been developed for ST theories, focusing on the derivation of a ST-EOB Hamiltonian [35,36]. Numerical works have shown that compact binaries in scalar-tensor theories can undergo a dynamical scalarization phenomenon [37,38], similar to the spontaneous scalarization effect for individual stars [39,40]. This phenomenon happens during the late-inspiral phase, where the post-Newtonian approximation is

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[1]As usual, we refer to post-Newtonian order as $n\mathrm{PN} \equiv \mathcal{O}(v^2/c^2)^n$.

expected to break done. Recently, an analytical method has been proposed to capture dynamical scalarization, using resummation techniques [41].

In order to compute the scalar waveform and energy flux at 2PN order, the equations of motion at 3PN order are required. In the present paper, we pursue this aim by constructing a Fokker action of point particles in harmonic coordinates. This method has recently been developed to successfully derive the 4PN equations of motion in GR [11]. Here, we adapt this approach to the specificities of scalar-tensor theories. We use dimensional regularization to treat both the infrared and ultraviolet divergences. We show that some tail effects appear at 3PN in ST theories, associated to the scalar dipole moment, while these effects start contributing only at 4PN in GR [42]. We then obtain a complete ambiguity-free result, as expected from the recent computation at 4PN in GR [13,14]. In the companion paper [43], we will study the conserved integrals of motion and the reduction to the center-of-mass frame.

In the following, we present in Sec. II our massless scalar-tensor theory, and derive the equations of motion. In Sec. III, we adapt the multipolar post-Newtonian formalism to ST theories. In particular, we consistently incorporate the tail effects that contribute to the conservative 3PN dynamics. In Sec. IV, we implement the post-Newtonian solution into the Fokker action, and explain the dimensional regularization method. Finally in Sec. V, we show the full 3PN equations of motion in harmonic coordinates for ST theories and conclude with some comments on our result.

Notations: We use boldface letter to represent three-dimensional Euclidean vectors. We denote by $\mathbf{y}_A(t)$ the two ordinary coordinate trajectories in a harmonic coordinate system $\{t, \mathbf{x}\}$, by $\mathbf{v}_A(t) = \mathrm{d}\mathbf{y}_A/\mathrm{d}t$ the two ordinary velocities and by $\mathbf{a}_A(t) = \mathrm{d}\mathbf{v}_A/\mathrm{d}t$ the two ordinary accelerations. The ordinary separation vector reads $\mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12}$, where $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$. Ordinary scalar products are denoted, e.g. $(n_{12}v_1) = \mathbf{n}_{12}\cdot\mathbf{v}_1$, while the two masses are indicated by m_1 and m_2. We note $\hat{n}_L$ the symmetric trace-free (STF) product of ℓ spatial vectors n_i, with $L = i_1 \cdots i_l$ a multi-index made of ℓ spatial indices.

II. MASSLESS SCALAR-TENSOR THEORIES

A. The field equations in ST theories

We consider a generic class of massless scalar-tensor theories in which a single massless scalar field ϕ minimally couples to the metric $g_{\mu\nu}$. It is described by the action

$$S_{\mathrm{st}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right] + S_{\mathrm{m}}(\mathfrak{m}, g_{\alpha\beta}), \tag{2.1}$$

where R and g are respectively the Ricci scalar and the determinant of the metric, ω is a function of the scalar field and $\mathfrak{m}$ stands generically for the matter fields. The action for the matter S_{m} is a function only of the matter fields and the metric. The action (2.1) is often called the "metric" or "Jordan"-frame action, as the matter does not couple directly to the scalar field.

We note ϕ_0 the value of the scalar field at spatial infinity and we assume that it is constant in time. We then define the rescaled scalar field $\varphi \equiv \frac{\phi}{\phi_0}$ and the conformally related metric,

$$\tilde{g}_{\mu\nu} \equiv \varphi g_{\mu\nu}. \tag{2.2}$$

In terms of these new variables, the action (2.1) can be rewritten as,

$$S_{\mathrm{st}} = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}} \times\left[\tilde{R} + \frac{3}{\varphi}\tilde{g}^{\alpha\beta}\nabla_\alpha\partial_\beta\varphi - \frac{9+2\omega(\phi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right] + S_{\mathrm{m}}(\mathfrak{m}, g_{\alpha\beta}), \tag{2.3}$$

Note that the matter fields still couple to the physical metric $g_{\mu\nu}$. As the scalar field is now minimally coupled to the metric, the action (2.3) is often called the "Einstein"-frame action, and we will do our calculation in this frame. Next, we perform some integrations by part to rewrite the action (2.3) into the Landau-Lifshitz form and we insert a harmonic gauge-fixing term $-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\Gamma}^\mu\tilde{\Gamma}^\nu$. The new action is fully equivalent to the previous one and reads,

$$S_{\mathrm{ST}} = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\Big[\tilde{g}^{\mu\nu}(\tilde{\Gamma}^\rho_{\mu\lambda}\tilde{\Gamma}^\lambda_{\nu\rho} - \tilde{\Gamma}^\rho_{\mu\nu}\tilde{\Gamma}^\lambda_{\rho\lambda}) - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\Gamma}^\mu\tilde{\Gamma}^\nu - \frac{3+2\omega(\phi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\Big] + S_{\mathrm{m}}(\mathfrak{m}, g_{\alpha\beta}), \tag{2.4}$$

where $\tilde{\Gamma}^\mu \equiv \tilde{g}^{\rho\sigma}\tilde{\Gamma}^\mu_{\rho\sigma}$ and $\tilde{\Gamma}^\mu_{\rho\sigma}$ are the Christoffel symbols of the conformal metric. Defining the inverse gothic metric by

$$\tilde{\mathfrak{g}}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}, \tag{2.5}$$

the action (2.3) can further be rewritten as

$$S_{\mathrm{ST}} = \frac{c^3\phi_0}{32\pi G}\int \mathrm{d}^4x\Big[-\frac{1}{2}\left(\tilde{\mathfrak{g}}_{\mu\rho}\tilde{\mathfrak{g}}_{\nu\sigma} - \frac{1}{2}\tilde{\mathfrak{g}}_{\mu\nu}\tilde{\mathfrak{g}}_{\rho\sigma}\right)\tilde{\mathfrak{g}}^{\lambda\tau}\partial_\lambda\tilde{\mathfrak{g}}^{\mu\nu}\partial_\tau\tilde{\mathfrak{g}}^{\rho\sigma} + \tilde{\mathfrak{g}}_{\mu\nu}\left(\partial_\rho\tilde{\mathfrak{g}}^{\mu\sigma}\partial_\sigma\tilde{\mathfrak{g}}^{\nu\rho} - \partial_\rho\tilde{\mathfrak{g}}^{\mu\rho}\partial_\sigma\tilde{\mathfrak{g}}^{\nu\sigma}\right) - \frac{3+2\omega}{\varphi^2}\tilde{\mathfrak{g}}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\Big] + S_{\mathrm{m}}(\mathfrak{m}, g_{\alpha\beta}). \tag{2.6}$$

Next, we expand the gothic metric around Minkowski space-time and define the metric and scalar perturbation variables $h^{\mu\nu}$ and ψ by

$$h^{\mu\nu} \equiv \tilde{\mathfrak{g}}^{\mu\nu} - \eta^{\mu\nu}, \quad \text{and} \quad \psi \equiv \varphi - 1. \tag{2.7}$$

The field equations derived from the gauge-fixed action (2.6) read,

$$\Box_\eta h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}, \tag{2.8a}$$

$$\Box_\eta \psi = -\frac{8\pi G}{c^4}\tau_s, \tag{2.8b}$$

with

$$\tau^{\mu\nu} = \frac{\varphi}{\phi_0}[(-g)T^{\mu\nu}] + \frac{c^4}{16\pi G}\Sigma^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda_{\mathrm{S}}^{\mu\nu}, \tag{2.9a}$$

$$\begin{aligned}\tau_s = &-\frac{\varphi}{\phi_0(3+2\omega)}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right)\\ &-\frac{c^4}{8\pi G}\left[-h^{\alpha\beta}\partial_\alpha\partial_\beta\psi - \partial_\alpha\psi\partial_\beta h^{\alpha\beta}\right.\\ &\left.+\left(\frac{1}{\varphi} - \frac{\phi_0\omega'}{3+2\omega}\right)\tilde{\mathfrak{g}}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right],\end{aligned} \tag{2.9b}$$

where $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}}$ is the matter stress-energy tensor and $T \equiv g_{\mu\nu}T^{\mu\nu}$. The scalar source term $\Lambda_{\mathrm{S}}^{\mu\nu}$ is given by

$$\Lambda_{\mathrm{S}}^{\mu\nu} = \frac{3+2\omega}{\varphi^2}\left(\tilde{\mathfrak{g}}^{\mu\alpha}\tilde{\mathfrak{g}}^{\nu\beta} - \frac{1}{2}\tilde{\mathfrak{g}}^{\mu\nu}\tilde{\mathfrak{g}}^{\alpha\beta}\right)\partial_\alpha\varphi\partial_\beta\varphi. \tag{2.10}$$

The gravitational source term $\Sigma^{\mu\nu} = \Lambda_{\mathrm{LL}}^{\mu\nu} + \Lambda_{\mathrm{H}}^{\mu\nu} + \Lambda_{\mathrm{gf}}^{\mu\nu}$, where $\Lambda_{\mathrm{LL}}^{\mu\nu}$ is the Landau-Lifshitz pseudoenergy tensor [44], is at least quadratic in the field h and its derivatives, with components given by

$$\begin{aligned}\Lambda_{\mathrm{LL}}^{\alpha\beta} = &\frac{1}{2}\tilde{\mathfrak{g}}^{\alpha\beta}\tilde{\mathfrak{g}}_{\mu\nu}\partial_\lambda h^{\mu\gamma}\partial_\gamma h^{\nu\lambda} - \tilde{\mathfrak{g}}^{\alpha\mu}\tilde{\mathfrak{g}}_{\nu\gamma}\partial_\lambda h^{\beta\gamma}\partial_\mu h^{\nu\lambda}\\ &-\tilde{\mathfrak{g}}^{\beta\mu}\tilde{\mathfrak{g}}_{\nu\gamma}\partial_\lambda h^{\alpha\gamma}\partial_\mu h^{\nu\lambda} + \tilde{\mathfrak{g}}_{\mu\nu}\tilde{\mathfrak{g}}^{\lambda\gamma}\partial_\lambda h^{\alpha\mu}\partial_\gamma h^{\beta\nu}\\ &+\frac{1}{8}(2\tilde{\mathfrak{g}}^{\alpha\mu}\tilde{\mathfrak{g}}^{\beta\nu} - \tilde{\mathfrak{g}}^{\alpha\beta}\tilde{\mathfrak{g}}^{\mu\nu})(2\tilde{\mathfrak{g}}_{\lambda\gamma}\tilde{\mathfrak{g}}_{\tau\pi} - \tilde{\mathfrak{g}}_{\gamma\tau}\tilde{\mathfrak{g}}_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\gamma\tau},\end{aligned} \tag{2.11a}$$

$$\Lambda_{\mathrm{H}}^{\alpha\beta} = -h^{\mu\nu}\partial_\mu\partial_\nu h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu}, \tag{2.11b}$$

$$\begin{aligned}\Lambda_{\mathrm{gf}}^{\alpha\beta} = &-\partial_\lambda h^{\lambda\alpha}\partial_\sigma h^{\sigma\beta} - \partial_\lambda h^{\lambda\rho}\partial_\rho h^{\alpha\beta} - \frac{1}{2}\tilde{\mathfrak{g}}^{\alpha\beta}\tilde{\mathfrak{g}}_{\rho\sigma}\partial_\lambda h^{\lambda\rho}\partial_\gamma h^{\gamma\sigma}\\ &+2\tilde{\mathfrak{g}}_{\rho\sigma}\tilde{\mathfrak{g}}^{\lambda(\alpha}\partial_\lambda h^{\beta)\rho}\partial_\gamma h^{\gamma\sigma}.\end{aligned} \tag{2.11c}$$

Note that the gauge-fixing term (2.11c) contains the harmonicities $\partial_\nu h^{\mu\nu}$ which are not zero in general. However, this term will ensure that, on-shell, our results are in harmonic coordinates.

B. The action for matter

We now make precise the action describing the matter. As we are dealing with compact, self-gravitating objects in scalar-tensor theories, we have to take into account the internal gravity of each body. To do so, we follow the approach pioneered by Eardley [45] and consider that the total mass of each body may depend on the value of the scalar field at its location. The skeletonized matter action is then given by the classical action for point particles, but with a mass $m_A(\phi)$, namely

$$S_{\mathrm{m}} = -\sum_A \int \mathrm{d}t\, m_A(\phi)c^2\sqrt{-(g_{\alpha\beta})_A\frac{v_A^\alpha v_A^\beta}{c^2}}, \tag{2.12}$$

where $v_A^\mu \equiv \frac{\mathrm{d}y_A^\mu}{\mathrm{d}t} = (c, \mathbf{v}_A)$ is the coordinate velocity of particle A, $y_A^\mu = (ct, \mathbf{y}_A)$ its trajectory and $(g_{\alpha\beta})_A$ is the physical metric evaluated at the position of particle A using the dimensional regularization scheme. We recall that the physical metric is related to the conformal one through $g_{\alpha\beta} = \frac{\tilde{g}_{\alpha\beta}}{\phi}$. Note that the scalar-field dependence of the mass is responsible for the term $\frac{\partial T}{\partial\varphi}$ in Eq. (2.9). In the absence of such a dependence, e.g. in GR, the matter stress-energy tensor should depend only on the matter variables and the metric.

We then define the sensitivity of each body with respect to the scalar field as

$$s_A \equiv \left.\frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln\phi}\right|_{\phi=\phi_0}. \tag{2.13}$$

In the calculation at 3PN, we will also need the higher order sensitivities, defined in Sec. V. The sensitivities of neutron stars are around $s_{\mathrm{NS}} \sim 0.2$, depending on the mass and the equation of state. Due to dynamical scalarization, neutron star sensitivities can dramatically grow during the late-inspiral. As we are working in the post-Newtonian formalism and we assume that the sensitivities are constant, we will not describe this effect in our work. Hawking's theorem states that stationary black holes have no hair in Brans-Dicke theory [46] and this result has been extended to generalized scalar-tensor theories [47]. Thus, for stationary black holes, the sensitivity is exactly $s_{\mathrm{BH}} = \frac{1}{2}$. Another way to see it is to define the scalar charges [39,48],

$$\alpha_A \equiv \frac{1 - 2s_A}{\sqrt{3 + 2\omega_0}}, \tag{2.14}$$

where $\omega_0 \equiv \omega(\phi_0)$. We see that $s_{\mathrm{BH}} = \frac{1}{2}$ implies $\alpha_{\mathrm{BH}} = 0$, i.e. that stationary black holes have no hair. However, in the case of nonstationary black-holes, i.e. for a time-varying scalar background, it has been shown that a scalar hair can arise [49,50]. A similar result has been obtain in the presence of a constant scalar gradient in the background [51].

C. The Fokker action

The Fokker action is then computed by replacing into the original action the gravitational and scalar degrees of freedom by their solution, obtained by resolving the field equations (2.8),

$$S_{\mathrm{Fokker}}[\mathbf{y}_A(t),\mathbf{v}_A(t),\ldots] \equiv S_{\mathrm{ST}}[\tilde{g}^{(\mathrm{sol})}_{\mu\nu}(\mathbf{y}_B(t),\mathbf{v}_B(t),\ldots),\varphi^{(\mathrm{sol})}(\mathbf{y}_B(t),\mathbf{v}_B(t),\ldots),\mathbf{v}_A(t)]. \tag{2.15}$$

This procedure only applies to the conservative dynamics.[2] In general relativity, starting at 2PN order, the Lagrangian depends linearly in the accelerations [53], and as expected, we recover this feature in ST theories [29]. At 3PN order, we first obtain a Lagrangian that also contains terms quadratic or of higher order in the accelerations and derivatives of the accelerations. By implementing the *double-zero* method [54] and adding total time-derivatives, that do not contribute to the dynamics, we can reduce our original result to a Lagrangian linear in the accelerations. The equations of motion for the particles are then obtained by writing the generalized Euler-Lagrange equations,

$$\frac{\delta S_{\mathrm{Fokker}}}{\delta \mathbf{y}_A} \equiv \frac{\partial L_{\mathrm{F}}}{\partial \mathbf{y}_{\mathbf{A}}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathrm{F}}}{\partial \mathbf{v}_{\mathbf{A}}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L_{\mathrm{F}}}{\partial \mathbf{a}_{\mathbf{A}}}\right) + \cdots, \tag{2.16}$$

where L_{F} is the Lagrangian corresponding to the action, $S_{\mathrm{Fokker}} = \int \mathrm{d}t L_{\mathrm{F}}$. Only once we have constructed the equations of motion using Eq. (2.16), do we order reduce them by replacing the accelerations by their lower order value.

III. THE MULTIPOLAR POST-NEWTONIAN FORMALISM IN SCALAR-TENSOR THEORIES

A. The separation between the near and wave zones

We generically denote $(\bar{h},\bar{\psi})$ the PN solution of the field equations in the near-zone of the compact source, i.e. in a region of small extent compared to the gravitational wavelength. It is obtained by a PN iteration of the field equations (2.8). In the exterior region of the source, including the wave zone, the multipolar solution is obtained by a post-Minkowskian iteration of the field equations in vacuum and is denoted $\mathcal{M}(h,\psi)$. As we are dealing with a post-Newtonian source, i.e. a compact weakly-stressed and slowly moving source, there exists a buffer region where the two expansions are valid. The complete solution is then obtained by a careful matching of the two solutions in the exterior part of the near zone, using the method of matched asymptotic expansions [6]. In particular, we impose the matching equation,

$$\overline{\mathcal{M}(h,\psi)} = \mathcal{M}(\bar{h},\bar{\psi}), \tag{3.1}$$

i.e. that the multipolar expansion of the PN solution is equal to the PN expansion of the multipolar solution. We emphasize that Eq. (3.1) is valid everywhere and not only in the buffer zone. The careful implementation of Eq. (3.1) is crucial when calculating the tail contribution to the 3PN equations of motion.

The gravitational part of the action $S_{\mathrm{g}} = \int \mathrm{d}t L_{\mathrm{g}}$ can be decomposed according to

$$L_{\mathrm{g}} = \int \mathrm{d}^d x \overline{\mathcal{L}_{\mathrm{g}}} + \int \mathrm{d}^d x \mathcal{M}(\mathcal{L}_{\mathrm{g}}), \tag{3.2}$$

where $\mathcal{L}_{\mathrm{g}}$ is the Lagrangian density. We use dimensional regularization to treat the infrared divergences of the post-Newtonian solution at infinity and the ultraviolet divergences of the multipolar solution at zero. The proof of this equation can be found in Appendix A.[3] It uses the formal structure of the multipolar expansion $\mathcal{M}(\mathcal{L}_{\mathrm{g}}) \sim \sum \hat{n}_L r^a (\ln r)^b F(t)$ and the fact that the integral over space of such generic terms is always zero by analytic continuation in $\varepsilon \equiv d-3$. Next, we investigate the second term in (3.2). In [11], it was shown that this integral is zero for instantaneous terms, namely

$$\int \mathrm{d}^d x\, \mathcal{M}(\mathcal{L}_{\mathrm{g}})|_{\mathrm{inst}} = 0. \tag{3.3}$$

Thus, the only contributions come from hereditary terms, that have the formal structure

$$\mathcal{M}(\mathcal{L}_{\mathrm{g}})|_{\mathrm{hered}} = \sum \frac{\hat{n}_L}{r^k}(\ln r)^q H(u) \times \int_{-\infty}^{u} \mathrm{d}v \mathcal{Q}\left(1+\frac{u-v}{r}\right) K(v), \tag{3.4}$$

where $u = t - r/c$ is the retarded time, and H and K are functions of the source multipole moments I_L and J_L. In ST theories, the multipole expansion of the Lagrangian density has the following formal structure after some integrations by part,

$$\mathcal{M}(\mathcal{L}_{\mathrm{g}}) \sim \mathcal{M}(h)\Box\mathcal{M}(h) + \mathcal{M}(\psi)\Box\mathcal{M}(\psi) + \mathcal{M}(h,\psi)\partial\mathcal{M}(h,\psi)\partial\mathcal{M}(h,\psi) + \cdots. \tag{3.5}$$

As $\mathcal{M}(h)$ and $\mathcal{M}(\psi)$ are solutions of the vacuum field equations, their source is at least quadratic in the fields, that is

[2] An effective field theory method to compute the dissipative effects in the dynamics from a Lagrangian, consisting in doubling the matter variables, has been developed for GR [52].

[3] The proof is similar to the one that one can find in Sec. II. B of [11]. The only difference lies in the fact that we are now dealing with dimensional regularization while in [11], the proof was done using a Hadamard-type regularization.

$$\Box\mathcal{M}(h) \sim \partial\mathcal{M}(h,\psi)\partial\mathcal{M}(h,\psi), \quad \text{and} \\ \Box\mathcal{M}(\psi) \sim \partial\mathcal{M}(h,\psi)\partial\mathcal{M}(h,\psi). \tag{3.6}$$

Inserting Eq. (3.6) into Eq. (3.5) we see that $\mathcal{M}(\mathcal{L}_g)$ is at least cubic in the gravitational fields, and will be at least of order $\mathcal{O}(G^3)$. As we know that $\mathcal{M}(\mathcal{L}_g)$ should contain at least one hereditary term, the dominant effect corresponds to an interaction of the type $M \times M \times I_L$, the so-called "tails-of-tails." In GR, when the scalar field is absent, these terms arise at least at 5.5PN order corresponding to an interaction between two mass-monopoles and one mass-quadrupole [55]. In ST theories, in addition to this effect we can also have an interaction between two mass-monopoles and one scalar mass-dipole, giving a first contribution at 4.5PN order. We conclude that the second term in the RHS of Eq. (3.2) is at least of order 4.5PN, and will not contribute to the dynamics at 3PN order.

Thus, the gravitational part of the Lagrangian has to be computed only using the PN solution only, namely

$$L_g = \int \mathrm{d}^d x \overline{\mathcal{L}_g}. \tag{3.7}$$

The post-Newtonian solutions $(\bar{h}, \bar{\psi})$, obtained by solving the field equations (2.8), read

$$\bar{h}^{\mu\nu} = \frac{16\pi G}{c^4}\overline{\Box^{-1}_{\text{ret}}}[r^\eta \bar{\tau}^{\mu\nu}] + \mathcal{H}^{\mu\nu}, \tag{3.8a}$$

$$\bar{\psi} = -\frac{8\pi G}{c^4}\overline{\Box^{-1}_{\text{ret}}}[r^\eta \bar{\tau}_s] + \Psi, \tag{3.8b}$$

where an overline denotes a PN expansion. The first terms in Eqs. (3.8) are particular retarded solutions of the PN-expanded field equations (2.8). They read

$$\overline{\Box^{-1}_{\text{ret}}}[r^\eta \bar{\tau}^{\mu\nu}] = -\frac{\tilde{k}}{4\pi}\int \mathrm{d}^d\mathbf{x}' |\mathbf{x}'|^\eta \\ \times \overline{\int_1^{+\infty} \mathrm{d}z\, \gamma_{\frac{1-d}{2}}(z) \frac{\bar{\tau}^{\mu\nu}(\mathbf{x}', t - z|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|^{d-2}}}, \tag{3.9a}$$

$$\overline{\Box^{-1}_{\text{ret}}}[r^\eta \bar{\tau}_s] = -\frac{\tilde{k}}{4\pi}\int \mathrm{d}^d\mathbf{x}' |\mathbf{x}'|^\eta \\ \times \overline{\int_1^{+\infty} \mathrm{d}z\, \gamma_{\frac{1-d}{2}}(z) \frac{\bar{\tau}_s(\mathbf{x}', t - z|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|^{d-2}}}, \tag{3.9b}$$

where $\tilde{k} = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}$, Γ is the Eulerian function, and the function $\gamma_{\frac{1-d}{2}}(z)$ is defined by

$$\gamma_s(z) = \frac{2\sqrt{\pi}}{\Gamma(s+1)\Gamma(-s-\frac{1}{2})}(z^2-1)^s, \tag{3.10}$$

with the normalization $\int_1^{+\infty}\gamma_s(z) = 1$. The retarded Green's function of the scalar wave equation $G_{\text{ret}}(\mathbf{x}, t)$, solution of $\Box G_{\text{ret}} = \delta(t)\delta^{(d)}(\mathrm{x})$, is then given by

$$G_{\text{ret}}(\mathbf{x}, t) = -\frac{\tilde{k}}{4\pi}\frac{\theta(t-r)}{r^{d-1}}\gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right), \tag{3.11}$$

where $\theta(t-r)$ is the usual Heaviside step function. In Eq. (3.9), we have used the so-called "$\varepsilon\eta$" regularization scheme, which is the equivalent for dimensional regularization of the finite part procedure of Hadamard regularization. It has recently been successfully used to compute the ambiguities at 4PN in GR [13,14]. We have introduced a factor r^η multiplying the PN source term, that acts as a regulator acting on top of dimensional regularization. In practice, we shall first take the limit $\eta \to 0$ in generic d dimensions and then take the limit $\varepsilon = d - 3 \to 0$. Although some poles in $1/\eta$ may appear in some individual terms, it should not be the case when considering the sum of all terms. In Sec. IV, we shall see in practice how to compute the particular PN solution.

B. The tail effects at 3PN order in scalar-tensor theories

We now focus on the second terms, $\mathcal{H}^{\mu\nu}$ and Ψ, in the Eqs. (3.8), that are the source of the tail effect. They are homogeneous solutions of the wave equation. We follow the algorithm developed in [13,14] to compute the near-zone expansion of homogeneous solutions of the wave equation in d dimensions. The result for $\mathcal{H}^{\mu\nu}$ still stays the same in ST theories. In particular, it starts contributing to the conservative dynamics at 4PN order. Thus, we only consider the scalar field homogeneous solution Ψ. As we are interested in the 3PN contribution, it is sufficient to restrict to the quadratic order in the expansion of the scalar field, $\psi = G\psi_1 + G^2\psi_2 + \mathcal{O}(G^3)$. The equation we want to solve is

$$\Box\psi_2 = N_{s,2}[h_1, \psi_1], \tag{3.12}$$

where $\Box$ is the flat d'Alembertian operator and $N_{s,2}$ is the quadratic part of the source, explicitly given by

$$N_{s,2}[h_1,\psi_1] = \left(1 - \frac{2\phi_0\omega_0'}{d^2 - d + 4\omega_0}\right)\eta^{\mu\nu}\partial_\mu\psi_1\partial_\nu\psi_1 \\ - h_1^{\mu\nu}\partial_{\mu\nu}\psi_1 - \partial_\mu\psi_1\partial_\nu h_1^{\mu\nu}, \tag{3.13}$$

where we have also expanded h at quadratic order, $h^{\mu\nu} = Gh_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \mathcal{O}(G^3)$. We know that the tail effect will result from an interaction between the constant ADM mass M of the system and one time-varying low multipole moment. Thus, we decompose the linearized field as

$$h_1^{\mu\nu} = h_{1,M}^{\mu\nu} + h_{1,I_{kl}}^{\mu\nu}, \tag{3.14a}$$

$$\psi_1 = \psi_{1,M} + \psi_{1,I_j}, \tag{3.14b}$$

with

$$h^{00}_{1,M} = -\frac{4}{c^2}\tilde{I}, \qquad h^{0i}_{1,M} = 0, \qquad h^{ij}_{1,M} = 0, \qquad \text{and} \qquad \psi_{1,M} = -\frac{2}{c^2}\tilde{I}_{\mathrm{s}}, \tag{3.15a}$$

$$h_{1,I_{kl}} = -\frac{2}{c^2}\partial_{ij}\tilde{I}_{ij}, \qquad h^{0i}_{1,I_{kl}} = \frac{2}{c^3}\partial_j\tilde{I}^{(1)}_{ij}, \qquad h^{ij}_{1,I_{kl}} = -\frac{2}{c^4}\tilde{I}^{(2)}_{ij}, \qquad \text{and} \qquad \psi_{1,I_j} = \frac{2}{c^2}\partial_i\tilde{I}^i_{\mathrm{s}}, \tag{3.15b}$$

where

$$\tilde{I}_L(t,r) = \frac{\tilde{k}}{r^{d-2}}\int_1^{+\infty} \mathrm{d}z\gamma_{\frac{1-d}{2}}(z) I_L\left(t - \frac{zr}{c}\right), \tag{3.16}$$

is the homogeneous retarded solution of the d'Alembertian operator. Note that the lowest time-varying multipole moment in ST theories is the dipole moment, instead of the quadrupole moment in GR. The static mass monopoles are given by

$$\tilde{I} = \frac{\tilde{k}I}{r^{d-2}}, \quad \text{and} \quad \tilde{I}_{\mathrm{s}} = \frac{\tilde{k}I_{\mathrm{s}}}{r^{d-2}}. \tag{3.17}$$

Inserting the decomposition (3.14a) into Eq. (3.13) and keeping only the terms contributing to the tails, we get

$$N^{\mathrm{tail}}_{\mathrm{s},2} = 2\left(1 - \frac{2\phi_0\omega_0'}{d^2 - d + 4\omega_0}\right)\partial_\alpha\psi_{1,M}\partial^\alpha\psi_{1,I_j} - \frac{1}{c^2}h^{00}_{1,M}\partial_t^2\psi_{1,I_j} - h^{\alpha\beta}_{1,I_{kl}}\partial_{\alpha\beta}\psi_{1,M}. \tag{3.18}$$

Using Eqs. (3.15), we see that $N^{\mathrm{tail}}_{\mathrm{s},2}$ admits the decomposition

$$N^{\mathrm{tail}}_{\mathrm{s},2} = \sum_{l=0}^{+\infty}\hat{n}_L N^{\mathrm{s,tail}}_{2,L}, \tag{3.19}$$

with

$$N^{\mathrm{s,tail}}_{2,L} = \sum r^{-k-2\varepsilon}\int_1^{+\infty}\mathrm{d}y y^p\gamma_{\frac{1-d}{2}}(y) F_L\left(t - \frac{yr}{c}\right), \tag{3.20}$$

where the function F_L is made of products of mass multipole moments. The tail contribution to the scalar field is then given by

$$\Psi_{2,\mathrm{tail}} = G^2\sum_{j=0}^{+\infty}\frac{1}{c^{2j}}\Delta^{-j}\hat{x}_L f^{(2j)}_{2,L}, \tag{3.21}$$

where

$$\Delta^{-j}\hat{x}_L \equiv \frac{\Gamma(\ell+\frac{d}{2})}{\Gamma(\ell+j+\frac{d}{2})}\frac{r^{2j}\hat{x}_L}{2^{2j}j!}. \tag{3.22}$$

The function $f_{2,L}$ can be factorized into the compact form:

$$f_{2,L} = \sum\frac{(-)^{\ell+k}C^{p,k}_\ell}{2\ell+1+\varepsilon}\frac{\Gamma(2\varepsilon-\eta)}{\Gamma(\ell+k-1+2\varepsilon-\eta)} \times\int_0^{+\infty}\mathrm{d}\tau\,\tau^{-2\varepsilon+\eta}F_L^{(\ell+k-1)\mu\nu}(t-\tau), \tag{3.23}$$

where the dimensionless coefficients $C^{p,k}_\ell$ are

$$C^{p,k}_\ell = \int_1^{+\infty}\mathrm{d}y\, y^p\gamma_{-1-\frac{\varepsilon}{2}}(y) \times\int_1^{+\infty}\mathrm{d}z(y+z)^{\ell+k-2+2\varepsilon-\eta}\gamma_{-\ell-1-\frac{\varepsilon}{2}}(z). \tag{3.24}$$

These coefficients have been computed and an analytic closed form expression can be found in the Appendix D of [13]. Plugging the formulas into the tail equation (3.21), carefully applying the "$\varepsilon\eta$" regularization procedure and expanding everything at 3PN order, we obtain the scalar tail,

$$\Psi_{2,\mathrm{tail}} = -\frac{8G^2M}{3c^8\phi_0}x^i\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\tau\sqrt{\bar{q}\phi_0^{1/2}}}{2\ell_0}\right) - \frac{1}{2\varepsilon} + \frac{11}{12}\right](I^{(5)}_{\mathrm{s},i}(t-\tau) - I^{(5)}_{\mathrm{s},i}(t+\tau)), \tag{3.25}$$

where $\bar{q} \equiv 4\pi\mathrm{e}^{\gamma_{\mathrm{E}}}$ and ℓ_0 is the characteristic length associated to dimensional regularization. Note the appearance of a pole $1/\varepsilon$. Finally, inserting it into the Fokker action, we obtain the tail part of the action

$$S^{\mathrm{tail}}_{\mathrm{F}} = \frac{2G^2M}{3c^6\phi_0}(3+2\omega_0)\int\mathrm{d}t I_{\mathrm{s},i}(t)\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\tau\sqrt{\bar{q}}}{2\ell_0}\right) - \frac{1}{2\varepsilon} - \frac{5}{4(3+2\omega_0)} + \frac{11}{12}\right](I^{(5)}_{\mathrm{s},i}(t-\tau) - I^{(5)}_{\mathrm{s},i}(t+\tau)). \tag{3.26}$$

Performing some integrations by part and using the Hadamard *partie finie* (Pf) notation,[4] we can rewrite the tail part of the action in a symmetric way,

$$\begin{aligned} S_{\mathrm{F}}^{\mathrm{tail}} &= \frac{2G^2M}{3c^6\phi_0}(3+2\omega_0)\int \mathrm{d}t\, I_{\mathrm{s},i}^{(2)}(t)\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\tau\sqrt{\bar{q}}}{2\ell_0}\right)-\frac{1}{2\varepsilon}-\frac{5}{4(3+2\omega_0)}+\frac{11}{12}\right](I_{\mathrm{s},i}^{(3)}(t-\tau)-I_{\mathrm{s},i}^{(3)}(t+\tau)) \\ &= \frac{2G^2M}{3c^6\phi_0}(3+2\omega_0)\mathop{\mathrm{Pf}}_{\tau_0}\int\int\frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|}I_{\mathrm{s},i}^{(2)}(t)I_{\mathrm{s},i}^{(2)}(t'). \end{aligned} \tag{3.27}$$

where we have defined the constant $\tau_0 \equiv \frac{2\ell_0}{c\sqrt{\bar{q}}}\mathrm{e}^{\frac{1}{2\varepsilon}+\frac{5}{4(3+2\omega_0)}-\frac{11}{12}}$.

IV. THE FOKKER LAGRANGIAN IN ST THEORIES

A. The "$n+2$" method

We now focus on the particular solution $(\bar{h}_{\mathrm{part}}, \bar{\psi}_{\mathrm{part}})$. It is obtained by a PN iteration of the field equations. Due to some cancellations between the gravitational and matter parts in the Fokker action, it is sufficient to know the metric at roughly half the order we would have expected. This is the so-called "$n+2$" method, that was developed in [11] for general relativity. Here, we generalize this method to scalar-tensor theories where we have one additional degree of freedom. As we are only interested in the dynamics at 3PN order, we do the reasoning for odd PN orders and in d dimensions. We reason by induction and we will see that the proof follows the one of [11], as the scalar field behaves similarly as h^{00ii}. First, we decompose the metric perturbation as

$$\bar{h}^{\mu\nu} \to \begin{cases} \bar{h}^{00ii} \equiv 2\,\frac{(d-2)\bar{h}^{00}+\bar{h}^{ii}}{d-1}, \\ \bar{h}^{0i}, \\ \bar{h}^{ij}. \end{cases} \tag{4.1}$$

At leading order in $(\bar{h},\bar{\psi})$, the gravitational action reads

$$S_g = \frac{c^4\phi_0^{\frac{d-1}{2}}}{128\pi G}\int\mathrm{d}t\int\mathrm{d}^d\mathbf{x}\left[\frac{d-1}{2(d-2)}\bar{h}^{00ii}\Box\bar{h}^{00ii}-4\bar{h}^{0i}\Box\bar{h}^{0i}+2\bar{h}^{ij}\Box\bar{h}^{ij}-\frac{2}{d-2}\bar{h}^{ii}\Box\bar{h}^{jj}+2(d(d-1)+4\omega_0)\bar{\psi}\Box\bar{\psi}+\mathcal{O}(\bar{h}^3,\bar{\psi}^3)\right], \tag{4.2}$$

while the matter action is given by

$$S_m = \sum_A m_A c^2\int\mathrm{d}t\left[-1+\frac{v_A^2}{2c^2}-\frac{1}{4}\bar{h}_A^{00ii}-(1-2s_A)\bar{\psi}+\frac{v_A^i}{c}\bar{h}_A^{0i}-\frac{v_A^iv_A^j}{2c^2}\bar{h}_A^{ij}+\frac{v_A^2}{2(d-2)c^2}\bar{h}_A^{ii}+\mathcal{O}(\bar{h}_A^2, c^{-2}\bar{h}_A, c^{-2}\bar{\psi}_A)\right]. \tag{4.3}$$

Varying this action with respect to the metric and scalar fields, we can see that the leading order of the PN solution is

$$(\bar{h}^{00ii},\bar{h}^{0i},\bar{h}^{ij};\bar{\psi}) = \mathcal{O}(2,3,4;2). \tag{4.4}$$

Consider now a solution of the field equations,

$$\bar{h}_n \equiv (\bar{h}_n^{00ii},\bar{h}_n^{0i},\bar{h}_n^{ij};\bar{\psi}_n) = \mathcal{O}(n+1,n+2,n+1;n+1), \tag{4.5}$$

where n is an odd number and the orders are included. As we schematically have $\frac{\delta S_{\mathrm{F}}}{\delta h} \sim c^4(\Box h - \bar{\Sigma} - \bar{T})$, we have the estimates

$$\frac{\delta S_{\mathrm{F}}}{\delta\bar{h}^{00ii}}[\bar{h}_n[\mathbf{y}_B],\mathbf{y}_A] = \mathcal{O}(n-1), \tag{4.6a}$$

$$\frac{\delta S_{\mathrm{F}}}{\delta\bar{h}^{0i}}[\bar{h}_n[\mathbf{y}_B],\mathbf{y}_A] = \mathcal{O}(n), \tag{4.6b}$$

$$\frac{\delta S_{\mathrm{F}}}{\delta\bar{h}^{ij}}[\bar{h}_n[\mathbf{y}_B],\mathbf{y}_A] = \mathcal{O}(n-1), \tag{4.6c}$$

$$\frac{\delta S_{\mathrm{F}}}{\delta\bar{\psi}}[\bar{h}_n[\mathbf{y}_B],\mathbf{y}_A] = \mathcal{O}(n-1). \tag{4.6d}$$

We now define the rest of the complete PN solution by

$$(\bar{h},\bar{\psi}) = \bar{h}_n + \bar{r}_{n+2}, \tag{4.7}$$

[4]For any regular function $f(t)$ tending towards zero sufficiently rapidly when $t\to+\infty$, the Hadamard *partie finie* is defined as

$$\mathop{\mathrm{Pf}}_{\tau_0}\int\mathrm{d}t' f(t') \equiv \int_0^{+\infty}\mathrm{d}\tau\ln\left(\frac{\tau}{\tau_0}\right)[f^{(1)}(t-\tau)-f^{(1)}(t+\tau)]$$

with

$$\bar{r}_{n+2} = (\bar{r}^{00ii}_{n+3}, \bar{r}^{0i}_{n+4}, \bar{r}^{ij}_{n+3}, \bar{r}^{\mathrm{s}}_{n+3}) = \mathcal{O}(n+3, n+4, n+3; n+3), \tag{4.8}$$

and we expand the Fokker action around the nth order PN solution,

$$\begin{aligned} S_{\mathrm{F}}[\bar{h}[\boldsymbol{y}_B], \boldsymbol{y}_A] = S_{\mathrm{F}}[\bar{h}_n[\boldsymbol{y}_B], \boldsymbol{y}_A] + \int \mathrm{d}t \int \mathrm{d}^d\mathbf{x} \bigg[\frac{\delta S_{\mathrm{F}}}{\delta \bar{h}^{00ii}} [\bar{h}_n[\boldsymbol{y}_B], \boldsymbol{y}_A] \bar{r}^{00ii}_{n+3} \\ + \frac{\delta S_{\mathrm{F}}}{\delta \bar{h}^{0i}} [\bar{h}_n[\boldsymbol{y}_B], \boldsymbol{y}_A] \bar{r}^{0i}_{n+4} + \frac{\delta S_{\mathrm{F}}}{\delta \bar{h}^{ij}} [\bar{h}_n[\boldsymbol{y}_B], \boldsymbol{y}_A] \bar{r}^{ij}_{n+3} + \frac{\delta S_{\mathrm{F}}}{\delta \bar{\psi}} [\bar{h}_n[\boldsymbol{y}_B], \boldsymbol{y}_A] \bar{r}^{\mathrm{s}}_{n+3} + \cdots \bigg], \end{aligned} \tag{4.9}$$

where the ellipsis stand for quadratic or higher order terms. Inserting the estimates (4.6) in Eq. (4.9), we have

$$S_{\mathrm{F}}[\bar{h}[\boldsymbol{y}_B], \boldsymbol{y}_A] = S_{\mathrm{F}}[\bar{h}_n[\boldsymbol{y}_B], \boldsymbol{y}_A] + \mathcal{O}(2n+2). \tag{4.10}$$

The action is thus known at nPN order as wanted. Note that the quadratic and higher order terms, generically denoted by the ellipsis in (4.9), do not change the result as they contribute to a higher order in the action. The reasoning in the case of n even is very similar. Summarizing our result, the ST "$n+2$" method is given by the rule: In order to control the Fokker action at the nth PN order, it is sufficient to know the metric at the order

$$\bar{h}_n = \begin{cases} \mathcal{O}(n+2, n+1, n+2; n+2) & \text{included} \quad \text{when } n \text{ is even}, \\ \mathcal{O}(n+1, n+2, n+1; n+1) & \text{included} \quad \text{when } n \text{ is odd}. \end{cases} \tag{4.11}$$

B. Iteration of the post-Newtonian solution

We now perform the iteration of the post-Newtonian solution. At 3PN order, according to the "$n+2$" method, we need to know the metric at the order $(4, 5, 4; 4)$. As we will use dimensional regularization to treat all the divergences, we already define all the quantities in d dimensions. We use the decomposition of the metric given by Eq. (4.1), and define the usual PN potentials

$$\bar{h}^{00ii} = -\frac{4}{c^2} V - \frac{4}{c^4} \left[\frac{d-1}{d-2} V^2 - 2 \frac{d-3}{d-2} K \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.12a}$$

$$\bar{h}^{0i} = -\frac{4}{c^3} V_i - \frac{4}{c^5} \left(2\hat{R}_i + \frac{d-1}{d-2} V V_i \right) + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{4.12b}$$

$$\bar{h}^{ij} = -\frac{4}{c^4} \left(\hat{W}_{ij} - \frac{1}{2} \delta_{ij} \hat{W} \right) + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.12c}$$

$$\bar{\psi} = -\frac{2}{c^2} \psi_{(0)} + \frac{2}{c^4} \left(1 - \frac{2\phi_0 \omega_0'}{d(d-1) + 4\omega_0} \right) \psi_{(0)}^2 + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.12d}$$

with $\hat{W} = \hat{W}_{ii}$. Each PN potential obeys a flat space-time wave equation, sourced by matter source densities and some lower order PN potentials. They read

$$\Box V = -4\pi G \sigma, \tag{4.13a}$$

$$\Box \psi_{(0)} = 4\pi G \sigma_{\mathrm{s}}, \tag{4.13b}$$

$$\Box K = -4\pi G \sigma V, \tag{4.13c}$$

$$\Box V_i = -4\pi G \sigma_i, \tag{4.13d}$$

$$\begin{aligned} \Box \hat{R}_i = &-\frac{4\pi G}{d-2} \left[\frac{5-d}{2} V \sigma_i - \frac{d-1}{2} V_i \sigma \right] - \frac{d-1}{d-2} \partial_k V \partial_i V_k \\ &- \frac{d(d-1)}{4(d-2)^2} \partial_t V \partial_i V + \frac{d(d-1) + 4\omega_0}{4} \partial_i \psi_{(0)} \partial_t \psi_{(0)}, \end{aligned} \tag{4.13e}$$

$$\begin{aligned} \Box \hat{W}_{ij} = &-4\pi G \left(\sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{d-2} \right) - \frac{d-1}{2(d-2)} \partial_i V \partial_j V \\ &- \frac{d(d-1) + 4\omega_0}{2} \partial_i \psi_{(0)} \partial_j \psi_{(0)}. \end{aligned} \tag{4.13f}$$

The gravitational constant G appearing in these equations is linked to the usual Newton constant G_{N} through the relation

$$G = G_{\mathrm{N}} \ell_0^{d-3}, \tag{4.14}$$

where ℓ_0 is the characteristic length associated to dimensional regularization. The matter source densities are constructed from the components of the stress-energy tensor for point particles,

$$T^{\mu\nu} = \sum_A \frac{m_A(\phi)}{\sqrt{-g}} \frac{v_A^\mu v_A^\nu}{\sqrt{-[g_{\rho\sigma}]_A v_A^\rho v_A^\sigma / c^2}} \delta^{(d)}(\mathbf{x} - \mathbf{y}_A). \tag{4.15}$$

They read

$$\sigma = 2\left(\frac{\varphi}{\phi_0}\right)\frac{(d-2)T^{00} + T^{ii}}{(d-1)c^2}, \qquad \sigma_i = \left(\frac{\varphi}{\phi_0}\right)\frac{T^{0i}}{c}, \qquad \sigma_{ij} = \left(\frac{\varphi}{\phi_0}\right)T^{ij},$$
$$\sigma_s = -\frac{2}{c^2\phi_0(d(d-1)+4\omega_0)}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right). \tag{4.16}$$

Note that, in addition to the new scalar density, we have slightly changed the definition of the usual densities with respect to the GR result [6] by adding the scalar field in factor. In the Appendix B, we give the explicit expressions of the matter source densities as a function of the potentials. Finally, the harmonicity conditions $\partial_\nu h^{\mu\nu} = 0$ read

$$\partial_t\left\{\frac{d-1}{2(d-2)}V + \frac{1}{2c^2}\left[\hat{W} + \left(\frac{d-1}{d-2}\right)^2 V^2 - \frac{2(d-1)(d-3)}{(d-2)^2}K\right]\right\} + \partial_i\left\{V_i + \frac{2}{c^2}\left[\hat{R}_i + \frac{d-1}{2(d-2)}VV_i\right]\right\} = \mathcal{O}\left(\frac{1}{c^4}\right), \tag{4.17a}$$

$$\partial_t V_i + \partial_j \hat{W}_{ij} - \frac{1}{2}\partial_i \hat{W} = \mathcal{O}\left(\frac{1}{c^2}\right). \tag{4.17b}$$

We emphasize that the gravitational field $\bar{h}$ should only verify the harmonicity conditions (4.17) when *on-shell*.

C. Dimensional regularization

The computation of the Lagrangian involves noncompact support integrals of the type

$$I = \int \mathrm{d}^3\mathbf{x} F(\mathbf{x}), \tag{4.18}$$

where $F(\mathbf{x})$ represents a generic function resulting from the PN iteration of the potentials carried out in the previous section, taken in the limit when $d \to 3$. The integration of such a function leads to two different types of divergences. First, the ultraviolet divergences result from the point-particle approximation that causes the function F to be singular at the points $\mathbf{y}_1$ and $\mathbf{y}_2$. Then, the infrared divergences come from the fact that the PN solution diverges at infinity. In the present work, we use dimensional regularization (DR) [56] to treat both the infrared and ultraviolet divergences appearing in the integrals of the type (4.18). Following the procedure used in previous works in general relativity, the regularization scheme will proceed in several steps. First, we perform the integration in 3 dimensions using Hadamard regularization (HR) [57] for both UV and IR divergences. In a second step, we compute the difference between HR and DR in the case of the ultraviolet divergences, resulting in the appearance of a pole. Finally, we add the difference between HR and DR for infrared divergences. The pole that appears after this step should exactly compensate the one coming from the tail term computed in Sec. III B.

1. Ultraviolet divergences

When $r_1 \to 0$, the 3-dimensional function F admits the following expansion, valid for any $\mathcal{N} \in \mathbb{N}$,

$$F(\mathbf{x}) = \sum_{a_0 \le a \le \mathcal{N}} r_1^a \underset{1}{f}_a(\mathbf{n}_1) + o(r_1^{\mathcal{N}}). \tag{4.19}$$

The Hadamard regularization of the spatial integral (4.18) is then given by

$$\begin{aligned} I^{\mathrm{HR}} &\equiv \underset{\ell_1,\ell_2}{\mathrm{Pf}} \int \mathrm{d}^3\mathbf{x} F(\mathbf{x}) \\ &= \lim_{s\to 0}\left\{\int_{\mathcal{S}(s)} \mathrm{d}^3\mathbf{x} F(\mathbf{x}) + 4\pi \sum_{a+3<0} \frac{s^{a+3}}{a+3}\left(\frac{F}{r_1^a}\right)_1 \right. \\ &\quad \left. + 4\pi \ln\left(\frac{s}{\ell_1}\right)(r_1^3 F)_1 + 1 \leftrightarrow 2\right\}. \end{aligned} \tag{4.20}$$

where ℓ_1 and ℓ_2 are two constants of regularizations. The integral on the second line is performed on the domain of integration $\mathcal{S}(s) \equiv \mathbb{R}^3 \backslash \mathcal{B}(\mathbf{y}_1, s) \cup \mathcal{B}(\mathbf{y}_2, s)$, where $\mathcal{B}(\mathbf{y}_A, s)$ is the sphere centered in $\mathbf{y}_A$ of radius s. When implementing it in the calculation of the Fokker Lagrangian, we obtain a result that depends on the two constants ℓ_1 and ℓ_2. We now turn on implementing dimensional regularization. In $d = 3 + \varepsilon$ spatial dimensions, the expansion (4.19) of the function $F^{(d)}$ becomes,

$$F^{(d)}(\mathbf{x}) = \sum_{\substack{p_0 \le p \le \mathcal{N} \\ q_0 \le q \le q_1}} r_1^{p+q\varepsilon} \underset{1}{f}{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1) + o(r_1^{\mathcal{N}}). \tag{4.21}$$

We further assume that the function $F^{(d)}$ does not have any pole when $\varepsilon \to 0$. It implies the following relation between the d-dimensional and the 3-dimensional coefficients,

$$\sum_{q=q_0}^{q_1} \underset{1}{f}{}^{(\varepsilon=0)}_{p,q}(\mathbf{n}_1) = \underset{1}{f}_p(\mathbf{n}_1). \tag{4.22}$$

To obtain the dimensionally regularized version of the integral (4.18), we only need to compute the difference between the d-dimensional integral $I^{\mathrm{DR}} \equiv \int \mathrm{d}^d\mathbf{x} F^d(\mathbf{x})$ and the HR integral (4.20), and add this result to the previous one. As when $\varepsilon \to 0$ the two regularization procedures give identical results outside the particles' position, these contributions will cancel out in the difference. Thus, we only have to carry-out the calculation locally, i.e., in the vicinity of the particles. Denoting $\mathcal{D}I \equiv I^{\mathrm{DR}} - I^{\mathrm{HR}}$ the difference between the two regularized integrals, we have the formula,

$$\mathcal{D}I = \frac{1}{\varepsilon} \sum_{q=q_0}^{q_1} \left[\frac{1}{q+1} + \varepsilon \ln \ell_1\right] \int \mathrm{d}\Omega_{2+\varepsilon}(\mathbf{n}_1) \underset{1}{f}{}^{(\varepsilon)}_{-3,q}(\mathbf{n}_1) + 1 \leftrightarrow 2 + \mathcal{O}(\varepsilon). \tag{4.23}$$

Due to the presence of the pole in Eq. (4.23), it is very important to perform the angular integration over the $(d-1)$-dimensional sphere, with volume element $\mathrm{d}\Omega_{2+\varepsilon}(\mathbf{n}_1)$, up to linear order in ε. Note the presence of the offending value $q = -1$ in the sum over q in Eq. (4.23). An important test of our calculation, and in turn of the validity of dimensional regularization, consists in checking that the spherical angular integrals are always zero for $q = -1$. By construction, the constants ℓ_1 and ℓ_2 will be absent from the final result, i.e. after adding Eq. (4.20) and Eq. (4.23), as these are pure HR constants.

2. Infrared divergences

Next, we carry out the regularization of the infrared divergences. In 3 dimensions, the expansion of the function F, when $r \to \infty$, is given by

$$F(\mathbf{x}) = \sum_{p=-p_0}^{N} \frac{1}{r^p} f_p(\mathbf{n}) + o\left(\frac{1}{r^N}\right). \tag{4.24}$$

The regularized value of the integral is then

$$I^{\mathrm{HR}} = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x}), \tag{4.25}$$

where we have introduced the regulator $(r/r_0)^B$, with $B \in \mathbb{C}$ and r_0 is a regularization constant. The finite part (FP) at $B = 0$ means that we take the zeroth power of B in the Laurent expansion when $B \to 0$ of the integrand $(r/r_0)^B F(\mathbf{x})$. Similarly, the d-dimensional function $F^{(d)}$ admits the following expansion near infinity

$$F^{(d)}(\mathbf{x}) = \sum_{p \ge -p_0} \sum_{q=-q_0}^{q_1} \frac{1}{r^p} \left(\frac{\ell_0}{r}\right)^{q\varepsilon} f^{(\varepsilon)}_{p,q}(\mathbf{n}). \tag{4.26}$$

Assuming that the coefficients $f^{(\varepsilon)}_{p,q}$ admit a well-defined limit when $\varepsilon \to 0$, which is the case at 3PN order, we have the following relation,

$$f_p(\mathbf{n}) = \sum_{q=-q_0}^{q_1} f^{(\varepsilon=0)}_{p,q}(\mathbf{n}). \tag{4.27}$$

The difference between the DR and HR integrals is entirely determined by the coefficients $f^{(\varepsilon)}_{p,q}$ in the expansion at infinity of the function $F^{(d)}$. At leading order in $\varepsilon \to 0$, we have

$$\mathcal{D}I = \sum_q \left[\frac{1}{(q-1)\varepsilon} - \ln\left(\frac{r_0}{\ell_0}\right)\right] \int \mathrm{d}\Omega_{2+\varepsilon} f^{(\varepsilon)}_{3,q}(\mathbf{n}) + \mathcal{O}(\varepsilon), \tag{4.28}$$

As for the ultraviolet regularization procedure, the presence of the pole in Eq. (4.28) implies that the spherical angular integral has to be performed in d dimensions up to linear order in ε. Note also the problematic case $q = 1$ in the sum over q. During the calculation one should check that the corresponding terms do not appear in our end result.

D. Implementation of the calculation

Once the Fokker Lagrangian has been computed using dimensional regularization, we can add the Lagrangian describing the tail computed in Sec. III B. We rewrite Eq. (3.27) by dividing the logarithmic kernel as,

$$\ln\left(\frac{\tau}{\tau_0}\right) = \ln\left(\frac{c\tau}{2r_{12}}\right) + \ln\left(\frac{2r_{12}}{c\tau_0}\right), \tag{4.29}$$

where we recall that $\tau_0 = \frac{2\ell_0}{c\sqrt{\bar{q}}} \mathrm{e}^{\frac{1}{2\varepsilon} + \frac{5}{4(3+2\omega_0)} - \frac{11}{12}}$, with $\bar{q} = 4\pi \mathrm{e}^{\gamma_{\mathrm{E}}}$. Thanks to this rewriting, one can see that the pole coming from the tails (3.27) directly cancels the one coming from the dimensional regularization of the infrared divergences.

Finally, the last step consists in renormalizing our result by absorbing the ultraviolet pole through some redefinition of the trajectory of the particles. The complete 3PN shift on the trajectories of the particle that allows to remove the pole $\propto 1/\varepsilon$ is given by,

$$\boldsymbol{\delta}\mathbf{y}_{3\mathrm{PN}} = \frac{\alpha^3\tilde{G}^3 m_1^2 m_2}{24c^6\varepsilon r_{12}^2}\mathbf{n}_{12}(44+44\bar{\gamma}+11\bar{\gamma}^2-4\bar{\delta}_1) \times\left(-2+6\varepsilon\ln\left(\frac{\sqrt{4\pi\phi_0 e^{\gamma_E}}r_1'}{\ell_0}\right)\right), \tag{4.30}$$

where the scalar-tensor PN parameters $\bar{\gamma}$ and $\bar{\delta}_1$ are defined in Eqs. (5.4) and (5.5). Following previous works in general relativity, we have introduced the gauge constant r_1' and r_2' to replace the characteristic length scale ℓ_0, such that the logarithmic dependence in our result only appears through the combination $\ln(r_{12}/r_1')$ and $\ln(r_{12}/r_2')$. At the end, our result is thus both IR and UV finite.

V. RESULTS

A. The 3PN acceleration in scalar-tensor theories

The 3PN Lagrangian in harmonic coordinates is a generalized one, meaning that it depends not only on the positions $\mathbf{y}_A$ and velocities $\mathbf{v}_A$ of the particles, but also on the accelerations $\mathbf{a}_A$ and their higher order derivatives.

The accelerations of the particles are obtained by writing the generalized Euler-Lagrange equations, see Eq. (2.16). Following [29], we express them using a finite number of parameters. We define the scalar-tensor parameters:

$$\tilde{G} \equiv \frac{G(4+2\omega_0)}{\phi_0(3+2\omega_0)}, \qquad \zeta \equiv \frac{1}{(4+2\omega_0)},$$
$$\lambda_1 \equiv \frac{\zeta^2}{(1-\zeta)}\frac{d\omega}{d\varphi}\bigg|_0, \qquad \lambda_2 \equiv \frac{\zeta^3}{(1-\zeta)}\frac{d^2\omega}{d\varphi^2}\bigg|_0, \qquad \lambda_3 \equiv \frac{\zeta^4}{(1-\zeta)}\frac{d^3\omega}{d\varphi^3}\bigg|_0, \tag{5.1}$$

as well as the zeroth and higher order sensitivities,

$$s_A \equiv \frac{d\ln m_A(\phi)}{d\ln\phi}\bigg|_0, \qquad s_A' \equiv \frac{d^2\ln m_A(\phi)}{d\ln\phi^2}\bigg|_0, \qquad s_A'' \equiv \frac{d^3\ln m_A(\phi)}{d\ln\phi^3}\bigg|_0, \qquad s_A''' \equiv \frac{d^4\ln m_A(\phi)}{d\ln\phi^4}\bigg|_0. \tag{5.2}$$

At Newtonian order, one additional parameter is sufficient to describe the dynamics,

$$\alpha \equiv 1-\zeta+\zeta(1-2s_1)(1-2s_2), \tag{5.3}$$

while at 1PN three new parameters were introduced. They all read,

$$\bar{\gamma} \equiv -\frac{2\zeta}{\alpha}(1-2s_1)(1-2s_2),$$
$$\bar{\beta}_1 \equiv \frac{\zeta}{\alpha^2}(1-2s_2)^2(\lambda_1(1-2s_1)+2\zeta s_1'), \qquad \bar{\beta}_2 \equiv \frac{\zeta}{\alpha^2}(1-2s_1)^2(\lambda_1(1-2s_2)+2\zeta s_2'). \tag{5.4}$$

Note that they are not all independent, as we have the relation $\alpha(2+\bar{\gamma}) = 2(1-\zeta)$. Then at 2PN, four new parameters were introduced,

$$\bar{\delta}_1 \equiv \frac{\zeta(1-\zeta)}{\alpha^2}(1-2s_1)^2, \qquad \bar{\delta}_2 \equiv \frac{\zeta(1-\zeta)}{\alpha^3}(1-2s_2)^3,$$
$$\bar{\chi}_1 \equiv \frac{\zeta}{\alpha^3}(1-2s_2)^3[(\lambda_2-4\lambda_1^2+\zeta\lambda_1)(1-2s_1)-6\zeta\lambda_1 s_1'+2\zeta^2 s_1''],$$
$$\bar{\chi}_2 \equiv \frac{\zeta}{\alpha^3}(1-2s_1)^3[(\lambda_2-4\lambda_1^2+\zeta\lambda_1)(1-2s_2)-6\zeta\lambda_1 s_2'+2\zeta^2 s_2'']. \tag{5.5}$$

Once again, these parameters are not all independent, as we have the relation $16\bar{\delta}_1\bar{\delta}_2 = \bar{\gamma}^2(2+\bar{\gamma})^2$. Finally at 3PN order we introduce two new parameters,

$$\bar{\kappa}_1 \equiv \frac{\zeta}{\alpha^4}(1-2s_2)^4[(\lambda_3-13\lambda_1\lambda_2+28\lambda_1^3+\zeta(3\lambda_2-13\lambda_1^2)+\lambda_1\zeta^2)(1-2s_1) +2\zeta(19\lambda_1^2-4\lambda_2-4\lambda_1\zeta)s_1'-12\zeta^2\lambda_1 s_1''+2\zeta^3 s_1'''],$$
$$\bar{\kappa}_2 \equiv \frac{\zeta}{\alpha^4}(1-2s_1)^4[(\lambda_3-13\lambda_1\lambda_2+28\lambda_1^3+\zeta(3\lambda_2-13\lambda_1^2)+\lambda_1\zeta^2)(1-2s_2) +2\zeta(19\lambda_1^2-4\lambda_2-4\lambda_1\zeta)s_2'-12\zeta^2\lambda_1 s_2''+2\zeta^3 s_2''']. \tag{5.6}$$

We write the full 3PN equations of motion in the following form:

$$\mathbf{a}_1 = \mathbf{a}_1^{\mathrm{N}} + \mathbf{a}_1^{\mathrm{1PN}} + \mathbf{a}_1^{\mathrm{2PN}} + \mathbf{a}_1^{\mathrm{3PN}}. \tag{5.7}$$

The 3PN piece is then decomposed into a local part and a non-local one,

$$\mathbf{a}_1^{\mathrm{3PN}} = \mathbf{a}_1^{\mathrm{3PN,inst}} + \mathbf{a}_1^{\mathrm{3PN,tail}}, \tag{5.8}$$

and the local part is further split into its increasing power of $\tilde{G}$:

$$\mathbf{a}_1^{\mathrm{3PN,inst}} = \tilde{G}\mathbf{a}_1^{\mathrm{3PN,(1)}} + \tilde{G}^2\mathbf{a}_1^{\mathrm{3PN,(2)}} + \tilde{G}^3\mathbf{a}_1^{\mathrm{3PN,(3)}} + \tilde{G}^4\mathbf{a}_1^{\mathrm{3PN,(4)}}. \tag{5.9}$$

We have

$$\mathbf{a}_1^{\mathrm{N}} = -\frac{\tilde{G}\alpha m_2}{r_{12}^2}\mathbf{n}_{12}, \tag{5.10a}$$

$$\begin{aligned}
\mathbf{a}_1^{\mathrm{1PN}} = {} & \frac{\tilde{G}^2\alpha^2}{r_{12}^3}\mathbf{n}_{12}\left[(5+2\bar{\gamma}+2\bar{\beta}_2)m_1m_2 + 2(2+\bar{\gamma}+\bar{\beta}_1)m_2^2\right] \\
& + \frac{\tilde{G}\alpha m_2}{r_{12}^2}\left(\mathbf{n}_{12}\left[\frac{3}{2}(n_{12}v_2)^2 + 2(2+\bar{\gamma})(v_1v_2) + (-1-\bar{\gamma})v_1^2 + (-2-\bar{\gamma})v_2^2\right]\right. \\
& \left. + \mathbf{v}_1[2(2+\bar{\gamma})(n_{12}v_1) + (-3-2\bar{\gamma})(n_{12}v_2)] + \mathbf{v}_2[-2(2+\bar{\gamma})(n_{12}v_1) + (3+2\bar{\gamma})(n_{12}v_2)]\right),
\end{aligned} \tag{5.10b}$$

$$\begin{aligned}
\mathbf{a}_1^{\mathrm{2PN}} = {} & \frac{\tilde{G}^3\alpha^3}{r_{12}^4}\mathbf{n}_{12}\left[m_1m_2^2\left(\frac{1}{2}(-69-48\bar{\gamma}-8\bar{\gamma}^2) - 4(3+\bar{\gamma})\bar{\beta}_2 + \bar{\beta}_1\left(-15-4\bar{\gamma}+\frac{24\bar{\beta}_2}{\bar{\gamma}}\right)\right)\right. \\
& \left. + m_2^3\left(-\frac{9}{4}(2+\bar{\gamma})^2 - 4(2+\bar{\gamma})\bar{\beta}_1 - \bar{\delta}_2 + 2\bar{\chi}_1\right) + m_1^2m_2\left(\frac{1}{4}(-57-44\bar{\gamma}-9\bar{\gamma}^2) - 4(3+\bar{\gamma})\bar{\beta}_2 - \bar{\delta}_1 + 2\bar{\chi}_2\right)\right] \\
& + \frac{\tilde{G}^2\alpha^2}{r_{12}^3}\left[\mathbf{v}_2\left(m_1m_2\left[\left(\frac{1}{4}(63+40\bar{\gamma}+2\bar{\gamma}^2) - 2\bar{\beta}_2 + 2\bar{\delta}_1\right)(n_{12}v_1) + \left(\frac{1}{4}(-55-40\bar{\gamma}-2\bar{\gamma}^2) + 4\bar{\beta}_2 - 2\bar{\delta}_1\right)(n_{12}v_2)\right]\right.\right. \\
& \left. + m_2^2\left[\left(\frac{1}{2}(2+\bar{\gamma})^2 + 2\bar{\delta}_2\right)(n_{12}v_1) + \left(-\frac{1}{2}(-2+\bar{\gamma})(2+\bar{\gamma}) + 2\bar{\beta}_1 - 2\bar{\delta}_2\right)(n_{12}v_2)\right]\right) \\
& + \mathbf{v}_1\left(m_1m_2\left[\left(\frac{1}{4}(-63-40\bar{\gamma}-2\bar{\gamma}^2) + 2\bar{\beta}_2 - 2\bar{\delta}_1\right)(n_{12}v_1) + \left(\frac{1}{4}(55+40\bar{\gamma}+2\bar{\gamma}^2) - 4\bar{\beta}_2 + 2\bar{\delta}_1\right)(n_{12}v_2)\right]\right. \\
& \left. + m_2^2\left[\left(-\frac{1}{2}(2+\bar{\gamma})^2 - 2\bar{\delta}_2\right)(n_{12}v_1) + \left(\frac{1}{2}(-2+\bar{\gamma})(2+\bar{\gamma}) - 2\bar{\beta}_1 + 2\bar{\delta}_2\right)(n_{12}v_2)\right]\right) \\
& + \mathbf{n}_{12}\left(m_2^2\left[\left(\frac{1}{2}(2+\bar{\gamma})^2 + 2\bar{\delta}_2\right)(n_{12}v_1)^2 + (-(2+\bar{\gamma})^2 - 4\bar{\delta}_2)(n_{12}v_1)(n_{12}v_2)\right.\right. \\
& \left. + \left(\frac{1}{2}(-6+\bar{\gamma})(2+\bar{\gamma}) - 4\bar{\beta}_1 + 2\bar{\delta}_2\right)(n_{12}v_2)^2 - 4(2+\bar{\gamma})(v_1v_2) - 2\bar{\beta}_1v_1^2 + 2(2+\bar{\gamma})v_2^2\right] \\
& + m_1m_2\left[\left(\frac{1}{2}(39+26\bar{\gamma}+\bar{\gamma}^2) - 4\bar{\beta}_2 + 2\bar{\delta}_1\right)(n_{12}v_1)^2 + (-39-26\bar{\gamma}-\bar{\gamma}^2+8\bar{\beta}_2-4\bar{\delta}_1)(n_{12}v_1)(n_{12}v_2)\right. \\
& \left.\left.\left. + \left(\frac{1}{2}(1+\bar{\gamma})(17+\bar{\gamma}) - 8\bar{\beta}_2 + 2\bar{\delta}_1\right)\left(n_{12}v_2\right)^2 + \left(-\frac{5}{2} - 2\bar{\beta}_2\right)(v_1v_2) + \left(\frac{1}{4}(-15-8\bar{\gamma}) - \bar{\beta}_2\right)v_1^2 + \left(\frac{5}{4}+\bar{\beta}_2\right)v_2^2\right]\right)\right] \\
& + \frac{\tilde{G}\alpha m_2}{r_{12}^2}\left(\mathbf{v}_1\left[\frac{3}{2}(3+2\bar{\gamma})(n_{12}v_2)^3 + (1+\bar{\gamma})(n_{12}v_2)v_1^2 + (n_{12}v_2)\left(2(2+\bar{\gamma})(v_1v_2) + (-5-3\bar{\gamma})v_2^2\right)\right.\right. \\
& \left. + (n_{12}v_1)\left(-3(2+\bar{\gamma})(n_{12}v_2)^2 - 2(2+\bar{\gamma})(v_1v_2) + 2(2+\bar{\gamma})v_2^2\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{v}_2\bigg[-\frac{3}{2}(3+2\bar{\gamma})(n_{12}v_2)^3+(-1-\bar{\gamma})(n_{12}v_2)v_1^2+(n_{12}v_1)\bigg(3(2+\bar{\gamma})(n_{12}v_2)^2+2(2+\bar{\gamma})(v_1v_2)-2(2+\bar{\gamma})v_2^2\bigg)\\
&\quad+(n_{12}v_2)\bigg(-2(2+\bar{\gamma})(v_1v_2)+(5+3\bar{\gamma})v_2^2\bigg)\bigg]\\
&+\mathbf{n}_{12}\bigg[-\frac{15}{8}(n_{12}v_2)^4+(-2-\bar{\gamma})(v_1v_2)^2+\frac{3}{2}(1+\bar{\gamma})(n_{12}v_2)^2v_1^2+2(2+\bar{\gamma})(v_1v_2)v_2^2\\
&\quad+(n_{12}v_2)^2\bigg(-3(2+\bar{\gamma})(v_1v_2)+\frac{3}{2}(3+\bar{\gamma})v_2^2\bigg)+(-2-\bar{\gamma})v_2^4\bigg]\bigg).
\end{aligned}
\tag{5.10c}
$$

At 2PN order, we recover the result from [29]. The instantaneous 3PN terms are then given by

$$
\begin{aligned}
\mathbf{a}_1^{3\mathrm{PN},(1)}=\frac{\alpha m_2}{r_{12}^2}\bigg\{&\mathbf{v}_1\bigg(-\frac{15}{8}(3+2\bar{\gamma})(n_{12}v_2)^5+(n_{12}v_2)^3\bigg(-3(2+\bar{\gamma})(v_1v_2)+\frac{3}{2}(8+5\bar{\gamma})v_2^2\bigg)\\
&\quad+v_1^2\bigg(-\frac{3}{2}(1+\bar{\gamma})(n_{12}v_2)^3+(1+\bar{\gamma})(n_{12}v_2)v_2^2\bigg)+(n_{12}v_2)\bigg((-2-\bar{\gamma})(v_1v_2)^2+4(2+\bar{\gamma})(v_1v_2)v_2^2\\
&\quad+(-7-4\bar{\gamma})v_2^4\bigg)+(n_{12}v_1)\bigg[\frac{15}{4}(2+\bar{\gamma})(n_{12}v_2)^4-2(2+\bar{\gamma})(v_1v_2)v_2^2\\
&\quad+(n_{12}v_2)^2(3(2+\bar{\gamma})(v_1v_2)-6(2+\bar{\gamma})v_2^2)+2(2+\bar{\gamma})v_2^4\bigg]\bigg)\\
&+\mathbf{v}_2\bigg(\frac{15}{8}(3+2\bar{\gamma})(n_{12}v_2)^5+(n_{12}v_2)^3\bigg(3(2+\bar{\gamma})(v_1v_2)-\frac{3}{2}(8+5\bar{\gamma})v_2^2\bigg)\\
&\quad+v_1^2\bigg(\frac{3}{2}(1+\bar{\gamma})(n_{12}v_2)^3+(-1-\bar{\gamma})(n_{12}v_2)v_2^2\bigg)+(n_{12}v_1)\bigg[-\frac{15}{4}(2+\bar{\gamma})(n_{12}v_2)^4+2(2+\bar{\gamma})(v_1v_2)v_2^2\\
&\quad+(n_{12}v_2)^2(-3(2+\bar{\gamma})(v_1v_2)+6(2+\bar{\gamma})v_2^2)-2(2+\bar{\gamma})v_2^4\bigg]\\
&\quad+(n_{12}v_2)\bigg((2+\bar{\gamma})(v_1v_2)^2-4(2+\bar{\gamma})(v_1v_2)v_2^2+(7+4\bar{\gamma})v_2^4\bigg)\bigg)\\
&+\mathbf{n}_{12}\bigg[\frac{35}{16}(n_{12}v_2)^6+(-2-\bar{\gamma})(v_1v_2)^2v_2^2+(n_{12}v_2)^4\bigg(\frac{15}{4}(2+\bar{\gamma})(v_1v_2)-\frac{15}{8}(4+\bar{\gamma})v_2^2\bigg)\\
&\quad+v_1^2\bigg(-\frac{15}{8}(1+\bar{\gamma})(n_{12}v_2)^4+\frac{3}{2}(1+\bar{\gamma})(n_{12}v_2)^2v_2^2\bigg)+2(2+\bar{\gamma})(v_1v_2)v_2^4+(n_{12}v_2)^2\bigg(\frac{3}{2}(2+\bar{\gamma})(v_1v_2)^2\\
&-6(2+\bar{\gamma})(v_1v_2)v_2^2+\frac{3}{2}(5+2\bar{\gamma})v_2^4\bigg)+(-2-\bar{\gamma})v_2^6\bigg]\bigg\},
\end{aligned}
\tag{5.11a}
$$

$$
\begin{aligned}
\mathbf{a}_1^{3\mathrm{PN},(2)}=\frac{\alpha^2}{r_{12}^3}\bigg\{&\mathbf{v}_2\bigg(m_1m_2\bigg[\bigg(\frac{1}{12}(729+888\bar{\gamma}+226\bar{\gamma}^2)-12\bar{\beta}_2+\frac{10}{3}\bar{\delta}_1\bigg)(n_{12}v_1)^3\\
&\quad+\bigg(\frac{1}{4}(-565-728\bar{\gamma}-192\bar{\gamma}^2)+32\bar{\beta}_2-8\bar{\delta}_1\bigg)(n_{12}v_1)^2(n_{12}v_2)+\bigg(\frac{1}{12}(95-168\bar{\gamma}-112\bar{\gamma}^2)\\
&\quad+\frac{8}{3}\bar{\delta}_1\bigg)(n_{12}v_2)^3+\bigg(\frac{1}{8}(137+208\bar{\gamma}+50\bar{\gamma}^2)-10\bar{\beta}_2+\bar{\delta}_1\bigg)(n_{12}v_2)v_1^2\\
&\quad+(n_{12}v_2)\bigg(\bigg(\frac{1}{4}(-27-128\bar{\gamma}-46\bar{\gamma}^2)+12\bar{\beta}_2+2\bar{\delta}_1\bigg)(v_1v_2)+\bigg(\frac{1}{8}(-83+48\bar{\gamma}+42\bar{\gamma}^2)-2\bar{\beta}_2-3\bar{\delta}_1\bigg)v_2^2\bigg)\\
&\quad+(n_{12}v_1)\bigg(\bigg(\frac{1}{4}(269+488\bar{\gamma}+154\bar{\gamma}^2)-24\bar{\beta}_2+2\bar{\delta}_1\bigg)(n_{12}v_2)^2+(2(18+29\bar{\gamma}+8\bar{\gamma}^2)-16\bar{\beta}_2)(v_1v_2)\\
&\quad+\bigg(\frac{1}{8}(-207-272\bar{\gamma}-66\bar{\gamma}^2)+9\bar{\beta}_2-\bar{\delta}_1\bigg)v_1^2+\bigg(\frac{1}{8}(-81-192\bar{\gamma}-62\bar{\gamma}^2)+7\bar{\beta}_2+\bar{\delta}_1\bigg)v_2^2\bigg)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+m_2^2\left[\left(\frac{1}{2}(2+\bar{\gamma})^2+2\bar{\delta}_2\right)(n_{12}v_1)^2(n_{12}v_2)+\left(\frac{1}{2}(2+\bar{\gamma})(-2+3\bar{\gamma})-4\bar{\beta}_1+6\bar{\delta}_2\right)(n_{12}v_2)^3\right.\\
&\quad-2\bar{\beta}_1(n_{12}v_2)v_1^2+(n_{12}v_2)\left(((-2+\bar{\gamma})(2+\bar{\gamma})+4\bar{\delta}_2)(v_1v_2)+(-(-2+\bar{\gamma})(2+\bar{\gamma})+2\bar{\beta}_1-4\bar{\delta}_2)v_2^2\right)\\
&\quad\left.+(n_{12}v_1)\left((-2(2+\bar{\gamma})^2-8\bar{\delta}_2)(n_{12}v_2)^2+\left(-\frac{1}{2}(2+\bar{\gamma})^2-2\bar{\delta}_2\right)(v_1v_2)+\left(\frac{1}{2}(2+\bar{\gamma})^2+2\bar{\delta}_2\right)v_2^2\right)\right]\Bigg)\\
&+\mathbf{v}_1\Bigg(m_1m_2\left[\left(\frac{1}{12}(-729-888\bar{\gamma}-226\bar{\gamma}^2)+12\bar{\beta}_2-\frac{10}{3}\bar{\delta}_1\right)(n_{12}v_1)^3+\left(\frac{1}{4}(565+728\bar{\gamma}+192\bar{\gamma}^2)\right.\right.\\
&\quad\left.-32\bar{\beta}_2+8\bar{\delta}_1\right)(n_{12}v_1)^2(n_{12}v_2)+\left(\frac{1}{12}(-95+168\bar{\gamma}+112\bar{\gamma}^2)-\frac{8}{3}\bar{\delta}_1\right)(n_{12}v_2)^3\\
&\quad+\left(\frac{1}{8}(-137-208\bar{\gamma}-50\bar{\gamma}^2)+10\bar{\beta}_2-\bar{\delta}_1\right)(n_{12}v_2)v_1^2+(n_{12}v_1)\left(\left(\frac{1}{4}(-269-488\bar{\gamma}-154\bar{\gamma}^2)+24\bar{\beta}_2\right.\right.\\
&\quad\left.-2\bar{\delta}_1\right)(n_{12}v_2)^2+(-2(18+29\bar{\gamma}+8\bar{\gamma}^2)+16\bar{\beta}_2)(v_1v_2)+\left(\frac{1}{8}(207+272\bar{\gamma}+66\bar{\gamma}^2)-9\bar{\beta}_2+\bar{\delta}_1\right)v_1^2\\
&\quad\left.+\left(\frac{1}{8}(81+192\bar{\gamma}+62\bar{\gamma}^2)-7\bar{\beta}_2-\bar{\delta}_1\right)v_2^2\right)+(n_{12}v_2)\left(\left(\frac{1}{4}(27+128\bar{\gamma}+46\bar{\gamma}^2)-12\bar{\beta}_2-2\bar{\delta}_1\right)(v_1v_2)\right.\\
&\quad\left.\left.+\left(\frac{1}{8}(83-48\bar{\gamma}-42\bar{\gamma}^2)+2\bar{\beta}_2+3\bar{\delta}_1\right)v_2^2\right)\right]\\
&+m_2^2\left[\left(-\frac{1}{2}(2+\bar{\gamma})^2-2\bar{\delta}_2\right)(n_{12}v_1)^2(n_{12}v_2)+\left(-\frac{1}{2}(2+\bar{\gamma})(-2+3\bar{\gamma})+4\bar{\beta}_1-6\bar{\delta}_2\right)(n_{12}v_2)^3\right.\\
&\quad+2\bar{\beta}_1(n_{12}v_2)v_1^2+(n_{12}v_1)\left((2(2+\bar{\gamma})^2+8\bar{\delta}_2)(n_{12}v_2)^2+\left(\frac{1}{2}(2+\bar{\gamma})^2+2\bar{\delta}_2\right)(v_1v_2)\right.\\
&\quad\left.+\left(-\frac{1}{2}(2+\bar{\gamma})^2-2\bar{\delta}_2\right)v_2^2\right)+(n_{12}v_2)\Big((-(-2+\bar{\gamma})(2+\bar{\gamma})-4\bar{\delta}_2)(v_1v_2)\\
&\quad\left.+((-2+\bar{\gamma})(2+\bar{\gamma})-2\bar{\beta}_1+4\bar{\delta}_2)v_2^2\Big)\right]\Bigg)\\
&+\mathbf{n}_{12}\Bigg[m_2^2\left(\left(-\frac{3}{2}(2+\bar{\gamma})^2-6\bar{\delta}_2\right)(n_{12}v_1)^2(n_{12}v_2)^2+\left(-\frac{3}{2}(-2+\bar{\gamma})(2+\bar{\gamma})+6\bar{\beta}_1-6\bar{\delta}_2\right)(n_{12}v_2)^4\right.\\
&\quad+2(2+\bar{\gamma})(v_1v_2)^2+4\bar{\beta}_1(n_{12}v_2)^2v_1^2-4(2+\bar{\gamma})(v_1v_2)v_2^2+(n_{12}v_2)^2\Big((-(-6+\bar{\gamma})(2+\bar{\gamma})\\
&\quad-4\bar{\delta}_2)(v_1v_2)+((-6+\bar{\gamma})(2+\bar{\gamma})-4\bar{\beta}_1+4\bar{\delta}_2)v_2^2\Big)+(n_{12}v_1)\Big[(3(2+\bar{\gamma})^2+12\bar{\delta}_2)(n_{12}v_2)^3\\
&\quad\left.+(n_{12}v_2)\left(((2+\bar{\gamma})^2+4\bar{\delta}_2)(v_1v_2)+(-(2+\bar{\gamma})^2-4\bar{\delta}_2)v_2^2\right)\Big]+2(2+\bar{\gamma})v_2^4\right)\\
&+m_1m_2\left(\left(\frac{1}{8}(-171-54\bar{\gamma}-20\bar{\gamma}^2)-18\bar{\beta}_2-10\bar{\delta}_1\right)(n_{12}v_1)^4+\left(\frac{1}{2}(171+54\bar{\gamma}+20\bar{\gamma}^2)+72\bar{\beta}_2\right.\right.\\
&\quad\left.+40\bar{\delta}_1\right)(n_{12}v_1)^3(n_{12}v_2)+\left(\frac{1}{8}(-455-294\bar{\gamma}-32\bar{\gamma}^2)-16\bar{\delta}_1\right)(n_{12}v_2)^4+\left(\frac{1}{4}(-177-152\bar{\gamma}-40\bar{\gamma}^2)\right.\\
&\quad\left.-8\bar{\beta}_2-8\bar{\delta}_1\right)(v_1v_2)^2+\left(\frac{1}{8}(-91-76\bar{\gamma}-20\bar{\gamma}^2)-\frac{9}{4}\bar{\beta}_2-2\bar{\delta}_1\right)v_1^4+(43+38\bar{\gamma}+10\bar{\gamma}^2\\
&\quad+7\bar{\beta}_2+8\bar{\delta}_1)(v_1v_2)v_2^2+v_1^2\left(\left(\frac{1}{4}(191+124\bar{\gamma}+33\bar{\gamma}^2)+20\bar{\beta}_2+13\bar{\delta}_1\right)(n_{12}v_2)^2\right.\\
&\quad\left.+\left(\frac{1}{2}(91+76\bar{\gamma}+20\bar{\gamma}^2)+9\bar{\beta}_2+8\bar{\delta}_1\right)(v_1v_2)+\left(\frac{1}{4}(-91-76\bar{\gamma}-20\bar{\gamma}^2)-\frac{9}{2}\bar{\beta}_2-4\bar{\delta}_1\right)v_2^2\right)
\end{aligned}
$$

$$+(n_{12}v_1)^2\bigg(\bigg(-\frac{3}{4}(241+102\bar{\gamma}+22\bar{\gamma}^2)-96\bar{\beta}_2-66\bar{\delta}_1\bigg)(n_{12}v_2)^2+\bigg(\frac{1}{2}(-229-176\bar{\gamma}-49\bar{\gamma}^2)$$
$$-36\bar{\beta}_2-26\bar{\delta}_1\bigg)(v_1v_2)+\bigg(\frac{1}{4}(229+176\bar{\gamma}+49\bar{\gamma}^2)+18\bar{\beta}_2+13\bar{\delta}_1\bigg)v_1^2+\bigg(\frac{1}{4}(229+176\bar{\gamma}+49\bar{\gamma}^2)$$
$$+18\bar{\beta}_2+13\bar{\delta}_1\bigg)v_2^2\bigg)+(n_{12}v_2)^2\bigg(\bigg(-\frac{5}{2}(45+32\bar{\gamma}+7\bar{\gamma}^2)-24\bar{\beta}_2-30\bar{\delta}_1\bigg)(v_1v_2)$$
$$+\bigg(\frac{1}{4}(259+196\bar{\gamma}+37\bar{\gamma}^2)+4\bar{\beta}_2+17\bar{\delta}_1\bigg)v_2^2\bigg)+(n_{12}v_1)\bigg[\bigg(\frac{1}{2}(383+198\bar{\gamma}+26\bar{\gamma}^2)$$
$$+48\bar{\beta}_2+52\bar{\delta}_1\bigg)(n_{12}v_2)^3+\bigg(\frac{1}{2}(-205-148\bar{\gamma}-41\bar{\gamma}^2)-36\bar{\beta}_2-26\bar{\delta}_1\bigg)(n_{12}v_2)v_1^2$$
$$+(n_{12}v_2)\bigg((2(122+87\bar{\gamma}+21\bar{\gamma}^2)+64\bar{\beta}_2+56\bar{\delta}_1)(v_1v_2)+\bigg(\frac{1}{2}(-283-200\bar{\gamma}-43\bar{\gamma}^2)$$
$$-28\bar{\beta}_2-30\bar{\delta}_1\bigg)v_2^2\bigg)\bigg]+\bigg(\frac{1}{8}(-81-76\bar{\gamma}-20\bar{\gamma}^2)-\frac{5}{4}\bar{\beta}_2-2\bar{\delta}_1\bigg)v_2^4\bigg)\bigg]\bigg\}, \tag{5.11b}$$

$$\mathbf{a}_1^{3\mathrm{PN},(3)}=\frac{\alpha^2m_1^2m_2}{r_{12}^4}\bigg[\mathbf{v}_1\bigg(\bigg(-\frac{11}{4}\bar{\gamma}(2+\bar{\gamma})+\frac{(10+\bar{\gamma})\bar{\delta}_1}{2+\bar{\gamma}}\bigg)(n_{12}v_1)+\bigg(\frac{11}{4}\bar{\gamma}(2+\bar{\gamma})-\frac{(10+\bar{\gamma})\bar{\delta}_1}{2+\bar{\gamma}}\bigg)(n_{12}v_2)\bigg)$$
$$+\mathbf{v}_2\bigg(\bigg(\frac{11}{4}\bar{\gamma}(2+\bar{\gamma})-\frac{(10+\bar{\gamma})\bar{\delta}_1}{2+\bar{\gamma}}\bigg)(n_{12}v_1)+\bigg(-\frac{11}{4}\bar{\gamma}(2+\bar{\gamma})+\frac{(10+\bar{\gamma})\bar{\delta}_1}{2+\bar{\gamma}}\bigg)(n_{12}v_2)\bigg)$$
$$+\mathbf{n}_{12}\bigg(\bigg(\frac{55}{8}\bar{\gamma}(2+\bar{\gamma})-\frac{5(10+\bar{\gamma})\bar{\delta}_1}{2(2+\bar{\gamma})}\bigg)(n_{12}v_1)^2+\bigg(-\frac{55}{4}\bar{\gamma}(2+\bar{\gamma})+\frac{5(10+\bar{\gamma})\bar{\delta}_1}{2+\bar{\gamma}}\bigg)(n_{12}v_1)(n_{12}v_2)$$
$$+\bigg(\frac{55}{8}\bar{\gamma}(2+\bar{\gamma})-\frac{5(10+\bar{\gamma})\bar{\delta}_1}{2(2+\bar{\gamma})}\bigg)(n_{12}v_2)^2+\bigg(\frac{11}{4}\bar{\gamma}(2+\bar{\gamma})-\frac{(10+\bar{\gamma})\bar{\delta}_1}{2+\bar{\gamma}}\bigg)(v_1v_2)$$
$$+\bigg(-\frac{11}{8}\bar{\gamma}(2+\bar{\gamma})+\frac{(10+\bar{\gamma})\bar{\delta}_1}{2(2+\bar{\gamma})}\bigg)v_1^2+\bigg(-\frac{11}{8}\bar{\gamma}(2+\bar{\gamma})+\frac{(10+\bar{\gamma})\bar{\delta}_1}{2(2+\bar{\gamma})}\bigg)v_2^2\bigg)\bigg]$$
$$+\frac{\alpha^3}{r_{12}^4}\bigg[\mathbf{v}_1\bigg(m_1m_2^2\bigg[\bigg(\frac{1}{24}(-921-1040\bar{\gamma}-234\bar{\gamma}^2+24\bar{\gamma}^3)+\frac{2}{3}(35+9\bar{\gamma})\bar{\delta}_1+\bar{\beta}_1\bigg(\frac{1}{2}(65+44\bar{\gamma})$$
$$+\frac{24\bar{\beta}_2}{\bar{\gamma}}-\frac{24\bar{\delta}_1}{\bar{\gamma}}\bigg)+\frac{1}{3}(53+18\bar{\gamma})\bar{\delta}_2-\frac{24\bar{\beta}_2\bar{\delta}_2}{\bar{\gamma}}+\pi^2\bigg(-\frac{3}{128}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)$$
$$-\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_1-\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_2\bigg)\bigg)(n_{12}v_1)+\bigg(\frac{1}{24}(1437+1232\bar{\gamma}+222\bar{\gamma}^2-24\bar{\gamma}^3)-\frac{2}{3}(35+9\bar{\gamma})\bar{\delta}_1$$
$$+\bar{\beta}_1\bigg(\frac{1}{2}(-43-44\bar{\gamma})-\frac{48\bar{\beta}_2}{\bar{\gamma}}+\frac{24\bar{\delta}_1}{\bar{\gamma}}\bigg)+\frac{1}{3}(-59-18\bar{\gamma})\bar{\delta}_2+\bar{\beta}_2\bigg(4(3+\bar{\gamma})+\frac{24\bar{\delta}_2}{\bar{\gamma}}\bigg)$$
$$+\pi^2\bigg(\frac{3}{128}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)+\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_1+\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_2\bigg)\bigg)(n_{12}v_2)\bigg]$$
$$+m_2^3\bigg(\bigg(\frac{1}{2}(2+\bar{\gamma})^3+2(2+\bar{\gamma})\bar{\delta}_2\bigg)(n_{12}v_1)+\bigg(-\frac{1}{4}(2+\bar{\gamma})^2(-5+2\bar{\gamma})+4(2+\bar{\gamma})\bar{\beta}_1$$
$$+(-3-2\bar{\gamma})\bar{\delta}_2-2\bar{\chi}_1\bigg)(n_{12}v_2)\bigg)$$
$$+m_1^2m_2\bigg[\bigg(\frac{1}{12}(1325+1328\bar{\gamma}+411\bar{\gamma}^2+39\bar{\gamma}^3)+\frac{1}{2}(63+40\bar{\gamma})\bar{\beta}_2+(5+\bar{\gamma})\bar{\delta}_1+4\bar{\chi}_2$$
$$+\bigg(\frac{33}{2}(2+\bar{\gamma})^2-6\bar{\delta}_1\bigg)\ln(r_1')+\bigg(-\frac{33}{2}(2+\bar{\gamma})^2+6\bar{\delta}_1\bigg)\ln(r_{12})\bigg)(n_{12}v_1)$$

$$
\begin{aligned}
&+\Bigg(\frac{1}{12}(-1463-1484\bar{\gamma}-438\bar{\gamma}^2-39\bar{\gamma}^3)+\frac{1}{2}(-43-40\bar{\gamma})\bar{\beta}_2+(-6-\bar{\gamma})\bar{\delta}_1-6\bar{\chi}_2\\
&+\Bigg(-\frac{33}{2}(2+\bar{\gamma})^2+6\bar{\delta}_1\Bigg)\ln(r'_1)+\Bigg(\frac{33}{2}(2+\bar{\gamma})^2-6\bar{\delta}_1\Bigg)\ln(r_{12})\Bigg)(n_{12}v_2)\Bigg]\Bigg)\\
&+\mathbf{v}_2\Bigg(m_1m_2^2\Bigg[\Bigg(\frac{1}{24}(921+1040\bar{\gamma}+234\bar{\gamma}^2-24\bar{\gamma}^3)-\frac{2}{3}(35+9\bar{\gamma})\bar{\delta}_1+\bar{\beta}_1\Bigg(\frac{1}{2}(-65-44\bar{\gamma})-\frac{24\bar{\beta}_2}{\bar{\gamma}}+\frac{24\bar{\delta}_1}{\bar{\gamma}}\Bigg)\\
&+\frac{1}{3}(-53-18\bar{\gamma})\bar{\delta}_2+\frac{24\bar{\beta}_2\bar{\delta}_2}{\bar{\gamma}}+\pi^2\Bigg(\frac{3}{128}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)+\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_1\\
&+\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_2\Bigg)\Bigg)(n_{12}v_1)+\Bigg(\frac{1}{24}(-1437-1232\bar{\gamma}-222\bar{\gamma}^2+24\bar{\gamma}^3)+\frac{2}{3}(35+9\bar{\gamma})\bar{\delta}_1\\
&+\bar{\beta}_1\Bigg(\frac{1}{2}(43+44\bar{\gamma})+\frac{48\bar{\beta}_2}{\bar{\gamma}}-\frac{24\bar{\delta}_1}{\bar{\gamma}}\Bigg)+\frac{1}{3}(59+18\bar{\gamma})\bar{\delta}_2+\bar{\beta}_2\Bigg(-4(3+\bar{\gamma})-\frac{24\bar{\delta}_2}{\bar{\gamma}}\Bigg)\\
&+\pi^2\Bigg(-\frac{3}{128}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)-\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_1-\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_2\Bigg)\Bigg)(n_{12}v_2)\Bigg]\\
&+m_2^3\Bigg(\Bigg(-\frac{1}{2}(2+\bar{\gamma})^3-2(2+\bar{\gamma})\bar{\delta}_2\Bigg)(n_{12}v_1)+\Bigg(\frac{1}{4}(2+\bar{\gamma})^2(-5+2\bar{\gamma})-4(2+\bar{\gamma})\bar{\beta}_1\\
&+(3+2\bar{\gamma})\bar{\delta}_2+2\bar{\chi}_1\Bigg)(n_{12}v_2)\Bigg)\\
&+m_1^2m_2\Bigg[\Bigg(\frac{1}{12}(-1325-1328\bar{\gamma}-411\bar{\gamma}^2-39\bar{\gamma}^3)+\frac{1}{2}(-63-40\bar{\gamma})\bar{\beta}_2+(-5-\bar{\gamma})\bar{\delta}_1-4\bar{\chi}_2\\
&+\Bigg(-\frac{33}{2}(2+\bar{\gamma})^2+6\bar{\delta}_1\Bigg)\ln(r'_1)+\Bigg(\frac{33}{2}(2+\bar{\gamma})^2-6\bar{\delta}_1\Bigg)\ln(r_{12})\Bigg)(n_{12}v_1)\\
&+\Bigg(\frac{1}{12}(1463+1484\bar{\gamma}+438\bar{\gamma}^2+39\bar{\gamma}^3)+\frac{1}{2}(43+40\bar{\gamma})\bar{\beta}_2+(6+\bar{\gamma})\bar{\delta}_1+6\bar{\chi}_2\\
&+\Bigg(\frac{33}{2}(2+\bar{\gamma})^2-6\bar{\delta}_1\Bigg)\ln(r'_1)+\Bigg(-\frac{33}{2}(2+\bar{\gamma})^2+6\bar{\delta}_1\Bigg)\ln(r_{12})\Bigg)(n_{12}v_2)\Bigg]\Bigg)\\
&+\mathbf{n}_{12}\Bigg(m_2^3\Bigg(\Bigg(-\frac{1}{4}(1+\bar{\gamma})(2+\bar{\gamma})^2+(-1-\bar{\gamma})\bar{\delta}_2\Bigg)(n_{12}v_1)^2+\Bigg(\frac{1}{2}(1+\bar{\gamma})(2+\bar{\gamma})^2\\
&+2(1+\bar{\gamma})\bar{\delta}_2\Bigg)(n_{12}v_1)(n_{12}v_2)+\Bigg(-\frac{1}{8}\Bigg(2+\bar{\gamma})^2(-43+2\bar{\gamma})+10(2+\bar{\gamma})\bar{\beta}_1+\frac{1}{2}(3-2\bar{\gamma})\bar{\delta}_2-5\bar{\chi}_1\Bigg)(n_{12}v_2)^2\\
&+\Bigg(\frac{9}{2}(2+\bar{\gamma})^2+8(2+\bar{\gamma})\bar{\beta}_1+2\bar{\delta}_2\Bigg)(v_1v_2)-2\bar{\chi}_1v_1^2+\Bigg(-\frac{9}{4}(2+\bar{\gamma})^2-4(2+\bar{\gamma})\bar{\beta}_1-\bar{\delta}_2\Bigg)v_2^2\Bigg)\\
&+m_1m_2^2\Bigg[\Bigg(\frac{1}{24}(1245+878\bar{\gamma}-264\bar{\gamma}^2-192\bar{\gamma}^3)+\frac{1}{3}(-181-18\bar{\gamma})\bar{\delta}_1+\bar{\beta}_1\Bigg(\frac{1}{4}(-299-192\bar{\gamma})\\
&-\frac{40\bar{\beta}_2}{\bar{\gamma}}+\frac{60\bar{\delta}_1}{\bar{\gamma}}\Bigg)+\frac{1}{3}(-83-18\bar{\gamma})\bar{\delta}_2+\bar{\beta}_2\Bigg(-8(2+\bar{\gamma})+\frac{60\bar{\delta}_2}{\bar{\gamma}}\Bigg)\\
&+\pi^2\Bigg(\frac{15}{256}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)+\frac{105}{128}(2+\bar{\gamma})\bar{\delta}_1+\frac{105}{128}(2+\bar{\gamma})\bar{\delta}_2\Bigg)\Bigg)(n_{12}v_1)^2\\
&+\Bigg(\frac{1}{12}(-1125-806\bar{\gamma}+270\bar{\gamma}^2+192\bar{\gamma}^3)+\frac{2}{3}(181+18\bar{\gamma})\bar{\delta}_1+\bar{\beta}_1\Bigg(\frac{1}{2}(299+192\bar{\gamma})+\frac{80\bar{\beta}_2}{\bar{\gamma}}-\frac{120\bar{\delta}_1}{\bar{\gamma}}\Bigg)\\
&+\frac{4}{3}(43+9\bar{\gamma})\bar{\delta}_2+\bar{\beta}_2\Bigg(16(2+\bar{\gamma})-\frac{120\bar{\delta}_2}{\bar{\gamma}}\Bigg)+\pi^2\Bigg(-\frac{15}{128}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{105}{64}(2+\bar{\gamma})\bar{\delta}_1-\frac{105}{64}(2+\bar{\gamma})\bar{\delta}_2\Big)\Big)(n_{12}v_1)(n_{12}v_2)+\Big(\frac{1}{24}(3339+2294\bar{\gamma}-36\bar{\gamma}^2-192\bar{\gamma}^3)\\
&+\frac{1}{3}(-181-18\bar{\gamma})\bar{\delta}_1+\bar{\beta}_1\Big(\frac{1}{4}(-129-152\bar{\gamma})-\frac{100\bar{\beta}_2}{\bar{\gamma}}+\frac{60\bar{\delta}_1}{\bar{\gamma}}\Big)+\frac{1}{3}(-89-18\bar{\gamma})\bar{\delta}_2\\
&+\bar{\beta}_2\Big(2(7+\bar{\gamma})+\frac{60\bar{\delta}_2}{\bar{\gamma}}\Big)+\pi^2\Big(\frac{15}{256}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)+\frac{105}{128}(2+\bar{\gamma})\bar{\delta}_1\\
&+\frac{105}{128}(2+\bar{\gamma})\bar{\delta}_2\Big)\Big)(n_{12}v_2)^2+\Big(\frac{1}{6}(198+17\bar{\gamma}-117\bar{\gamma}^2-36\bar{\gamma}^3)-\frac{70}{3}\bar{\delta}_1+\bar{\beta}_1\Big(-\frac{5}{2}(5+4\bar{\gamma})\\
&-\frac{8\bar{\beta}_2}{\bar{\gamma}}+\frac{24\bar{\delta}_1}{\bar{\gamma}}\Big)-\frac{20}{3}\bar{\delta}_2+\bar{\beta}_2\Big(4(1+\bar{\gamma})+\frac{24\bar{\delta}_2}{\bar{\gamma}}\Big)+\pi^2\Big(\frac{3}{128}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)\\
&+\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_1+\frac{21}{64}(2+\bar{\gamma})\bar{\delta}_2\Big)\Big)(v_1v_2)+\Big(\frac{1}{12}(216+271\bar{\gamma}+165\bar{\gamma}^2+36\bar{\gamma}^3)+\frac{35}{3}\bar{\delta}_1+\bar{\beta}_1\Big(\frac{1}{4}(85+36\bar{\gamma})\\
&-\frac{20\bar{\beta}_2}{\bar{\gamma}}-\frac{12\bar{\delta}_1}{\bar{\gamma}}\Big)+\frac{10}{3}\bar{\delta}_2+\bar{\beta}_2\Big(2(5+\bar{\gamma})-\frac{12\bar{\delta}_2}{\bar{\gamma}}\Big)+\pi^2\Big(-\frac{3}{256}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)\\
&-\frac{21}{128}(2+\bar{\gamma})\bar{\delta}_1-\frac{21}{128}(2+\bar{\gamma})\bar{\delta}_2\Big)\Big)v_1^2+\Big(\frac{1}{12}(-198-17\bar{\gamma}+117\bar{\gamma}^2+36\bar{\gamma}^3)\\
&+\frac{35}{3}\bar{\delta}_1+\bar{\beta}_1\Big(\frac{5}{4}(5+4\bar{\gamma})+\frac{4\bar{\beta}_2}{\bar{\gamma}}-\frac{12\bar{\delta}_1}{\bar{\gamma}}\Big)+\frac{10}{3}\bar{\delta}_2+\bar{\beta}_2\Big(-2(1+\bar{\gamma})-\frac{12\bar{\delta}_2}{\bar{\gamma}}\Big)\\
&+\pi^2\Big(-\frac{3}{256}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2)-\frac{21}{128}(2+\bar{\gamma})\bar{\delta}_1-\frac{21}{128}(2+\bar{\gamma})\bar{\delta}_2\Big)\Big)v_2^2\Big]\\
&+m_1^2m_2\Big[\Big(\frac{1}{24}(-8959-9568\bar{\gamma}-2865\bar{\gamma}^2-171\bar{\gamma}^3)+\frac{1}{4}(-187-144\bar{\gamma})\bar{\beta}_2+\frac{1}{2}(-23+3\bar{\gamma})\bar{\delta}_1\\
&-10\bar{\chi}_2+\Big(-\frac{165}{4}(2+\bar{\gamma})^2+15\bar{\delta}_1\Big)\ln(r'_1)+\Big(\frac{165}{4}(2+\bar{\gamma})^2-15\bar{\delta}_1\Big)\ln(r_{12})\Big)(n_{12}v_1)^2\\
&+\Big(\frac{1}{12}(9268+9760\bar{\gamma}+2871\bar{\gamma}^2+171\bar{\gamma}^3)+\frac{1}{2}(179+144\bar{\gamma})\bar{\beta}_2+(25-3\bar{\gamma})\bar{\delta}_1+20\bar{\chi}_2\\
&+\Big(\frac{165}{2}(2+\bar{\gamma})^2-30\bar{\delta}_1\Big)\ln(r'_1)+\Big(-\frac{165}{2}(2+\bar{\gamma})^2+30\bar{\delta}_1\Big)\ln(r_{12})\Big)(n_{12}v_1)(n_{12}v_2)\\
&+\Big(\frac{1}{24}(-8386-9148\bar{\gamma}-2742\bar{\gamma}^2-171\bar{\gamma}^3)+\frac{1}{4}(-31-104\bar{\gamma})\bar{\beta}_2+\frac{1}{2}(-22+3\bar{\gamma})\bar{\delta}_1-15\bar{\chi}_2\\
&+\Big(-\frac{165}{4}(2+\bar{\gamma})^2+15\bar{\delta}_1\Big)\ln(r'_1)+\Big(\frac{165}{4}(2+\bar{\gamma})^2-15\bar{\delta}_1\Big)\ln(r_{12})\Big)(n_{12}v_2)^2\\
&+\Big(\frac{1}{12}(-1463-1634\bar{\gamma}-513\bar{\gamma}^2-33\bar{\gamma}^3)+\frac{1}{2}(5-8\bar{\gamma})\bar{\beta}_2+(-7+\bar{\gamma})\bar{\delta}_1-4\bar{\chi}_2\\
&+\Big(-\frac{33}{2}(2+\bar{\gamma})^2+6\bar{\delta}_1\Big)\ln(r'_1)+\Big(\frac{33}{2}(2+\bar{\gamma})^2-6\bar{\delta}_1\Big)\ln(r_{12})\Big)(v_1v_2)+\Big(\frac{1}{24}(1805+1898\bar{\gamma}\\
&+567\bar{\gamma}^2+33\bar{\gamma}^3)+\frac{1}{4}(43+24\bar{\gamma})\bar{\beta}_2+\frac{1}{2}(9-\bar{\gamma})\bar{\delta}_1+\Big(\frac{33}{4}(2+\bar{\gamma})^2-3\bar{\delta}_1\Big)\ln(r'_1)\\
&+\Big(-\frac{33}{4}(2+\bar{\gamma})^2+3\bar{\delta}_1\Big)\ln(r_{12})\Big)v_1^2+\Big(\frac{1}{24}(1463+1634\bar{\gamma}+513\bar{\gamma}^2+33\bar{\gamma}^3)+\frac{1}{4}(-5+8\bar{\gamma})\bar{\beta}_2\\
&+\frac{1}{2}(7-\bar{\gamma})\bar{\delta}_1+2\bar{\chi}_2+\Big(\frac{33}{4}(2+\bar{\gamma})^2-3\bar{\delta}_1\Big)\ln(r'_1)+\Big(-\frac{33}{4}(2+\bar{\gamma})^2+3\bar{\delta}_1\Big)\ln(r_{12})\Big)v_2^2\Big]\Big)\Big],
\end{aligned}
\tag{5.11c}
$$

$$
\begin{aligned}
\mathbf{a}_1^{3\mathrm{PN},(4)} &= \frac{\alpha^3 \mathbf{n}_{12}}{r_{12}^5}\bigg[\left(\frac{11}{12}\bar{\gamma}(2+\bar{\gamma}) - \frac{(10+\bar{\gamma})\bar{\delta}_1}{3(2+\bar{\gamma})}\right)m_1^3 m_2 + \left(\frac{11}{2}\bar{\gamma}(2+\bar{\gamma}) - \frac{2\bar{\gamma}\bar{\delta}_1}{2+\bar{\gamma}}\right)m_1^2 m_2^2 \\
&\quad + \left(\frac{55}{12}\bar{\gamma}(2+\bar{\gamma}) - \frac{5(-2+\bar{\gamma})\bar{\delta}_2}{3(2+\bar{\gamma})}\right)m_1 m_2^3\bigg] \\
&\quad + \frac{\alpha^4 \mathbf{n}_{12}}{r_{12}^5}\Bigg(m_2^4\bigg[\frac{8}{3}(2+\bar{\gamma})\bar{\delta}_2 + \bar{\beta}_1\left(\frac{14}{3}(2+\bar{\gamma})^2 + \frac{8}{3}\bar{\delta}_2\right) + \frac{2}{3}(24+36\bar{\gamma}+18\bar{\gamma}^2+3\bar{\gamma}^3+2\bar{\kappa}_1) - 4(2+\bar{\gamma})\bar{\chi}_1\bigg] \\
&\quad + m_1^2 m_2^2\bigg[\frac{1}{36}(6168+5240\bar{\gamma}+1085\bar{\gamma}^2-27\bar{\gamma}^3) + 8(\bar{\beta}_2)^2 + \frac{11}{9}(-5+3\bar{\gamma})\bar{\delta}_1 - 4\bar{\delta}_2 \\
&\qquad + \bar{\beta}_2\left(2(65+34\bar{\gamma}+4\bar{\gamma}^2) + \frac{16\bar{\delta}_2}{\bar{\gamma}}\right) + \pi^2\left(\frac{1}{64}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2) + \frac{7}{32}(2+\bar{\gamma})\bar{\delta}_1 + \frac{7}{32}(2+\bar{\gamma})\bar{\delta}_2\right) \\
&\qquad - 4(4+\bar{\gamma})\bar{\chi}_2 + \bar{\beta}_1\left(\frac{1}{3}(149+80\bar{\gamma}+14\bar{\gamma}^2) - \frac{4(44+9\bar{\gamma})\bar{\beta}_2}{\bar{\gamma}} + \frac{64(\bar{\beta}_2)^2}{\bar{\gamma}^2} + \frac{8(2+\bar{\gamma})\bar{\delta}_1}{3\bar{\gamma}} + \frac{32\bar{\chi}_2}{\bar{\gamma}}\right)\bigg] \\
&\quad + m_1^3 m_2\bigg[4(\bar{\beta}_2)^2 + \frac{1}{3}(16+9\bar{\gamma})\bar{\delta}_1 + \bar{\beta}_2\left(\frac{1}{3}(119+74\bar{\gamma}+14\bar{\gamma}^2) + \frac{8}{3}\bar{\delta}_1\right) + \frac{1}{36}(-563-614\bar{\gamma}-72\bar{\gamma}^2 \\
&\qquad + 39\bar{\gamma}^3 + 48\bar{\kappa}_2) - \frac{2}{3}(19+6\bar{\gamma})\bar{\chi}_2 + \left(-\frac{11}{2}(2+\bar{\gamma})^2 + 2\bar{\delta}_1\right)\ln(r_1') + \left(\frac{11}{2}(2+\bar{\gamma})^2 - 2\bar{\delta}_1\right)\ln(r_{12})\bigg] \\
&\quad + m_1 m_2^3\bigg[\frac{1}{36}(6668+6514\bar{\gamma}+1619\bar{\gamma}^2+6\bar{\gamma}^3) + (\bar{\beta}_1)^2\left(10 + \frac{64\bar{\beta}_2}{\bar{\gamma}^2}\right) - \frac{86}{9}\bar{\delta}_1 + \bar{\beta}_1\bigg(117+66\bar{\gamma}+8\bar{\gamma}^2 \\
&\qquad - \frac{2(68+21\bar{\gamma})\bar{\beta}_2}{\bar{\gamma}} + \frac{16\bar{\delta}_1}{\bar{\gamma}}\bigg) + \frac{5}{9}(5+6\bar{\gamma})\bar{\delta}_2 + \pi^2\bigg(\frac{1}{64}(2+\bar{\gamma})(-82-34\bar{\gamma}+7\bar{\gamma}^2) + \frac{7}{32}(2+\bar{\gamma})\bar{\delta}_1 \\
&\qquad + \frac{7}{32}(2+\bar{\gamma})\bar{\delta}_2\bigg) - 2(9+2\bar{\gamma})\bar{\chi}_1 + \bar{\beta}_2\left(\frac{1}{3}(89+68\bar{\gamma}+14\bar{\gamma}^2) + \frac{8(2+\bar{\gamma})\bar{\delta}_2}{3\bar{\gamma}} + \frac{32\bar{\chi}_1}{\bar{\gamma}}\right) \\
&\qquad + \left(\frac{11}{2}(2+\bar{\gamma})^2 - 2\bar{\delta}_2\right)\ln(r_2') + \left(-\frac{11}{2}(2+\bar{\gamma})^2 + 2\bar{\delta}_2\right)\ln(r_{12})\bigg]\Bigg). \qquad (5.11\mathrm{d})
\end{aligned}
$$

Finally, the nonlocal part of the acceleration is given by

$$
\begin{aligned}
a_1^{i3\mathrm{PN,tail}} &= -\frac{4G^2M}{3c^6\phi_0}(1-2s_1)\int_0^{+\infty}\mathrm{d}\tau\ln\left(\frac{c\tau}{2r_{12}}\right)[I_{si}^{(5)}(t-\tau) - I_{si}^{(5)}(t+\tau)] \\
&\quad + \frac{8G^2M}{3c^6\phi_0}(1-2s_1)([\ln r_{12} I_{(s)i}^{(2)}]^{(2)} - \ln r_{12} I_{si}^{(4)}) - \frac{4G^2M}{3c^6 m_1}(3+2\omega_0)\frac{n_{12}^i}{r_{12}}(I_{si}^{(2)})^2, \qquad (5.12)
\end{aligned}
$$

where $M = m_1 + m_2$ is the ADM mass. The instantaneous terms on the second line come from the introduction of the time-varying scale r_{12} in the decomposition (4.29). The term on the first line is the nonlocal tail term. Replacing the scalar dipole moment by its explicit expression,

$$
I_\mathrm{s}^i(t) = -\frac{1}{\phi_0(3+2\omega_0)}[m_1(1-2s_1)y_1^i + m_2(1-2s_2)y_2^i], \qquad (5.13)
$$

and using the ST parameters to express the instantaneous terms, we get

$$
\begin{aligned}
a_1^{i3\mathrm{PN,tail}} &= -\frac{4G^2M}{3c^6\phi_0}(1-2s_1)\int_0^{+\infty}\mathrm{d}\tau\ln\left(\frac{c\tau}{2r_{12}}\right)[I_{si}^{(5)}(t-\tau) - I_{si}^{(5)}(t+\tau)] \\
&\quad + \frac{8\tilde{G}^3\alpha^3 M m_1 m_2}{3c^6 r_{12}^4}\left(\bar{\delta}_1 + \frac{\bar{\gamma}(2+\bar{\gamma})}{4}\right)\left[2(n_{12}v_{12})v_{12}^i - 8(n_{12}v_{12})^2 n_{12}^i + v_{12}^2 n_{12}^i - \frac{\tilde{G}\alpha M}{r_{12}}n_{12}^i\right] \\
&\quad - \frac{16\tilde{G}^4\alpha^4 M m_1 m_2^2}{3c^6 r_{12}^5}\left(\frac{\bar{\delta}_1}{4} + \frac{\bar{\gamma}(2+\bar{\gamma})}{2} + \frac{\bar{\delta}_2}{4}\right)n_{12}^i. \qquad (5.14)
\end{aligned}
$$

B. Discussions

1. General comments

We have verified that our result is manifestly Lorentz invariant, as it is expected because we are in harmonic coordinates and dimensional regularization does not break the Lorentz-Poincaré symmetry. Then in the GR limit, i.e. when $\omega_0 \to \infty$ and $\phi_0 \to 1$, we recover the 3PN acceleration of GR, up to an unphysical shift of the trajectories of the particles. The presence of such a shift only reflects the freedom we have when performing the redefinition of the trajectories of the particles in order to remove the pole. Finally, up to 2PN, the equations of motion depend only on the constant α through the combination $\tilde{G}\alpha m_A$. At 3PN, this is no more the case and an additional dependence on α appears in some terms. This is a new and unexpected result. One way of seeing it is to rewrite it as a dependence on ζ, through the relation $1-\zeta=\alpha(1+\bar{\gamma}/2)$. It is then clear that it introduces an explicit dependence on the function ω_0. However, depending on the compact objects we are considering such a particularity may disappear, and thus it may be difficult to see the observational consequence of such a dependence.

2. The binary black hole limit

An important test of our result consists in studying the binary black hole limit. We have seen that the sensitivity of a stationary black hole is exactly given by $s=1/2$. If we assume that $s_A=1/2$ still holds for each black hole in a binary system, our result is indistinguishable from GR, up to a simple rescaling of the mass. In particular, the nonlocal tail part of the acceleration does not contribute and the explicit dependence in ζ disappears. This result confirms that Hawking's theorem may hold also for binary black holes, which is *a priori* not a stationary system, at least up to 3PN order. However, the 3PN dynamics only describes the early-inspiral phase of the coalescence. In particular, it does not tell us anything about the late-inspiral phase where strong-field effects appear and Hawking's theorem may break down. A correct implementation of such hypothetical effects can only be done using the ST EOB formalism coupled to full numerical relativity results for ST theories. Some numerical results [58] have shown that, unless an external mechanism activates the dynamics of the scalar field, binary black holes in ST theories and GR are indistinguishable.

3. Black hole–neutron star binary

We now consider the case when one of the compact object is a black hole, say $s_1=1/2$, while the other one is a neutron star, with $s_2\approx 0.2$. First, we find that the explicit dependence in ζ also disappears for this configuration, up to an unphysical shift. Then, as we have $\bar{\gamma}=\bar{\delta}_1=\bar{\beta}_i=\bar{\kappa}_i=\bar{\chi}_i=0$, the final result depends only on one single parameter,

$$\bar{\delta}_2=\frac{\zeta}{1-\zeta}(1-2s_2)^2. \tag{5.15}$$

It means that the 3PN equations differs from GR only through this only parameter. Thus, if this result still holds for the gravitational waveforms,[5] the black hole–neutron star system may not allow to distinguish between Brans-Dicke theory (with constant function ω), and general scalar-tensor theories. Of course, this conclusion does not apply when dynamical scalarization takes place [37,38], a situation that is not described by our prescription for the matter through a skeletonized action [41].

4. Concluding remarks

In the companion paper [43], we compute the conserved integrals of motion and the reduction to the center-of-mass frame. Due to the presence of the nonlocal term in the action (3.27), the computation of the conserved energy and angular momentum has to be treated carefully, as some extra contributions may appear [12].

Finally, in scalar-tensor theories, the finite-size effects are expected to start contributing to the dynamics at 3PN order [27].[6] They may prove very usefull to constrain the theory as such effects can have a different signature in the signal. Thus, if we want to capture the full gravitational waveform at 2PN order in ST theories, the tidal effects should be properly included in the 3PN dynamics. As it is a work on its own, we have not considered these effects in this paper, and have left it for a future work.

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APPENDIX A: DEMONSTRATION OF EQ. (3.2)

In this Appendix, we give the proof of Eq. (3.2) in the case of dimensional regularization. It mainly follows the

[5]It has already been shown that it is the case for the tensor gravitational waveform [31].

[6]The tidal effects may even start at a lower order (1PN) due to some dynamical scalarization phenomenon that could be responsible for the large value of some coefficients in the expansion of the mass with respect to the scalar field [27].

proof done for Hadamard regularization in [11]. We consider the difference

$$\Delta_g = L_g - \int d^d x \mathcal{M}(\mathcal{L}_g), \tag{A1}$$

where $L_g = \int d^3 x \mathcal{L}_g$ involves only the complete solution. As it is perfectly regular everywhere, we do not need any regularization. Thus, we can add a regulator in the integral without altering the result,

$$\Delta_g = \int d^d x [\mathcal{L}_g - \mathcal{M}(\mathcal{L}_g)]. \tag{A2}$$

Now, as the complete solution $\mathcal{L}_g$ coincide with $\mathcal{M}(\mathcal{L}_g)$ outside the source, the integrand of (A2) is zero in the exterior region. Thus, it is of compact support around the source and we can PN expand Eq. (A2) without changing the result,

$$\Delta_g = \int d^d x [\overline{\mathcal{L}_g} - \overline{\mathcal{M}(\mathcal{L}_g)}]. \tag{A3}$$

Then, the matching equation (3.1) implies a common structure of the Lagrangian densities, namely

$$\overline{\mathcal{M}(\mathcal{L}_g)} = \mathcal{M}(\overline{\mathcal{L}}_g) \sim \sum \hat{n}_L r^a (\ln r)^b F(t), \tag{A4}$$

where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and the functions $F(t)$ are functions of the source multipole moments. Inserting Eq. (A4) into the integral involving $\overline{\mathcal{M}(\mathcal{L}_g)}$ in Eq. (A3), one can see that it involves integrals of the type $\int d^d x \hat{n}_L r^a (\ln r)^b F(t)$. After performing the angular integration, one is left with the simple radial integrals, $\int dr\, r^{a+2+\varepsilon} (\ln r)^b$, where we have written the dimension $d = 3 + \varepsilon$. These integrals are all zero by analytic continuation in $\varepsilon \in \mathbb{C}$. To show this, we split this integral into a near-zone integral, $\int_{r<\mathcal{R}}$, and a far-zone integral, $\int_{r>\mathcal{R}}$. The near-zone integral is computed for $\mathrm{Re}(\varepsilon) > -a - 3$, and analytically continued for $\varepsilon \in \mathbb{C}$, except for the value $\varepsilon = -a - 3$. Similarly the far-zone integral is computed for $\mathrm{Re}(\varepsilon) < -a - 3$, and analytically continued for $\varepsilon \in \mathbb{C}$, except for the value $\varepsilon = -a - 3$. Then, summing the two analytic continuations, one find that they cancel each other and the total integral is zero for any $\varepsilon \in \mathbb{C}$. Finally, one gets that $\int d^d x \overline{\mathcal{M}(\mathcal{L}_g)} = 0$, and as a consequence,

$$\Delta_g = \int d^d x \overline{\mathcal{L}_g}. \tag{A5}$$

This ends our proof.

APPENDIX B: THE MATTER SOURCE DENSITIES IN SCALAR-TENSOR THEORIES

In this Appendix, we write the explicit expressions of the matter source densities (4.16) as a function of the PN potentials at the required order:

$$\sigma = \frac{2(d-2)}{\phi_0^{\frac{d-1}{2}}(d-1)} m_1 \left[1 + \frac{1}{c^2}\left((1-2s_1)(\psi_{(0)})_1 + \frac{d}{2(d-2)} v_1^2 - \frac{4-d}{d-2}(V)_1\right)\right] \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + [1 \leftrightarrow 2], \tag{B1a}$$

$$\sigma_i = \frac{1}{\phi_0^{\frac{d-1}{2}}} m_1 \left[1 + \frac{1}{c^2}\left((1-2s_1)(\psi_{(0)})_1 + \frac{1}{2} v_1^2 - \frac{4-d}{d-2}(V)_1\right)\right] v_1^i \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + [1 \leftrightarrow 2], \tag{B1b}$$

$$\sigma_{ij} = \frac{1}{\phi_0^{\frac{d-1}{2}}} m_1 \left[v_1^i v_1^j - \frac{1}{d-2} \delta_{ij} v_1^2\right] \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + [1 \leftrightarrow 2], \tag{B1c}$$

$$\begin{aligned}\sigma_s = \frac{2}{\phi_0^{\frac{d-1}{2}}(d(d-1)+4\omega_0)} m_1 \Bigg[(1-2s_1) + \frac{1}{c^2}\Bigg(\left((1-2s_1)^2 + 4s_1' + \frac{4\phi_0\omega_0'}{d(d-1)+4\omega_0}(1-2s_1)\right)(\psi_{(0)})_1 \\ - \frac{1}{2}(1-2s_1)v_1^2 - (1-2s_1)(V)_1\Bigg)\Bigg] \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + [1 \leftrightarrow 2]. \end{aligned} \tag{B1d}$$

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