Lorentz and *CPT* tests with clock-comparison experiments

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Clock-comparison experiments are among the sharpest existing tests of Lorentz symmetry in matter. We characterize signals in these experiments arising from modifications to electron or nucleon propagators and involving Lorentz- and *CPT*-violating operators of arbitrary mass dimension. The spectral frequencies of the atoms or ions used as clocks exhibit perturbative shifts that can depend on the constituent-particle properties and can display sidereal and annual variations in time. Adopting an independent-particle model for the electronic structure and the Schmidt model for the nucleus, we determine observables for a variety of clock-comparison experiments involving fountain clocks, comagnetometers, ion traps, lattice clocks, entangled states, and antimatter. The treatment demonstrates the complementarity of sensitivities to Lorentz and *CPT* violation among these different experimental techniques. It also permits the interpretation of some prior results in terms of bounds on nonminimal coefficients for Lorentz violation, including first constraints on nonminimal coefficients in the neutron sector. Estimates of attainable sensitivities in future analyses are provided. Two technical appendices collect relationships between spherical and Cartesian coefficients for Lorentz violation and provide explicit transformations converting Cartesian coefficients in a laboratory frame to the canonical Sun-centered frame.

I. INTRODUCTION

Among the best laboratory tests of rotation invariance are experiments measuring the ticking rate of a clock as its orientation changes, often as it rotates with the Earth. A spatial anisotropy in the laws of nature would be revealed if the clock frequency varies in time at harmonics of the rotation frequency. Detecting any time variation requires a reference clock that either is insensitive to the anisotropy or responds differently to it. Typically, the two clock frequencies in these experiments are transition frequencies in atoms or ions, and the spatial orientation of a clock is the quantization axis established by an applied magnetic field. These clock-comparison experiments can attain impressive sensitivities to rotation violations, as originally shown by Hughes *et al.* and Drever [1].

Rotation invariance is a key component of Lorentz symmetry, the foundation of relativity. Tests of this symmetry have experienced a revival in recent years, stimulated by the possibility that minuscule violations could arise from a unification of quantum physics with gravity such as string theory [2]. Using techniques from different subfields of physics, numerous searches for Lorentz violation have now reached sensitivities to physical effects originating at the Planck scale $M_P \simeq 10^{19}$ GeV [3]. Since the three boost generators of the Lorentz group close under commutation into the three rotation generators, any deviations from Lorentz symmetry in nature must necessarily come with violations of rotation invariance. Searches for rotation violations therefore offer crucial tests of Lorentz symmetry. In the present work, we pursue this line of reasoning by developing and applying a theoretical treatment for the analysis of clock-comparison experiments searching for Lorentz violation.

To date, no compelling experimental evidence for Lorentz violation has been adduced. Even if Lorentz violation does occur in nature, identifying the correct realistic model among a plethora of options in the absence of positive experimental guidance seems a daunting and improbable prospect. An alternative is instead to adopt a general theoretical framework for Lorentz violation that encompasses specific models and permits a comprehensive study of possible effects. Since any Lorentz violation is expected to be small, it is reasonable to use effective field theory [4] for this purpose. A realistic treatment then starts from well-established physics, which can be taken as the action formed by coupling general relativity to the Standard Model of particle physics, and adds all possible Lorentz-violating operators to yield the framework known as the Standard-Model Extension (SME) [5,6]. Each Lorentz-violating operator in the SME is contracted

with a coefficient that determines the magnitude of its physical effects while preserving coordinate independence of the theory. The operators can be classified according to their mass dimension d, with larger values of d associated with greater suppression at low energies. The limiting case with $d \leq 4$ is power-counting renormalizable in Minkowski spacetime and is called the minimal SME. Since CPT violation in effective field theory is concurrent with Lorentz violation [5,7], the SME also characterizes general effects from CPT violation. Experimental constraints on the parameters of any Lorentz-violating model that is consistent with realistic effective field theory can be found by identifying the model parameters with specific SME coefficients and their known constraints [3,8].

Signals arising from Lorentz and CPT violation are predicted by the minimal SME to appear in clock-comparison experiments with atoms or ions [9]. The signals include observable modifications of the spectra that can exhibit time variations and that depend on the electron and nucleon composition of the species used as clocks. Null results from early clock-comparison experiments [10–13] can be reinterpreted as bounds on coefficients for Lorentz violation in the minimal SME [9]. Many minimal-SME coefficients have been directly constrained in recent experiments, including clock comparisons performed using a hydrogen maser [14,15], ^{133}Cs and ^{87}Rb fountain clocks [16], trapped ultracold neutrons and ^{199}Hg atoms [17], ^{3}He-K and ^{21}Ne-Rb-K comagnetometers [18,19], ^{133}Cs and ^{199}Hg magnetometers [20], transitions in ^{162}Dy and ^{164}Dy atoms [21], ^{129}Xe and ^{3}He atoms [22–25], and entangled states of ^{40}Ca^{+} ions [26]. The results represent competitive tests of Lorentz and CPT symmetry [3,27], and additional constraints on minimal-SME coefficients have been extracted by detailed theoretical analyses [28–34].

In this work, we extend the existing theoretical treatment of Lorentz and CPT violation in clock-comparison experiments to include SME operators of nonminimal mass dimension $d > 4$ that modify the Dirac propagators of the constituent electrons, protons, and neutrons in atoms and ions. At an arbitrary given value of d, all Lorentz- and CPT-violating operators affecting the propagation have been identified and classified [35], which in the present context permits a perturbative analysis of the effects of general Lorentz and CPT violation on the spectra of the atoms or ions used in clock-comparison experiments. Nonminimal SME operators are of direct interest in various theoretical contexts associated with Lorentz-violating quantum field theories including, for instance, formal studies of the underlying Riemann-Finsler geometry [36] or of causality and stability [37] and phenomenological investigations of supersymmetric models [38] or noncommutative quantum field theories [39,40]. They are also of interest in experimental searches for geometric forces, such as torsion [41] and nonmetricity [42]. Only a comparatively few constraints on nonminimal SME coefficients for

Lorentz violation in the electron and proton sectors have been derived from laboratory experiments to date [35,43–48], while the neutron sector is unexplored in the literature. Here, we seek to improve this situation by developing techniques for analyzing clock-comparison experiments and identifying potential signals from nonminimal Lorentz and CPT violation. Several clock modalities are considered in this context for the first time, including ones yielding measurements of nonminimal SME coefficients from atomic fountains, comagnetometers, ion traps, lattice clocks, and antimatter spectroscopy. We adopt existing results to deduce numerous first constraints on nonminimal coefficients in the neutron sector, and we estimate sensitivities to electron, proton, and neutron nonminimal coefficients that are attainable in future analyses.

The organization of this work is as follows. In Sec. II, we present the theoretical techniques that enable a perturbative treatment of the effects of Lorentz and CPT violation on the spectra of atoms and ions. A description of the perturbation induced by Lorentz- and CPT-violating operators of arbitrary mass dimension d is provided in Sec. II A. The perturbative shifts in energy levels are discussed in Sec. II B along with generic features of the resulting spectra, and some useful formulas for subsequent calculations are derived. In Sec. II C, we consider methods for determining expectation values of electronic states, with emphasis on an independent-particle model. The corresponding techniques for nucleon states are presented in Sec. II D, primarily in the context of a comparatively simple nuclear model. We then turn to evaluating the time variations in the spectrum due to the noninertial nature of the laboratory frame, first examining effects induced by the rotation of the Earth about its axis in Sec. II E and next discussing ones induced by the revolution of the Earth about the Sun in Sec. II F. The latter section also considers related issues associated with space-based missions.

Applications of these theoretical results in the context of various clock-comparison experiments are addressed in Sec. III. Searches for Lorentz and CPT violation using fountain clocks are considered in Sec. III A, and estimates for attainable sensitivities are obtained. Studies with comagnetometers are investigated in Sec. III B, and first sensitivities to many nonminimal coefficients for Lorentz violation are deduced from existing data. Optical transitions in ion-trap and lattice clocks are discussed in Sec. III C, and potential sensitivities in available systems are considered. Some comments about prospects for antimatter experiments are offered in Sec. III D, where the first SME constraints from antihydrogen spectroscopy are presented. A summary of the work is provided in Sec. IV. Two appendices are also included. Appendix A describes the general relationship between spherical and Cartesian coefficients for Lorentz violation and tabulates explicit expressions for $3 \leq d \leq 8$. Appendix B presents techniques for transforming Cartesian coefficients for Lorentz violation

from the laboratory frame to the Sun-centered frame and collects explicit results for $3 \leq d \leq 8$. In this work, we use conventions and notation matching those of Ref. [35] except as indicated. Note that natural units with $c = \hbar = 1$ are adopted throughout.

II. THEORY

This section discusses the theoretical techniques for the perturbative treatment of Lorentz and *CPT* violation in atoms and ions. The perturbation is described using a framework that encompasses all Lorentz-violating quantum operators affecting the motion of the component particles in the atom. Generic restrictions on the induced energy shifts arising from symmetries of the system are considered. The perturbative calculation of the energy shift is formulated, and expressions useful for application to experiments are obtained. Simple models are selected for the electronic and nuclear structure so that derivations of the relevant expectation values can be performed for a broad range of atomic species used in experiments. The conversion from the laboratory frame to the Sun-centered frame is provided, accounting both for the rotation of the Earth about its axis and for the revolution of the Earth about the Sun at first order in the boost parameter.

A. Description of the perturbation

The experiments of interest here involve comparisons of transitions in atoms or ions, seeking shifts in energy levels due to Lorentz and *CPT* violation. All possible shifts are controlled by SME coefficients, which can be viewed as a set of background fields in the vacuum. The energy-level shifts arise from the coupling of these background fields to the elementary particles and interactions comprising the atom or ion. An exact theoretical treatment of the shifts is prohibitive. However, since any Lorentz and *CPT* violation is small, a perturbative analysis is feasible and sufficient to establish the dominant effects.

From the perspective of perturbation theory, the interaction between the electrons and the protons inside an atom or ion has some common features with the interaction between the nucleons inside the nucleus. In both cases, the magnitude $|\boldsymbol{p}|$ of the momentum $\boldsymbol{p}$ of a fermion of flavor w in the zero-momentum frame is smaller than its rest mass m_w. One consequence is that the dominant contribution due to a perturbation added to the Hamiltonian of the system can be obtained by expanding the perturbation in terms of the ratio $|\boldsymbol{p}|/m_w$, keeping only leading terms in the power series. In most cases, it suffices to treat the system as effectively nonrelativistic. Another feature of interest is that the energy per particle due to the interaction between the nucleons and the interaction between the electrically charged particles in the bound states is comparable to the nonrelativistic kinetic energy of the particles. The nonrelativistic kinetic energy is second order in the ratio

$|\boldsymbol{p}|/m_w$, so the corrections to the propagation of the particles at order $(|\boldsymbol{p}|/m_w)^0$ and $(|\boldsymbol{p}|/m_w)^1$ and at leading order in Lorentz violation dominate any effects due to Lorentz-violating operators coupled to the interactions between the fermions.

With these considerations in mind, it is reasonable to proceed under the usual assumption that the dominant Lorentz-violating shifts of the spectrum of the atom or ion arise from corrections to the propagation of the constituent particles. For most purposes, these particles can be taken as electrons e, protons p, and neutrons n, so that w takes the values e, p, and n. Applications to exotic atoms or ions can be accommodated by extending appropriately the values of w. The relevant Lorentz-violating terms in the Lagrange density are then quadratic in the fermion fields for the constituent particles. All terms of this type have been classified and enumerated [35], and applications to hydrogen and hydrogen-like systems have been established [43]. For convenience, we reproduce in this section the key results relevant in the present context.

For a Dirac fermion ψ_w of flavor w and mass m_w, all quadratic terms in the Lagrange density $\mathcal{L}$ can be expressed as [35]

$$\mathcal{L} \supset \frac{1}{2}\overline{\psi_w}(\gamma^\mu i\partial_\mu - m_w + \hat{\mathcal{Q}}_w)\psi_w + \text{H.c.} \qquad (1)$$

Here, $\hat{\mathcal{Q}}_w$ is a spinor matrix describing modifications of the standard fermion propagator, including all Lorentz-invariant and Lorentz-violating contributions obtained by contracting SME coefficients with operators formed from derivatives $i\partial_\mu$. The matrix $\hat{\mathcal{Q}}_w$ can be decomposed in a basis of Dirac matrices and can be converted to momentum space with the identification $i\partial_\mu \to p_\mu$. Individual operators with definite mass dimension d in the Lagrange density incorporate $d-3$ momentum factors, and the corresponding SME coefficients have dimension $4-d$. The Lagrange density (1) has been extended to include operators at arbitrary d in the photon sector [45,49,50]. Analogous constructions exist for the neutrino [51] and gravity [52] sectors.

For present purposes, the SME coefficients can be assumed uniform and time independent within the solar system [5] and so can be taken as constants when specified in the canonical Sun-centered frame [53]. Using standard procedures, an effective nonrelativistic one-particle Hamiltonian that includes the leading-order correction due to Lorentz- and *CPT*-violation to the propagation of a fermion of flavor w can be derived from the Lagrange density (1). This Hamiltonian can be separated into the conventional Hamiltonian for a free nonrelativistic fermion and a perturbation term δh_w^{NR} containing the Lorentz- and *CPT*-violating contributions. The perturbation δh_w^{NR} is thus a 2×2 matrix, with each component being a function of the momentum operator and independent of the position. It can

be expanded in terms of the identity matrix and the vector $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ of Pauli matrices. For convenience, this expansion can be performed using a helicity basis instead of a Cartesian one. The corresponding three basis vectors can be taken as $\hat{\boldsymbol{\epsilon}}_r = \hat{\boldsymbol{p}} \equiv \boldsymbol{p}/|\boldsymbol{p}|$ and $\hat{\boldsymbol{\epsilon}}_{\pm} = (\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}})/\sqrt{2}$, where $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the usual unit vectors for the polar angle θ and azimuthal angle ϕ in momentum space, with $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. In this helicity basis, the perturbation δh_w^{NR} takes the form [35]

$$\delta h_w^{\mathrm{NR}} = h_{w0} + h_{wr}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_r + h_{w+}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_- + h_{w-}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\epsilon}}_+, \quad (2)$$

where h_{w0} contains spin-independent effects and the remaining terms describe spin-dependent ones.

Many experiments searching for Lorentz and *CPT* violation focus on testing the rotation subgroup of the Lorentz group. To facilitate the analysis of rotation properties, it is useful to express the components h_{w0}, h_{wr}, $h_{w\pm}$ of the perturbation in spherical coordinates. It is opportune to express the spherical decomposition of the operators in the perturbation δh_w^{NR} in terms of spin-weighted spherical harmonics ${}_sY_{jm}(\hat{\boldsymbol{p}})$ of spin weight s, as these harmonics capture in a comparatively elegant form the essential properties of the perturbation under rotations. The usual spherical harmonics are spin-weighted harmonics with spin-weight $s = 0$, $Y_{jm}(\theta, \phi) \equiv {}_0Y_{jm}(\theta, \phi)$. Definitions and some useful features of spin-weighted spherical harmonics are presented in Appendix A of Ref. [50].

In terms of the spherical decomposition, the components of the perturbation (2) can be expanded as

$$h_{w0} = -\sum_{kjm}|\boldsymbol{p}|^k {}_0Y_{jm}(\hat{\boldsymbol{p}})\mathcal{V}_{wkjm}^{\mathrm{NR}} \quad (3)$$

for the spin-independent terms, and

$$h_{wr} = -\sum_{kjm}|\boldsymbol{p}|^k {}_0Y_{jm}(\hat{\boldsymbol{p}})\mathcal{T}_{wkjm}^{\mathrm{NR}(0B)},$$

$$h_{w\pm} = \sum_{kjm}|\boldsymbol{p}|^k {}_{\pm 1}Y_{jm}(\hat{\boldsymbol{p}})(i\mathcal{T}_{wkjm}^{\mathrm{NR}(1E)} \pm \mathcal{T}_{wkjm}^{\mathrm{NR}(1B)}) \quad (4)$$

for the spin-dependent ones. The coefficients $\mathcal{V}_{wkjm}^{\mathrm{NR}}$, $\mathcal{T}_{wkjm}^{\mathrm{NR}(qP)}$, where qP takes values $0B$, $1B$, or $1E$, are nonrelativistic spherical coefficients for Lorentz violation. These effective coefficients are linear combinations of SME coefficients for Lorentz violation that emerge naturally in the nonrelativistic limit of the one-particle Hamiltonian obtained from the Lagrange density (1).

For applications, it is useful to perform a further decomposition of the components of the perturbation Hamiltonian according to their *CPT* handedness. In particular, each nonrelativistic spherical coefficient can be separated into two pieces characterized by the *CPT* handedness of the corresponding operator. This decomposition can be expressed as [35]

$$\mathcal{V}_{wkjm}^{\mathrm{NR}} = c_{wkjm}^{\mathrm{NR}} - a_{wkjm}^{\mathrm{NR}},$$
$$\mathcal{T}_{wkjm}^{\mathrm{NR}(qP)} = g_{wkjm}^{\mathrm{NR}(qP)} - H_{wkjm}^{\mathrm{NR}(qP)}, \quad (5)$$

where the a- and g-type coefficients are contracted with *CPT*-odd operators and the c- and H-type coefficients with *CPT*-even ones. The notation here parallels the standard assignments in the minimal SME [5]. Each nonrelativistic coefficient on the right-hand side of this equation can be expressed as a sum of SME coefficients in the Lagrange density, suitably weighted by powers of m_w. The explicit expressions for these sums are given in Eqs. (111) and (112) of Ref. [35]. The allowed ranges of values for the indices k, j, m and the numbers of independent components for each coefficient are listed in Table IV of Ref. [35]. Note that in the present work we follow the convention of Ref. [43] and adopt the subscript index k instead of n, to avoid confusion with the principal quantum number of the atom or ion.

Given the perturbation δh_w^{NR} affecting the propagation of each fermion in an atom or ion, we can formally express the perturbation δh^{atom} of the system as a whole as

$$\delta h^{\mathrm{atom}} = \sum_w \sum_{a=1}^{N_w} (\delta h_w^{\mathrm{NR}})_a, \quad (6)$$

where $a = 1, \ldots, N_w$ labels the fermions of given flavor w in the atom or ion. The Lorentz-violating operators considered in this work are functions of the momentum and the spin of the particle, so $(\delta h_w^{\mathrm{NR}})_a = \delta h_w^{\mathrm{NR}}(\boldsymbol{p}_a, \boldsymbol{\sigma}_a)$ depends on the momentum operator $\boldsymbol{p}_a$ and the spin operator $\boldsymbol{\sigma}_a$ for the ath fermion. Each term $(\delta h_w^{\mathrm{NR}})_a$ is understood to be the tensor product of the perturbation (2) acting on the states of the ath fermion of flavor w with the identity operator acting on the Hilbert space of all other fermions. Note that the index a is tied to the momentum and spin, whereas the index w controlling the flavor of the particle is contained in the coefficient for Lorentz violation. Note also that the perturbation (6) can be separated according to operator flavor as

$$\delta h^{\mathrm{atom}} = \delta h_e^{\mathrm{atom}} + \delta h_p^{\mathrm{atom}} + \delta h_n^{\mathrm{atom}}, \quad (7)$$

where $\delta h_w^{\mathrm{atom}}$ is the sum of all operators of flavor w that contribute to δh^{atom}. For example, the expression for $\delta h_e^{\mathrm{atom}}$ is given by $\delta h_e^{\mathrm{atom}} = \sum_{a=1}^{N_e}(\delta h_e^{\mathrm{NR}})_a$, where a ranges over the N_e electrons in the atom.

B. Energy shifts

The corrections to the spectrum of the atom or ion due to Lorentz and *CPT* violation can be obtained from the perturbation δh_{atom} using the Raleigh-Schrödinger perturbation theory. At first order, the shift of an energy level is obtained from the matrix elements of the perturbation

evaluated in the subspace spanned by the degenerate unperturbed energy eigenstates, as usual. In typical applications of relevance here, the degeneracy in the energy levels is lifted by an external field such as an applied magnetic field. In this scenario, the first-order shift of an energy level is obtained from the expectation value of the perturbation with respect to the unperturbed state. Since the exact unperturbed energy states for multielectron atoms or ions are typically unknown, approximations to these states must be used to obtain the first-order Lorentz-violating shift of the spectrum. However, the symmetries of the unperturbed system place restrictions on the expectation values of the perturbation. In this section, we describe some of these constraints and establish the general form of the perturbative energy shifts.

Assuming that the degeneracy of the energy levels is broken by an applied magnetic field, parity is a symmetry of the system and so the states of the atom or ion must be parity eigenstates. As a result, the expectation values of parity-odd operators with respect to parity eigenstates must vanish, so only parity-even operators can affect the spectrum. This prevents some terms in the perturbation δh_{atom} from contributing to the energy shift. Each operator in the spherical decomposition (3) and (4) of the perturbation δh_{atom} is either odd or even under parity, with handedness determined by the indices j and k of the corresponding coefficient for Lorentz violation. The coefficients a_{wkjm}^{NR} and c_{wkjm}^{NR} can contribute to energy shift at first-order in perturbation theory only for even values of j and k, while the coefficients $g_{wkjm}^{\text{NR}(qB)}$ and $H_{wkjm}^{\text{NR}(qB)}$ can contribute only if j is odd and k even.

Another constraint arises from time-reversal invariance and the Wigner-Eckart theorem [54], and it concerns the expectation value in any angular-momentum eigenstate of Lorentz-violating operators controlled by spin-dependent coefficients with $P = E$. It can be shown that this expectation value must vanish when the Lorentz-violating operators transform as spherical operators under rotations generated by the angular-momentum operator [43], which is the case for the perturbation δh^{atom} of interest here. As a result, none of the spin-dependent coefficients with $P = E$ contribute to the perturbative shift of the spectrum for any values of k and j. This result applies for all atoms and ions considered in the present work.

In the absence of Lorentz violation, the total angular momentum $\mathbf{F}$ of the atom or ion commutes with the Hamiltonian of the system. When a magnetic field $\mathbf{B} = B\hat{\mathbf{B}}$ is applied, the rotational symmetry is broken. If the perturbative shift due to the magnetic field is smaller than the hyperfine structure, both the quantum number F corresponding to $\mathbf{F}$ and the quantum number m_F corresponding to $\mathbf{F} \cdot \hat{\mathbf{B}}$ can be approximated as good quantum numbers. Suppose the states $|\alpha' m_F\rangle$ represent a basis of eigenstates of the Hamiltonian, where α' is a set of quantum numbers including F that together with m_F forms a complete set of quantum numbers. Using the Wigner-Eckart theorem, the energy shift due to the propagation of the fermions in the atom or ion can then be written as

$$\delta\epsilon = \langle \alpha' m_F | \delta h^{\text{atom}} | \alpha' m_F \rangle = \sum_j A_{j0} \langle F m_F j0 | F m_F \rangle, \quad (8)$$

where $\langle j_2 m_2 j_3 m_3 | j_1 m_1 \rangle$ denote Clebsch-Gordan coefficients. The factors $A_{j0} = A_{j0}(\alpha')$ are independent of m_F. The sum over j in Eq. (8) involves the index j labeling the coefficients for Lorentz violation in δh^{atom}, in parallel with the sums over j in Eqs. (3) and (4). Note that the Clebsch-Gordan coefficient $\langle F m_F j0 | F m_F \rangle$ vanishes when $j > 2F$, implying that no operator with $j > 2F$ contributes to the energy shift.

To find useful expressions for the factors A_{j0}, we make some additional assumptions that are broadly valid for the systems considered in this work. Except where stated otherwise, we suppose that both the magnitude J of the total angular momentum $\mathbf{J}$ of the electrons and the magnitude I of the nuclear spin $\mathbf{I}$ are good quantum numbers for the system. We also assume that the states $|\alpha' m_F\rangle$ can be expressed as a tensor product $|\Psi(\alpha')\rangle \otimes |F m_F\rangle$, where

$$|F m_F\rangle = \sum_{m_J m_I} \langle I m_I J m_J | F m_F \rangle |I m_I\rangle \otimes |J m_J\rangle. \quad (9)$$

Here, the kets $|F m_F\rangle$, $|J m_J\rangle$, and $|I m_I\rangle$ are associated with the angular momenta $\mathbf{F}$, $\mathbf{J}$, and $\mathbf{I}$, respectively. These states also depend on other quantum numbers that are suppressed in the notation. For example, the ket $|F m_F\rangle$ depends on J, on I, and also on other quantum numbers established by the couplings of the orbital angular momenta and spins of the component particles to form $\mathbf{F}$. For later use, it is also convenient to introduce the notation $|\alpha' m_J\rangle = |\Psi(\alpha')\rangle \otimes |J m_J\rangle$ and $|\alpha' m_I\rangle = |\Psi(\alpha')\rangle \otimes |I m_I\rangle$. Under these assumptions, we can expand the factors A_{j0} appearing in Eq. (8) in the form

$$A_{j0} = C_j^e W_{j0}^e + C_j^p W_{j0}^p + C_j^n W_{j0}^n, \quad (10)$$

where $C_j^w(FJI)$ are weights for the quantities $W_{j0}^w(\alpha')$ containing the expectation values of δh_w^{atom} with respect to the states $|\alpha' m_J\rangle$ and $|\alpha' m_I\rangle$.

The analytical expressions for the factors $C_j^w(FJI)$ in terms of Clebsch-Gordan coefficients are

$$C_j^n = C_j^p = \sum_{m_J m_I} \frac{\langle I m_I J m_J | FF \rangle^2}{\langle FF j0 | FF \rangle} \langle I m_I j0 | I m_I \rangle,$$

$$C_j^e = \sum_{m_J m_I} \frac{\langle I m_I J m_J | FF \rangle^2}{\langle FF j0 | FF \rangle} \langle J m_J j0 | J m_J \rangle. \quad (11)$$

Their numerical values can be obtained for any given allowed values of F, J, and I. Some of the properties of $C_j^w(FJI)$ are induced by features of the Clebsch-Gordan coefficients. For example, $C_0^w = 1$ for any values of F, J, and I because $\langle Km_K 00|Km_K \rangle = 1$ for K equal to F, J, or I and because $\sum_{m_J m_I} \langle Im_I Jm_J|FF \rangle^2 = 1$. As another example, $C_j^e(FJI) = 0$ whenever $j > 2J$ and $C_j^p(FJI) = C_j^n(FJI) = 0$ whenever $j > 2I$ because $\langle Km_K j0|Km_K \rangle = 0$ if $j > 2K$.

The explicit relationships between the expectation values of the perturbations δh_w^{atom} and the $W_{j0}^w(\alpha')$ in Eq. (10) can be written as

$$\langle \alpha' m_J|\delta h_e^{\text{atom}}|\alpha' m_J \rangle = \sum_j W_{j0}^e \langle Jm_J j0|Jm_J \rangle,$$

$$\langle \alpha' m_I|\delta h_p^{\text{atom}}|\alpha' m_I \rangle = \sum_j W_{j0}^p \langle Im_I j0|Im_I \rangle,$$

$$\langle \alpha' m_I|\delta h_n^{\text{atom}}|\alpha' m_I \rangle = \sum_j W_{j0}^n \langle Im_I j0|Im_I \rangle. \quad (12)$$

These expressions are based on using the Wigner-Eckart theorem, which is valid because the single-particle operators in the spherical decomposition of δh^{atom} transform as spherical operators with respect to rotations generated by $\boldsymbol{I}$ and $\boldsymbol{J}$. The W_{j0}^w are combinations of coefficients for Lorentz violation with expectation values of the one-particle operators in Eqs. (3) and (4). The combinations take the form

$$W_{j0}^w = \sum_{k=j-1}^{\infty} (\mathcal{T}_{wkj0}^{\text{NR}(0B)} \Lambda_{wkj}^{(0B)} + \mathcal{T}_{wkj0}^{\text{NR}(1B)} \Lambda_{wkj}^{(1B)})$$
$$+ \sum_{k=j}^{\infty} \mathcal{V}_{wkj0}^{\text{NR}} \Lambda_{wkj}^{(0E)}, \quad (13)$$

where the indicated restrictions of the values of k in the sums originate in the properties of the nonrelativistic coefficients provided in Table IV of Ref. [35]. Generic expressions for the quantities $\Lambda_{wkj}^{(qP)}$ can be found in terms of expectation values of the states $|\alpha' m_J \rangle$ and $|\alpha' m_I \rangle$. For the electron operators, we have

$$\Lambda_{ekj}^{(0B)} = -\sum_{a=1}^{N_e} \frac{\langle \alpha' J||\boldsymbol{p}_a|^k{}_0 Y_{j0}(\hat{\boldsymbol{p}}_a)\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{p}}_a|\alpha' J \rangle}{\langle JJ|j0JJ \rangle},$$

$$\Lambda_{ekj}^{(1B)} = \sum_{a=1}^{N_e} \frac{\langle \alpha' J||\boldsymbol{p}_a|^k{}_{+1} Y_{j0}(\hat{\boldsymbol{p}}_a)\boldsymbol{\sigma}_a \cdot (\hat{\boldsymbol{\epsilon}}_{+a} + \hat{\boldsymbol{\epsilon}}_{-a})|\alpha' J \rangle}{\langle JJ|j0JJ \rangle},$$

$$\Lambda_{ekj}^{(0E)} = -\sum_{a=1}^{N_e} \frac{\langle \alpha' J||\boldsymbol{p}_a|^k{}_0 Y_{j0}(\hat{\boldsymbol{p}}_a)|\alpha' J \rangle}{\langle JJ|j0JJ \rangle}, \quad (14)$$

where the sum on a ranges over the electrons in the atom. For the nucleon operators, we find

$$\Lambda_{wkj}^{(0B)} = -\sum_{a=1}^{N_w} \frac{\langle \alpha' I||\boldsymbol{p}_a|^k{}_0 Y_{j0}(\hat{\boldsymbol{p}}_a)\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{p}}_a|\alpha' I \rangle}{\langle II|j0II \rangle},$$

$$\Lambda_{wkj}^{(1B)} = \sum_{a=1}^{N_w} \frac{\langle \alpha' I||\boldsymbol{p}_a|^k{}_{+1} Y_{j0}(\hat{\boldsymbol{p}}_a)\boldsymbol{\sigma}_a \cdot (\hat{\boldsymbol{\epsilon}}_{+a} + \hat{\boldsymbol{\epsilon}}_{-a})|\alpha' I \rangle}{\langle II|j0II \rangle},$$

$$\Lambda_{wkj}^{(0E)} = -\sum_{a=1}^{N_w} \frac{\langle \alpha' I||\boldsymbol{p}_a|^k{}_0 Y_{j0}(\hat{\boldsymbol{p}}_a)|\alpha' I \rangle}{\langle II|j0II \rangle}, \quad (15)$$

where the sum on a ranges over all particles with flavors $w \in \{p, n\}$.

Explicit determination of the nonvanishing expectation values in Eqs. (14) and (15) requires models for the electronic states and for the nuclear states, as discussed below in Secs. II C and II D, respectively. However, certain components of $\Lambda_{wkj}^{(qP)}$ vanish. We saw above that only coefficients with even values of k can contribute due to parity invariance. This implies that $\Lambda_{wkj}^{(qP)}(\alpha')$ vanishes if k is even. For $\Lambda_{wkj}^{(0B)}$ and $\Lambda_{wkj}^{(1B)}$ it follows that j must be odd, while for $\Lambda_{wkj}^{(0E)}$ we find j must be even. These results are a consequence of the relationships between the indices k and j of the nonrelativistic coefficients, as listed in Table IV of Ref. [35].

Collecting the results discussed in this section yields a set of constraints determining which coefficients for Lorentz violation can affect the shift of an energy level in an atom or ion. Table I compiles information about the nonrelativistic spherical coefficients that can contribute to spectral shifts. The first column of the table lists the coefficients, which we denote generically by $\mathcal{K}_{wkjm}^{\text{NR}}$. The flavor of the operator associated to the coefficient is specified in the second column. The third column gives the angular momenta K that restrict the values of the j index on the coefficient according to the constraint $2K \geq j$. For electron coefficients these angular momenta are the total angular momentum F of the system and the total angular momentum J of the electronic shells, while for nucleon coefficients they are F and the nuclear spin I. The next two columns provide conditions on the values of j for the cases of integer K and of half-integer K. The value of j must be even for coefficients in the first two rows and odd for other coefficients. This can constrain the maximum allowed value of j. For example, for even j and half-integer K the equality in the condition $2K \geq j$ cannot be satisfied because $2K$ is odd, so the maximum allowed value of j is $2K - 1$. The final column in the table displays the allowed values of k. Note that the appearance of a coefficient in the table is necessary but not sufficient for it to contribute to a theoretical energy shift because some $\Lambda_{wkj}^{(qP)}$ may vanish for other reasons when a particular model is used to compute the expectation values.

TABLE I. Contributing nonrelativistic spherical coefficients.

$\mathcal{K}_{wkj0}^{\text{NR}}$	w	K	j values, integer K	j values, half-integer K	k values
$\mathcal{V}_{wkj0}^{\text{NR}}$, a_{wkj0}^{NR}, c_{wkj0}^{NR}	e	F, J	even, $2K \geq j \geq 0$	even, $2K-1 \geq j \geq 0$	even, $k \geq j$
$\mathcal{V}_{wkj0}^{\text{NR}}$, a_{wkj0}^{NR}, c_{wkj0}^{NR}	n, p	F, I	even, $2K \geq j \geq 0$	even, $2K-1 \geq j \geq 0$	even, $k \geq j$
$\mathcal{T}_{wkj0}^{\text{NR}(0B)}$, $g_{wkj0}^{\text{NR}(0B)}$, $H_{wkj0}^{\text{NR}(0B)}$	e	F, J	odd, $2K-1 \geq j \geq 1$	odd, $2K \geq j \geq 1$	even, $k \geq j-1$
$\mathcal{T}_{wkj0}^{\text{NR}(0B)}$, $g_{wkj0}^{\text{NR}(0B)}$, $H_{wkj0}^{\text{NR}(0B)}$	n, p	F, I	odd, $2K-1 \geq j \geq 1$	odd, $2K \geq j \geq 1$	even, $k \geq j-1$
$\mathcal{T}_{wkj0}^{\text{NR}(1B)}$, $g_{wkj0}^{\text{NR}(1B)}$, $H_{wkj0}^{\text{NR}(1B)}$	e	F, J	odd, $2K-1 \geq j \geq 1$	odd, $2K \geq j \geq 1$	even, $k \geq j-1$
$\mathcal{T}_{wkj0}^{\text{NR}(1B)}$, $g_{wkj0}^{\text{NR}(1B)}$, $H_{wkj0}^{\text{NR}(1B)}$	n, p	F, I	odd, $2K-1 \geq j \geq 1$	odd, $2K \geq j \geq 1$	even, $k \geq j-1$

C. Electron expectation values

In this section, the calculation of the electronic expectation values (14) is discussed. The situation where F or J vanish is considered first. We then outline an approach to more complicated cases that is general enough to cover systems of interest here while yielding a sufficient approximation to the effects of Lorentz and *CPT* violation. This involves modeling the electromagnetic interaction between the electrons and the nucleus via a central Coulomb potential and treating the repulsion between the electrons using a mean-field approximation. The approach provides enough information about the states $|\alpha' m_J\rangle$ to permit a reasonable estimation of the perturbative energy shift due to Lorentz- and *CPT*-violating effects on the electron propagators.

1. Case F = 0 or J = 0

The ground states of many atoms and ions considered in this work have quantum numbers $F = 0$ or $J = 0$. For example, this holds for the ground state of any noble gas and any IIB transition metal such as Hg. It also holds for the ground states of many ions of interest, including ^{27}Al$^+$, ^{113}Cd$^+$, ^{115}In$^+$, ^{171}Yb$^+$, and ^{199}Hg$^+$. The excited states of some systems of interest also have these quantum numbers, such as the P_0 state in ^{27}Al$^+$ or ^{115}In$^+$.

If either of the quantum numbers F or J vanishes, then the electron coefficients for Lorentz violation that can contribute to the energy shift must have $j = 0$. These coefficients control isotropic Lorentz- and *CPT*-violating effects. The discussion in Sec. II B reveals that the only relevant isotropic coefficients for electrons are $\mathcal{V}_{ek00}^{\text{NR}}$. These special coefficients are commonly denoted as $\overset{\circ}{\mathcal{V}}_{e,k}^{\text{NR}}$, where $\overset{\circ}{\mathcal{V}}_{e,k}^{\text{NR}} \equiv \mathcal{V}_{ek00}^{\text{NR}}/\sqrt{4\pi}$.

Since only $\overset{\circ}{\mathcal{V}}_{e,k}^{\text{NR}}$ can affect the energy shift, the quantities $\Lambda_{ekj}^{(0B)}$ and $\Lambda_{ekj}^{(1B)}$ cannot contribute to Eq. (13) and so become irrelevant. Moreover, when only the isotropic coefficients for electrons can provide nonvanishing contributions, we can simplify the expression (14) for $\Lambda_{ek0}^{(0E)}$. The values of the relevant Clebsch-Gordan coefficient and spherical harmonic are $\langle JJ00|JJ\rangle = 1$ and $_0Y_{00} = 1/\sqrt{4\pi}$. This yields

$$\Lambda_{ek0}^{(0E)} = -\sum_{a=1}^{N_e} \frac{1}{\sqrt{4\pi}} \langle |\boldsymbol{p}_a|^k \rangle, \quad (16)$$

where the sum on a ranges over all the electrons in the atom. The quantities $\langle |\boldsymbol{p}_a|^k \rangle$ are the expectation values of powers of the momentum magnitude.

Denote the contribution to the energy shift due to the electron isotropic coefficients by $\delta\overset{\circ}{\epsilon}_e$. Recalling that $C_0^w = 1$ for any value of F, J, and I, it follows from Eqs. (8) and (10) that $\delta\overset{\circ}{\epsilon}_e$ takes the simple form $\delta\overset{\circ}{\epsilon}_e = W_{00}^e$. Using Eqs. (13) and (16) then yields an expression for the energy shift due to the electron isotropic coefficients,

$$\delta\overset{\circ}{\epsilon}_e = -\sum_q \overset{\circ}{\mathcal{V}}_{e,2q}^{\text{NR}} \sum_{a=1}^{N_e} \langle |\boldsymbol{p}_a|^{2q} \rangle, \quad (17)$$

where the index q is related to the index k of the coefficients for Lorentz violation by $2q = k$. This enforces the condition that only even values of k can contribute to the energy shift.

2. One open subshell with one electron

For atoms or ions with all electronic subshells closed except for a single subshell occupied by one electron, we can find closed-form expressions for the expectation values $\Lambda_{ekj}^{(qP)}$ under suitable simplifying approximations. Treating the electrons in the closed subshells as forming states with zero total angular momentum, the value of J for the whole system can be identified with the total angular momentum of the electron in the open subshell. It follows that the only contribution to $\Lambda_{ekj}^{(qP)}$ with $j \neq 0$ can arise from the valence electron. Contributions from isotropic coefficients with $j = 0$ are given by Eq. (17).

The closed shells produce a spherically symmetric electronic distribution. For present purposes, the effective potential acting on the valence electron due to the repulsion from the closed-shell electrons can be approximated as central. One consequence of this is that the magnitude L of the orbital angular momentum $\boldsymbol{L}$ of the valence electron is a good quantum number for the system. It then becomes

feasible to obtain explicit expressions for the quantities $\Lambda_{ekj}^{(qP)}$ defined in Eq. (14). We find

$$\Lambda_{ekj}^{(0E)} = i^j \frac{(j-1)!!}{j!!} \frac{M_j^J}{\Lambda_j^J} \langle |\boldsymbol{p}|^k \rangle,$$

$$\Lambda_{ekj}^{(0B)} = i^{j-1} \frac{j!!}{(j-1)!!} M_j^J \Lambda_j^J \langle |\boldsymbol{p}|^k \rangle,$$

$$\Lambda_{ekj}^{(1B)} = i^{j-1} \frac{2J+1}{L-J} M_j^J \Lambda_j^J \sqrt{\frac{j!!(j-2)!!}{2(j+1)!!(j-1)!!}} \langle |\boldsymbol{p}|^k \rangle.$$

$$(18)$$

In this equation, $\boldsymbol{p}$ is the momentum of the valence electron, and we define

$$M_j^J = \sqrt{\frac{2j+1}{4\pi(2J+1)}},$$

$$\Lambda_j^J = \sqrt{\frac{(2J+j)!!(2J-j-1)!!}{(2J-j)!!(2J+j+1)!!}}. \qquad (19)$$

Note that the spin-independent operators in Eq. (3) transform as spherical operators with respect to rotations generated by $\boldsymbol{L}$, which suffices to exclude contributions from $\mathcal{V}_{ekjm}^{\mathrm{NR}}$ to the energy shift when $j > 2L$. However, this requirement is already implied in the present context by the condition $2J - 1 \geq j$ presented in Table I, because the lowest value of L for a given J is $L = J - 1/2$.

The results (18) can be applied to alkali-metal atoms and to singly ionized alkaline-earth ions. In both cases, the electrons in the closed subshells belong to closed shells, so the approximations made above are comparatively good. This can be illustrated by comparing our results with detailed calculations for specific states of particular systems. For example, consider the numerical results presented in Table 1 of Ref. [55] for the $D_{3/2}$ and $D_{5/2}$ states in Ca$^+$, Ba$^+$, and Yb$^+$. The table provides the reduced matrix elements of the operator

$$T_0^{(2)} = -\sqrt{\frac{16\pi}{5}} |\boldsymbol{p}|_0^2 Y_{20}(\hat{\boldsymbol{p}}), \qquad (20)$$

calculated using several many-body techniques and defined in terms of Wigner 3-j symbols instead of Clebsch-Gordan coefficients. The ratio of the reduced matrix elements for the two states $D_{3/2}$ and $D_{5/2}$ is 0.77 for Ca$^+$ and 0.79 for Ba$^+$. Converting the notation appropriately, we find that Eq. (18) predicts a ratio of 0.76 for both systems, in reasonable agreement with the many-body calculations. However, for Yb$^+$ the results obtained in Ref. [55] give a ratio of 0.82, revealing a greater deviation from our prediction. This is unsurprising because in this ion some electrons in a closed subshell lie outside the closed shells, so the accuracy of our approximation is expected to be reduced.

The results (17) and (18) involve expectation values of powers of the magnitude of the electron momentum. An analytical evaluation of these expectation values is impractical, even for comparatively simple cases such as the expectation values $\langle |\boldsymbol{p}|^k \rangle$ for a valence electron. Numerical methods can be adopted to resolve this issue, in conjunction with techniques such as a self-consistent mean-field approximation. However, the principle goal of this work is to serve as a guide to search for Lorentz and *CPT* violation. In this context, a precise determination of these expectation values is often inessential. For example, some transitions studied here are hyperfine or Zeeman transitions. These involve two levels with similar momentum expectation values, and the difference leaves unaffected the qualitative form of experimental signals for Lorentz and *CPT* violation. For these and many other transitions, estimates of the expectation values of the electron momentum suffice as a guide to the sensitivity of experiments across a broad range of systems. An accurate determination of the expectation values relevant to a given experimental setup may become useful once enough data are collected and a detailed analysis is being performed to extract the coefficients for Lorentz violation. For a few transitions used in experiments, estimates may be inadequate even as a guide to the sensitivity. For example, for optical transitions the difference between the expectation values in the two states can be significant and must be included in the treatment, as described in Sec. III C below.

For atoms or ions with more than one electron in an open subshell, it is typically infeasible to obtain closed-form expressions like Eqs. (17) and (18). These systems can have substantial many-body effects, and their treatment requires a more sophisticated and individualized approach. Investigations of such systems are likely also to be of interest in searches for Lorentz and *CPT* violation, but a discussion along these lines lies beyond the scope of this work.

D. Nucleon expectation values

Next, we turn to the evaluation of the nucleon expectation values (15). The simplest situation arises when F or I vanishes, for which a compact expression for the energy shift can be presented. For more complicated scenarios, a model accounting for the strong nuclear interactions is required. The central effective potential and mean-field approximation used above for the electronic structure are inappropriate to describe the nucleon interactions. Instead, we adopt here a simple nuclear shell model that permits analytical evaluation of the quantities $\Lambda_{wkj}^{(qP)}$. This enables an evaluation of the effects of Lorentz and *CPT* violation from nucleon propagators on spectral shifts in a broad range of systems.

1. Case F = 0 or I = 0

A number of atoms or ions have either vanishing total angular momentum $F = 0$ or vanishing nuclear spin $I = 0$.

The latter situation arises in nuclei with an even number of neutrons and an even number of protons. In these cases, independently of the nuclear model adopted, the energy shift $\delta\epsilon_w$ due to a nucleon of flavor $w = p$ or n receives contributions only from isotropic coefficients for Lorentz violation. The arguments here parallel those in Sec. II C 1.

Introducing the special isotropic coefficients $\overset{\circ}{\mathcal{V}}{}^{\mathrm{NR}}_{w,k} \equiv \mathcal{V}^{\mathrm{NR}}_{wk00}/\sqrt{4\pi}$, the expression for $\delta\overset{\circ}{\epsilon}_w$ is found to be

$$\delta\overset{\circ}{\epsilon}_w = -\sum_{q=0}^{N_w} \overset{\circ}{\mathcal{V}}{}^{\mathrm{NR}}_{w,2q} \sum_{a=1}^{N_w} \langle |\boldsymbol{p}_a|^{2q}\rangle, \qquad (21)$$

where the sum over a spans the N_w nucleons of flavor w in the nucleus. Like the electron case, this isotropic shift can also affect other energy levels having $F \neq 0$ or $I \neq 0$ through its contribution to Eq. (13).

2. Schmidt model for one unpaired nucleon

The Schmidt model [56,57] offers a comparatively simple description of a broad range of nuclei. The model assumes a shell structure for the nucleus in which any pair of nucleons of a given flavor combines to form states with total angular momentum equal to zero. If only one unpaired nucleon exists in the nucleus, then it is treated as a single-particle state with total angular momentum equal to the spin I of the nucleus. The magnitude L of the orbital angular momentum of the unpaired nucleon is treated as a good quantum number. The model can be expected to deviate significantly from observation for nuclei lying away from a magic number.

Mathematically, the Schmidt model represents a setup equivalent to the one described in Sec. II C 2 for a valence electron outside closed subshells. The contribution to the perturbative energy shift involving isotropic coefficients is obtained from Eq. (21). When $j > 0$, the expressions for the quantities $\Lambda^{(qP)}_{wkj}$ can be calculated in closed form and are given by

$$\Lambda^{(0E)}_{wkj} = i^j \frac{(j-1)!!}{j!!} \frac{M^I_j}{\Lambda^I_j} \langle |\boldsymbol{p}|^k\rangle,$$

$$\Lambda^{(0B)}_{wkj} = i^{j-1} \frac{j!!}{(j-1)!!} M^I_j \Lambda^I_j \langle |\boldsymbol{p}|^k\rangle,$$

$$\Lambda^{(1B)}_{wkj} = i^{j-1} \frac{2I+1}{L-I} M^I_j \Lambda^I_j \sqrt{\frac{j!!(j-2)!!}{2(j+1)!!(j-1)!!}} \langle |\boldsymbol{p}|^k\rangle,$$

$$(22)$$

where $\boldsymbol{p}$ is the linear momentum of the unpaired nucleon of flavor w. The factors M^I_j and Λ^I_j are defined as

$$M^I_j = \sqrt{\frac{2j+1}{4\pi(2I+1)}},$$

$$\Lambda^I_j = \sqrt{\frac{(2I+j)!!(2I-j-1)!!}{(2I-j)!!(2I+j+1)!!}}. \qquad (23)$$

The primary advantage of the Schmidt model in the present context is its application to a broad range of systems for which the quantities $\Lambda^{(qP)}_{wkj}$ can be evaluated using Eq. (22). The model has previously been used to determine signals arising from Lorentz- and *CPT*-violating operators in the minimal SME for numerous experiments comparing atomic or ionic transitions [9]. A significant limitation of the Schmidt model in this respect is that only one flavor of nucleon is assumed to contribute to transitions in any given atom or ion, which implies the corresponding experiment is sensitive only to coefficients for Lorentz violation in that flavor sector. A more realistic treatment can be expected to reveal dependence on coefficients for both values of w. This was illustrated in Ref. [9] using more detailed wave functions for the nuclei of ⁷Li and ⁹Be. Recently, calculations using semiempirical models [32] and many-body methods [58] have obtained improved values for the coefficients $\Lambda^{(qP)}_{wkj}$, particularly for the coefficient $\Lambda^{(0E)}_{w22}$. These improved values emphasize the disadvantages of using a single-valence model to study Lorentz- and *CPT*-violating effects involving the nucleus. Nonetheless, to maintain generality in this work and to permit the discussion of a broad range of atoms and ions, we adopt the Schmidt model throughout, commenting where appropriate on the likely consequences of using improved nuclear modeling. We remark also that it suffices to estimate the expectation values of the magnitude of the linear nucleon momentum for all experiments considered here because no nuclear transitions are involved.

E. Energy shift at zeroth boost order

In any Cartesian inertial frame within the solar system, the coefficients for Lorentz violation can reasonably be taken as constant in both time and space [5,6]. However, the energy shift (8) is calculated in a laboratory frame. Laboratories on the surface of the Earth or on orbiting satellites generically correspond to noninertial frames, so most coefficients appearing in Eq. (8) vary with time due to the laboratory rotation and boost [59]. Moreover, the explicit forms of the coefficients for Lorentz violation differ in distinct inertial frames. To permit meaningful comparison of different experiments, experimental coefficient values must therefore be reported in a canonical inertial frame. The standard frame adopted in the literature for this purpose is the Sun-centered celestial-equatorial frame [53], with Cartesian coordinates denoted (T, X, Y, Z). In this frame, the origin of the time coordinate T is defined as the vernal equinox 2000. The X axis points from the location of

the Earth at this equinox to the Sun, the Z axis is aligned with the Earth's rotation axis, and the X, Y, Z axes form a right-handed coordinate system. The Sun-centered frame is appropriate for reporting measurements of coefficients because it is inertial to an excellent approximation over the time scale of typical experiments.

The observer Lorentz transformation $\Lambda^\mu{}_\nu(\boldsymbol{\theta}, \boldsymbol{\beta})$ between the laboratory frame and the Sun-centered frame can be expressed as the composition of an observer rotation $\mathcal{R}^\mu{}_\nu(\boldsymbol{\theta})$ with an observer boost $\mathcal{B}^\mu{}_\nu(\boldsymbol{\beta})$,

$$\Lambda^\mu{}_\nu(\boldsymbol{\theta}, \boldsymbol{\beta}) = \mathcal{R}^\mu{}_\alpha(\boldsymbol{\theta}) \mathcal{B}^\alpha{}_\nu(\boldsymbol{\beta}). \qquad (24)$$

The boost parameter $\boldsymbol{\beta}$ is the velocity of the laboratory frame with respect to the Sun-centered frame, while the rotation parameter $\boldsymbol{\theta}$ fixes the relative orientation between the laboratory frame and the frame obtained via the boost. The magnitude β of $\boldsymbol{\beta}$ is small compared to the speed of light. For example, the speed of the Earth in the Sun-centered frame in natural units is $\beta \simeq 10^{-4}$. At zeroth order in β the boost transformation is simply the identity map, so the Lorentz transformation between the two frames becomes a pure rotation. In this section, we consider the energy shift (8) at zeroth boost order. Effects at linear boost order are discussed in Sec. II F.

In the laboratory frame, only the nonrelativistic coefficients $\mathcal{K}^{\mathrm{NR,lab}}_{wkjm}$ for Lorentz and CPT violation with $m = 0$ contribute to the energy shift. At zeroth boost order and for a laboratory on the Earth, these coefficients can be converted to coefficients $\mathcal{K}^{\mathrm{NR,Sun}}_{wkjm}$ in the Sun-centered frame via

$$\mathcal{K}^{\mathrm{NR,lab}}_{wkj0} = \sum_m e^{im\omega_\oplus T_L} d^j_{0m}(-\vartheta) \mathcal{K}^{\mathrm{NR,Sun}}_{wkjm}. \qquad (25)$$

Here, ϑ is the angle between the applied magnetic field and the Earth's rotation axis Z, and the quantities $d^j_{mm'}$ are the little Wigner matrices given in Eq. (136) of Ref. [50]. The conversion (25) reveals the time variations of the laboratory-frame coefficients, which occur at harmonics of the Earth's sidereal frequency $\omega_\oplus \simeq 2\pi/(23\,\mathrm{h}\,56\,\mathrm{min})$. The local sidereal time T_L is a convenient local Earth sidereal time with origin chosen as the time when the magnetic field lies in the XZ plane in the Sun-centered frame. This choice yields the comparatively simple expression (25). For some applications below it is preferable instead to adopt a different local sidereal time $T_\oplus$, which is associated with the laboratory frame introduced in Ref. [53] and has as origin the time at which the tangential velocity of the laboratory frame points along the Y axis. The relationship between T_L and $T_\oplus$ is

$$\omega_\oplus T_\oplus = \omega_\oplus T_L - \varphi, \qquad (26)$$

where φ is the angle between the X axis and the projection of the magnetic field on the XY plane at $T_\oplus = 0$. Note that both T_L and $T_\oplus$ are offset from the standard time T in the Sun-centered frame by an amount that depends on the longitude of the laboratory, given explicitly for $T_\oplus$ in Eq. (43) of Ref. [45].

The factors $A_{j0} \equiv A^{\mathrm{lab}}_{j0}$ appearing in Eq. (8) are defined in the laboratory frame. They transform in the same way under rotations as the nonrelativistic coefficients for Lorentz violation, so we can convert them to factors A^{Sun}_{j0} defined in the Sun-centered frame via the relation

$$A^{\mathrm{lab}}_{j0} = \sum_m e^{im\omega_\oplus T_\oplus} d^j_{0m}(-\vartheta) A^{\mathrm{Sun}}_{jm}. \qquad (27)$$

The energy shift (8) can therefore be expressed in the Sun-centered frame as

$$\begin{aligned}
\delta\epsilon = &\sum_{jm} d^{(j)}_{0|m|}(-\vartheta) \langle Fm_F j0|Fm_F\rangle \\
&\times [\mathrm{Re} A^{\mathrm{Sun}}_{j|m|} \cos(|m|\omega_\oplus T_\oplus) \\
&- \mathrm{Im}^{\mathrm{Sun}}_{j|m|} \sin(|m|\omega_\oplus T_\oplus)],
\end{aligned} \qquad (28)$$

thereby explicitly demonstrating the time variation of the spectrum of the atom or ion at harmonics of the sidereal frequency $\omega_\oplus$.

For any m, a given factor $A^{\mathrm{Sun}}_{j|m|}$ contains coefficients for Lorentz violation labeled with the same index j. However, as summarized in Table I, only restricted values of j for nonrelativistic coefficients can contribute to a specific energy shift. Since the highest harmonic that can contribute to the sidereal variation is determined by the maximum value of $|m|$, which in turn is fixed by the largest allowed value of the index j, we can use the information in Table I to deduce constraints on the possible harmonics contributing to the time variation of any particular energy level. Table II summarizes these constraints for various conditions on the quantum numbers F, J, and I. The first column of the table lists the conditions, while the second column displays the range of allowed values of $|m|$, which corresponds to the possible harmonics of $\omega_\oplus$ that can appear. For example, the first row of the table shows that the time variation of an energy level with quantum numbers $F = 3$, $I = 7/2$, and $J = 3/2$ can in principle contain up to the sixth harmonic of $\omega_\oplus$. Note, however, that special

TABLE II. Possible harmonics of $\omega_\oplus$ at zeroth boost order.

Conditions on F, J, I	Possible harmonics
$F \leq J$ or $F \leq I$ or both	$2F \geq |m| \geq 0$
$F \geq J$, $F \geq I$, $J \geq I$	$2J \geq |m| \geq 0$
$F \geq J$, $F \geq I$, $I \geq J$	$2I \geq |m| \geq 0$

circumstances might contrive to lower the maximum harmonic affecting a given transition frequency. For example, a factor $A_{j|m|}^{\text{Sun}}$ might vanish identically, or the two energy levels involved in the transition might have identical contributions at a particular harmonic so that the transition frequency is unaffected. Note also that time variations at higher harmonics than those displayed in Table II can become allowed when effects at linear or higher order in the boost are incorporated, but any such variations are suppressed by powers of the boost.

F. Energy shift at linear boost order

Since the magnitude of the boost between the laboratory frame and the Sun-centered frame is small, it is reasonable to expand the boost transformation $\mathcal{B}^{\mu}{}_{\nu}(\boldsymbol{\beta})$ of Eq. (24) in powers of the relative speed β. In this section, we consider contributions to the energy shift that appear at linear order in the boost. At this order, the components of the observer Lorentz transformation $\Lambda^{\mu}{}_{\nu}(\boldsymbol{\theta},\boldsymbol{\beta})$ take the form

$$\Lambda^{0}{}_{T}=1, \quad \Lambda^{0}{}_{J}=-\boldsymbol{\beta}^{J}, \quad \Lambda^{j}{}_{T}=-\mathcal{R}^{j}{}_{J}\boldsymbol{\beta}^{J}, \quad \Lambda^{j}{}_{J}=\mathcal{R}^{j}{}_{J}, \tag{29}$$

where lowercase and uppercase indices represent spatial Cartesian coordinates in the laboratory frame and in the Sun-centered frame, respectively.

Given an expression for the energy shift in the laboratory frame in terms of spherical coefficients for Lorentz violation, converting to the Sun-centered frame at linear boost order can be performed in two steps. First, the spherical coefficients in the laboratory frame can be rewritten in terms of Cartesian coefficients in the same frame. The transformation (29) can then be applied to express the Cartesian coefficients in the laboratory frame in terms of Cartesian coefficients in the Sun-centered frame.

Explicit expressions relating spherical coefficients to Cartesian coefficients in any inertial frame are given in Appendix A. To implement the conversion to the Sun-centered frame, note that only spherical coefficients for Lorentz violation with $m = 0$ contribute to the energy shift (8) in the laboratory frame. This implies that all uncontracted spatial indices on the corresponding Cartesian coefficients are in the x^3 direction. The relevant part of the rotation matrix $\mathcal{R}^{j}{}_{J}$ converting the Cartesian components between the laboratory frame and the Sun-centered frame therefore involves the row $\mathcal{R}^{3}{}_{J}$. This row can be viewed as the components of a unit vector along x^3. Since by construction this is the quantization axis of the atom or ion, which points in the direction $\hat{\boldsymbol{B}}$ of the applied magnetic field, it follows that $\mathcal{R}^{3}{}_{J} = \hat{B}^{J}$.

To illustrate this idea with an example, consider the spherical coefficient $g_{210}^{(4)(1B)}$ given in the laboratory frame. From Appendix A we see that the spherical coefficient

$g_{210}^{(4)(1B)}$ is proportional to the combination $\tilde{g}_{\text{eff}}^{(4)j3j}$ of Cartesian coefficients in the laboratory frame. This combination can be converted to Cartesian coefficients $\tilde{g}_{\text{eff,rot}}^{(4)\mu\nu\alpha}$ in the rotated frame as

$$\tilde{g}_{\text{eff}}^{(4)j3j} = \mathcal{R}^{j}{}_{K}\mathcal{R}^{3}{}_{L}\mathcal{R}^{j}{}_{M}\tilde{g}_{\text{eff,rot}}^{(4)KLM} = \mathcal{R}^{3}{}_{K}\tilde{g}_{\text{eff,rot}}^{(4)JKJ} = \tilde{g}_{\text{eff,rot}}^{(4)JKJ}\hat{B}^{K}, \tag{30}$$

where the second equality follows from the identity $\mathcal{R}^{j}{}_{K}\mathcal{R}^{j}{}_{L} = \delta_{KL}$.

The above discussion shows that the number of factors of $\hat{B}^{J}$ appearing in a given term contributing to the energy shift at linear boost order is determined by the index structure of the corresponding coefficient for Lorentz violation. To keep the explicit tables appearing in Sec. III of reasonable size, we limit attention below to Lorentz- and *CPT*-violating operators of mass dimension $d \leq 8$. Expressions relating all the corresponding Cartesian coefficients in the laboratory frame to those in the Sun-centered frame at linear boost order are given in Appendix B. Inspection of these results reveals that the form of the shift $\delta\nu$ in a transition frequency for an atom or ion takes the generic form

$$\delta\nu = \sum_{d=3}^{8} \sum_{s=0}^{5} V^{(d)JJ_1\ldots J_s}\hat{B}^{J_1}\ldots\hat{B}^{J_s}\boldsymbol{\beta}^{J} \tag{31}$$

at linear boost order, where the quantities $V^{(d)JJ_1\ldots J_s}$ are linear combinations of Cartesian coefficients for Lorentz violation in the Sun-centered frame. The explicit forms of $\boldsymbol{\beta}$ and $\hat{\boldsymbol{B}}$ in this equation depend on the choice of laboratory frame. We consider here in turn two types of laboratory, one located on the surface of the Earth and another on a spacecraft orbiting the Earth.

For a laboratory on the surface of the Earth, the boost velocity $\boldsymbol{\beta}$ in Eq. (31) can be taken as

$$\boldsymbol{\beta} = \boldsymbol{\beta}_{\oplus} + \boldsymbol{\beta}_{L}, \tag{32}$$

where $\boldsymbol{\beta}_{\oplus}$ is the instantaneous Earth orbital velocity in the Sun-centered frame and $\boldsymbol{\beta}_{L}$ is the instantaneous tangential velocity of the laboratory relative to the Earth's rotation axis. Approximating the Earth's orbit as circular, the velocity $\boldsymbol{\beta}_{\oplus}$ can be written as

$$\boldsymbol{\beta}_{\oplus} = \beta_{\oplus}\sin\Omega_{\oplus}T\hat{X} - \beta_{\oplus}\cos\Omega_{\oplus}T(\cos\eta\hat{Y}+\sin\eta\hat{Z}), \tag{33}$$

where $\beta_{\oplus} \simeq 10^{-4}$ is the Earth's orbital speed, $\Omega_{\oplus} \simeq 2\pi/(365.26\text{ d})$ is the Earth's orbital angular frequency, and $\eta \simeq 23.4°$ is the angle between the XY plane and the Earth's orbital plane. Also, treating the Earth as spherical, the tangential velocity $\boldsymbol{\beta}_{L}$ takes the form

$$\boldsymbol{\beta}_L = -\beta_L \sin \omega_\oplus T_\oplus \hat{X} + \beta_L \cos \omega_\oplus T_\oplus \hat{Y}, \qquad (34)$$

where $\beta_L \approx r_\oplus \omega_\oplus \sin \chi$ is determined by the colatitude χ of the laboratory the radius $r_\oplus$ of the Earth, and the sidereal frequency $\omega_\oplus$. The unit vector $\hat{\boldsymbol{B}}$ in Eq. (31) can conveniently be expressed in an instantaneous Earth-centered coordinate system with axes parallel to those of the Sun-centered frame,

$$\hat{\boldsymbol{B}} = \sin \vartheta \cos (\omega_\oplus T_\oplus + \varphi) \hat{X}$$
$$+ \sin \vartheta \sin (\omega_\oplus T_\oplus + \varphi) \hat{Y} + \cos \vartheta \hat{Z}, \qquad (35)$$

where ϑ and φ are the polar and azimuthal angles of the magnetic field at $T_\oplus = 0$.

Next, consider a laboratory located on a space-based platform orbiting the Earth. Examples include experiments on board the International Space Station (ISS) such as the Atomic Clock Ensemble in Space (ACES) [60] and the Quantum Test of the Equivalence Principle and Space Time (QTEST) [61], or dedicated missions searching for Lorentz violation such as the Space-Time Explorer and Quantum Equivalence Space Test (STE-QUEST) [62] and the Boost Symmetry Test (BOOST) [63]. We adopt the coordinates depicted in Fig. 2 of Ref. [64]. Assuming for definiteness a trajectory with negligible eccentricity, the parameters for the orbit are the mean orbital radius r_s, the mean orbital angular speed ω_s, the angle ζ between the satellite orbital axis and the Earth's rotation axis, and the azimuthal angle α between the Earth and satellite orbital planes. In this scenario, the boost velocity $\boldsymbol{\beta}$ in Eq. (31) can be written as the vector sum

$$\boldsymbol{\beta} = \boldsymbol{\beta}_\oplus + \boldsymbol{\beta}_s \qquad (36)$$

of the Earth's orbital velocity $\boldsymbol{\beta}_\oplus$ in the Sun-centered frame and the satellite velocity $\boldsymbol{\beta}_s$ relative to an instantaneous Earth-centered frame. Explicitly, the components of the satellite velocity $\boldsymbol{\beta}_s$ in the Sun-centered frame take the form

$$\boldsymbol{\beta}_s = \begin{pmatrix} -\beta_s \cos \alpha \sin \omega_s T_s - \beta_s \cos \zeta \sin \alpha \cos \omega_s T_s \\ -\beta_s \sin \alpha \sin \omega_s T_s + \beta_s \cos \alpha \cos \zeta \cos \omega_s T_s \\ \beta_s \sin \zeta \cos \omega_s T_s \end{pmatrix}, \qquad (37)$$

where $\beta_s = r_s \omega_s$ and the local satellite time T_s has origin fixed as the satellite crosses the equatorial plane on an ascending orbit. Obtaining also an explicit expression for the unit vector $\hat{\boldsymbol{B}}$ in Eq. (31) requires a further specification of the orientation of the space-based laboratory relative to the Earth. For example, when this orientation is fixed then an instantaneous satellite frame can be defined with x axis pointing radially towards the Earth and z axis aligned along $\boldsymbol{\beta}_s$. The components of the corresponding unit spatial vectors $\hat{\boldsymbol{x}}_s, \hat{\boldsymbol{y}}_s, \hat{\boldsymbol{z}}_s$ can be expressed in the Sun-centered frame as

$$\hat{\boldsymbol{x}}_s = \begin{pmatrix} -\cos \alpha \cos \omega_s T_s + \cos \zeta \sin \alpha \sin \omega_s T_s \\ -\sin \alpha \cos \omega_s T_s - \cos \alpha \cos \zeta \sin \omega_s T_s \\ -\sin \zeta \sin \omega_s T_s \end{pmatrix},$$

$$\hat{\boldsymbol{y}}_s \equiv \frac{\boldsymbol{\beta}_s \times \hat{\boldsymbol{x}}_s}{\beta_s} = \begin{pmatrix} \sin \alpha \sin \zeta \\ -\cos \alpha \sin \zeta \\ \cos \zeta \end{pmatrix},$$

$$\hat{\boldsymbol{z}}_s = \frac{\boldsymbol{\beta}_s}{\beta_s}. \qquad (38)$$

Using this basis, the direction $\hat{\boldsymbol{B}}$ of the magnetic field in the space-based experiment can be expressed as

$$\hat{\boldsymbol{B}} = \sin \theta_s \cos \phi_s \hat{\boldsymbol{x}}_s + \sin \theta_s \sin \phi_s \hat{\boldsymbol{y}}_s + \cos \theta_s \hat{\boldsymbol{z}}_s, \qquad (39)$$

where $\cos \theta_s = \boldsymbol{\beta}_s \cdot \hat{\boldsymbol{B}}/\beta_s$ and $\cos \phi_s = \hat{\boldsymbol{x}}_s \cdot \hat{B}$.

III. APPLICATIONS

In this section, we comment on some applications of the formulas derived above. Many existing searches for Lorentz and *CPT* violation are based on the study of transitions in fountain clocks, in comagnetometers, and in trapped ions or lattice clocks. Each of these experimental approaches is considered in turn. We present expressions relevant to the analysis of data from a variety of experiments, and we estimate the attainable sensitivities to coefficients for Lorentz violation along with actual constraints from existing data where possible.

A. Fountain clocks

Fountain clocks using ^{133}Cs atoms have been widely adopted as primary time and frequency standards. The standard transition in these clocks, $|F = 3, m_F = 0\rangle \leftrightarrow |F = 4, m_F = 0\rangle$, is insensitive to the Lorentz- and *CPT*-violating spectral shifts discussed above. This implies that the ^{133}Cs frequency standard can be used as a reference in experimental studies searching for Lorentz violation, in parallel with the hydrogen-maser standard [43]. Searches for violations of Lorentz and *CPT* symmetry using a ^{133}Cs fountain clock can instead be performed by studying the frequencies ν_{m_F} for transitions $|F = 3, m_F\rangle \leftrightarrow |F = 4, m_F\rangle$ with $m_F \neq 0$. These transitions are individually sensitive to the linear Zeeman shift, and hence their precision is limited by systematic effects. However, the systematics can be significantly reduced by measuring the observable $\nu_c = \nu_{+3} + \nu_{-3} - 2\nu_0$ [16,65].

The total electronic angular momentum for the states with $m_F = \pm 1$ is $J = 1/2$. Consulting Table I reveals that only electron operators with $j \leq 1$ can in principle shift the frequencies $\nu_{\pm 3}$. However, the observable ν_c remains unaffected by these shifts. To evaluate the nucleon contributions to ν_c we adopt the Schmidt model as discussed above, in which the nuclear spin $I = 7/2$ of ^{133}Cs is

assigned to the unpaired proton. With this assumption, the Lorentz-violating shift $\delta\nu_c$ of the observable ν_c is given by

$$2\pi\delta\nu_c = -\frac{3}{14}\sqrt{\frac{5}{\pi}}(\langle|\boldsymbol{p}|^2\rangle\mathcal{V}_{p220}^{\mathrm{NR}} + \langle|\boldsymbol{p}|^4\rangle\mathcal{V}_{p420}^{\mathrm{NR}})$$
$$+ \frac{45}{77\sqrt{\pi}}\langle|\boldsymbol{p}|^4\rangle\mathcal{V}_{p440}^{\mathrm{NR}}, \qquad (40)$$

where $\boldsymbol{p}$ is the momentum of the valence proton. Note that the results presented in Refs. [16,65] based on the minimal SME analysis in Ref. [9] can be recovered from the above expression by excluding contributions from nonminimal Lorentz-violating operators. In practice, this corresponds to making the replacements

$$\mathcal{V}_{p220}^{\mathrm{NR}} \to \frac{1}{3m_p}\sqrt{\frac{4\pi}{5}}(3c_{pzz}^{(4)} - c_{pjj}^{(4)}),$$
$$\mathcal{V}_{p420}^{\mathrm{NR}} \to 0, \qquad \mathcal{V}_{p440}^{\mathrm{NR}} \to 0 \qquad (41)$$

in Eq. (40).

To convert the above expression to the Sun-centered frame, consider first the frequency shift $\delta\nu_{c,0}$ at zeroth boost order. Applying the transformation rule (25) for nonrelativistic coefficients to the result (40) yields a somewhat lengthy form for $\delta\nu_{c,0}$ in the Sun-centered frame. The result is presented in tabular form in Table III. In each row of the table, the first entry contains the harmonic dependence on the sidereal frequency $\omega_\oplus$ and the local sidereal time T_L. The second entry describes the dependence on the orientation of the magnetic field in the laboratory frame. The third entry provides the relevant expectation value of the proton momentum magnitude $|\boldsymbol{p}|$, while the fourth entry contains the numerical factor and the coefficient for Lorentz violation. To obtain the frequency shift $\delta\nu_{c,0}$, it suffices to multiply the columns and add the rows in the table. For example, the contributions to $\delta\nu_{c,0}$ from the first and second rows are

$$-\frac{3}{56}\sqrt{\frac{5}{\pi}}\langle|\boldsymbol{p}|^2\rangle\mathcal{V}_{p220}^{\mathrm{NR,Sun}} - \frac{9}{56}\sqrt{\frac{5}{\pi}}\cos 2\vartheta\langle|\boldsymbol{p}|^2\rangle\mathcal{V}_{p220}^{\mathrm{NR,Sun}}.$$
$$(42)$$

Note that the corresponding expression for $\delta\nu_{c,0}$ in the minimal SME can be obtained by making the replacements

$$\mathcal{V}_{p4jm}^{\mathrm{NR,Sun}} \to 0,$$
$$\mathcal{V}_{p220}^{\mathrm{NR,Sun}} \to -\frac{1}{3m_p^2}\sqrt{\frac{4\pi}{5}}\tilde{c}_{pQ}^{(4)},$$
$$\mathrm{Re}\mathcal{V}_{p221}^{\mathrm{NR,Sun}} \to -\frac{1}{m_p^2}\sqrt{\frac{2\pi}{15}}\tilde{c}_{pY}^{(4)},$$
$$\mathrm{Im}\mathcal{V}_{p221}^{\mathrm{NR,Sun}} \to \frac{1}{m_p^2}\sqrt{\frac{2\pi}{15}}\tilde{c}_{pX}^{(4)},$$
$$\mathrm{Re}\mathcal{V}_{p222}^{\mathrm{NR,Sun}} \to \frac{1}{m_p^2}\sqrt{\frac{2\pi}{15}}\tilde{c}_{p-}^{(4)},$$
$$\mathrm{Im}\mathcal{V}_{p222}^{\mathrm{NR,Sun}} \to -\frac{1}{m_p^2}\sqrt{\frac{2\pi}{15}}\tilde{c}_{pZ}^{(4)} \qquad (43)$$

in the entries in the fourth column of Table III.

Next, consider the contribution to the frequency shift $\delta\nu_{c,1}$ at linear order in the boost. Applying the transformation (29) to Eq. (40) and writing the result in the form (31) yields

$$2\pi\delta\nu_{c,1} = \sum_d \frac{\langle|\boldsymbol{p}|^2\rangle}{m_p^{5-d}}(V_{\mathrm{Cs},2}^{(d)J}\boldsymbol{\beta}^J + V_{\mathrm{Cs},2}^{(d)JKL}\hat{B}^K\hat{B}^L\boldsymbol{\beta}^J)$$
$$+ \sum_d \frac{\langle|\boldsymbol{p}|^4\rangle}{m_p^{7-d}}(V_{\mathrm{Cs},4}^{(d)J}\boldsymbol{\beta}^J + V_{\mathrm{Cs},4}^{(d)JKL}\hat{B}^K\hat{B}^L\boldsymbol{\beta}^J)$$
$$+ \sum_d \frac{\langle|\boldsymbol{p}|^4\rangle}{m_p^{7-d}}V_{\mathrm{Cs},4}^{(d)JKLMN}\hat{B}^K\hat{B}^L\hat{B}^M\hat{B}^N\boldsymbol{\beta}^J. \qquad (44)$$

Expressions for the quantities $V_{\mathrm{Cs},k}^{(d)JJ_1\ldots J_s}$ in terms of the effective Cartesian coefficients can be deduced from the results presented in Appendix B and are displayed in Table IV for mass dimensions $5 \leq d \leq 8$. In each row of this table, the first entry lists a specific quantity $V_{\mathrm{Cs},k}^{(d)JJ_1\ldots J_s}$, while the second entry gives its expression as a combination of effective Cartesian coefficients in the Sun-centered frame.

Note that the minimal-SME limit of the result (44) can be obtained by setting to zero all the quantities $V_{\mathrm{Cs},k}^{(d)JJ_1\ldots J_s}$ except for

$$V_{\mathrm{Cs},2}^{(4)J} = -\frac{2}{7}c_{p\mathrm{eff}}^{(4)TJ}, \qquad V_{\mathrm{Cs},2}^{(4)JKL} = \frac{6}{7}c_{p\mathrm{eff}}^{(4)TK}\delta^{JL}, \qquad (45)$$

where the coefficients $c_{\mathrm{eff}}^{(4)\mu\nu}$ are defined as the symmetric combination $\frac{1}{2}(c^{(4)\mu\nu} + c^{(4)\nu\mu})$, as in Ref. [35]. In contrast to the minimal-SME case, the nonminimal terms introduce sidereal variations incorporating the third, fourth, and fifth harmonics of the sidereal frequency. For example, the contribution to $\delta\nu_{c,1}$ from the fifth harmonic is given by

TABLE III. Frequency shift $\delta\nu_{c,0}$ at zeroth boost order.

$\omega_\oplus T_L$	ϑ	$\langle|\boldsymbol{p}|^k\rangle$	Coefficient
1	1	$\langle|\boldsymbol{p}|^2\rangle$	$-\frac{3}{56}\sqrt{\frac{5}{\pi}}\mathcal{V}_{p\,220}^{\mathrm{NR,Sun}}$
1	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^2\rangle$	$-\frac{9}{56}\sqrt{\frac{5}{\pi}}\mathcal{V}_{p\,220}^{\mathrm{NR,Sun}}$
1	1	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{3}{56}\sqrt{\frac{5}{\pi}}\mathcal{V}_{p\,420}^{\mathrm{NR,Sun}}+\frac{405}{4928\sqrt{\pi}}\mathcal{V}_{p\,440}^{\mathrm{NR,Sun}}$
1	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{9}{56}\sqrt{\frac{5}{\pi}}\mathcal{V}_{p\,420}^{\mathrm{NR,Sun}}+\frac{225}{1232\sqrt{\pi}}\mathcal{V}_{p\,440}^{\mathrm{NR,Sun}}$
1	$\cos 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{225}{704\sqrt{\pi}}\mathcal{V}_{p\,440}^{\mathrm{NR,Sun}}$
$\sin \omega_\oplus T_L$	$\sin 2\vartheta$	$\langle|\boldsymbol{p}|^2\rangle$	$\frac{3}{14}\sqrt{\frac{15}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,221}^{\mathrm{NR,Sun}}$
$\sin \omega_\oplus T_L$	$\sin 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{3}{14}\sqrt{\frac{15}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,421}^{\mathrm{NR,Sun}}-\frac{45}{616}\sqrt{\frac{5}{\pi}}\mathrm{Im}\mathcal{V}_{p\,441}^{\mathrm{NR,Sun}}$
$\sin \omega_\oplus T_L$	$\sin 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{45}{176}\sqrt{\frac{5}{\pi}}\mathrm{Im}\mathcal{V}_{p\,441}^{\mathrm{NR,Sun}}$
$\cos \omega_\oplus T_L$	$\sin 2\vartheta$	$\langle|\boldsymbol{p}|^2\rangle$	$-\frac{3}{14}\sqrt{\frac{15}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,221}^{\mathrm{NR,Sun}}$
$\cos \omega_\oplus T_L$	$\sin 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{3}{14}\sqrt{\frac{15}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,421}^{\mathrm{NR,Sun}}+\frac{45}{616}\sqrt{\frac{5}{\pi}}\mathrm{Re}\mathcal{V}_{p\,441}^{\mathrm{NR,Sun}}$
$\cos \omega_\oplus T_L$	$\sin 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{45}{176}\sqrt{\frac{5}{\pi}}\mathrm{Re}\mathcal{V}_{p\,441}^{\mathrm{NR,Sun}}$
$\sin 2\omega_\oplus T_L$	1	$\langle|\boldsymbol{p}|^2\rangle$	$\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,222}^{\mathrm{NR,Sun}}$
$\sin 2\omega_\oplus T_L$	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^2\rangle$	$-\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,222}^{\mathrm{NR,Sun}}$
$\sin 2\omega_\oplus T_L$	1	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,422}^{\mathrm{NR,Sun}}-\frac{135}{1232}\sqrt{\frac{5}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$
$\sin 2\omega_\oplus T_L$	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,422}^{\mathrm{NR,Sun}}-\frac{45}{308}\sqrt{\frac{5}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$
$\sin 2\omega_\oplus T_L$	$\cos 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{45}{176}\sqrt{\frac{5}{2\pi}}\mathrm{Im}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$
$\cos 2\omega_\oplus T_L$	1	$\langle|\boldsymbol{p}|^2\rangle$	$-\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,222}^{\mathrm{NR,Sun}}$
$\cos 2\omega_\oplus T_L$	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^2\rangle$	$\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,222}^{\mathrm{NR,Sun}}$
$\cos 2\omega_\oplus T_L$	1	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,422}^{\mathrm{NR,Sun}}+\frac{135}{1232}\sqrt{\frac{5}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$
$\cos 2\omega_\oplus T_L$	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{3}{28}\sqrt{\frac{15}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,422}^{\mathrm{NR,Sun}}+\frac{45}{308}\sqrt{\frac{5}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$
$\cos 2\omega_\oplus T_L$	$\cos 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{45}{176}\sqrt{\frac{5}{2\pi}}\mathrm{Re}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$
$\sin 3\omega_\oplus T_L$	$\sin 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{45}{88}\sqrt{\frac{5}{7\pi}}\mathrm{Im}\mathcal{V}_{p\,443}^{\mathrm{NR,Sun}}$
$\sin 3\omega_\oplus T_L$	$\sin 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{45}{176}\sqrt{\frac{5}{7\pi}}\mathrm{Im}\mathcal{V}_{p\,443}^{\mathrm{NR,Sun}}$
$\cos 3\omega_\oplus T_L$	$\sin 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{45}{88}\sqrt{\frac{5}{7\pi}}\mathrm{Re}\mathcal{V}_{p\,443}^{\mathrm{NR,Sun}}$
$\cos 3\omega_\oplus T_L$	$\sin 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{45}{176}\sqrt{\frac{5}{7\pi}}\mathrm{Re}\mathcal{V}_{p\,443}^{\mathrm{NR,Sun}}$
$\sin 4\omega_\oplus T_L$	1	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{135}{352}\sqrt{\frac{5}{14\pi}}\mathrm{Im}\mathcal{V}_{p\,444}^{\mathrm{NR,Sun}}$
$\sin 4\omega_\oplus T_L$	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{45}{88}\sqrt{\frac{5}{14\pi}}\mathrm{Im}\mathcal{V}_{p\,444}^{\mathrm{NR,Sun}}$
$\sin 4\omega_\oplus T_L$	$\cos 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{45}{352}\sqrt{\frac{5}{14\pi}}\mathrm{Im}\mathcal{V}_{p\,444}^{\mathrm{NR,Sun}}$
$\cos 4\omega_\oplus T_L$	1	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{135}{352}\sqrt{\frac{5}{14\pi}}\mathrm{Re}\mathcal{V}_{p\,444}^{\mathrm{NR,Sun}}$
$\cos 4\omega_\oplus T_L$	$\cos 2\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$-\frac{45}{88}\sqrt{\frac{5}{14\pi}}\mathrm{Re}\mathcal{V}_{p\,444}^{\mathrm{NR,Sun}}$
$\cos 4\omega_\oplus T_L$	$\cos 4\vartheta$	$\langle|\boldsymbol{p}|^4\rangle$	$\frac{45}{352}\sqrt{\frac{5}{14\pi}}\mathrm{Re}\mathcal{V}_{p\,442}^{\mathrm{NR,Sun}}$

TABLE IV. The quantities $V_{\mathrm{Cs},k}^{(d)JJ_1\ldots J_s}$ for $5 \leq d \leq 8$.

$V_{\mathrm{Cs},k}^{(d)JJ_1\ldots J_s}$	Combination
$V_{\mathrm{Cs},2}^{(5)J}$	$\frac{3}{7}\left(a_{p\,\mathrm{eff}}^{(5)JKK} + 2a_{p\,\mathrm{eff}}^{(5)JTT}\right)$
$V_{\mathrm{Cs},2}^{(5)JJ_1J_2}$	$-\frac{9}{7}\left(a_{p\,\mathrm{eff}}^{(5)JJ_1J_2} + 2\delta^{JJ_1}a_{p\,\mathrm{eff}}^{(5)TTJ_2}\right)$
$V_{\mathrm{Cs},2}^{(6)J}$	$-\frac{12}{7}\left(c_{p\,\mathrm{eff}}^{(6)JTKK} + c_{p\,\mathrm{eff}}^{(6)JTTT}\right)$
$V_{\mathrm{Cs},2}^{(6)JJ_1J_2}$	$\frac{36}{7}\left(c_{p\,\mathrm{eff}}^{(6)JTJ_1J_2} + \delta^{JJ_1}c_{p\,\mathrm{eff}}^{(6)TTTJ_2}\right)$
$V_{\mathrm{Cs},2}^{(7)J}$	$\frac{10}{7}\left(3a_{p\,\mathrm{eff}}^{(7)JTTKK} + 2a_{p\,\mathrm{eff}}^{(7)JTTTT}\right)$
$V_{\mathrm{Cs},2}^{(7)JJ_1J_2}$	$-\frac{30}{7}\left(3a_{p\,\mathrm{eff}}^{(7)JTTJ_1J_2} + 2\delta^{JJ_1}a_{p\,\mathrm{eff}}^{(7)TTTTJ_2}\right)$
$V_{\mathrm{Cs},4}^{(7)J}$	$\frac{60}{77}\left(a_{p\,\mathrm{eff}}^{(7)JKKLL} + 4a_{p\,\mathrm{eff}}^{(7)JTTKK}\right)$
$V_{\mathrm{Cs},4}^{(7)JJ_1J_2}$	$-\frac{540}{77}\left(a_{p\,\mathrm{eff}}^{(7)JTTJ_1J_2} + \delta^{JJ_1}a_{p\,\mathrm{eff}}^{(7)TTJ_1KK}\right)$ $-\frac{270}{77}a_{p\,\mathrm{eff}}^{(7)JJ_1J_2KK}$
$V_{\mathrm{Cs},4}^{(7)JJ_1J_2J_3J_4}$	$\frac{150}{77}\left(a_{p\,\mathrm{eff}}^{(7)JJ_1J_2J_3J_4} + 4\delta^{JJ_1}a_{p\,\mathrm{eff}}^{(7)TTJ_2J_3J_4}\right)$
$V_{\mathrm{Cs},2}^{(8)J}$	$-\frac{30}{7}\left(2c_{p\,\mathrm{eff}}^{(8)JTTTKK} + c_{p\,\mathrm{eff}}^{(8)JTTTTT}\right)$
$V_{\mathrm{Cs},2}^{(8)JJ_1J_2}$	$\frac{90}{7}\left(2c_{p\,\mathrm{eff}}^{(8)JTTTJ_1J_2} + \delta^{JJ_1}c_{p\,\mathrm{eff}}^{(8)TTTTTJ_2}\right)$
$V_{\mathrm{Cs},4}^{(8)J}$	$-\frac{360}{77}\left(c_{p\,\mathrm{eff}}^{(8)JTKKLL} + 2c_{p\,\mathrm{eff}}^{(8)JTTTKK}\right)$
$V_{\mathrm{Cs},4}^{(8)JJ_1J_2}$	$\frac{1620}{77}\left(c_{p\,\mathrm{eff}}^{(8)JTTTJ_1J_2} + \delta^{JJ_1}c_{p\,\mathrm{eff}}^{(8)TTTJ_2KK}\right)$ $+\frac{1620}{77}c_{p\,\mathrm{eff}}^{(8)JTJ_1J_2KK}$
$V_{\mathrm{Cs},4}^{(8)JJ_1J_2J_3J_4}$	$-\frac{900}{77}c_{p\,\mathrm{eff}}^{(8)JTJ_1J_2J_3J_4} - \frac{1800}{77}\delta^{JJ_1}c_{p\,\mathrm{eff}}^{(7)TTTJ_2J_3J_4}$

$$2\pi\delta\nu_{c,1,5\omega_\oplus} = \frac{1}{16}\beta_L \sin^4\vartheta \sum_d \frac{\langle|\boldsymbol{p}|^4\rangle}{m_p^{7-d}}$$
$$\times \left[\sin 5\omega_\oplus T\left(10V_{\mathrm{Cs},4}^{(d)XXXYY}\right.\right.$$
$$\left.- V_{\mathrm{Cs},4}^{(d)XXXXX} - 5V_{\mathrm{Cs},4}^{(d)XYYYY}\right)$$
$$+ \cos 5\omega_\oplus T\left(5V_{\mathrm{Cs},4}^{(d)XXXXY}\right.$$
$$\left.\left.- 10V_{\mathrm{Cs},4}^{(d)XXYYY} + V_{\mathrm{Cs},4}^{(d)YYYYY}\right)\right] \quad (46)$$

and is suppressed by the boost factor β_L, in agreement with the discussion following Eq. (28).

Taken together, the above results permit estimates of the potential sensitivity to Lorentz and *CPT* violation that is attainable in experiments with ^{133}Cs fountain clocks via studies of sidereal and annual variations. Adopting as a benchmark the measurements of minimal-SME coefficients reported in Ref. [65], it is reasonable to expect sensitivities in the Sun-centered frame of the orders of magnitude listed in Table V. The first four lines of this table provides estimated sensitivities to the nonrelativistic coefficients $\mathcal{V}_{pk2m}^{\mathrm{NR}}$, while the remainder of the table concerns the effective Cartesian coefficients $V_{\mathrm{Cs},k}^{(d)J}$. For the entries involving the latter, the uncontracted Cartesian spatial index J represents any of the possible values X, Y, Z. These estimated attainable sensitivities are competitive, so

extracting real constraints from data would be of definite interest.

Further developments of these results are also possible. Corrections at second boost order that are sensitive to isotropic coefficients in the minimal SME are analyzed in Ref. [65]. Generalizing this analysis to the nonminimal sector is a worthwhile open project. Another line of reasoning extending the above results would involve replacing the Schmidt model with a more realistic nuclear model for the ^{133}Cs nucleus. In the minimal SME, this replacement reveals that neutron Lorentz-violating operators with $j = 2$ also contribute to the frequency shift, thereby leading to constraints on coefficients for Lorentz violation in the neutron sector [65]. For the nonrelativistic coefficients, the neutron-sector corrections can be incorporated into the expressions given above via the replacement

$$\mathcal{V}_{p22m}^{\mathrm{NR}} \to \mathcal{V}_{p22m}^{\mathrm{NR}} + 0.021\mathcal{V}_{n22m}^{\mathrm{NR}}. \quad (47)$$

We can therefore estimate the attainable sensitivities to these neutron-sector coefficients by reducing by about 2 orders of magnitude the corresponding proton-sector estimates given in Table V. Note that neutron Lorentz-violating operators with $j = 2$, $k = 4$ or with $j = 4$ may also affect the energy shift, but this possibility remains unexplored in the literature to date.

Atomic clocks placed on orbiting satellites or other spacecraft offer qualitatively different experimental opportunities for studying Lorentz and *CPT* symmetry. Since typical space missions involve different clock trajectories than those relevant to Earth-based laboratories, they provide access to different combinations of coefficients for Lorentz violation [64]. For example, the orbital plane of space-based laboratories like the International Space

TABLE V. Potential sensitivities to coefficients in the Sun-centered frame from sidereal and annual variations in a ^{133}Cs fountain clock.

Coefficient	Sensitivity
$\lvert a_{p221}^{\mathrm{NR}}\rvert,\ \lvert c_{p221}^{\mathrm{NR}}\rvert$	10^{-24} GeV^{-1}
$\lvert a_{p421}^{\mathrm{NR}}\rvert,\ \lvert c_{p421}^{\mathrm{NR}}\rvert$	10^{-22} GeV^{-3}
$\lvert a_{p222}^{\mathrm{NR}}\rvert,\ \lvert c_{p222}^{\mathrm{NR}}\rvert$	10^{-24} GeV^{-1}
$\lvert a_{p422}^{\mathrm{NR}}\rvert,\ \lvert c_{p422}^{\mathrm{NR}}\rvert$	10^{-22} GeV^{-3}
$\lvert c_{p\,\mathrm{eff}}^{(4)TJ}\rvert$	10^{-20}
$\lvert a_{p\,\mathrm{eff}}^{(5)TTJ}\rvert,\ \lvert a_{p\,\mathrm{eff}}^{(5)KKJ}\rvert$	10^{-20} GeV^{-1}
$\lvert c_{p\,\mathrm{eff}}^{(6)TTTJ}\rvert,\ \lvert c_{p\,\mathrm{eff}}^{(6)TKKJ}\rvert$	10^{-21} GeV^{-2}
$\lvert a_{p\,\mathrm{eff}}^{(7)TTTTJ}\rvert,\ \lvert a_{p\,\mathrm{eff}}^{(7)TTKKJ}\rvert$	10^{-21} GeV^{-3}
$\lvert a_{p\,\mathrm{eff}}^{(7)KKLLJ}\rvert$	10^{-18} GeV^{-3}
$\lvert c_{p\,\mathrm{eff}}^{(8)TTTTTJ}\rvert,\ \lvert c_{p\,\mathrm{eff}}^{(8)TTTKKJ}\rvert$	10^{-21} GeV^{-4}
$\lvert c_{p\,\mathrm{eff}}^{(8)TKKLLJ}\rvert$	10^{-19} GeV^{-4}

Station (ISS) is inclined relative to the equatorial plane and precesses, thereby sampling orientation-dependent effects in a unique way. The satellite orbital velocity can also exceed the rotational velocity of the Earth, which can enhance some signals for Lorentz violation and can permit faster data accumulation. For instance, the period of the ISS is approximately 92 minutes, so over 15 orbits are completed during a sidereal day.

A ^{133}Cs cold-atom clock is a component of the ACES platform on the ISS [60]. The proposed STE-QUEST mission [62] may also involve a ^{133}Cs clock. Frequency data obtained from operating a ^{133}Cs clock in the spin-polarized mode can be converted to the Sun-centered frame using Eqs. (36)–(39) or similar expressions, leading to bounds on combinations of coefficients for Lorentz violation that are inaccessible to Earth-based experiments. For example, Table III shows that the coefficients $\mathcal{V}_{pkj0}^{\mathrm{NR}}$ produce no sidereal effects in an Earth-based laboratory, but they can be measured on a space platform.

Fountain clocks using ^{87}Rb atoms have been considered as interesting alternatives for a primary frequency standard [66,67] and for studying Lorentz symmetry via the proposed space-based mission QTEST [61]. A double ^{133}Cs and ^{87}Rb fountain clock has been used to search for Lorentz and CPT violation [16,65]. Like its ^{133}Cs and H analogues, the ^{87}Rb clock transition $|F = 1, m_F = 0\rangle \leftrightarrow |F = 2, m_F = 0\rangle$ is insensitive to Lorentz and CPT violation and can thus be used as a reference in experiments. However, the frequencies $\nu_{m_F}^{\mathrm{Rb}}$ associated with the transitions $|F=1, m_F\rangle \leftrightarrow |F=2, m_F\rangle$ with $m_F \neq 0$ do experience Lorentz- and CPT-violating shifts. The systematics associated with the linear Zeeman shift can in this case be reduced by considering the observable $\nu_c^{\mathrm{Rb}} = \nu_{+1}^{\mathrm{Rb}} + \nu_{-1}^{\mathrm{Rb}} - 2\nu_0^{\mathrm{Rb}}$. Coefficients in the electron sector leave ν_c^{Rb} unaffected. In the context of the Schmidt model the valence nucleon is a proton with spin $I = 3/2$, so the shift $\delta\nu_c^{\mathrm{Rb}}$ in the observable ν_c^{Rb} depends on coefficients for Lorentz violation in the proton sector. We find

$$2\pi\delta\nu_c^{\mathrm{Rb}} = -\frac{1}{\sqrt{5\pi}}(\langle|\boldsymbol{p}|^2\rangle \mathcal{V}_{p220}^{\mathrm{NR}} + \langle|\boldsymbol{p}|^4\rangle \mathcal{V}_{p420}^{\mathrm{NR}}). \quad (48)$$

Since the nuclear spin of ^{87}Rb is smaller than that of ^{133}Cs, fewer coefficients appear in Eq. (48) than in Eq. (40). All results for ^{133}Cs fountains discussed in the present section can be transcribed to results for ^{87}Rb fountains by matching the changes between Eqs. (40) and (48).

B. Comagnetometers

Comagnetometers form another category of sensitive tools used for studies of Lorentz and CPT symmetry. High-sensitivity searches for Lorentz and CPT violation in both sidereal and annual variations have been achieved using ^{129}Xe-^{3}He comagnetometers [22–25]. The experiments

compared the angular frequencies ω_{Xe} and ω_{He} of Larmor transitions in the ground states of ^{129}Xe and ^{3}He atoms by measuring the observable

$$\omega = \omega_{\mathrm{He}} - \frac{\gamma_{\mathrm{He}}}{\gamma_{\mathrm{Xe}}}\omega_{\mathrm{Xe}}, \quad (49)$$

which is insensitive to the linear Zeeman shift. Here, γ_{Xe} is the gyromagnetic ratio for the ground state of ^{129}Xe and γ_{He} is that for the ground state of ^{3}He.

Since the total electronic angular momentum of the noble gases in the ground state is $J = 0$, the Larmor transitions are unaffected by electron coefficients for Lorentz violation. The contributions from the nucleon coefficients can be estimated using the Schmidt model, in which the nuclear spin $I = 1/2$ of each species is assigned to the unpaired neutron. The analysis in Sec. II then yields the Lorentz-violating shift $\delta\omega$ of the observable ω as

$$\delta\omega = -\frac{1}{\sqrt{3\pi}}\sum_{q=0}^{2}\left(\langle|\boldsymbol{p}|^{2q}\rangle_{\mathrm{He}} - \frac{\gamma_{\mathrm{He}}}{\gamma_{\mathrm{Xe}}}\langle|\boldsymbol{p}|^{2q}\rangle_{\mathrm{Xe}}\right)$$
$$\times (\mathcal{T}_{n(2q)10}^{\mathrm{NR}(0B)} + 2\mathcal{T}_{n(2q)10}^{\mathrm{NR}(1B)}), \quad (50)$$

evaluated in the laboratory frame. In this expression, $\langle|\boldsymbol{p}|^k\rangle_{\mathrm{He}}$ and $\langle|\boldsymbol{p}|^k\rangle_{\mathrm{Xe}}$ are the expectation values of the Schmidt neutron in ^{3}He and ^{129}Xe. These quantities can reasonably be taken as roughly the same order of magnitude, $\langle|\boldsymbol{p}|^k\rangle_{\mathrm{He}} \sim \langle|\boldsymbol{p}|^k\rangle_{\mathrm{Xe}}$, so the shift $\delta\omega$ can be written as

$$\delta\omega = \sum_{q=0}^{2}\left(\frac{\gamma_{\mathrm{He}}}{\gamma_{\mathrm{Xe}}} - 1\right)\frac{\langle|\boldsymbol{p}|^{2q}\rangle}{\sqrt{3\pi}}(\mathcal{T}_{n(2q)10}^{\mathrm{NR}(0B)} + 2\mathcal{T}_{n(2q)10}^{\mathrm{NR}(1B)}). \quad (51)$$

This result reduces to the minimal-SME expressions presented in Refs. [22–25] based on the theoretical treatment of Ref. [9], by taking the limit

$$\mathcal{T}_{n010}^{\mathrm{NR}(0B)} + 2\mathcal{T}_{n010}^{\mathrm{NR}(1B)} \to 2\sqrt{3\pi}\tilde{b}_3^n,$$
$$\mathcal{T}_{n210}^{\mathrm{NR}(0B)} + 2\mathcal{T}_{n210}^{\mathrm{NR}(1B)} \to 0,$$
$$\mathcal{T}_{n410}^{\mathrm{NR}(0B)} + 2\mathcal{T}_{n410}^{\mathrm{NR}(1B)} \to 0, \quad (52)$$

as expected.

Conversion of Eq. (51) to the Sun-centered frame reveals the time variations in the observable ω. At zeroth boost order, the nonminimal terms produce time variations at the first harmonic of the sidereal frequency, which can be explicitly obtained using Eq. (25). We can then translate existing bounds on the minimal SME coefficients $\tilde{b}_X^n$ and $\tilde{b}_Y^n$ obtained from studies of this harmonic to constraints on

TABLE VI. Constraints on the moduli of the real and imaginary parts of neutron nonrelativistic coefficients determined from ^{129}Xe-^{3}He comparisons using Eq. (54).

Coefficient $\mathcal{K}$	Constraint on $|\text{Re}\mathcal{K}|, |\text{Im}\mathcal{K}|$
$H_{n011}^{\text{NR}(0B),\text{Sun}}, \; g_{n011}^{\text{NR}(0B),\text{Sun}}$	$< 4 \times 10^{-33}$ GeV
$H_{n011}^{\text{NR}(1B),\text{Sun}}, \; g_{n011}^{\text{NR}(1B),\text{Sun}}$	$< 2 \times 10^{-33}$ GeV
$H_{n211}^{\text{NR}(0B),\text{Sun}}, \; g_{n211}^{\text{NR}(0B),\text{Sun}}$	$< 4 \times 10^{-31}$ GeV^{-1}
$H_{n211}^{\text{NR}(1B),\text{Sun}}, \; g_{n211}^{\text{NR}(1B),\text{Sun}}$	$< 2 \times 10^{-31}$ GeV^{-1}
$H_{n411}^{\text{NR}(0B),\text{Sun}}, \; g_{n411}^{\text{NR}(0B),\text{Sun}}$	$< 4 \times 10^{-29}$ GeV^{-3}
$H_{n411}^{\text{NR}(1B),\text{Sun}}, \; g_{n411}^{\text{NR}(1B),\text{Sun}}$	$< 2 \times 10^{-29}$ GeV^{-3}

nonminimal coefficients for Lorentz violation. For this purpose, it suffices to implement the identifications

$$\tilde{b}_X^n \to -\frac{1}{\sqrt{6\pi}} \sum_{q=0}^{2} \langle|\boldsymbol{p}|^{2q}\rangle \text{Re}[\mathcal{T}_{n(2q)11}^{\text{NR}(0B)} + 2\mathcal{T}_{n(2q)11}^{\text{NR}(1B)}],$$

$$\tilde{b}_Y^n \to \frac{1}{\sqrt{6\pi}} \sum_{q=0}^{2} \langle|\boldsymbol{p}|^{2q}\rangle \text{Im}[\mathcal{T}_{n(2q)11}^{\text{NR}(0B)} + 2\mathcal{T}_{n(2q)11}^{\text{NR}(1B)}] \quad (53)$$

on existing minimal-SME limits. For example, the bound on the coefficient $\tilde{b}_\perp^n$ reported in Ref. [25] then yields the constraint

$$\left| \sum_{q=0}^{2} \langle|\boldsymbol{p}|^{2q}\rangle (\mathcal{T}_{n(2q)11}^{\text{NR}(0B),\text{Sun}} + 2\mathcal{T}_{n(2q)11}^{\text{NR}(1B),\text{Sun}}) \right|$$
$$< 3.7 \times 10^{-33} \text{ GeV} \quad (54)$$

at the one sigma level. Following the standard procedure in the literature of taking one coefficient to be nonzero at a time [3], we find the maximal sensitivities to non-relativistic coefficients shown in Table VI. These are the first constraints on neutron nonrelativistic coefficients in the literature. They correspond to substantially greater sensitivities to Lorentz and *CPT* violation than those attained to date on electron or proton nonrelativistic coefficients, and they exceed even the comparatively tight constraints on muon nonminimal coefficients obtained from laboratory measurements of the muon anomalous magnetic moment [68,69].

At linear boost order in the Sun-centered frame, the Lorentz-violating shift $\delta\omega_1$ in ω follows the generic structure (31) and can be written as

$$\delta\omega_1 = \sum_{d=3}^{8} \sum_{q=0}^{2} \left(\frac{\gamma_{\text{He}}}{\gamma_{\text{Xe}}} - 1\right) \frac{\langle|\boldsymbol{p}|^{2q}\rangle}{m_n^{3+2q-d}} T_{\text{HeXe},(2q)}^{(d)JK} \hat{B}^J \beta^K. \quad (55)$$

The quantities $T_{\text{HeXe},k}^{(d)JK}$ are the linear combinations of effective Cartesian coefficients displayed in Table VII. In this table, parentheses around indices are understood to represent symmetrization with a suitable factor, e.g., $\tilde{g}_{n\text{eff}}^{(4)J(TK)} = (\tilde{g}_{n\text{eff}}^{(4)JTK} + \tilde{g}_{n\text{eff}}^{(4)JKT})/2!$. Also, repeated dummy indices are understood to be summed, e.g., $\tilde{H}_{n\text{eff}}^{(5)TJTJ} = \tilde{H}_{n\text{eff}}^{(5)TXTX} + \tilde{H}_{n\text{eff}}^{(5)TYTY} + \tilde{H}_{n\text{eff}}^{(5)TZTZ}$. The explicit form of the result (55) can be displayed by substituting Eqs. (32)–(35) given in Sec. II F for the boost velocity of the laboratory and for the direction of the magnetic field. This ensuing expression takes the form

TABLE VII. The quantities $T_{\text{HeXe},k}^{(d)JK}$ for $3 \leq d \leq 8$.

$T_{\text{HeXe},k}^{(d)JK}$	Combination
$T_{\text{HeXe},0}^{(3)JK}$	$-2\tilde{H}_{n\text{eff}}^{(3)JK}$
$T_{\text{HeXe},0}^{(4)JK}$	$4\tilde{g}_{n\text{eff}}^{(4)J(TK)}$
$T_{\text{HeXe},0}^{(5)JK}$	$-6\tilde{H}_{n\text{eff}}^{(5)J(TTK)}$
$T_{\text{HeXe},0}^{(6)JK}$	$8\tilde{g}_{n\text{eff}}^{(6)J(TTTK)}$
$T_{\text{HeXe},0}^{(7)JK}$	$-10\tilde{H}_{n\text{eff}}^{(7)J(TTTTK)}$
$T_{\text{HeXe},0}^{(8)JK}$	$12\tilde{g}_{n\text{eff}}^{(8)J(TTTTTK)}$
$T_{\text{HeXe},2}^{(5)JK}$	$-\frac{4}{3}\tilde{H}_{n\text{eff}}^{(5)TLTL}\delta^{JK} - 4\tilde{H}_{n\text{eff}}^{(5)J(TTK)} - 2\tilde{H}_{n\text{eff}}^{(5)J(LLK)}$
$T_{\text{HeXe},2}^{(6)JK}$	$2\tilde{g}_{n\text{eff}}^{(6)TLTTL}\delta^{JK} + 8\tilde{g}_{n\text{eff}}^{(6)J(TTTK)} + 8\tilde{g}_{n\text{eff}}^{(6)J(TLLK)}$
$T_{\text{HeXe},2}^{(7)JK}$	$-\frac{8}{3}\tilde{H}_{n\text{eff}}^{(7)TLTTTL}\delta^{JK} - \frac{40}{3}\tilde{H}_{n\text{eff}}^{(7)J(TTTTK)} - 20\tilde{H}_{n\text{eff}}^{(7)J(TTLLK)}$
$T_{\text{HeXe},2}^{(8)JK}$	$\frac{10}{3}\tilde{g}_{n\text{eff}}^{(8)TLTTTTL}\delta^{JK} + 20\tilde{g}_{n\text{eff}}^{(8)J(TTTTTK)} + 40\tilde{g}_{n\text{eff}}^{(8)J(TTTLLK)}$
$T_{\text{HeXe},4}^{(7)JK}$	$-\frac{8}{5}\tilde{H}_{n\text{eff}}^{(7)TLTMML}\delta^{JK} - 8\tilde{H}_{n\text{eff}}^{(7)J(TTLLK)} - 2\tilde{H}_{n\text{eff}}^{(7)J(LLMMK)}$
$T_{\text{HeXe},4}^{(8)JK}$	$4\tilde{g}_{n\text{eff}}^{(8)TMTTLLM}\delta^{JK} + 24\tilde{g}_{n\text{eff}}^{(8)J(TTTLLK)} + 12\tilde{g}_{n\text{eff}}^{(8)J(TLLMMK)}$

TABLE VIII. The quantities $\lambda_*^{(d)k}$ in terms of $T_{\mathrm{HeXe},k}^{(d)JK}$.

$\lambda_*^{(d)k}$	Combination
$\lambda_\oplus^{(d)k}$	$\cos\phi\, T_{\mathrm{HeXe},(2q)}^{(d)[XY]} + \frac{1}{2}\sin\phi\,(T_{\mathrm{HeXe},k}^{(d)XX} + T_{\mathrm{HeXe},k}^{(d)YY})$
$\lambda_L^{(d)k}$	$T_{\mathrm{HeXe},k}^{(d)ZX}$
$\lambda_{c\Omega}^{(d)k}$	$-(\cos\eta\, T_{\mathrm{HeXe},k}^{(d)ZY} + \sin\eta\, T_{\mathrm{HeXe},k}^{(d)ZZ})$
$\lambda_{s\Omega}^{(d)k}$	$T_{\mathrm{HeXe},k}^{(d)ZX}$
$\lambda_{c\omega}^{(d)k}$	$T_{\mathrm{HeXe},k}^{(d)ZY}$
$\lambda_{s\omega}^{(d)k}$	$-T_{\mathrm{HeXe},k}^{(d)ZX}$
$\lambda_{c2\omega}^{(d)k}$	$\cos\phi\, T_{\mathrm{HeXe},k}^{(d)(XY)} + \sin\phi\,\frac{1}{2}(T_{\mathrm{HeXe},k}^{(d)YY} - T_{\mathrm{HeXe},k}^{(d)XX})$
$\lambda_{s2\omega}^{(d)k}$	$-\sin\phi\, T_{\mathrm{HeXe},k}^{(d)(XY)} - \cos\phi\,\frac{1}{2}(T_{\mathrm{HeXe},k}^{(d)YY} - T_{\mathrm{HeXe},k}^{(d)XX})$
$\lambda_{c\Omega s\omega}^{(d)k}$	$\cos\eta(\sin\phi\, T_{\mathrm{HeXe},k}^{(d)XY} - \cos\phi\, T_{\mathrm{HeXe},k}^{(d)YY}) + \sin\eta(\sin\phi\, T_{\mathrm{HeXe},k}^{(d)XZ} - \cos\phi\, T_{\mathrm{HeXe},k}^{(d)YZ})$
$\lambda_{c\Omega c\omega}^{(d)k}$	$-\sin\eta(\cos\phi\, T_{\mathrm{HeXe},k}^{(d)XZ} + \sin\phi\, T_{\mathrm{HeXe},k}^{(d)YZ}) - \cos\eta(\cos\phi\, T_{\mathrm{HeXe},k}^{(d)XY} + \sin\phi\, T_{\mathrm{HeXe},k}^{(d)YY})$
$\lambda_{s\Omega s\omega}^{(d)k}$	$\cos\phi\, T_{\mathrm{HeXe},k}^{(d)YX} - \sin\phi\, T_{\mathrm{HeXe},k}^{(d)XX}$
$\lambda_{s\Omega c\omega}^{(d)k}$	$\sin\phi\, T_{\mathrm{HeXe},k}^{(d)YX} + \cos\phi\, T_{\mathrm{HeXe},k}^{(d)XX}$

$$\begin{aligned}
\delta\omega_1 &= \beta_\oplus \sin\vartheta\lambda_\oplus + \beta_L\cos\vartheta\lambda_L + \beta_\oplus\cos\vartheta\cos(\Omega_\oplus T)\lambda_{c\Omega} + \beta_\oplus\cos\vartheta\sin(\Omega_\oplus T)\lambda_{s\Omega} + \beta_L\cos\vartheta\cos(\omega_\oplus T_\oplus)\lambda_{c\omega} \\
&\quad + \beta_L\cos\vartheta\sin(\omega_\oplus T_\oplus)\lambda_{s\omega} + \beta_\oplus\sin\vartheta\cos(\omega_\oplus T_\oplus)\cos(\Omega_\oplus T)\lambda_{c\omega c\Omega} + \beta_\oplus\sin\vartheta\sin(\omega_\oplus T_\oplus)\cos(\Omega_\oplus T)\lambda_{s\omega c\Omega} \\
&\quad + \beta_\oplus\sin\vartheta\cos(\omega_\oplus T_\oplus)\sin(\Omega_\oplus T)\lambda_{c\omega s\Omega} + \beta_\oplus\sin\vartheta\sin(\omega_\oplus T_\oplus)\sin(\Omega_\oplus T)\lambda_{s\omega s\Omega} + \beta_L\sin\vartheta\cos(2\omega_\oplus T_\oplus)\lambda_{c2\omega} \\
&\quad + \beta_L\sin\vartheta\sin(2\omega_\oplus T_\oplus)\lambda_{s2\omega},
\end{aligned} \tag{56}$$

where the twelve quantities λ_* with subscripts $*$ ranging over the values $\oplus$, L, $c\Omega$, $s\Omega$, $c\omega$, $s\omega$, $c\omega c\Omega$, $s\omega c\Omega$, $c\omega s\Omega$, $s\omega s\Omega$, $c2\omega$, $s2\omega$ can be decomposed in terms of quantities $\lambda_*^{(d)k}$ with fixed values of k and d via the relation

$$\lambda_* = \sum_{d=3}^{8}\sum_{q=0}^{2}\left(\frac{\gamma_{\mathrm{He}}}{\gamma_{\mathrm{Xe}}} - 1\right)\frac{\langle|\boldsymbol{p}|^{2q}\rangle}{m_n^{3+2q-d}}\lambda_*^{(d)(2q)}. \tag{57}$$

Expressions for the quantities $\lambda_*^{(d)k}$ in terms of $T_{\mathrm{HeXe},k}^{(d)JK}$ are given in Table VIII.

An experiment using a dual ^{129}Xe-^{3}He maser to study sidereal variations in the observable ω at different times of the year was performed at the Harvard-Smithsonian Center for Astrophysics [23]. In this experiment, the magnetic field was oriented west to east, corresponding to $\vartheta = 0°$ and $\phi = 90°$. We can use the bounds on $\delta\omega_1$ reported in Ref. [23] to determine limits on nonminimal effective Cartesian coefficients for the neutron. The published analysis neglected contributions proportional to the laboratory velocity β_L in the Sun-centered frame, so we can deduce the four bounds

$$\begin{aligned}
\lambda_{c\omega c\Omega} &= (-3.9 \pm 3.5) \times 10^{-27} \text{ GeV,} \\
\lambda_{c\omega s\Omega} &= (0.7 \pm 6.3) \times 10^{-27} \text{ GeV,} \\
\lambda_{s\omega s\Omega} &= (-6.3 \pm 6.7) \times 10^{-27} \text{ GeV,} \\
\lambda_{s\omega c\Omega} &= (-3.9 \pm 2.8) \times 10^{-27} \text{ GeV.}
\end{aligned} \tag{58}$$

Note that the dependence of the quantities λ_* on the angle ϕ means that these bounds hold only at $\phi = 90°$. Using the results in Tables VII and VIII and the bounds (58), we can extract maximal sensitivities to many nonminimal effective Cartesian coefficients for the neutron. These constraints are listed in Table IX. They are the first of their kind reported in the literature for neutrons.

Improvements over the results in Table IX are within reach of existing experiments. The sensitivity recently attained in the Heidelberg apparatus described in Ref. [25] represents a gain of about 2 orders of magnitude, so sufficient sidereal data accumulated at the annual frequency with this apparatus could in principle better the constraints in Table IX by a similar factor. Moreover, with the sidereal data already in hand, the time variations at the second harmonic of the sidereal frequency appearing in Eq. (56) could in principle be studied and would be

TABLE IX.　Constraints on the moduli of neutron effective Cartesian coefficients determined from ^{129}Xe-^{3}He comparisons using Eq. (58).

Coefficient	Constraint	Coefficient	Constraint
$\tilde{H}_{n\mathrm{eff}}^{(5)X(TXT)}$	$<1 \times 10^{-27}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)X(TXTT)}$	$<9 \times 10^{-28}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)X(TYT)}$	$<8 \times 10^{-28}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)X(TYTT)}$	$<7 \times 10^{-28}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)X(TZT)}$	$<2 \times 10^{-27}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)X(TZTT)}$	$<2 \times 10^{-27}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)Y(TXT)}$	$<8 \times 10^{-28}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)Y(TXTT)}$	$<6 \times 10^{-28}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)Y(TYT)}$	$<8 \times 10^{-28}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)Y(TYTT)}$	$<7 \times 10^{-28}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)Y(TZT)}$	$<2 \times 10^{-27}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)Y(TZTT)}$	$<2 \times 10^{-27}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)X(JXJ)}$	$<4 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)X(JXJT)}$	$<9 \times 10^{-26}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)X(JYJ)}$	$<3 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)X(JYJT)}$	$<7 \times 10^{-26}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)X(JZJ)}$	$<6 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)X(JZJT)}$	$<2 \times 10^{-25}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)Y(JXJ)}$	$<2 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)Y(JXJT)}$	$<2 \times 10^{-25}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)Y(JYJ)}$	$<3 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)Y(JYJT)}$	$<7 \times 10^{-26}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)Y(JZJ)}$	$<6 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)Y(JZJT)}$	$<2 \times 10^{-25}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(5)TJTJ}$	$<6 \times 10^{-25}$ GeV^{-1}	$\tilde{g}_{n\mathrm{eff}}^{(6)TJTJT}$	$<5 \times 10^{-25}$ GeV^{-2}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(TXTTT)}$	$<8 \times 10^{-28}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(TXTTTT)}$	$<7 \times 10^{-28}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(TYTTT)}$	$<6 \times 10^{-28}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(TYTTTT)}$	$<5 \times 10^{-28}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(TZTTT)}$	$<2 \times 10^{-27}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(TZTTTT)}$	$<1 \times 10^{-27}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(TXTTT)}$	$<6 \times 10^{-28}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(TXTTTT)}$	$<5 \times 10^{-28}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(TYTTT)}$	$<6 \times 10^{-28}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(TYTTTT)}$	$<5 \times 10^{-28}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(TZTTT)}$	$<2 \times 10^{-27}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(TZTTTT)}$	$<1 \times 10^{-27}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(JXJTT)}$	$<4 \times 10^{-26}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(JXJTTT)}$	$<2 \times 10^{-26}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(JYJTT)}$	$<3 \times 10^{-26}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(JYJTTT)}$	$<1 \times 10^{-26}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(JZJTT)}$	$<7 \times 10^{-26}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(JZJTTT)}$	$<4 \times 10^{-26}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(JXJTT)}$	$<3 \times 10^{-26}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(JXJTTT)}$	$<1 \times 10^{-26}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(JYJTT)}$	$<3 \times 10^{-26}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(JYJTTT)}$	$<1 \times 10^{-26}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(JZJTT)}$	$<7 \times 10^{-26}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(JZJTTT)}$	$<3 \times 10^{-26}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)TJTJTT}$	$<2 \times 10^{-25}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)TJTJTTT}$	$<4 \times 10^{-25}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(JXJKK)}$	$<4 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(JXJTKK)}$	$<7 \times 10^{-24}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(JYJKK)}$	$<3 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(JYJTKK)}$	$<5 \times 10^{-24}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)X(JZJKK)}$	$<7 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)X(JZJTKK)}$	$<1 \times 10^{-23}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(JXJKK)}$	$<3 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(JXJTKKK)}$	$<5 \times 10^{-24}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(JYJKK)}$	$<3 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(JYJTKK)}$	$<5 \times 10^{-24}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)Y(JZJKK)}$	$<7 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)Y(JZJTKK)}$	$<1 \times 10^{-23}$ GeV^{-4}
$\tilde{H}_{n\mathrm{eff}}^{(7)TJTJKK}$	$<6 \times 10^{-23}$ GeV^{-3}	$\tilde{g}_{n\mathrm{eff}}^{(8)TJTJTKK}$	$<2 \times 10^{-23}$ GeV^{-4}

expected to yield additional measurements of interest. Although this signal is suppressed by about 2 orders of magnitude compared to annual-variation effects, the greater sensitivity of the Heidelberg apparatus suggests constraints of the same order of magnitude as those in Table IX could be obtained. Note also that direct measurements of the annual modulation would lead to new constraints on SME

coefficients, as sidereal variations are insensitive to the combinations $\lambda_{c\Omega}$ and $\lambda_{s\Omega}$ even when monitored throughout the year.

Another avenue offering potential improvements is the adoption of better nuclear models beyond the Schmidt model. These techniques have already been used to show that contributions from proton coefficients to Eq. (51) are

significant, being suppressed only by a factor of about 5 for coefficients with $k = 0$ [30]. If a similar relationship for coefficients with $k = 2, 4$ can be demonstrated, then the constraints on the neutron coefficients listed in Tables VI and IX could be extended to bounds on the corresponding proton coefficients by multiplying by a factor of 5. This would represent a striking gain in sensitivity to the proton nonrelativistic coefficients compared to the existing results obtained using data from a hydrogen maser [43].

Other comagnetometers can also be used to test Lorentz and CPT symmetry and may offer sensitivities to additional coefficients. One potential example is the ^{21}Ne-Rb-K comagnetometer described in Ref. [19], which is designed to extend the reach achieved earlier by a ^{3}He-K self-compensating comagnetometer [18]. The addition of ^{21}Ne to the system is of particular interest here because the nuclear spin of ^{21}Ne is $I = 3/2$ and so this comagnetometer can access more coefficients for Lorentz violation. A glance at Table I reveals that there are prospects for measuring the coefficients with $j = 2$ and $j = 3$. The underlying physics of this comagnetometer system differs significantly from that of the other systems discussed in this work, so the results obtained in Sec. II cannot be directly applied to estimate sensitivities. However, some of the bounds presented in Ref. [19] can be converted to constraints on nonrelativistic coefficients for the neutron by applying the relationship (43) between the nonrelativistic coefficients and the coefficients $c^{(4)}_{\mu\nu}$. Table X lists the corresponding maximal sensitivities achieved, which are the first of this kind in the literature. As before, these results can be expected to extend to constraints on nonrelativistic coefficients for the proton because nuclear models beyond the Schmidt model are known to allow contributions from proton operators to Lorentz-violating expectation values with $j = 2$ [33]. It is also plausible that a similar situation holds for coefficients with $j = 3$. All these interesting issues are open for future investigation.

TABLE X. Constraints on neutron nonrelativistic coefficients determined using data from the ^{21}Ne-Rb-K comagnetometer.

Coefficient	Constraint
$\mathrm{Re}\,a^{\mathrm{NR}}_{n221}$, $\mathrm{Re}\,c^{\mathrm{NR}}_{n221}$	$-(3.3 \pm 3.0) \times 10^{-29}$ GeV^{-1}
$\mathrm{Im}\,a^{\mathrm{NR}}_{n221}$, $\mathrm{Im}\,c^{\mathrm{NR}}_{n221}$	$-(1.9 \pm 2.3) \times 10^{-29}$ GeV^{-1}
$\mathrm{Re}\,a^{\mathrm{NR}}_{n222}$, $\mathrm{Re}\,c^{\mathrm{NR}}_{n222}$	$(1.0 \pm 1.2) \times 10^{-29}$ GeV^{-1}
$\mathrm{Im}\,a^{\mathrm{NR}}_{n222}$, $\mathrm{Im}\,c^{\mathrm{NR}}_{n222}$	$(0.83 \pm 0.96) \times 10^{-29}$ GeV^{-1}
$\mathrm{Re}\,a^{\mathrm{NR}}_{n421}$, $\mathrm{Re}\,c^{\mathrm{NR}}_{n421}$	$-(3.7 \pm 3.4) \times 10^{-27}$ GeV^{-3}
$\mathrm{Im}\,a^{\mathrm{NR}}_{n421}$, $\mathrm{Im}\,c^{\mathrm{NR}}_{n421}$	$-(2.2 \pm 2.6) \times 10^{-27}$ GeV^{-3}
$\mathrm{Re}\,a^{\mathrm{NR}}_{n422}$, $\mathrm{Re}\,c^{\mathrm{NR}}_{n422}$	$(1.1 \pm 1.3) \times 10^{-27}$ GeV^{-3}
$\mathrm{Im}\,a^{\mathrm{NR}}_{n422}$, $\mathrm{Im}\,c^{\mathrm{NR}}_{n422}$	$(0.9 \pm 1.1) \times 10^{-27}$ GeV^{-3}

C. Trapped ions and lattice clocks

The stability and accuracy of optical frequency standards currently exceeds the performance of fountain clocks. It is thus natural to consider the prospects for testing Lorentz and CPT symmetry using optical transitions. However, sensitivities to many coefficients for Lorentz violation depend on the absolute uncertainty of the frequency measurement rather than on its relative precision. As the absolute uncertainties of fountain clocks still surpass those of optical clocks, the advantages of the latter lie primarily in their ability to access distinct Lorentz- and CPT-violating effects. In particular, optical clocks offer sensitivities to coefficients for Lorentz violation in the electron sector that are unattainable in other clock-comparison experiments. In this section, we study sensitivities to electron coefficients in trapped-ion and lattice optical clocks. Since any signals from proton and neutron coefficients are better accessed via other techniques, we disregard nucleon contributions in what follows.

The transition $^1S_0$-$^3P_0$ is commonly used in optical frequency standards. It has been studied with trapped ions, including ^{27}Al$^+$ [70–73] and ^{115}In$^+$ [74–76], and also in the context of optical lattice clocks based on ^{87}Sr [77–82], ^{171}Yb [83–86], and ^{199}Hg [87–89]. For this transition, the total electronic angular momentum of the two states involved is $J = 0$, so only isotropic electron Lorentz-violating operators can contribute. The Lorentz-violating shift $\delta\nu$ in the transition frequency ν therefore involves only coefficients with $jm = 00$. In the independent-particle model discussed in Sec. II C, the shift in the laboratory frame is given by

$$2\pi\delta\nu = -\frac{1}{\sqrt{4\pi}}(\Delta p^2 \mathcal{V}^{\mathrm{NR}}_{e200} + \Delta p^4 \mathcal{V}^{\mathrm{NR}}_{e400}), \qquad (59)$$

where Δp^k is the difference in the expectation values $\langle |\boldsymbol{p}|^k \rangle$ of the energy levels involved in the transition.

Some optical frequency standards involve transitions between energy levels with $J \neq 0$. For example, the transition $^2S_{1/2}$-$^2D_{5/2}$ is used as a frequency standard in ion-trap clocks based on ^{40}Ca$^+$ [90–93] and ^{88}Sr$^+$ [94–96]. For these systems, certain systematic effects can be minimized by measuring transitions involving different Zeeman sublevels. These techniques typically also eliminate sensitivity to some Lorentz-violating effects, as is to be expected given that the coefficients for Lorentz violation behave in many ways as effective external fields.

One common technique to remove the linear Zeeman shift of the clock transition is averaging over the Zeeman pair $^2S_{1/2,1/2} - ^2D_{5/2,m_J}$ and $^2S_{1/2,-1/2} - ^2D_{5/2,-m_J}$. Similarly, the electric quadrupole shift can be removed by averaging over three different Zeeman pairs. Implementing this process eliminates any contributions from Lorentz-violating operators with $j \neq 0$ at linear order in perturbation theory, due to the identity

$$\sum_{m_J=-5/2}^{5/2} \left\langle \frac{5}{2} m_J j 0 \Big| \frac{5}{2} m_J \right\rangle = 6\delta_{j0}. \tag{60}$$

As a result, the Lorentz- and *CPT*-violating frequency shift $\delta\nu$ measurable in these systems is still given by Eq. (59), despite the nonzero value of J.

Another technique to remove the quadrupole shift uses instead two Zeeman pairs to interpolate the value of the frequency at $m_J^2 = 35/12$, which corresponds to zero quadrupole shift because the shift is proportional to $35/12 - m_J^2$. This method eliminates contributions involving coefficients for Lorentz violation with $j = 2$, but it retains contributions with $j = 4$. In this scenario, the Lorentz- and *CPT*-violating shift (59) is replaced by the expression

$$2\pi\delta\nu = -\frac{1}{\sqrt{4\pi}}(\Delta p^2 \mathcal{V}_{e200}^{\mathrm{NR}} + \Delta p^4 \mathcal{V}_{e400}^{\mathrm{NR}})$$
$$+ \frac{7}{27\sqrt{\pi}}\langle|\boldsymbol{p}|^4\rangle \mathcal{V}_{e440}^{\mathrm{NR}}, \tag{61}$$

where the expectation value $\langle|\boldsymbol{p}|^4\rangle$ is evaluated in the state $^2D_{5/2}$.

The clock transition $^2S_{1/2}(F=0)$-$^2D_{3/2}(F=2)$ with $\Delta m_F = 0$ in $^{171}\mathrm{Yb}^+$ has also been used as a frequency standard [97,98]. In the context of the independent-particle model described in Sec. II C, the Lorentz- and *CPT*-violating frequency shift $\delta\nu$ for this system is given in the laboratory frame by

$$2\pi\delta\nu = -\frac{1}{\sqrt{4\pi}}(\Delta p^2 \mathcal{V}_{e200}^{\mathrm{NR}} + \Delta p^4 \mathcal{V}_{e400}^{\mathrm{NR}})$$
$$+ \frac{1}{2\sqrt{5\pi}}(\langle|\boldsymbol{p}|^2\rangle \mathcal{V}_{e420}^{\mathrm{NR}} + \langle|\boldsymbol{p}|^4\rangle \mathcal{V}_{e420}^{\mathrm{NR}}), \tag{62}$$

where the expectation value $\langle|\boldsymbol{p}|^4\rangle$ is evaluated in the state $^2D_{3/2}$. However, to suppress the contribution from the electric quadrupole shift, an averaging of the frequency over three orthogonal directions of the magnetic field is performed. This procedure suppresses the contribution from coefficients with $j = 2$. As a result, in the limit that the three directions are exactly orthogonal, the shift (62) reduces to the expression (59).

Other frequency standards are provided by the electric octopole transitions in $^{171}\mathrm{Yb}^+$ [98,99] and $^{199}\mathrm{Hg}^+$ [100,101]. The clock transition used in these systems is the transition $\Delta m_F = 0$, which is insensitive to *B*-type coefficients for Lorentz violation. As before, the contribution to the Lorentz- and *CPT*-violating frequency shift $\delta\nu$ arising from coefficients with $j = 0$ is given by Eq. (59). The contribution from coefficients with $j = 2$ is again eliminated by the averaging procedure over three different directions of the magnetic field, which is designed to cancel

the electric quadrupole shift. It is conceivable that coefficients with $j = 4$ contribute to the frequency shift, but establishing this lies outside our present scope.

The coefficients in the above expressions are in the laboratory frame and hence may vary with time. In converting to the Sun-centered frame, the isotropic frequency shift (59) receives contributions that depend on the boost velocity of the laboratory frame. At linear boost order, we find that the shift $\delta\nu_1$ is given by

$$2\pi\delta\nu_1 = -\sum_{d,k} \frac{\Delta p^k}{\sqrt{4\pi}}[\beta_\oplus \sin\Omega_\oplus T V_{e,k}^{(d)X}$$
$$- \beta_\oplus \cos\Omega_\oplus T(\cos\eta V_{e,k}^{(d)Y} + \sin\eta V_{e,k}^{(d)Z})$$
$$+ \beta_L(\cos\omega_\oplus T_\oplus V_{e,k}^{(d)Y} - \sin\omega_\oplus T_\oplus V_{e,k}^{(d)X})], \tag{63}$$

where expressions for the quantities $V_{e,k}^{(d)J}$ in terms of effective Cartesian coefficients are given in Table XI.

The result (63) predicts annual and sidereal variations of the transition frequency, which can in principle be detected by comparison to a reference. Since optical clocks can outperform other frequency standards, an effective way to search for the effects predicted by Eq. (63) is to compare two optical clocks and search for a sidereal or annual modulation of their frequency difference. For systems with long-term stability, studying annual variations is preferable because the speed $\beta_\oplus$ is typically about 2 orders of magnitude bigger than β_L. Note also that the two clocks can be located in different laboratories. Using Eq. (63), we see that the annual and sidereal modulations of the frequency difference between clocks A and B are given by

$$2\pi\delta\nu_{\mathrm{AB}} = \sum_{d,k} \frac{\Delta p_{\mathrm{B}}^k - \Delta p_{\mathrm{A}}^k}{\sqrt{4\pi}}[\beta_\oplus \sin\Omega_\oplus T V_{e,k}^{(d)X}$$
$$- \beta_\oplus \cos\Omega_\oplus T(\cos\eta V_{e,k}^{(d)Y} + \sin\eta V_{e,k}^{(d)Z})]$$
$$+ \sum_{d,k} \frac{\Delta p_{\mathrm{B}}^k \beta_{L,\mathrm{B}} - \Delta p_{\mathrm{A}}^k \beta_{L,\mathrm{A}}}{\sqrt{4\pi}}(\cos\omega_\oplus T_\oplus V_{e,k}^{(d)Y}$$
$$- \sin\omega_\oplus T_\oplus V_{e,k}^{(d)X}), \tag{64}$$

TABLE XI. The quantities $V_{e,k}^{(d)J}$ for $5 \le d \le 8$.

$V_{e,k}^{(d)J}$	Combination
$V_{e,2}^{(5)J}$	$-2a_{\mathrm{eff}}^{(5)JTT} - a_{\mathrm{eff}}^{(5)JKK}$
$V_{e,2}^{(6)J}$	$4c_{\mathrm{eff}}^{(6)JTTT} + 4c_{\mathrm{eff}}^{(6)JTKK}$
$V_{e,2}^{(7)J}$	$-\frac{10}{3}(2a_{\mathrm{eff}}^{(7)JTTTT} + 3a_{\mathrm{eff}}^{(7)JTTKK})$
$V_{e,2}^{(8)J}$	$10c_{\mathrm{eff}}^{(8)JTTTTT} + 20c_{\mathrm{eff}}^{(8)JTTTKK}$
$V_{e,4}^{(7)J}$	$-a_{\mathrm{eff}}^{(7)JKKLL} - 4a_{\mathrm{eff}}^{(7)JTTKK}$
$V_{e,4}^{(8)J}$	$6c_{\mathrm{eff}}^{(8)JTKKLL} + 12c_{\mathrm{eff}}^{(8)JTTTKK}$

where Δp_A^k is the expectation value $\langle |\boldsymbol{p}|^k \rangle$ for the transitions in clock A, $\beta_{L,A}$ is the speed of the laboratory containing clock A, and Δp_B^k, $\beta_{L,B}$ are defined similarly for clock B.

Several laboratories have the potential to compare two clocks at the same location, searching for the effects predicted in Eq. (64) in a scenario with $\beta_{L,A} = \beta_{L,B}$. For example, scanning the literature cited above suggests that comparisons of any two lattice clocks based on ^{87}Sr, ^{171}Yb, or ^{199}Hg could in principle be performed at Rikagaku Kenkyūsho (RIKEN) in Japan. Similarly, ^{87}Sr and ^{199}Hg lattice clocks can be compared at the Système de Référence Temps-Espace (SYRTE) in France, ones based on ^{87}Sr and ^{171}Yb can be compared at the National Metrology Institute of Japan (NMIJ), and the ^{27}Al$^+$ ion clock could be compared to the ^{171}Yb lattice clock at the National Institute of Standards and Technology (NIST) in the United States. Many individual comparisons between clocks located at different institutions are also possible in principle, by using Eq. (64) with $\beta_{L,A} \neq \beta_{L,B}$. Moreover, some Lorentz- and *CPT*-violating effects that are absent in Eq. (64) and hence cannot be studied with any of these clock combinations might become accessible given suitable care for the treatment of systematics and its implication for cancellations of signals. Some examples of such experiments with clocks at a single location might include comparison of the ^{88}Sr$^+$ and ^{171}Yb$^+$ ion clocks at the National Physical Laboratory (NPL) in England, the ^{87}Sr lattice clock and the ^{171}Yb$^+$ ion clock at the Physikalisch-Technische Bundesanstalt (PTB) in Germany, or the ^{27}Al$^+$ and ^{199}Hg$^+$ ion clocks at NIST.

A qualitatively different approach to testing Lorentz and *CPT* symmetry is to create an entangled state and monitor its time evolution. In Ref. [26], the entangled state combines the states $(|\pm 5/2\rangle |\mp 5/2\rangle + |\pm 1/2\rangle |\mp 1/2\rangle)/\sqrt{2}$ of two ^{40}Ca$^+$ ions, where the kets $|m_F\rangle$ represent the m_F Zeeman level of the energy state $^2D_{5/2}$. The experimental observable $\bar{f}$ is obtained by averaging the energy difference between the product states $|\pm 5/2\rangle |\mp 5/2\rangle$ and $|\pm 1/2\rangle |\mp 1/2\rangle$.

Following the approach in Sec. II C, we assign angular momenta $J = 5/2$ and $L = 2$ to the valence electron. The Lorentz- and *CPT*-violating shift $\delta\bar{f}$ of the observable $\bar{f}$ in the laboratory frame is found to be

$$2\pi\delta\bar{f} = \frac{18}{7\sqrt{5\pi}}(\langle |\boldsymbol{p}|^2 \rangle \mathcal{V}_{e220}^{NR} + \langle |\boldsymbol{p}|^4 \rangle \mathcal{V}_{e420}^{NR}) + \frac{1}{7\sqrt{\pi}}\langle |\boldsymbol{p}|^4 \rangle \mathcal{V}_{p440}^{NR}. \quad (65)$$

This expression has a structure similar to that of the frequency shift (40) in fountain clocks, so we can adapt the results presented in Sec. III A to convert the expression (65) to the Sun-centered frame. The expression for the shift $\delta\bar{f}_0$ at zeroth boost order is therefore given by Table III with the replacements

$$\mathcal{V}_{pk2m}^{NR} \rightarrow -\frac{12}{5}\mathcal{V}_{ek2m}^{NR}, \qquad \mathcal{V}_{pk4m}^{NR} \rightarrow \frac{11}{45}\mathcal{V}_{ek4m}^{NR}. \quad (66)$$

At linear boost order, the contribution $\delta\bar{f}_1$ is

$$2\pi\delta\bar{f}_1 = \sum_d \frac{\langle |\boldsymbol{p}|^2 \rangle}{m_p^{5-d}}(V_{Ca,2}^{(d)J}\boldsymbol{\beta}^J + V_{Ca,2}^{(d)JKL}\hat{B}^K\hat{B}^L\boldsymbol{\beta}^J)$$

$$+ \sum_d \frac{\langle |\boldsymbol{p}|^4 \rangle}{m_p^{7-d}}(V_{Ca,4}^{(d)J}\boldsymbol{\beta}^J + V_{Ca,4}^{(d)JKL}\hat{B}^K\hat{B}^L\boldsymbol{\beta}^J)$$

$$+ \sum_d \frac{\langle |\boldsymbol{p}|^4 \rangle}{m_p^{7-d}}V_{Ca,4}^{(d)JKLMN}\hat{B}^K\hat{B}^L\hat{B}^M\hat{B}^N\boldsymbol{\beta}^J, \quad (67)$$

where expressions for the quantities $V_{Ca,k}^{(d)JJ_1...J_s}$ in terms of effective Cartesian coefficients are displayed in Table XII.

For the nonminimal terms considered in this work, the result (67) incorporates time variation at the first five harmonics of the sidereal frequency along with annual variations. At the sidereal frequency, the dominant contributions to the variations in the first four harmonics are given by Table III with the substitutions (66). The variation at the fifth harmonic is suppressed by β_L, and it is given by Eq. (46) with the replacement $V_{Cs,k}^{(d)J...K} \rightarrow V_{Ca,k}^{(d)J...K}$. Using these results, we can estimate the sensitivities of the ^{40}Ca$^+$ experiment [26] to the nonrelativistic coefficients. Table XIII displays these sensitivities. In deriving them, we take $\langle |\boldsymbol{p}|^2 \rangle \sim 10^{-11}$ GeV^2 and $\langle |\boldsymbol{p}|^4 \rangle \sim 10^{-22}$ GeV^4. We also suppose the experimental reach is 0.03 Hz. With sufficient stability and data collection over a long time

TABLE XII. The quantities $V_{Ca,k}^{(d)JJ_1...J_s}$ for $5 \leq d \leq 8$.

$V_{Ca,k}^{(d)JJ_1...J_s}$	Combination
$V_{Ca,2}^{(5)J}$	$-\frac{36}{35}(a_{e\mathrm{eff}}^{(5)JKK} + 2a_{e\mathrm{eff}}^{(5)JTT})$
$V_{Ca,2}^{(5)JJ_1J_2}$	$\frac{108}{35}(a_{e\mathrm{eff}}^{(5)JJ_1J_2} + 2\delta^{JJ_1}a_{e\mathrm{eff}}^{(5)TTJ_2})$
$V_{Ca,2}^{(6)J}$	$\frac{144}{35}(c_{e\mathrm{eff}}^{(6)JTKK} + c_{e\mathrm{eff}}^{(6)JTTT})$
$V_{Ca,2}^{(6)JJ_1J_2}$	$-\frac{432}{35}(c_{e\mathrm{eff}}^{(6)JTJ_1J_2} + \delta^{JJ_1}c_{e\mathrm{eff}}^{(6)TTTJ_2})$
$V_{Ca,2}^{(7)J}$	$-\frac{24}{7}(3a_{e\mathrm{eff}}^{(7)JTTKK} + 2a_{e\mathrm{eff}}^{(7)JTTTT})$
$V_{Ca,2}^{(7)JKL}$	$\frac{72}{7}(3a_{p\mathrm{eff}}^{(7)JTTJ_1J_2} + 2\delta^{JJ_1}a_{p\mathrm{eff}}^{(7)TTTTJ_2})$
$V_{Ca,4}^{(7)J}$	$-\frac{10}{7}(a_{e\mathrm{eff}}^{(7)JKKLL} + 4a_{e\mathrm{eff}}^{(7)JTTKK})$
$V_{Ca,4}^{(7)JJ_1J_2}$	$8(a_{e\mathrm{eff}}^{(7)JTTJ_1J_2} + \delta^{JJ_1}a_{e\mathrm{eff}}^{(7)TTJ_2KK}) + 4a_{e\mathrm{eff}}^{(7)JJ_1J_2KK}$
$V_{Ca,4}^{(7)JJ_1J_2J_3J_4}$	$\frac{10}{21}(a_{e\mathrm{eff}}^{(7)JJ_1J_2J_3J_4} + 4\delta^{JJ_1}a_{e\mathrm{eff}}^{(7)TTJ_2J_3J_4})$
$V_{Ca,2}^{(8)J}$	$\frac{72}{7}(2c_{e\mathrm{eff}}^{(8)JTTTKK} + c_{e\mathrm{eff}}^{(8)JTTTTT})$
$V_{Ca,2}^{(8)JJ_1J_2}$	$-\frac{216}{7}(2c_{e\mathrm{eff}}^{(8)JTTTJ_1J_2} + \delta^{JJ_1}c_{e\mathrm{eff}}^{(8)TTTTTJ_2})$
$V_{Ca,4}^{(8)J}$	$\frac{60}{7}(c_{e\mathrm{eff}}^{(8)JTKKLL} + 2c_{e\mathrm{eff}}^{(8)JTTTKK})$
$V_{Ca,4}^{(8)JJ_1J_2}$	$-24(c_{e\mathrm{eff}}^{(8)JTTTJ_1J_2} + \delta^{JJ_1}c_{e\mathrm{eff}}^{(8)TTTJ_2KK}) - 24c_{e\mathrm{eff}}^{(8)JTJ_1J_2KK}$
$V_{Ca,4}^{(8)JJ_1J_2J_3J_4}$	$-\frac{20}{7}(c_{e\mathrm{eff}}^{(8)JTJ_1J_2J_3J_4} - 2\delta^{JJ_1}c_{e\mathrm{eff}}^{(7)TTTJ_2J_3J_4})$

TABLE XIII. Potential sensitivities to coefficients in the Sun-centered frame from sidereal variations in entangled ^{40}Ca$^+$ ions.

Coefficient	Sensitivity
$|a_{e22m}^{\text{NR}}|$, $|c_{e22m}^{\text{NR}}|$	10^{-14} GeV^{-1}
$|a_{e42m}^{\text{NR}}|$, $|c_{e42m}^{\text{NR}}|$	10^{-3} GeV^{-3}
$|a_{e44m}^{\text{NR}}|$, $|c_{e44m}^{\text{NR}}|$	10^{-3} GeV^{-3}

period, constraints could also be placed on coefficients for Lorentz violation associated with the annual variation signal predicted by Eq. (67).

Related experiments have been proposed using Zeeman transitions of the $F_{7/2}$ state in Yb$^+$ [55]. Experiments using a dynamical decoupling technique have also been proposed for a broad class of trapped ions and lattice clocks [102]. All are expected to permit significant improvements over existing constraints on the coefficients $c_{\mu\nu}^{(4)}$. The energy shift produced by these minimal-SME coefficients can be generalized to incorporate also the contributions from the nonminimal coefficients $\mathcal{V}_{e22m}^{\text{NR}}$ in the nonrelativistic limit. The proposed experiments can therefore be expected to yield substantial improvements over the estimated sensitivities to the coefficients a_{e22m}^{NR} and c_{e22m}^{NR} given in Table XIII.

D. Antimatter clocks

In this final section on applications, we offer some comments about the prospects for spectroscopic experiments using antimatter. Comparisons of the properties of matter and antimatter are of particular interest for testing the *CPT* symmetry of quantum field theory. The line of reasoning outlined in the Introduction reveals that effective field theory also provides the general model-independent framework for analyzing antimatter systems, so the results of experiments testing *CPT* symmetry can be expressed in a model-independent way as constraints on SME coefficients.

A diverse set of such constraints has already been obtained via precision spectroscopy of positrons and antiprotons confined in a Penning trap [45–47,103–105]. Studying antimatter instead of antiparticles offers advantages in searches for *CPT* violation [43,106], and several collaborations are developing techniques for the precision spectroscopy of antihydrogen. Recently, the Antihydrogen Laser Physics Apparatus (ALPHA) Collaboration has measured the antihydrogen ground-state hyperfine transitions [107] and the 1S-2S transition [108], heralding an era of precision antimatter spectroscopy. Other collaborations pursuing this goal include the Atomic Spectroscopy and Collisions Using Slow Antiprotons (ASACUSA) Collaboration [109,110], and the Antihydrogen Trap (ATRAP) Collaboration [111]. Experiments investigating the gravitational response of antihydrogen are also being developed, including the Antihydrogen Experiment: Gravity, Interferometry, Spectroscopy (AEGIS)

Collaboration [112], the ALPHA Collaboration [113], and the Gravitational Behavior of Antihydrogen at Rest (GBAR) Collaboration [114], and the corresponding techniques may also enhance future spectroscopic studies of antihydrogen.

One signal for nonzero *CPT* violation would be a measured difference $\Delta\bar{\nu}_{1S2S} \equiv \nu_{1S2S} - \bar{\nu}_{1S2S}$ between the resonance frequency ν_{1S2S} of the 1S-2S transition in hydrogen and the analogous resonance frequency $\bar{\nu}_{1S2S}$ in antihydrogen. Performing a general analysis in the context of effective field theory [43] reveals that *CPT*-violating effects contributing to a nonzero value of $\Delta\bar{\nu}_{1S2S}$ can be classified as spin independent or spin dependent and as isotropic or anisotropic, and they can exhibit time variations induced by the noninertial nature of the experimental laboratory. It turns out that the spin-dependent effects are more readily studied using ground-state hyperfine transitions, while the time variations are better explored by directly studying modulations of $\bar{\nu}_{1S2S}$. However, the difference $\Delta\bar{\nu}_{1S2S}$ is particularly sensitive to isotropic, spin-independent, and time-constant *CPT* violation controlled by the coefficients $\mathring{a}_{e,2}^{\text{NR}}$, $\mathring{a}_{e,4}^{\text{NR}}$, $\mathring{a}_{p,2}^{\text{NR}}$, $\mathring{a}_{p,4}^{\text{NR}}$. An explicit expression for $\Delta\bar{\nu}_{1S2S}$ in terms of these coefficients is given by Eq. (86) of Ref. [43], with the correction $8 \rightarrow 16$ in the denominator. Note that these nonrelativistic coefficients incorporate effects from *CPT*-violating operators of arbitrary mass dimension [35].

Based on an analysis that assumes no spin-, geometry-, or time-dependent *CPT* violation, the ALPHA Collaboration reported agreement between the 1S-2S resonance frequencies of hydrogen and antihydrogen at a precision of 2×10^{-12} [108]. We can therefore deduce the constraint

$$\left| \mathring{a}_{e,2}^{\text{NR}} + \mathring{a}_{p,2}^{\text{NR}} + \frac{67}{12}(\alpha m_r)^2 (\mathring{a}_{e,4}^{\text{NR}} + \mathring{a}_{p,4}^{\text{NR}}) \right| < 1 \times 10^{-9} \text{ GeV}^{-1},$$

(68)

where α is the fine-structure constant and m_r is the reduced mass of hydrogen. The result (68) represents the first constraint on SME coefficients extracted from antihydrogen spectroscopy. Table XIV lists the corresponding maximal sensitivities obtained by taking each coefficient to be nonzero in turn, following the standard procedure in the literature [3]. Note that several factors currently limit the precision of the measurement of $\bar{\nu}_{1S2S}$, including the comparatively smaller

TABLE XIV. Constraints on electron and proton nonrelativistic coefficients determined from 1S-2S hydrogen and antihydrogen spectroscopy.

Coefficient	Constraint
$|\mathring{a}_{e,2}^{\text{NR}}|$, $|\mathring{a}_{p,2}^{\text{NR}}|$	1×10^{-9} GeV^{-1}
$|\mathring{a}_{e,4}^{\text{NR}}|$, $|\mathring{a}_{p,4}^{\text{NR}}|$	14 GeV^{-3}

number and higher temperature of atoms in antihydrogen experiments relative to hydrogen ones. However, there is every reason to expect improvements in the future. One proposal along these lines is to trap the ultracold antiatoms from the GBAR antihydrogen beam in an optical lattice [115], which could enable measurements of the 1S-2S transition in antihydrogen at a level approaching the precision of 4.2×10^{-15} already obtained with hydrogen [116].

Other signals for *CPT* violation can appear in comparisons of the hyperfine structure of hydrogen and antihydrogen [43,106]. High-precision measurements of the hyperfine transition of hydrogen can be obtained using a hydrogen maser [117], but these methods are impractical for antihydrogen due, for example, to collisions with the walls in the maser bulb. One different approach already realized by the ALPHA Collaboration is to perform hyperfine spectroscopy on trapped antihydrogen [107]. An alternative option being pursued by the ASACUSA Collaboration involves using instead an antihydrogen beam [110]. Testing the latter method with hydrogen has demonstrated a precision only 3 orders of magnitude below that achieved via the hydrogen maser. The prospects are excellent for further substantial improvements in hyperfine spectroscopy using advanced tools such as ultracold antihydrogen beams, and perhaps ultimately adapting techniques similar to those used for atomic fountain clocks.

In the longer term, antiatom spectroscopy could conceivably evolve to include also experiments with heavier antiatoms. The simplest candidate system is antideuterium, which has the antideuteron as its nucleus. Unlike antiprotonic deuterium, antideuterium is expected to be stable and is therefore in principle a candidate for precision spectroscopy and hence for *CPT* tests. Deuterium spectroscopy is known to be many orders of magnitude more sensitive than hydrogen spectroscopy to certain kinds of Lorentz and *CPT* violation [43], and the same arguments hold for the comparative sensitivities of antideuterium and antihydrogen spectroscopy. The production of a single heavier antiion is also of real interest, as it could in principle be confined in an ion trap and repeatedly interrogated to perform high-precision spectroscopy.

Whatever the future of antimatter experiments with heavier systems than antihydrogen, the theoretical treatments presented in Sec. II and in Ref. [43] can readily be adapted to antiatoms and antiions. In particular, the expression for the shift in an energy level of an antiatom or antiion can be obtained from the corresponding expression for an atom or ion by implementing the substitutions

$$a_{w\,jkm}^{\mathrm{NR}} \to -a_{w\,jkm}^{\mathrm{NR}}, \qquad c_{w\,jkm}^{\mathrm{NR}} \to c_{w\,jkm}^{\mathrm{NR}},$$
$$H_{w\,jkm}^{\mathrm{NR}(sB)} \to -H_{w\,jkm}^{\mathrm{NR}(sB)}, \qquad g_{w\,jkm}^{\mathrm{NR}(sB)} \to g_{w\,jkm}^{\mathrm{NR}(sB)} \qquad (69)$$

for the SME coefficients. For example, an expression for the frequency shift $\delta\nu_{\bar{\mathrm{D}}}$ of the $nL\text{-}n'L'$ transition in antideuterium due to isotropic Lorentz and *CPT* violation can be obtained from the corresponding expression for the shift $\delta\nu_{\mathrm{D}}$ in deuterium given as Eq. (103) of Ref. [43], yielding the result

$$2\pi\delta\nu_{\bar{\mathrm{D}}} = \frac{\bar{m}_{\mathrm{r}}}{\sqrt{\pi}}(\varepsilon_{n'} - \varepsilon_n)\left[\bar{\mathcal{V}}_{e200}^{\mathrm{NR}} + \frac{1}{4}(\bar{\mathcal{V}}_{p200}^{\mathrm{NR}} + \bar{\mathcal{V}}_{n200}^{\mathrm{NR}}) \right.$$
$$\left. + \langle \boldsymbol{p}_{\bar{p}\,\bar{d}}^2 \rangle (\bar{\mathcal{V}}_{p400}^{\mathrm{NR}} + \bar{\mathcal{V}}_{n400}^{\mathrm{NR}}) \right]$$
$$- \frac{2\bar{m}_{\mathrm{r}}^2}{\sqrt{\pi}}\left[\varepsilon_{n'}^2\left(\frac{8n'}{2L'+1} - 3 \right) - \varepsilon_n^2\left(\frac{8n}{2L+1} - 3 \right) \right]$$
$$\times \left(\bar{\mathcal{V}}_{e400}^{\mathrm{NR}} + \frac{1}{16}(\bar{\mathcal{V}}_{p400}^{\mathrm{NR}} + \bar{\mathcal{V}}_{n400}^{\mathrm{NR}}) \right), \qquad (70)$$

where $\bar{\mathcal{V}}_{wkjm}^{\mathrm{NR}} = c_{wkjm}^{\mathrm{NR}} + a_{wkjm}^{\mathrm{NR}}$, $\bar{m}_{\mathrm{r}}$ is the reduced mass of antideuterium, $\varepsilon_n \equiv -\alpha^2\bar{m}_{\mathrm{r}}/2n^2$, and $\langle \boldsymbol{p}_{\bar{p}\,\bar{d}}^2 \rangle \simeq 10^4 \ \mathrm{MeV}^2$.

IV. SUMMARY

This work studies Lorentz and *CPT* violation in clock-comparison experiments by incorporating effects on electron and nucleon propagators arising from SME operators of arbitrary mass dimension d. It begins with a discussion of theoretical issues in Sec. II. The general Lagrange density (1) for a fermion propagating in the presence of arbitrary Lorentz and *CPT* violation implies the perturbative result (2) for the corresponding nonrelativistic one-particle Hamiltonian. Combining the expressions for the constituent particles yields the Hamiltonian (6) for an atom or ion, which is the basis for our analysis of clock-comparison experiments.

The experimental observables are transition frequencies in atoms or ions. The Lorentz- and *CPT*-violating signals in these frequencies can be calculated from the perturbative shifts (8) in energy levels. These shifts involve products of Clebsch-Gordan coefficients with expectation values of the perturbative Hamiltonian. The symmetries of the system imply that contributions to the energy shifts can arise only from specific nonrelativistic spherical coefficients for Lorentz violation, as listed in Table I. Explicit computation of the expectation values requires modeling the electronic and nuclear states. Our approach for electrons adopts the independent-particle model described in Sec. II C, while for the nucleus we use the Schmidt model as discussed in Sec. II D.

A laboratory on the surface of the Earth or on an orbiting satellite typically represents a noninertial frame. As a result, most SME coefficients measurable in the laboratory acquire a dependence on time due to the laboratory rotation and boost relative to the canonical Sun-centered frame, which is an approximately inertial frame over the experimental timescale. Determining the time dependence induced by the rotation of the Earth is the subject of Sec. II E. This treatment is extended in Sec. II F to include effects at linear order in the Earth's boost as it orbits the Sun. The time

dependence in a space-based laboratory arising from the orbital motion of a satellite is also discussed.

The application of our results to the analysis of clock-comparison experiments is described in Sec. III. We first consider fountain clocks, deriving expressions for the frequency shift in terms of coefficients expressed in the Sun-centered frame. At zeroth boost order the frequency shift is given in Table III, while at linear boost order it is given by Eq. (44) and the entries in Table IV. Estimates for attainable sensitivities to SME coefficients using existing devices are provided in Table V. The discussion covers both ^{133}Cs and ^{87}Rb fountain clocks, and it is also applicable to clocks located on a space-based platform. The primary sensitivity in these systems is to coefficients for Lorentz violation in the proton sector.

We next consider the prospects for using comagnetometers to search for nonminimal violations of Lorentz and *CPT* symmetry. The methodology developed in Sec. II is well suited for application to investigations using ^{129}Xe and ^{3}He atoms as comagnetometers. Within the nuclear model adopted here, the Lorentz- and *CPT*-violating signals are affected predominantly by SME coefficients in the neutron sector. In Sec. III B, we determine the shift in the experimental frequency observable at zeroth boost order and extract the bound (54) by extending to arbitrary d the known results for the minimal SME. This leads to the constraints on neutron nonrelativistic coefficients listed in Table VI. We also establish the Lorentz- and *CPT*-violating shift at linear boost order, using existing data to place constraints on neutron effective Cartesian coefficients in Table IX. Other comagnetometers can also place competitive limits on the neutron sector of the SME. We derive a partial map from known minimal-SME bounds to nonminimal coefficients, which permits using data from a ^{21}Ne-Rb-K comagnetometer to place the additional constraints on the neutron sector given in Table X. All these constraints on neutron coefficients for Lorentz violation are the first of their kind reported in the literature.

As another application, we consider the attainable reach in clock-comparison experiments using trapped ions and lattice clocks. In this case, interesting sensitivities are in principle attainable to coefficients for Lorentz violation in the electron sector. Various transitions are considered for a range of atoms and ions. The expression (64) is found to describe the annual and sidereal modulations of the frequency difference between two clocks, including ones located in distinct laboratories. In this section, we also consider tests of Lorentz and *CPT* symmetry based on studying the time evolution of an entangled state. The shift in the experimental frequency observable is determined at both zeroth and first boost order and is used to estimate attainable sensitivities to electron nonrelativistic coefficients, as listed in Table XIII.

Our final application considers the prospects for experiments using antimatter. Signals for Lorentz and *CPT* violation in antihydrogen have previously been investigated

theoretically both in the minimal SME [106] and allowing for nonminimal terms of arbitrary mass dimension [43]. These treatments are combined with recent spectroscopic measurements of the 1*S*-2*S* transition in antihydrogen to extract first constraints on SME coefficients from this system, summarized in Table XIV. We also propose that in the long term it may become feasible to perform experiments with heavier antiatoms and antiions, with options possibly including the precision spectroscopy of antideuterium or of trapped antiions. A technique is presented to convert theoretical results for frequency shifts in atoms or ions to the corresponding ones in antiatoms or antiions.

The two appendices following the present summary collect some results that are useful in handling coefficients for Lorentz violation. Appendix A includes relations connecting spherical and Cartesian coefficients and provides explicit expressions between them for the cases $3 \leq d \leq 8$. Appendix B discusses the transformation between the laboratory frame and the Sun-centered frame and tabulates explicit results connecting Cartesian coefficients in the two frames for the cases $3 \leq d \leq 8$. The results in these appendices are generally applicable and so have implications extending outside the analysis of clock-comparison experiments.

Throughout this work, we have noted possibilities for pursuing investigations that go beyond our present scope while remaining within the context of Lorentz- and *CPT*-violating corrections to the propagators of the constituents of atoms and ions. In principle, our scope could also be extended by incorporating effects arising from other SME sectors. For instance, the Maxwell equations acquire modifications due to Lorentz and *CPT* violation in the pure-photon sector. Including these might further enhance the reach of clock-comparison experiments, though in practice most relevant photon-sector coefficients are already tightly bounded from analyses of other systems [3,118,119]. Effects involving U(1)-covariant Lorentz- and *CPT*-violating couplings between photons and fermions are of interest as well, with only a few SME coefficients currently constrained by experiment [3,45,120]. One could also envisage the inclusion of SME effects arising in the strong, electroweak, or gravitational sectors, although some of these are expected either to be suppressed or to be more readily studied by other means. An exception might be countershaded Lorentz and *CPT* violation [121], for which unexpectedly large effects can appear in the context of special measurements. For example, sensitivity to countershaded coefficients has been demonstrated using atom interferometry, which can be interpreted in terms of clock comparisons [122].

Overall, the content of this paper provides a broad methodology for exploring Lorentz and *CPT* symmetry using clock-comparison experiments. While our treatment has yielded many first constraints, numerous coefficients for Lorentz violation are unmeasured to date. The striking potential sensitivities attainable either from reanalysis of existing data or in future searches suggest that further work with clock-comparison experiments remains one of the

most interesting prospects for uncovering these novel physical effects in nature.

APPENDIX A: RELATION BETWEEN SPHERICAL AND CARTESIAN COEFFICIENTS

This Appendix presents relationships between spherical coefficients and effective Cartesian coefficients and tabulates explicit results for $d \leq 8$. We focus on spherical coefficients for Lorentz violation with even k and $m = 0$, which are centrally relevant to analyses of clock-comparison experiments. The coefficients $\mathcal{T}_{kjm}^{(d)(1E)}$ controlling spin-dependent operators of the E type are disregarded here as they leave unaffected the energy shifts. As elsewhere in this work, we follow Ref. [35] in using the symbol $\mathcal{V}$ with appropriate subscripts and superscripts to indicate the difference of c- and a-type coefficients and $\mathcal{T}$ to indicate the difference of g- and H-type coefficients. For instance, $\mathcal{V}_{kjm}^{(d)}$ represents the difference $\mathcal{V}_{kjm}^{(d)} = c_{kjm}^{(d)} - a_{kjm}^{(d)}$. The spherical coefficients are assigned indices $kj0$, while t, x, y, z are used for specific index values on the effective Cartesian coefficients in a chosen frame. Dummy spatial Cartesian indices are represented by l, m, n, and repeated Cartesian indices are summed. For example, $c_{\text{eff}}^{(4)ll}$ represents the sum $c_{\text{eff}}^{(4)ll} = c_{\text{eff}}^{(4)xx} + c_{\text{eff}}^{(4)yy} + c_{\text{eff}}^{(4)zz}$.

The single-particle Hamiltonian can be decomposed in either the spherical or the Cartesian bases. The connection between these decompositions is presented in Sec. IV of Ref. [35]. Consider first the spin-independent component of the Hamiltonian. The corresponding match between the Cartesian and spherical bases is fixed by

$$
\begin{aligned}
\hat{\mathcal{V}}_{\text{eff}}^{(d)\mu} p_\mu &= \mathcal{V}_{\text{eff}}^{(d)\mu\alpha_1\alpha_2\ldots\alpha_{d-3}} p_\mu p_{\alpha_1} p_{\alpha_2}\cdots p_{\alpha_{d-3}} \\
&= \sum_{kjm} E_0^{d-2-k}|\boldsymbol{p}|^k Y_{jm}(\hat{\boldsymbol{p}})\mathcal{V}_{kjm}^{(d)},
\end{aligned} \tag{A1}
$$

where $p_\mu = (E_0, -\boldsymbol{p})$. Using the orthogonality of the spherical harmonics, the connection between the Cartesian and spherical terms can be written as

$$
\int d\Omega Y_{jm}^*(\hat{\boldsymbol{p}})\hat{\mathcal{V}}_{\text{eff}}^{(d)\mu} p_\mu = \sum_{k=j}^{d-2} E_0^{d-2-k}|\boldsymbol{p}|^k \mathcal{V}_{kjm}^{(d)}, \tag{A2}
$$

where $d\Omega$ is the differential element of solid angle in momentum space. The upper and lower bounds for the summation index k are determined by the spherical-index relations listed in Table III of Ref. [35].

Using Eq. (A2), we can extract explicit expressions for the spin-independent spherical coefficients in terms of effective Cartesian coefficients. Table XV contains the results for spherical coefficients with $3 \leq d \leq 8$, $m = 0$, even values of j in the range $0 \leq j \leq k$, and even values of k in the range $0 \leq k \leq d - 2$. The table consists of two pairs of columns. In each pair, the first entry in a given row lists a spherical coefficient, while the second entry provides its equivalent as a linear combination of effective Cartesian coefficients.

Next, we consider the spin-dependent part of the single-particle Hamiltonian. For the component involving only the coefficients $\mathcal{T}_{kjm}^{(d)(0B)}$, the relation between the Cartesian and spherical terms is

$$
\begin{aligned}
\tilde{\tilde{\mathcal{T}}}_{\text{eff}}^{(d)\mu t} p_\mu &= \tilde{\mathcal{T}}_{\text{eff}}^{(d)\mu t \alpha_1\alpha_2\ldots\alpha_{d-3}} p_\mu p_{\alpha_1} p_{\alpha_2}\cdots p_{\alpha_{d-3}} \\
&= \sum_{kjm} E_0^{d-3-k}|\boldsymbol{p}|^{k+1}(k+1)Y_{jm}(\hat{\boldsymbol{p}})\mathcal{T}_{kjm}^{(d)(0B)}. \tag{A3}
\end{aligned}
$$

Using orthonormality of the spherical harmonics then yields

$$
\int d\Omega Y_{jm}^*(\hat{\boldsymbol{p}})\tilde{\tilde{\mathcal{T}}}_{\text{eff}}^{(d)\mu t} p_\mu = \sum_{k=j-1}^{d-3} E_0^{d-3-k}|\boldsymbol{p}|^{k+1}(k+1)\mathcal{T}_{kjm}^{(d)(0B)} \tag{A4}
$$

between effective Cartesian coefficients and spherical coefficients. This result permits the extraction of explicit expressions for the spin-dependent spherical coefficients $\mathcal{T}_{kjm}^{(d)(0B)}$ as linear combinations of effective Cartesian coefficients. Table XVI contains these expressions for spherical coefficients $\mathcal{T}_{kjm}^{(d)(0B)}$ with $3 \leq d \leq 8$, $m = 0$, odd values of j in the range $0 \leq j \leq k + 1$, and even values of k in the range $0 \leq k \leq d - 3$. The structure of this table parallels that of Table XV.

Determining the spherical coefficients $\mathcal{T}_{kjm}^{(d)(1B)}$ in terms of effective Cartesian coefficients requires more work because the relation containing $\mathcal{T}_{kjm}^{(d)(1B)}$ also incorporates the coefficients $\mathcal{T}_{kjm}^{(d)(0B)}$. We find

$$
\begin{aligned}
&\int d\Omega \tilde{\tilde{\mathcal{T}}}_{\text{eff}}^{(d)j\nu} \hat{\boldsymbol{\epsilon}}_+^j p_{\nu 1} Y_{jm}^*(\hat{\boldsymbol{p}}) \\
&= \sum_{k=j-1}^{d-2} E_0^{d-2-k}|\boldsymbol{p}|^k \left(\sqrt{\frac{j(j+1)}{2}}\mathcal{T}_{kjm}^{(d)(0B)} \right. \\
&\quad \left. + \mathcal{T}_{kjm}^{(d)(1B)} + i\mathcal{T}_{kjm}^{(d)(1E)} \right), \tag{A5}
\end{aligned}
$$

where $\hat{\boldsymbol{\epsilon}}_{\pm} = (\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}})/\sqrt{2}$. This result links three types of spherical coefficients with the effective Cartesian coefficients. It can be disentangled first by eliminating the

TABLE XV. Relations between spherical coefficients $\mathcal{V}_{kj0}^{(d)}$ and effective Cartesian coefficients for $3 \leq d \leq 8$.

Spherical	Cartesian	Spherical	Cartesian
$a_{000}^{(3)}$	$\sqrt{4\pi}\, a_{\text{eff}}^{(3)t}$	$c_{400}^{(6)}$	$\frac{1}{5}\sqrt{4\pi}\, c_{\text{eff}}^{(6)llmm}$
$a_{000}^{(5)}$	$\sqrt{4\pi}\, a_{\text{eff}}^{(5)ttt}$	$c_{220}^{(6)}$	$4\sqrt{\frac{\pi}{5}}(3c_{\text{eff}}^{(6)ttzz} - c_{\text{eff}}^{(6)ttmm})$
$a_{200}^{(5)}$	$\sqrt{4\pi}\, a_{\text{eff}}^{(5)tll}$	$c_{420}^{(6)}$	$\frac{4}{7}\sqrt{\frac{\pi}{5}}(3c_{\text{eff}}^{(6)llzz} - c_{\text{eff}}^{(6)llmm})$
$a_{220}^{(5)}$	$\sqrt{\frac{4\pi}{5}}(3a_{\text{eff}}^{(5)tzz} - a_{\text{eff}}^{(5)tll})$	$c_{440}^{(6)}$	$\frac{\sqrt{4\pi}}{21}(7c_{\text{eff}}^{(6)zzzz} - 6c_{\text{eff}}^{(6)zzll}) + \frac{\sqrt{4\pi}}{35}c_{\text{eff}}^{(6)mmll}$
$a_{000}^{(7)}$	$\sqrt{4\pi}\, a_{\text{eff}}^{(7)ttttt}$	$c_{000}^{(8)}$	$\sqrt{4\pi}\, c_{\text{eff}}^{(8)tttttt}$
$a_{200}^{(7)}$	$\frac{10}{3}\sqrt{4\pi}\, a_{\text{eff}}^{(7)tttll}$	$c_{200}^{(8)}$	$5\sqrt{4\pi}\, c_{\text{eff}}^{(8)ttttll}$
$a_{400}^{(7)}$	$\sqrt{4\pi}\, a_{\text{eff}}^{(7)tmmll}$	$c_{400}^{(8)}$	$3\sqrt{4\pi}\, c_{\text{eff}}^{(8)ttmmll}$
$a_{220}^{(7)}$	$\frac{4}{3}\sqrt{5\pi}(3a_{\text{eff}}^{(7)tttzz} - a_{\text{eff}}^{(7)tttmm})$	$c_{600}^{(8)}$	$\frac{1}{7}\sqrt{4\pi}\, c_{\text{eff}}^{(8)nnllmm}$
$a_{420}^{(7)}$	$\frac{4}{7}\sqrt{5\pi}(3a_{\text{eff}}^{(7)tllzz} - a_{\text{eff}}^{(7)tllmm})$	$c_{220}^{(8)}$	$2\sqrt{5\pi}(3c_{\text{eff}}^{(8)ttttzz} - c_{\text{eff}}^{(8)ttttmm})$
$a_{440}^{(7)}$	$\frac{\sqrt{100\pi}}{21}(7a_{\text{eff}}^{(7)tzzzz} - 6a_{\text{eff}}^{(7)tzzll}) + \frac{\sqrt{4\pi}}{7}a_{\text{eff}}^{(7)tmmll}$	$c_{420}^{(8)}$	$\frac{12}{7}\sqrt{5\pi}(3c_{\text{eff}}^{(8)ttllzz} - c_{\text{eff}}^{(8)ttllmm})$
$c_{000}^{(4)}$	$\sqrt{4\pi}\, c_{\text{eff}}^{(4)tt}$	$c_{620}^{(8)}$	$\frac{2}{21}\sqrt{5\pi}(3c_{\text{eff}}^{(8)nnllzz} - c_{\text{eff}}^{(8)nnllmm})$
$c_{200}^{(4)}$	$\frac{1}{3}\sqrt{4\pi}\, c_{\text{eff}}^{(4)ll}$	$c_{440}^{(8)}$	$\frac{\sqrt{100\pi}}{7}(7c_{\text{eff}}^{(8)ttzzzz} - 6c_{\text{eff}}^{(8)ttzzll}) + 3\frac{\sqrt{4\pi}}{7}c_{\text{eff}}^{(8)ttmmll}$
$c_{220}^{(4)}$	$\frac{2}{3}\sqrt{\frac{\pi}{5}}(3c_{\text{eff}}^{(4)zz} - c_{\text{eff}}^{(4)ll})$	$c_{640}^{(8)}$	$\frac{\sqrt{100\pi}}{77}(7c_{\text{eff}}^{(8)mmzzzz} - 6c_{\text{eff}}^{(8)mmzzll}) + \frac{\sqrt{4\pi}}{77}3c_{\text{eff}}^{(8)nnmmll}$
$c_{000}^{(6)}$	$\sqrt{4\pi}\, c_{\text{eff}}^{(6)tttt}$	$c_{660}^{(8)}$	$\frac{\sqrt{4\pi}}{231}(231c_{\text{eff}}^{(8)zzzzzz} - 5c_{\text{eff}}^{(8)nnmmll})$
$c_{200}^{(6)}$	$2\sqrt{4\pi}\, c_{\text{eff}}^{(6)ttll}$		$+\frac{\sqrt{100\pi}}{11}(c_{\text{eff}}^{(8)zzmmll} - 3c_{\text{eff}}^{(8)zzzzmm})$

TABLE XVI. Relations between spherical coefficients $\mathcal{T}_{kj0}^{(d)(0B)}$ and effective Cartesian coefficients for $3 \leq d \leq 8$.

Spherical	Cartesian	Spherical	Cartesian
$H_{010}^{(3)(0B)}$	$\sqrt{\frac{4\pi}{3}}\tilde{H}_{\text{eff}}^{(3)tz}$	$g_{010}^{(4)(0B)}$	$\sqrt{\frac{4\pi}{3}}\tilde{g}_{\text{eff}}^{(4)tzt}$
$H_{010}^{(5)(0B)}$	$\sqrt{\frac{4\pi}{3}}\tilde{H}_{\text{eff}}^{(5)tztt}$	$g_{010}^{(6)(0B)}$	$\sqrt{\frac{4\pi}{3}}\tilde{g}_{\text{eff}}^{(6)tzttt}$
$H_{210}^{(5)(0B)}$	$\frac{1}{15}\sqrt{\frac{4\pi}{3}}(2\tilde{H}_{\text{eff}}^{(5)tllz} + \tilde{H}_{\text{eff}}^{(5)tzll})$	$g_{210}^{(6)(0B)}$	$\frac{1}{5}\sqrt{\frac{4\pi}{3}}(2\tilde{g}_{\text{eff}}^{(6)tlltz} + \tilde{g}_{\text{eff}}^{(6)tztll})$
$H_{230}^{(5)(0B)}$	$\frac{2}{15}\sqrt{\frac{\pi}{7}}(5\tilde{H}_{\text{eff}}^{(5)tzzz} - \tilde{H}_{\text{eff}}^{(5)tzll}) - \frac{4}{15}\sqrt{\frac{\pi}{7}}\tilde{H}_{\text{eff}}^{(5)tllz}$	$g_{230}^{(6)(0B)}$	$\frac{2}{5}\sqrt{\frac{\pi}{7}}(5\tilde{g}_{\text{eff}}^{(6)tztzz} - \tilde{g}_{\text{eff}}^{(6)tztll}) - \frac{4}{5}\sqrt{\frac{\pi}{7}}\tilde{g}_{\text{eff}}^{(6)tltlz}$
$H_{010}^{(7)(0B)}$	$\sqrt{\frac{4\pi}{3}}\tilde{H}_{\text{eff}}^{(7)tzttt}$	$g_{010}^{(8)(0B)}$	$\sqrt{\frac{4\pi}{3}}\tilde{g}_{\text{eff}}^{(8)tzttttt}$
$H_{210}^{(7)(0B)}$	$\frac{2}{5}\sqrt{\frac{4\pi}{3}}(2\tilde{H}_{\text{eff}}^{(7)tmtttmz} + \tilde{H}_{\text{eff}}^{(7)tzttmm})$	$g_{210}^{(8)(0B)}$	$\frac{2}{3}\sqrt{\frac{4\pi}{3}}(2\tilde{g}_{\text{eff}}^{(8)tmttttmz} + \tilde{g}_{\text{eff}}^{(8)tzttttmm})$
$H_{410}^{(7)(0B)}$	$\frac{2}{175}\sqrt{3\pi}(4\tilde{H}_{\text{eff}}^{(7)tmmllz} + \tilde{H}_{\text{eff}}^{(7)tzllmm})$	$g_{410}^{(8)(0B)}$	$\frac{2}{35}\sqrt{3\pi}(4\tilde{g}_{\text{eff}}^{(8)tmtmllz} + \tilde{g}_{\text{eff}}^{(8)tztllmm})$
$H_{230}^{(7)(0B)}$	$\frac{4}{5}\sqrt{\frac{\pi}{7}}(5\tilde{H}_{\text{eff}}^{(7)tzttzz} - \tilde{H}_{\text{eff}}^{(7)tzttmm}) - \frac{8}{5}\sqrt{\frac{\pi}{7}}\tilde{H}_{\text{eff}}^{(7)tmttmz}$	$g_{230}^{(8)(0B)}$	$\frac{4}{3}\sqrt{\frac{\pi}{7}}(5\tilde{g}_{\text{eff}}^{(8)tztttzz} - \tilde{g}_{\text{eff}}^{(8)tztttmm}) - \frac{8}{3}\sqrt{\frac{\pi}{7}}\tilde{g}_{\text{eff}}^{(8)tmttttmz}$
$H_{430}^{(7)(0B)}$	$\frac{4}{45}\sqrt{\frac{\pi}{7}}(3\tilde{H}_{\text{eff}}^{(7)tzmmzz} + 2\tilde{H}_{\text{eff}}^{(7)tmmzzz})$ $-\frac{4}{75}\sqrt{\frac{\pi}{7}}(4\tilde{H}_{\text{eff}}^{(7)tmmllz} + \tilde{H}_{\text{eff}}^{(7)tzmmll})$	$g_{430}^{(8)(0B)}$	$\frac{4}{9}\sqrt{\frac{\pi}{7}}(3\tilde{g}_{\text{eff}}^{(8)tztmmzz} + 2\tilde{g}_{\text{eff}}^{(8)tmtmzzz})$ $-\frac{4}{15}\sqrt{\frac{\pi}{7}}(4\tilde{g}_{\text{eff}}^{(8)tmtmllz} + \tilde{g}_{\text{eff}}^{(8)tztmmll})$
$H_{450}^{(7)(0B)}$	$\frac{2}{15}\sqrt{\frac{\pi}{11}}(3\tilde{H}_{\text{eff}}^{(7)tzzzzz} - 2\tilde{H}_{\text{eff}}^{(7)tzmmzz})$ $+\frac{2}{105}\sqrt{\frac{\pi}{11}}(4\tilde{H}_{\text{eff}}^{(7)tmmllz} + \tilde{H}_{\text{eff}}^{(7)tzmmll})$ $-\frac{8}{45}\sqrt{\frac{\pi}{11}}\tilde{H}_{\text{eff}}^{(7)tmmzzz}$	$g_{450}^{(8)(0B)}$	$\frac{2}{3}\sqrt{\frac{\pi}{11}}(3\tilde{g}_{\text{eff}}^{(8)tztzzzz} - 2\tilde{g}_{\text{eff}}^{(8)tztmmzz})$ $+\frac{2}{21}\sqrt{\frac{\pi}{11}}(4\tilde{g}_{\text{eff}}^{(8)tmtmllz} + \tilde{g}_{\text{eff}}^{(8)tztmmll})$ $-\frac{8}{9}\sqrt{\frac{\pi}{11}}\tilde{g}_{\text{eff}}^{(8)trtrzzz}$

TABLE XVII. Relations between spherical coefficients $\mathcal{T}_{kj0}^{(d)(1B)}$ and effective Cartesian coefficients for $3 \leq d \leq 8$.

Spherical	Cartesian	Spherical	Cartesian
$H_{210}^{(5)(1B)}$	$\frac{2}{3}\sqrt{\frac{\pi}{3}}(3\tilde{H}_{\text{eff}}^{(5)nztn} - \tilde{H}_{\text{eff}}^{(5)tnnz}) + \frac{2}{3}\sqrt{\frac{\pi}{3}}\tilde{H}_{\text{eff}}^{(5)tznn}$	$g_{210}^{(8)(1B)}$	$\frac{5}{3}\sqrt{\frac{\pi}{3}}(4\tilde{g}_{\text{eff}}^{(8)tztttmm} - 4\tilde{g}_{\text{eff}}^{(8)tmtttmz}) + 5\sqrt{\frac{\pi}{3}}\tilde{g}_{\text{eff}}^{(8)mztttm}$
$H_{210}^{(7)(1B)}$	$4\sqrt{\frac{\pi}{3}}(\tilde{H}_{\text{eff}}^{(7)tzttnn} - \tilde{H}_{\text{eff}}^{(7)tntnz}) + 4\sqrt{\frac{\pi}{3}}\tilde{H}_{\text{eff}}^{(7)nzttn}$	$g_{410}^{(8)(1B)}$	$\frac{4}{5}\sqrt{3\pi}(\tilde{g}_{\text{eff}}^{(8)tztllmm} - \tilde{g}_{\text{eff}}^{(8)tmtmllz}) + 2\sqrt{3\pi}\tilde{g}_{\text{eff}}^{(8)mzttmll}$
$H_{410}^{(7)(1B)}$	$\frac{4}{25}\sqrt{3\pi}(\tilde{H}_{\text{eff}}^{(7)tzllnn} - \tilde{H}_{\text{eff}}^{(7)tnnllz}) + \frac{4}{5}\sqrt{3\pi}\tilde{H}_{\text{eff}}^{(7)nztnll}$	$g_{610}^{(8)(1B)}$	$\frac{1}{7}\sqrt{3\pi}\tilde{g}_{\text{eff}}^{(8)mzmnnll}$
$H_{430}^{(7)(1B)}$	$\frac{2}{5}\sqrt{\frac{6\pi}{7}}(5\tilde{H}_{\text{eff}}^{(7)nztnzz} + \tilde{H}_{\text{eff}}^{(7)tznnzz})$	$g_{430}^{(8)(1B)}$	$\sqrt{\frac{6\pi}{7}}(5\tilde{g}_{\text{eff}}^{(8)mzttmzz} + 2\tilde{g}_{\text{eff}}^{(8)tztmmzz})$
	$- \frac{2}{5}\sqrt{\frac{6\pi}{7}}(\tilde{H}_{\text{eff}}^{(7)tnnzzz} + \tilde{H}_{\text{eff}}^{(7)nztnll})$		$- \sqrt{\frac{6\pi}{7}}(2\tilde{g}_{\text{eff}}^{(8)tmtmzzz} + \tilde{g}_{\text{eff}}^{(8)mzttmll})$
	$+ \frac{2}{25}\sqrt{\frac{6\pi}{7}}(\tilde{H}_{\text{eff}}^{(7)tnnllz} - \tilde{H}_{\text{eff}}^{(7)tznnll})$		$+ \frac{2}{5}\sqrt{\frac{6\pi}{7}}(\tilde{g}_{\text{eff}}^{(8)tmtmllz} - \tilde{g}_{\text{eff}}^{(8)tztmmll})$
$g_{210}^{(4)(1B)}$	$\sqrt{\frac{\pi}{3}}\tilde{g}_{\text{eff}}^{(4)nzn}$	$g_{630}^{(8)(1B)}$	$\frac{1}{3}\sqrt{\frac{2\pi}{21}}(5\tilde{g}_{\text{eff}}^{(8)mzmllzz} - \tilde{g}_{\text{eff}}^{(8)mzmllnn})$
$g_{210}^{(6)(1B)}$	$\sqrt{\frac{4\pi}{3}}(\tilde{g}_{\text{eff}}^{(6)tztnn} - \tilde{g}_{\text{eff}}^{(6)tntnz}) + \sqrt{3\pi}\tilde{g}_{\text{eff}}^{(6)nzttn}$	$g_{650}^{(8)(1B)}$	$\frac{1}{3}\sqrt{\frac{5\pi}{33}}(3\tilde{g}_{\text{eff}}^{(8)mzmzzzz} - 2\tilde{g}_{\text{eff}}^{(8)mzmllzz})$
			$+ \frac{1}{21}\sqrt{\frac{5\pi}{33}}\tilde{g}_{\text{eff}}^{(8)mzmllnn}$
$g_{410}^{(6)(1B)}$	$\frac{1}{5}\sqrt{3\pi}\tilde{g}_{\text{eff}}^{(6)nznmm}$		
$g_{430}^{(6)(1B)}$	$\frac{1}{5}\sqrt{\frac{3\pi}{14}}(5\tilde{g}_{\text{eff}}^{(6)lzlzz} - \tilde{g}_{\text{eff}}^{(6)lzlnn})$		

$\mathcal{T}_{kjm}^{(d)(0B)}$ via Eq. (A4) and then by grouping the remaining terms according to powers of the momentum magnitude. The point is that the E-type and B-type coefficients are proportional to distinct powers of the momentum when j is fixed. For example, if j is odd then the terms involving B-type and E-type coefficients can only contain even and odd powers of the momentum magnitude, respectively.

For the particular case with $m = 0$, the spherical coefficients and the spin-weighted harmonics are all real numbers. It is therefore useful to separate the real and imaginary parts of Eq. (A5). The real part is

$$\int d\Omega \tilde{\bar{\mathcal{T}}}_{\text{eff}}^{(d)j\nu} \hat{\boldsymbol{\theta}}^j p_{\nu 1} Y_{j0}(\hat{\boldsymbol{p}})$$
$$= \sum_{k=j-1}^{d-2} E_0^{d-2-k}|\boldsymbol{p}|^k(\sqrt{j(j+1)}\mathcal{T}_{kj0}^{(d)(0B)} + \sqrt{2}\mathcal{T}_{kj0}^{(d)(1B)}),$$

(A6)

and it contains only B-type coefficients. The imaginary part of Eq. (A5) is given by

$$\int d\Omega \tilde{\bar{\mathcal{T}}}_{\text{eff}}^{(d)j\nu} \hat{\boldsymbol{\phi}}^j p_{\nu 1} Y_{j0}(\hat{\boldsymbol{p}}) = \sqrt{2}\sum_{k=j}^{d-2} E_0^{d-2-k}|\boldsymbol{p}|^k \mathcal{T}_{kj0}^{(d)(1E)}$$

(A7)

and contains only E-type coefficients. By combining Eqs. (A2) and (A6), we can extract explicit expressions for the coefficients $\mathcal{T}_{kjm}^{(d)(1B)}$ in terms of effective Cartesian components. Table XVII contains the results for spherical

coefficients with $4 \leq d \leq 8$, $m = 0$, odd values of j in the range $0 \leq j \leq k - 1$, and even values of k in the range $0 \leq k \leq d - 2$. The structure of this table again follows that of Table XV.

APPENDIX B: TRANSFORMATIONS TO THE SUN-CENTERED FRAME

Constraints on the coefficients for Lorentz violation are commonly reported in the Sun-centered frame [3]. This Appendix describes the conversion of coefficients for Lorentz violation in a laboratory frame into combinations of coefficients in the Sun-centered frame, including effects at zeroth and linear boost order. The primary focus here is on effective Cartesian coefficients, which are better suited for boost analyses. We use Greek indices to denote spacetime indices and Latin indices to represent spatial components. Generic indices in the laboratory frame are represented by lowercase letters, while indices in the Sun-centered frame are represented by uppercase ones. For definiteness, we label Cartesian components in the laboratory frame by 0,1,2,3 and assume that the Lorentz transformation is given by Eq. (29) with $\mathcal{R}^3{}_J = \hat{B}^J$. Cartesian components in the Sun-centered frame are denoted by T, X, Y, Z, and contractions of spatial uppercase indices imply summation over components in the Sun-centered frame.

Consider first the effective Cartesian coefficients associated with spin-independent Lorentz and CPT violation. The expressions for these effective Cartesian coefficients in the laboratory frame in terms of effective Cartesian coefficients in the Sun-centered frame can be reconstructed at linear boost order from the information contained in Table XVIII. The table limits attention to coefficients in

TABLE XVIII. Relations between spin-independent Cartesian coefficients in laboratory and Sun-centered frames for $3 \leq d \leq 8$.

Laboratory	Factor	Sun-centered	Laboratory	Factor	Sun-centered
$a_{\text{eff}}^{(3)0}$	1	$a_{\text{eff}}^{(3)T}$	$c_{\text{eff}}^{(6)0000}$	1	$c_{\text{eff}}^{(6)TTTT}$
	$-\beta^J$	$a_{\text{eff}}^{(3)J}$		$-4\beta^J$	$c_{\text{eff}}^{(6)TTTJ}$
$a_{\text{eff}}^{(5)000}$	1	$a_{\text{eff}}^{(5)TTT}$	$c_{\text{eff}}^{(6)00jj}$	1	$c_{\text{eff}}^{(6)TTJJ}$
	$-3\beta^J$	$a_{\text{eff}}^{(5)TTJ}$		$-2\beta^J$	$c_{\text{eff}}^{(6)TTTJ}$
$a_{\text{eff}}^{(5)0jj}$	1	$a_{\text{eff}}^{(5)TKK}$		$-2\beta^J$	$c_{\text{eff}}^{(6)TKKJ}$
	$-2\beta^J$	$a_{\text{eff}}^{(5)TTJ}$	$c_{\text{eff}}^{(6)0033}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(6)TTJ_1J_2}$
	$-\beta^J$	$a_{\text{eff}}^{(5)KKJ}$		$-2\hat{B}^{J_1}\hat{B}^{J_2}\beta^J$	$c_{\text{eff}}^{(6)TJ_1J_2J}$
$a_{\text{eff}}^{(5)033}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$a_{\text{eff}}^{(5)TJ_1J_2}$		$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(6)TTTJ_1}$
	$-\hat{B}^{J_1}\hat{B}^{J_2}\beta^J$	$a_{\text{eff}}^{(5)J_1J_2J}$	$c_{\text{eff}}^{(6)jjkk}$	1	$c_{\text{eff}}^{(6)KKLL}$
	$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$a_{\text{eff}}^{(5)TTJ_1}$		$-4\beta^J$	$c_{\text{eff}}^{(6)TKKJ}$
$a_{\text{eff}}^{(7)00000}$	1	$a_{\text{eff}}^{(7)TTTTT}$	$c_{\text{eff}}^{(6)33jj}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(6)KKJ_1J_2}$
	$-5\beta^J$	$a_{\text{eff}}^{(7)TTTTJ}$		$-2\hat{B}^{J_1}\hat{B}^{J_2}\beta^J$	$c_{\text{eff}}^{(6)TJ_1J_2J}$
$a_{\text{eff}}^{(7)000jj}$	1	$a_{\text{eff}}^{(7)TTTKK}$		$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(6)TKKJ_1}$
	$-2\beta^J$	$a_{\text{eff}}^{(7)TTTTJ}$	$c_{\text{eff}}^{(6)3333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}$	$c_{\text{eff}}^{(6)J_1J_2J_3J_4}$
	$-3\beta^J$	$a_{\text{eff}}^{(7)TTKKJ}$		$-4\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(6)TJ_1J_2J_3}$
$a_{\text{eff}}^{(7)00033}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$a_{\text{eff}}^{(7)TTTJ_1J_2}$	$c_{\text{eff}}^{(8)000000}$	1	$c_{\text{eff}}^{(8)TTTTTT}$
	$-3\hat{B}^{J_1}\hat{B}^{J_2}\beta^J$	$a_{\text{eff}}^{(7)TTJ_1J_2J}$		$-6\beta^J$	$c_{\text{eff}}^{(8)TTTTTJ}$
	$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$a_{\text{eff}}^{(7)TTTTJ_1}$	$c_{\text{eff}}^{(8)0000jj}$	1	$c_{\text{eff}}^{(8)TTTTKK}$
$a_{\text{eff}}^{(7)0jjkk}$	1	$a_{\text{eff}}^{(7)TKKLL}$		$-2\beta^J$	$c_{\text{eff}}^{(8)TTTTTJ}$
	$-4\beta^J$	$a_{\text{eff}}^{(7)TTKKJ}$		$-4\beta^J$	$c_{\text{eff}}^{(8)TTTKKJ}$
	$-\beta^J$	$a_{\text{eff}}^{(7)KKLLJ}$	$c_{\text{eff}}^{(8)000033}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(8)TTTTJ_1J_2}$
$a_{\text{eff}}^{(7)0jj33}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$a_{\text{eff}}^{(7)TKKJ_1J_2}$		$-4\beta^J\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(8)TTTJJ_1J_2}$
	$-2\hat{B}^{J_1}\hat{B}^{J_2}\beta^J$	$a_{\text{eff}}^{(7)TTJ_1J_2J}$		$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(8)TTTTTJ_1}$
	$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$a_{\text{eff}}^{(7)TTKKJ_1}$	$c_{\text{eff}}^{(8)00jjkk}$	1	$c_{\text{eff}}^{(8)TTKKLL}$
	$-\hat{B}^{J_1}\hat{B}^{J_2}\beta^{J_3}$	$a_{\text{eff}}^{(7)KKJ_1J_2J_3}$		$-4\beta^J$	$c_{\text{eff}}^{(8)TTTKKJ}$
$a_{\text{eff}}^{(7)03333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}$	$a_{\text{eff}}^{(7)TJ_1J_2J_3J_4}$		$-2\beta^J$	$c_{\text{eff}}^{(8)TKKLLJ}$
	$-\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\beta^J$	$a_{\text{eff}}^{(7)J_1J_2J_3J_4J}$	$c_{\text{eff}}^{(8)00jj33}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(8)TTKKJ_1J_2}$
	$-4\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}\cdot\boldsymbol{\beta}$	$a_{\text{eff}}^{(7)TTJ_1J_2J_3}$		$-2\beta^J\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(8)TTTJJ_1J_2}$
$c_{\text{eff}}^{(4)00}$	1	$c_{\text{eff}}^{(4)TT}$		$-2\beta^J\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(8)TKKJ_1J_2J}$
	$-2\beta^J$	$c_{\text{eff}}^{(4)TJ}$		$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(8)TTTKKJ_1}$
$c_{\text{eff}}^{(4)jj}$	1	$c_{\text{eff}}^{(4)KK}$	$c_{\text{eff}}^{(8)003333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}$	$c_{\text{eff}}^{(8)TTJ_1J_2J_3J_4}$
	$-2\beta^J$	$c_{\text{eff}}^{(4)TJ}$		$-2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\beta^J$	$c_{\text{eff}}^{(8)TJ_1J_2J_3J_4J}$
$c_{\text{eff}}^{(4)33}$	$\hat{B}^{J_1}\hat{B}^{J_2}$	$c_{\text{eff}}^{(4)J_1J_2}$		$-4\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(8)TTKKJ_1J_2J_3}$
	$-2\hat{B}^{J_1}\hat{B}\cdot\boldsymbol{\beta}$	$c_{\text{eff}}^{(4)TJ_1}$			

the laboratory frame that contribute to the spherical coefficients with $3 \leq d \leq 8$ discussed in Appendix A, which are the ones relevant to the clock-comparison experiments analyzed in this work. The table contains two triplets of columns. In each triplet, the first column lists the Cartesian components of interest in the laboratory frame. Entries in the second column are factors involving the boost $-\boldsymbol{\beta}^J$ and the direction $\hat{B}^J$ of the magnetic field. The third column lists the relevant Cartesian components in the Sun-centered frame. The expression converting a given coefficient from the laboratory frame to the Sun-centered frame is obtained by multiplying the entries in the second and third columns and adding the associated rows. For example, the first two rows of the table generate the equation $a_{\text{eff}}^{(3)t} = a_{\text{eff}}^{(3)T} - \beta^J a_{\text{eff}}^{(3)J}$. Using the contents of this table and the results in Appendix A, it is straightforward to convert spin-independent spherical coefficients in the laboratory frame to effective Cartesian coefficients in the Sun-centered frame at linear boost order.

To analyze the experiments discussed in this work, it is useful to find analogous expressions converting the nonrelativistic coefficients for Lorentz violation to the Sun-centered frame. The nonrelativistic coefficients are combinations of spherical coefficients for Lorentz violation of arbitrary mass dimension, as illustrated in Eqs. (111) and (112) of Ref. [35]. All the spherical coefficients contributing to a particular nonrelativistic coefficient behave the same way under rotations, so at zeroth boost order the conversion between frames is given by the comparatively simple result (25). However, the spherical coefficients transform differently under boosts, so converting nonrelativistic coefficients at linear boost order becomes involved. In contrast, the effective Cartesian coefficients have comparatively simple transformations under boosts and so are better suited for studying boost effects.

To circumvent this issue, we limit attention here to terms involving effective Cartesian coefficients that contribute at zeroth order in $|\boldsymbol{p}|/m_w$, which yields the dominant contributions at linear boost order and suffices for the experimental analyses of interest. With this assumption, the spin-independent nonrelativistic coefficients $\mathcal{V}_{kjm}^{\text{NR}}$ in the laboratory frame are expressed in terms of spherical coefficients as

$$\mathcal{V}_{kjm}^{\text{NR}} \approx \sum_d m_\psi^{d-3-k} \mathcal{V}_{kjm}^{(d)}. \tag{B1}$$

The spherical coefficients can then be translated into effective Cartesian coefficients in the laboratory frame using the results in Appendix A. To perform the conversion between the laboratory frame and the Sun-centered frame, we note that any nonrelativistic coefficient $\mathcal{K}_{kjm}^{\text{NR}}$ can be expanded to linear boost order as

$$\mathcal{K}_{kjm}^{\text{NR}} \approx \mathcal{K}_{kjm}^{\text{NR}}|_{\beta^J=0} + \left(\left.\frac{\partial \mathcal{K}_{kjm}^{\text{NR}}}{\partial \beta^J}\right|_{\beta^J=0}\right)\beta^J. \tag{B2}$$

For all coefficients, the zeroth-order term is given by Eq. (25).

At linear boost order, we are interested in the contribution $\mathcal{V}_{kj0}^{\text{NR}}(\mathcal{O}(\beta))$ to the nonrelativistic spin-independent coefficients with $m = 0$. Decomposing this contribution as a polynomial in the unit vector $\hat{B}^J$ along the magnetic field yields the result

$$\mathcal{V}_{kj0}^{\text{NR}}(\mathcal{O}(\beta)) = \left(\left.\frac{\partial \mathcal{V}_{kj0}^{\text{NR}}}{\partial \beta^J}\right|_{\beta^J=0}\right)\beta^J$$

$$= -\sqrt{\frac{4\pi}{2j+1}} m^{d-k-3}\left[\sum_{d=3}^{8} V_{kj}^{(d)J}\beta^J\right.$$

$$+ \sum_{d=5}^{8} V_{kj}^{(d)JJ_1J_2}\beta^J\hat{B}^{J_1}\hat{B}^{J_2}$$

$$\left. + \sum_{d=7}^{8} V_{kj}^{(d)JJ_1J_2J_3J_4}\beta^J\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\right]. \tag{B3}$$

The quantities $V_{kj}^{(d)JJ_1\ldots J_n}$ with $d - 3 - k < 0$ vanish. For $3 \leq d \leq 8$, Table XIX provides explicit expressions for many nonvanishing $V_{kj}^{(d)JJ_1\ldots J_n}$ in terms of combinations of effective Cartesian coefficients in the Sun-centered frame.

TABLE XIX. The quantities $V_{kj}^{(d)JK\ldots M}$ for $5 \leq d \leq 8$.

$V_{kj}^{(d)JK\ldots M}$	Combination
$V_{20}^{(5)J}$	$-2a_{\text{eff}}^{(5)TTJ} - a_{\text{eff}}^{(5)KKJ}$
$V_{20}^{(6)J}$	$4c_{\text{eff}}^{(6)TTTJ} + 4c_{\text{eff}}^{(6)TKKJ}$
$V_{20}^{(7)J}$	$-\frac{10}{3}(2a_{\text{eff}}^{(7)TTTTJ} + 3a_{\text{eff}}^{(7)TTKKJ})$
$V_{20}^{(8)J}$	$10c_{\text{eff}}^{(8)TTTTTJ} + 20c_{\text{eff}}^{(8)TTTTKKJ}$
$V_{40}^{(7)J}$	$-a_{\text{eff}}^{(7)LLKKJ} - 4a_{\text{eff}}^{(7)TTKKJ}$
$V_{40}^{(8)J}$	$6c_{\text{eff}}^{(8)TLLKKJ} + 12c_{\text{eff}}^{(8)TTTKKJ}$
$V_{22}^{(5)JJ_1J_2}$	$-3a_{\text{eff}}^{(5)JJ_1J_2} - 6\delta^{JJ_1}a_{\text{eff}}^{(5)TTJ_2}$
$V_{22}^{(6)JJ_1J_2}$	$12c_{\text{eff}}^{(6)TJJ_1J_2} + 12\delta^{JJ_1}c_{\text{eff}}^{(6)TTTJ_2}$
$V_{22}^{(7)JJ_1J_2}$	$-30a_{\text{eff}}^{(7)TTJJ_1J_2} - 20\delta^{JJ_1}a_{\text{eff}}^{(7)TTTTJ_2}$
$V_{22}^{(8)JJ_1J_2}$	$60c_{\text{eff}}^{(8)TTTJJ_1J_2} + 30\delta^{JJ_1}c_{\text{eff}}^{(8)TTTTTJ_2}$
$V_{42}^{(7)JJ_1J_2}$	$-\frac{60}{7}(a_{\text{eff}}^{(7)TTJJ_1J_2} + \delta^{JJ_1}a_{\text{eff}}^{(7)TTLLJ_2})$ $-\frac{30}{7}a^{(7)LLJJ_1J_2}$
$V_{42}^{(8)JJ_1J_2}$	$\frac{180}{7}(c_{\text{eff}}^{(8)TTTJJ_1J_2} + c_{\text{eff}}^{(8)TKKJJ_1J_2})$ $+\frac{180}{7}\delta^{JJ_1}c_{\text{eff}}^{(8)TTTKKJ_2}$
$V_{44}^{(7)JJ_1J_2J_3J_4}$	$-5(\delta^{JJ_1}a_{\text{eff}}^{(7)TTJ_2J_3J_4} + a_{\text{eff}}^{(7)JJ_1J_2J_3J_4})$
$V_{44}^{(8)JJ_1J_2J_3J_4}$	$60\delta^{JJ_1}c_{\text{eff}}^{(8)TTTJ_2J_3J_4} + 30c_{\text{eff}}^{(8)TJJ_1J_2J_3J_4}$

TABLE XX. Relations between spin-dependent Cartesian coefficients in laboratory and Sun-centered frames for $3 \leq d \leq 6$.

Laboratory	Factor	Sun-centered	Laboratory	Factor	Sun-centered
$\tilde{H}_{\rm eff}^{(3)03}$	$\hat{B}^{J_1}$	$\tilde{H}_{\rm eff}^{(3)TJ_1}$	$\tilde{g}_{\rm eff}^{(6)03000}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(6)TJ_1TTT}$
	$\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(3)J_1J}$		$4\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1(TTTJ)}$
$\tilde{H}_{\rm eff}^{(5)0300}$	$\hat{B}^{J_1}$	$\tilde{H}_{\rm eff}^{(5)TJ_1TT}$	$\tilde{g}_{\rm eff}^{(6)0j0j3}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(6)TKTKJ_1}$
	$3\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)J_1(TTJ)}$		$\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)JTTTJ_1}$
$\tilde{H}_{\rm eff}^{(5)j30j}$	$\hat{B}^{J_1}$	$\tilde{H}_{\rm eff}^{(5)KJ_1TK}$		$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)K(JTK)J_1}$
	$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)J_1(JT)T}$		$\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\rm eff}^{(6)KTTTK}$
	$\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)J_1KKJ}$	$\tilde{g}_{\rm eff}^{(6)030jj}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(6)TJ_1TKK}$
	$\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{H}_{\rm eff}^{(5)TKTK}$		$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1TTTJ}$
$\tilde{H}_{\rm eff}^{(5)0jj3}$	$\hat{B}^{J_1}$	$\tilde{H}_{\rm eff}^{(5)TKKJ_1}$		$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1(TJ)KK}$
	$\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)JTTJ_1}$	$\tilde{g}_{\rm eff}^{(6)j300j}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(6)KJ_1TTK}$
	$\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)KJKJ_1}$		$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1KTJK}$
	$\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{H}_{\rm eff}^{(5)KTKT}$		$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1(JT)TT}$
$\tilde{H}_{\rm eff}^{(5)03jj}$	$\hat{B}^{J_1}$	$\tilde{H}_{\rm eff}^{(5)TJ_1KK}$		$\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\rm eff}^{(6)TKTTK}$
	$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)J_1TTJ}$	$\tilde{g}_{\rm eff}^{(6)j3jkk}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(6)JJ_1JKK}$
	$\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)J_1JKK}$		$4\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1(KTKJ)}$
$\tilde{H}_{\rm eff}^{(5)0333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{H}_{\rm eff}^{(5)TJ_1J_2J_3}$		$\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\rm eff}^{(6)TKKLL}$
	$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{H}_{\rm eff}^{(5)J_1TTJ_2}$	$\tilde{g}_{\rm eff}^{(6)03033}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{g}_{\rm eff}^{(6)TJ_1TJ_2J_3}$
	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{H}_{\rm eff}^{(5)J_1JJ_2J_1}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\rm eff}^{(6)J_1TTTJ_2}$
$\tilde{g}_{\rm eff}^{(4)030}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(4)TJ_1T}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1(TJ)J_2J_3}$
	$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(4)J_1(TJ)}$	$\tilde{g}_{\rm eff}^{(6)jl3j33}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{g}_{\rm eff}^{(6)KJ_1KJ_2J}$
$\tilde{g}_{\rm eff}^{(4)j3j}$	$\hat{B}^{J_1}$	$\tilde{g}_{\rm eff}^{(4)KJ_1K}$		$-3\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\rm eff}^{(6)K(J_1TJ_2)K}$
	$2\hat{B}^{J_1}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(4)J_1(TJ)}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{g}_{\rm eff}^{(6)J_1(TJ)J_2J_3}$
	$\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\rm eff}^{(4)TKK}$			

The other quantities of relevance can be obtained from entries in this table using the relations

$$V_{22}^{(d)J} = -V_{20}^{(d)J}, \qquad V_{42}^{(d)J} = -\frac{10}{7}V_{40}^{(d)J},$$
$$V_{44}^{(d)J} = \frac{3}{7}V_{40}^{(d)J}, \qquad V_{44}^{(d)JJ_1J_2} = -V_{42}^{(d)JJ_1J_2}. \quad \text{(B4)}$$

With the above results for spin-independent coefficients in hand, we next consider spin-dependent effects. The relations connecting the sets of spin-dependent effective Cartesian coefficients in the laboratory frame and the Sun-centered frame up to linear boost order can be found using the information in Tables XX and XXI. These tables restrict

attention to coefficients with $3 \leq d \leq 6$ and $7 \leq d \leq 8$, respectively, which are the ones of relevance to our analysis of clock-comparison experiments. Each table contains two triplets of columns, and each triplet has the same structure as that of Table XVIII. Taking products of the second and third entries in a row and summing over rows relevant to the chosen laboratory-frame coefficient yields the desired equation converting the effective Cartesian coefficients from the laboratory to the Sun-centered frame, as before.

In parallel with the above discussion for spin-independent effects, the analysis of experiments is facilitated by translating nonrelativistic coefficients for Lorentz violation in the laboratory frame to expressions involving effective Cartesian coefficients in the Sun-centered frame. Adopting the assumptions leading to Eq. (B1), the spin-dependent

TABLE XXI.   Relations between spin-dependent Cartesian coefficients in laboratory and Sun-centered frames for $7 \leq d \leq 8$.

Laboratory	Factor	Sun-centered	Laboratory	Factor	Sun-centered
$\tilde{H}_{\text{eff}}^{(7)030000}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)TJ_1TTTT}$	$\tilde{g}_{\text{eff}}^{(8)0300000}$	$\hat{B}^{J_1}$	$\tilde{g}_{\text{eff}}^{(8)TJ_1TTTTT}$
	$5\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1(JTTTT)}$		$6\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(TTTTTJ)}$
$\tilde{H}_{\text{eff}}^{(7)0j00j3}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)TKTTKJ_1}$	$\tilde{g}_{\text{eff}}^{(8)0j000j3}$	$\hat{B}^{J_1}$	$\tilde{g}_{\text{eff}}^{(8)TKTTTKJ_1}$
	$\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)JTTTTJ_1}$		$\hat{B}\cdot\beta$	$\tilde{g}_{\text{eff}}^{(8)KTTTTTK}$
	$3\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)K(JTT)KJ_1}$		$\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)JTTTTTJ_1}$
	$\hat{B}\cdot\beta$	$\tilde{H}_{\text{eff}}^{(7)KTTTTK}$		$4\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)K(JTTT)J_1K}$
$\tilde{H}_{\text{eff}}^{(7)0300jj}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)TJ_1TTKK}$	$\tilde{g}_{\text{eff}}^{(8)03000jj}$	$\hat{B}^{J_1}$	$\tilde{g}_{\text{eff}}^{(8)TJ_1TTTKK}$
	$2\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1TTTTJ}$		$2\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1TTTTTJ}$
	$3\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1(JTT)KK}$		$4\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JTTT)KK}$
$\tilde{H}_{\text{eff}}^{(7)j3000j}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)KJ_1TTTK}$	$\tilde{g}_{\text{eff}}^{(8)j30000j}$	$\hat{B}^{J_1}$	$\tilde{g}_{\text{eff}}^{(8)KJ_1TTTTK}$
	$2\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1(JT)TTT}$		$2\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JT)TTTT}$
	$3\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1KTTJK}$		$4\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1KTTTKJ}$
	$\hat{B}\cdot\beta$	$\tilde{H}_{\text{eff}}^{(7)TKTTTK}$		$\hat{B}\cdot\beta$	$\tilde{g}_{\text{eff}}^{(8)TKTTTTK}$
$\tilde{H}_{\text{eff}}^{(7)j30jkk}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)JJ_1TJKK}$	$\tilde{g}_{\text{eff}}^{(8)j300jkk}$	$\hat{B}^{J_1}$	$\tilde{g}_{\text{eff}}^{(8)JJ_1TTJKK}$
	$2\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1(JT)TKK}$		$\hat{B}\cdot\beta$	$\tilde{g}_{\text{eff}}^{(8)TKTTKLL}$
	$\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1KJKLL}$		$4\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JTKK)TT}$
	$2\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1KTTKJ}$		$2\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1KTJKLL}$
	$\hat{B}\cdot\beta$	$\tilde{H}_{\text{eff}}^{(7)TKTKJJ}$	$\tilde{g}_{\text{eff}}^{(8)030jjkk}$	$\hat{B}^{J_1}$	$\tilde{g}_{\text{eff}}^{(8)TJ_1TKKJJ}$
$\tilde{H}_{\text{eff}}^{(7)03jjkk}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)TJ_1KKLL}$		$4\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1TTTJKK}$
	$\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1JLLKK}$		$2\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JT)LLKK}$
	$4\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1TTJKK}$	$\tilde{g}_{\text{eff}}^{(8)0j0jkk3}$	$\hat{B}^{J_1}$	$\tilde{g}^{(8)TKTKLLJ_1}$
$\tilde{H}_{\text{eff}}^{(7)0jjkk3}$	$\hat{B}^{J_1}$	$\tilde{H}_{\text{eff}}^{(7)TKKJJJ_1}$		$\hat{B}\cdot\beta$	$\tilde{g}_{\text{eff}}^{(8)KTTTKLL}$
	$\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)LJLKKJ_1}$		$2\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)K(JT)KLLJ_1}$
	$-3\hat{B}^{J_1}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)T(KKJ)J_1T}$		$-3\hat{B}^{J_1}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)T(KKJ)TTJ_1}$
	$\hat{B}\cdot\beta$	$\tilde{H}_{\text{eff}}^{(7)KTTKLL}$	$\tilde{g}_{\text{eff}}^{(8)j300j33}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{g}_{\text{eff}}^{(8)KJ_1TTKJ_2J_3}$
$\tilde{H}_{\text{eff}}^{(7)030033}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{H}_{\text{eff}}^{(7)TJ_1TTJ_2J_3}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1KTJ_2KJ_3}$
	$3\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1(JTT)J_2J_3}$		$-3\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\beta$	$\tilde{g}_{\text{eff}}^{(8)K(J_1J_2T)TTK}$
	$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\beta$	$\tilde{H}_{\text{eff}}^{(7)J_1TTTTJ_2}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JT)TTJ_2J_3}$
$\tilde{H}_{\text{eff}}^{(7)j30j33}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{H}_{\text{eff}}^{(7)KJ_1TKJ_2J_3}$	$\tilde{g}_{\text{eff}}^{(8)0j0j333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{g}_{\text{eff}}^{(8)TKTKJ_1J_2J_3}$
	$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1(JT)TJ_2J_3}$		$3\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\beta$	$\tilde{g}_{\text{eff}}^{(8)KTTTKJ_1J_2}$
	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{H}_{\text{eff}}^{(7)J_1KJKJ_2J_3}$		$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)JTTTTJ_1J_2J_3}$
	$-3\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\beta$	$\tilde{H}_{\text{eff}}^{(7)K(TJ_1J_2)TK}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\beta^J$	$\tilde{g}_{\text{eff}}^{(8)K(JT)KJ_1J_2J_3}$

(Table continued)

TABLE XXI. *(Continued)*

Laboratory	Factor	Sun-centered	Laboratory	Factor	Sun-centered
$\tilde{H}_{\text{eff}}^{(7)0jj333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{H}_{\text{eff}}^{(7)TKKJ_1J_2J_3}$	$\tilde{g}_{\text{eff}}^{(8)030jj33}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{g}_{\text{eff}}^{(8)TJ_1TKKJ_2J_3}$
	$3\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{H}_{\text{eff}}^{(7)KTTKJ_1J_2}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\text{eff}}^{(8)J_1TTTJ_2KK}$
	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{H}_{\text{eff}}^{(7)JTTJ_1J_2J_3}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{g}_{\text{eff}}^{(8)J_1TTTJ_2J_3J}$
	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{H}_{\text{eff}}^{(7)KJKJ_1J_2J_3}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JT)J_2J_3KK}$
$\tilde{H}_{\text{eff}}^{(7)03jj33}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{H}_{\text{eff}}^{(7)TJ_1KKJ_2J_3}$	$\tilde{g}_{\text{eff}}^{(8)0300033}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}$	$\tilde{g}_{\text{eff}}^{(8)TJ_1TTTJ_2J_3}$
	$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{H}_{\text{eff}}^{(7)J_1TTJ_2KK}$		$4\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(TTTJ)J_2J_3}$
	$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{H}_{\text{eff}}^{(7)J_1TTJJ_2J_3}$		$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\text{eff}}^{(8)J_1TTTTTJ_2}$
	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\boldsymbol{\beta}^J$	$\tilde{H}_{\text{eff}}^{(7)J_1JKKJ_2J_3}$	$\tilde{g}_{\text{eff}}^{(8)0303333}$	$\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\hat{B}^{J_5}$	$\tilde{g}_{\text{eff}}^{(8)TJ_1TJ_2J_3J_4J_5}$
				$2\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\hat{B}^{J_5}\boldsymbol{\beta}^J$	$\tilde{g}_{\text{eff}}^{(8)J_1(JT)J_2J_3J_4J_5}$
				$4\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\hat{B}\cdot\boldsymbol{\beta}$	$\tilde{g}_{\text{eff}}^{(8)J_1TTTJ_2J_3J_4}$

nonrelativistic coefficients can be approximated in terms of spherical coefficients as

$$\mathcal{T}_{kjm}^{\text{NR}(0B)} \approx \sum_d m_\psi^{d-3-k}(k+1)\mathcal{T}_{kjm}^{(d)(0B)},$$

$$\mathcal{T}_{kjm}^{\text{NR}(1B)} \approx \sum_d m_\psi^{d-3-k}\left(\mathcal{T}_{kjm}^{(d)(1B)} + \sqrt{\frac{j(j+1)}{2}}\mathcal{T}_{kjm}^{(d)(0B)}\right).$$

(B5)

The spherical coefficients can then in turn be converted to effective Cartesian coefficients using the results in Appendix A. The conversion can be implemented to linear boost order via Eq. (B2), where the zeroth-order term is again given by Eq. (25).

At linear boost order, the relevant spin-dependent non-relativistic coefficients $\mathcal{T}_{kj0}^{\text{NR}(qB)}(\mathcal{O}(\beta))$ have $m=0$. Expanding them in powers of the unit vector $\hat{B}^J$ along the magnetic field, we obtain

$$
\begin{aligned}
\mathcal{T}_{kj0}^{\text{NR}(qB)}(\mathcal{O}(\beta)) &= \left(\frac{\partial \mathcal{T}_{kj0}^{\text{NR}(qB)}}{\partial \boldsymbol{\beta}^J}\bigg|_{\beta^J=0}\right)\boldsymbol{\beta}^J \\
&= \sqrt{\frac{4\pi}{2j+1}}m^{d-k-3}\Bigg[\sum_{d=3}^{8} T_{sB,kj}^{(d)JJ_1}\boldsymbol{\beta}^J\hat{B}^{J_1} \\
&\quad + \sum_{d=5}^{8} T_{sB,kj}^{(d)JJ_1J_2J_3}\boldsymbol{\beta}^J\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3} \\
&\quad + \sum_{d=7}^{8} T_{sB,kj}^{(d)JJ_1J_2J_3J_4J_5}\boldsymbol{\beta}^J\hat{B}^{J_1}\hat{B}^{J_2}\hat{B}^{J_3}\hat{B}^{J_4}\hat{B}^{J_5}\Bigg],
\end{aligned}
$$

(B6)

where the quantities $T_{sB,kj}^{(d)JJ_1\ldots J_n}$ with $d-3-k<0$ vanish. Explicit expressions for nonvanishing $T_{0B,kj}^{(d)JJ_1\ldots J_n}$ in terms of effective Cartesian coefficients in the Sun-centered frame can be found in the first two columns of Table XXII and by using the relations

$$T_{0B,23}^{(d)JJ_1} = -T_{0B,21}^{(d)JJ_1}, \qquad T_{0B,43}^{(d)JJ_1} = -\frac{14}{9}T_{0B,41}^{(d)JJ_1},$$

$$T_{0B,45}^{(d)JJ_1J_2J_3} = -T_{0B,43}^{(d)JJ_1J_2J_3}, \qquad T_{0B,45}^{(d)JJ_1} = \frac{5}{9}T_{0B,41}^{(d)JJ_1}. \quad \text{(B7)}$$

The nonvanishing quantities $T_{1B,kj}^{(d)JJ_1\ldots J_n}$ are compiled in the second pair of columns of Table XXII.

In working with these results, the reader is cautioned that the coefficients $\mathcal{T}_{kjm}^{\text{NR}(0B)}$ and $\mathcal{T}_{kjm}^{\text{NR}(1B)}$ with $j=k+1$ are linearly dependent at zeroth order in $|\boldsymbol{p}|_w/m_w$ because the spherical coefficients $\mathcal{T}_{kjm}^{(d)(1B)}$ vanish for $j=k+1$. One implication of this, for instance, is the existence of the relationships

$$T_{1B,k(k+1)}^{(d)JJ_1} = \sqrt{\frac{k+2}{2(k+1)}}T_{0B,k(k+1)}^{(d)JJ_1},$$

$$T_{1B,k(k+1)}^{(d)JJ_1J_2J_3} = \sqrt{\frac{k+2}{2(k+1)}}T_{0B,k(k+1)}^{(d)J_1J_2J_3},$$

$$T_{1B,k(k+1)}^{(d)JJ_1J_2J_3J_4J_5} = \sqrt{\frac{k+2}{2(k+1)}}T_{0B,k(k+1)}^{(d)J_1J_2J_3J_4J_5} \quad \text{(B8)}$$

that link the quantities with subscripts $0B$ and $1B$.

TABLE XXII. The quantities $T_{0B,kj}^{(d)JK...M}$ and $T_{1B,kj}^{(d)JK...M}$ for $3 \le d \le 8$.

$T_{0B,kj}^{(d)JK...M}$	Combination	$T_{1B,kj}^{(d)JK...M}$	Combination
$T_{0B,01}^{(3)JJ_1}$	$-\tilde{H}_{\text{eff}}^{(3)J_1J}$	$T_{1B,21}^{(5)JJ_1}$	$-\frac{1}{5}(15\tilde{H}_{\text{eff}}^{(5)J_1(JTT)} + 2\tilde{H}_{\text{eff}}^{(5)T(J_1J)T}$ $+6\tilde{H}_{\text{eff}}^{(5)J_1(JKK)} - 2\tilde{H}_{\text{eff}}^{(5)K(J_1J)K}$ $+6\tilde{H}_{\text{eff}}^{(5)TKTK}\delta_{JJ_1})$
$T_{0B,01}^{(4)JJ_1}$	$2\tilde{g}_{\text{eff}}^{(4)J_1(JT)}$		
$T_{0B,01}^{(5)JJ_1}$	$-3\tilde{H}_{\text{eff}}^{(5)J_1(JTT)}$		
$T_{0B,01}^{(6)JJ_1}$	$4\tilde{g}_{\text{eff}}^{(6)J_1(JTTT)}$	$T_{1B,21}^{(6)JJ_1}$	$\frac{1}{10}(60\tilde{g}_{\text{eff}}^{(6)J_1(JTTT)} + 12\tilde{g}_{\text{eff}}^{(6)T(J_1J)TT}$ $+48\tilde{g}_{\text{eff}}^{(6)J_1(TJKK)} - 18\tilde{g}_{\text{eff}}^{(6)K(J_1JT)K}$ $+21\tilde{g}_{\text{eff}}^{(6)TKTTK}\delta_{JJ_1})$
$T_{0B,01}^{(7)JJ_1}$	$-5\tilde{H}_{\text{eff}}^{(7)J_1(JTTTT)}$		
$T_{0B,01}^{(8)JJ_1}$	$6\tilde{g}_{\text{eff}}^{(8)J_1(JTTTTT)}$		
$T_{0B,21}^{(5)JJ_1}$	$\frac{1}{5}(3\tilde{H}_{\text{eff}}^{(5)J(J_1KK)} + 4\tilde{H}_{\text{eff}}^{(5)T(J_1J)T}$ $+2\tilde{H}_{\text{eff}}^{(5)TKTK}\delta_{JJ_1})$	$T_{1B,21}^{(7)JJ_1}$	$-\frac{2}{5}(25\tilde{H}_{\text{eff}}^{(7)J_1(TTTTJ)} + 6\tilde{H}_{\text{eff}}^{(7)T(J_1J)TTT}$ $+30\tilde{H}_{\text{eff}}^{(7)J_1(TTJKK)} - 12\tilde{H}_{\text{eff}}^{(7)K(TTJJ_1)K}$ $+8\tilde{H}_{\text{eff}}^{(7)TKTTTK}\delta_{JJ_1})$
$T_{0B,21}^{(6)JJ_1}$	$-\frac{3}{5}(3\tilde{g}_{\text{eff}}^{(6)J(J_1KK)T} + 4\tilde{g}_{\text{eff}}^{(6)T(J_1J)TT}$ $+3\tilde{g}_{\text{eff}}^{(6)T(J_1KK)J} + 2\tilde{g}_{\text{eff}}^{(6)TKTTK}\delta_{JJ_1})$	$T_{1B,21}^{(8)JJ_1}$	$\frac{1}{2}(30\tilde{g}_{\text{eff}}^{(8)J_1(JTTTT)} + 8\tilde{g}_{\text{eff}}^{(8)T(J_1J)TTTT}$ $+48\tilde{g}_{\text{eff}}^{(8)J_1(TTTJKK)} - 20\tilde{g}_{\text{eff}}^{(8)K(J_1JTTT)K}$ $+9\tilde{g}_{\text{eff}}^{(8)TKTTTTK}\delta_{JJ_1})$
$T_{0B,21}^{(7)JJ_1}$	$\frac{6}{5}(3\tilde{H}_{\text{eff}}^{(7)J(J_1KK)TT} + 6\tilde{H}_{\text{eff}}^{(7)T(J_1KK)TJ}$ $+4\tilde{H}_{\text{eff}}^{(7)T(J_1J)TTT} + 2\tilde{H}_{\text{eff}}^{(7)TKTTTK}\delta_{JJ_1})$	$T_{1B,41}^{(7)JJ_1}$	$-\frac{3}{35}(70\tilde{H}_{\text{eff}}^{(7)J_1(JTTKK)} - 4\tilde{H}_{\text{eff}}^{(7)K(JJ_1)KLL}$ $+15\tilde{H}_{\text{eff}}^{(7)J_1(JLLKK)} + 8\tilde{H}_{\text{eff}}^{(7)T(JJ_1LL)T}$ $+16\tilde{H}_{\text{eff}}^{(7)TKKLLT}\delta^{JJ_1})$
$T_{0B,21}^{(8)JJ_1}$	$-2(3\tilde{g}_{\text{eff}}^{(8)J(J_1KK)TTT} + 9\tilde{g}_{\text{eff}}^{(8)T(J_1KK)TTJ}$ $+4\tilde{g}_{\text{eff}}^{(8)T(J_1J)TTTT} + 2\tilde{g}_{\text{eff}}^{(8)TKTTTTK}\delta_{JJ_1})$	$T_{1B,41}^{(8)JJ_1}$	$\frac{3}{7}(18\tilde{g}_{\text{eff}}^{(8)J_1(JTKKLL)} - 2\tilde{g}_{\text{eff}}^{(8)K(JJ_1T)KLL}$ $+8\tilde{g}_{\text{eff}}^{(8)T(KKJ_1J)TT} + 42\tilde{g}_{\text{eff}}^{(8)J_1(JLLTTT)}$ $+9\tilde{g}_{\text{eff}}^{(8)TKKLLTT}\delta^{JJ_1})$
$T_{0B,41}^{(7)JJ_1}$	$\frac{3}{35}(16\tilde{H}_{\text{eff}}^{(7)T(J_1JKK)T} + 5\tilde{H}_{\text{eff}}^{(7)J(J_1KKLL)}$ $+4\tilde{H}_{\text{eff}}^{(7)TKKLLT}\delta_{JJ_1})$	$T_{1B,43}^{(7)JJ_1}$	$\frac{\sqrt{6}}{15}(15\tilde{H}_{\text{eff}}^{(7)J_1(JKKTT)}$ $-2\tilde{H}_{\text{eff}}^{(7)K(JJ_1)KLL} + 5\tilde{H}_{\text{eff}}^{(7)J_1(JLLKK)}$ $-4\tilde{H}_{\text{eff}}^{(7)T(JJ_1LL)T} + 2\tilde{H}_{\text{eff}}^{(7)TKKLLT}\delta^{JJ_1})$
$T_{0B,41}^{(8)JJ_1}$	$-\frac{3}{7}(16\tilde{g}_{\text{eff}}^{(8)T(J_1JKK)TT} + 5\tilde{g}_{\text{eff}}^{(8)J(J_1KKLL)T}$ $+5\tilde{g}_{\text{eff}}^{(8)T(J_1KKLL)J} + 4\tilde{g}_{\text{eff}}^{(8)TKKLLTT}\delta_{JJ_1})$	$T_{1B,43}^{(8)JJ_1}$	$-\frac{1}{\sqrt{6}}(12\tilde{g}_{\text{eff}}^{(8)J_1(JTKKLL)}$ $+6\tilde{g}_{\text{eff}}^{(8)K(JJ_1T)KLL} - 8\tilde{g}_{\text{eff}}^{(8)T(KKJ_1J)TT}$ $+18\tilde{g}_{\text{eff}}^{(8)J_1(JLLTTT)} + \tilde{g}_{\text{eff}}^{(8)TKKLLTT}\delta^{JJ_1})$
$T_{0B,23}^{(5)JJ_1J_2J_3}$	$\tilde{H}_{\text{eff}}^{(5)JJ_1J_2J_3} + 2\delta^{JJ_1}\tilde{H}_{\text{eff}}^{(5)TJ_2TJ_3}$	$T_{1B,43}^{(7)JJ_1J_2J_3}$	$\frac{\sqrt{6}}{9}(12\tilde{H}_{\text{eff}}^{(7)K(JJ_1J_2J_3)K}$ $+5\tilde{H}_{\text{eff}}^{(7)J(J_1J_2J_3KK)} - 27\tilde{H}_{\text{eff}}^{(7)J_1(JTT)J_2J_3}$ $-4\tilde{H}_{\text{eff}}^{(7)T(JJ_1J_2J_3)T}$ $+12\delta^{JJ_1}\tilde{H}_{\text{eff}}^{(7)T(J_2J_3KK)T}$ $+36\delta^{JJ_1}\tilde{H}_{\text{eff}}^{(7)K(J_2J_3TT)K})$
$T_{0B,23}^{(6)JJ_1J_2J_3}$	$6(\tilde{g}_{\text{eff}}^{(6)J_1(JT)J_2J_3} - \delta^{JJ_1}\tilde{g}_{\text{eff}}^{(6)TJ_2J_3TT})$		
$T_{0B,23}^{(7)JJ_1J_2J_3}$	$-18\tilde{H}_{\text{eff}}^{(7)J_1(JTT)J_2J_3} + 12\delta^{JJ_1}\tilde{H}_{\text{eff}}^{(7)TJ_2J_3TTT}$		
$T_{0B,23}^{(8)JJ_1J_2J_3}$	$40\tilde{g}_{\text{eff}}^{(8)J_1(JTTT)J_2J_3}$ $-20\delta^{JJ_1}\tilde{g}_{\text{eff}}^{(8)T(J_2J_3)TTTT}$		
$T_{0B,43}^{(7)JJ_1J_2J_3}$	$\frac{2}{9}(8\tilde{H}_{\text{eff}}^{(7)T(JJ_1J_2J_3)T} + 5\tilde{H}_{\text{eff}}^{(7)J(J_1J_2J_3KK)}$ $+12\delta^{JJ_1}\tilde{H}_{\text{eff}}^{(7)T(J_2J_3KK)T})$	$T_{1B,43}^{(8)JJ_1J_2J_3}$	$-\frac{5}{3\sqrt{6}}(10\tilde{g}_{\text{eff}}^{(8)K(JJ_1J_2J_3T)K}$ $-8\tilde{g}_{\text{eff}}^{(8)T(JJ_1J_2J_3)TT} - 24\tilde{g}_{\text{eff}}^{(8)J_1(JKKT)J_2J_3}$ $-36\tilde{g}_{\text{eff}}^{(8)J_1(JTTT)J_2J_3}$ $+24\delta^{JJ_1}\tilde{g}_{\text{eff}}^{(8)T(J_2J_3KK)TT}$ $+45\delta^{JJ_1}\tilde{g}_{\text{eff}}^{(8)K(J_2J_3TTT)K})$
$T_{0B,43}^{(8)JJ_1J_2J_3}$	$-\frac{10}{9}(8\tilde{g}_{\text{eff}}^{(8)T(JJ_1J_2J_3)TT}$ $+5\tilde{g}_{\text{eff}}^{(8)J(J_1J_2J_3KK)T} + 5\tilde{g}_{\text{eff}}^{(8)T(J_1J_2J_3KK)J}$ $+12\delta^{JJ_1}\tilde{g}_{\text{eff}}^{(7)T(J_2J_3KK)TT})$		
$T_{0B,45}^{(7)JJ_1J_2J_3J_4J_5}$	$\tilde{H}_{\text{eff}}^{(7)JJ_1J_2J_3J_4J_5} + \delta^{JJ_1}\tilde{H}_{\text{eff}}^{(7)TJ_2J_3J_4J_5T}$		
$T_{0B,45}^{(8)JJ_1J_2J_3J_4J_5}$	$-5(\tilde{g}_{\text{eff}}^{(8)JJ_1J_2J_3J_4J_5T} + \tilde{g}_{\text{eff}}^{(8)TJ_1J_2J_3J_4J_5J}$ $+4\delta^{JJ_1}\tilde{g}_{\text{eff}}^{(8)TJ_2J_3J_4J_5TT})$		

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