Lorentz-violating scalar Hamiltonian and the equivalence principle in a static metric

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In this paper, we obtain a nonrelativistic Hamiltonian from the Lorentz-violating (LV) scalar Lagrangian in the minimal standard model extension (SME). The Hamiltonian is obtained by two different methods. One is through the usual ansatz $\Phi(t,\vec{r})=e^{-imt}\Psi(t,\vec{r})$ applied to the LV-corrected Klein-Gordon equation, and the other is the Foldy-Wouthuysen transformation. The consistency of our results is also partially supported by the comparison with the spin-independent part of the fermion Hamiltonian. In this comparison, we can also establish a relation between the set of scalar LV coefficients with their fermion counterparts. Using a pedagogical definition of the weak equivalence principle (WEP), we further point out that the LV Hamiltonian not only necessarily violates universal free fall, which is clearly demonstrated in the geodesic deviation, but also violates WEP in a semiclassical setting. As a bosonic complement, this method can be straightforwardly applicable to the spin-1 case, which shall be useful in the analysis of atomic tests of WEP, such as the case of the $^{87}\mathrm{Rb}_1$ atom.

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I. INTRODUCTION

Symmetry has been a main theme of physics in the previous century and may continue to be so in the 21st century. Of the various kinds of symmetries we know, local Lorentz symmetry (LLS) is the most fundamental. It is a cornerstone of the Standard Model (SM) in particle physics and General Relativity (GR). Though SM and GR have achieved impressive successes with various experimental verifications [1,2], there is still no concrete indication of a consistent theory of quantum gravity (QG) that may help to resolve longstanding puzzles in contemporary physics, such as the intriguing information paradox inside black holes [3]. On the other hand, there is a growing interest in searching for tiny violations of Lorentz symmetry both in theory [4] and in experiment [5]. Indeed, many candidate QG theories predict such a possibility [6]. If proven to be true, it will definitely be a concrete clue to the physics at Planck scale, an ultrahigh energy scale far beyond any direct experimental access. To thoroughly explore this possibility, Kostelecký and collaborators established an effective field theory called Standard Model Extension (SME) [7–9], which incorporates SM and GR, with various possible LV operators. This framework largely facilitates the study of Lorentz and charge, parity, and time reversal (CPT) symmetry and has already become a powerful toolbox in both theoretical and phenomenological investigations in this field [10].

As another conceptual bridge from special relativity to GR, the equivalence principle (EP), especially the Einstein equivalence principle (EEP), entails a close relationship to Lorentz symmetry and has also been broadly tested in various kinds of physical systems [11–14]. According to the famous statement by C.M. Will [15], LLS, local position invariance, and the weak equivalence principle (WEP) are the three key ingredients of EEP. So violation of LLS necessarily implies violation of EEP, whereas the contrary is not necessarily true. A thorough investigation of the relation between EP and LLS is still missing [16,17], though in view of Schiff's conjecture [18], WEP may imply the validity of LLS. Moreover, even in the Lorentz-invariant (LI) context, the debate as to whether EP holds true in the quantum domain seems far from complete [19,20]. In this paper, we do not focus too much on this debate. Instead, we adopt a relatively conservative point of view; *i.e.*, there is no conflict of WEP with nonrelativistic (NR) quantum mechanics [20,21]. In other words, the NR Hamiltonian derived from GR for the Schrödinger equation is compatible with WEP. For any nonrelativistic system, the well-known Bargmann's superselection rule prohibits mass from being a superposition parameter [22], thus superposition of different mass eigenstates, like neutrino oscillation in relativistic physics, is beyond the scope of this constrained assertion. Taking into account the fact [13,23,24] that most laboratory tests until now are still nonrelativistic, we think an appropriate test framework for WEP even in the quantum regime must go

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beyond GR (test of WEP in the classical domain necessarily to go beyond GR).

Many generalized theories of gravity [11,15,25,26] fit into this category, but in our viewpoint, the gravity sector of SME [8] is more suitable for such a task. In SME, WEP violation is associated with Lorentz and *CPT* violation because various LV coefficients can also be species dependent, which enables more exotic violation effects [27] and makes this framework as broad as it can be. Discussions of EP in this framework are also abundant [27–30], and most of them concentrate on fermion-gravity couplings because matter is composed of fermions. However, for an effective point of view, as the test particles can also be composite bosons made of fermions, such as ^{88}Sr or ^{133}Cs, we think it would be a valuable complement to discuss EP directly using boson fields instead, especially taking into account the recent trend in utilizing microscopic objects such as cold atoms as test particles [11,24,31]. In this sense, the boson LV coefficients can be totally effective; *i.e.*, microscopically, they must be certain combinations of the LV coefficients of the component fermions involved (*e.g.*, electron and proton). In this paper, for simplicity, we focus on the scalar.

The paper is organized as follows. In the next section, we briefly review the scalar LV Lagrangian and the corresponding canonical formalism. In Sec. III, by using the ansatz $\Phi(t,\vec{r}) = e^{-imt}\psi(t,\vec{r})$, we derive the NR Hamiltonian to first order in LV coefficients and metric perturbations from the LV-corrected Klein-Gordon equation. In Sec. IV, following the method of Ref. [32], we recast the Klein-Gordon equation into the Schrödinger form, then to the desired order of approximation, we get the NR Hamiltonian using the Foldy-Wouthuysen transformation (FWT) [33,34]. In Sec. V, we briefly discuss the test of EP and its possible relevance to the Hamiltonian we derived. Then we summarize our results in Sec. VI. The convention is the same as that in Ref. [8], where $\mathrm{diag}(\eta_{\mu\nu}) = (-1,1,1,1)$ and $\epsilon_{0123} = +1$.

II. HAMILTONIAN OF THE LORENTZ-VIOLATING SCALAR

In Ref. [8], by generalizing SME to Riemann-Cartan spacetime, Kostelecký introduced various LV operators both in the pure gravity sector and in the matter sector through minimal matter–gravity couplings. In the matter sector, the Higgs Lagrangian reads

$$\mathcal{L}_\Phi = -e\bigg\{[g^{\mu\nu} - (\tilde{k}_{\phi\phi})^{\mu\nu}]D_\mu\Phi^\dagger D_\nu\Phi + (m^2+\xi R)\Phi^\dagger\Phi \\ - [i(k_\phi)^\mu\Phi^\dagger D_\mu\Phi + \mathrm{H.c.}] + \frac{1}{2}k_{\phi A}{}^{\mu\nu}F_{\mu\nu}\Phi^\dagger\Phi\bigg\}, \quad (1)$$

where $D_\nu\Phi = (\nabla_\nu - iqA_\nu)\Phi$, and for completeness, we also included the nonminmal coupling ξR term. Note that for notational simplicity, we have introduced $(\tilde{k}_{\phi\phi})^{\mu\nu} \equiv \frac{1}{2}[(k_{\phi\phi})^{\mu\nu} + (k_{\phi\phi})^{\nu\mu *}]$, which can be considered to have a symmetric real part and an antisymmetric imaginary part. $(k_\phi)^\mu$ can also have complex values, though in flat spacetime, it must be real. For later convenience, we can further define $\tilde{k}_{\phi\phi}^{\mu\nu} \equiv (K^{\mu\nu} + iS^{\mu\nu})$ with $K^{\mu\nu} = K^{\nu\mu}$, $S^{\mu\nu} = -S^{\nu\mu}$, and $K^{\mu\nu}$, $S^{\mu\nu} \in \mathbb{R}$. Similarly, we can also define $(k_\phi)^\mu \equiv (a^\mu + ib^\mu)$ with a^μ, $b^\mu \in \mathbb{R}$. As mentioned before, here, $(k_\phi)^\mu$, $\tilde{k}_{\phi\phi}^{\mu\nu}$ can be regarded as effective LV coefficients of composite spin-0 bosons, not necessarily referring to the LV coefficients of the Higgs particle.

From the Lagrangian (1), we can define $\tilde{G}^{\mu\nu} \equiv [g^{\mu\nu} - (\tilde{k}_{\phi\phi})^{\mu\nu}]$. Then the Euler-Lagrangian equation is given by

$$\left[D_\mu + \frac{\partial_\mu e}{e}\right][\tilde{G}^{\mu\nu}D_\nu\Phi + ik_\phi{}^{\mu *}\Phi] + ik_\phi{}^\mu D_\mu\Phi \\ - \frac{1}{2}k_{\phi A}{}^{\mu\nu}F_{\mu\nu}\Phi - (m^2+\xi R)\Phi = 0. \quad (2)$$

This equation is intrinsically second order in time derivatives, so we cannot obtain a Schrödinger-like equation directly from (2). Instead, we turn to the canonical formalism. From

$$\pi_\Phi \equiv \frac{\partial\mathcal{L}_\Phi}{\partial\dot{\Phi}} = -e[\tilde{G}^{\rho 0}(D_\rho\Phi)^\dagger - ik_\phi{}^0\Phi^\dagger], \quad (3)$$

$$\pi_{\Phi^\dagger} \equiv \frac{\partial\mathcal{L}_\Phi}{\partial\dot{\Phi}^\dagger} = -e[\tilde{G}^{0\rho}D_\rho\Phi + ik_\phi{}^{0*}\Phi], \quad (4)$$

we can solve $\dot{\Phi}$, $\dot{\Phi}^\dagger$ in terms of π_Φ, $\pi_{\Phi^\dagger}$, *i.e.*,

$$\dot{\Phi}^\dagger = \frac{-1}{\tilde{G}^{00}}\left[\frac{\pi_\Phi}{e} - ik_\phi{}^0\Phi^\dagger + \tilde{G}^{i0}(D_i\Phi)^\dagger\right] - iqA_0\Phi^\dagger, \quad (5)$$

$$\dot{\Phi} = \frac{-1}{\tilde{G}^{00}}\left[\frac{\pi_{\Phi^\dagger}}{e} + ik_\phi{}^{0*}\Phi + \tilde{G}^{0i}D_i\Phi\right] + iqA_0\Phi. \quad (6)$$

Performing the canonical transformation on (1), we get the Hamiltonian density,

$$\mathcal{H} = -\frac{\pi_\Phi\pi_{\Phi^\dagger}}{e\tilde{G}^{00}} - \frac{1}{\tilde{G}^{00}}[\tilde{G}^{0j}\pi_\Phi D_j\Phi + \tilde{G}^{j0}(D_j\Phi)^\dagger\pi_{\Phi^\dagger}] \\ + i\left[\frac{k_\phi^0}{\tilde{G}^{00}} - qA_0\right]\Phi^\dagger\pi_{\Phi^\dagger} - i\left[\frac{k_\phi^{0*}}{\tilde{G}^{00}} - qA_0\right]\pi_\Phi\Phi \\ + e[\bar{G}^{ij}(D_i\Phi)^\dagger D_j\Phi + \bar{M}^2\Phi^\dagger\Phi] \\ + ie\bigg\{[k_\phi^{j*}(D_j\Phi)^\dagger\Phi - k_\phi^j\Phi^\dagger D_j\Phi] + \frac{1}{\tilde{G}^{00}}[k_\phi^0\Phi^\dagger\tilde{G}^{0j}D_j\Phi \\ - k_\phi^{0*}\tilde{G}^{j0}(D_j\Phi)^\dagger\Phi]\bigg\}, \quad (7)$$

where we have defined $\bar{G}^{ij} \equiv [\tilde{G}^{ij} - \frac{\tilde{G}^{i0}\tilde{G}^{0j}}{\tilde{G}^{00}}]$ and $\bar{M}^2 \equiv [m^2 + \xi R + \frac{1}{2} k_{A\phi} \cdot F - \frac{|k_\phi^0|^2}{\tilde{G}^{00}}]$. Also note $\bar{G}^{ij*} = \tilde{G}^{ji} - \frac{\tilde{G}^{0i}\tilde{G}^{j0}}{\tilde{G}^{00}} = \bar{G}^{ji}$ as $(\tilde{k}_{\phi\phi})^{\mu\nu*} = (\tilde{k}_{\phi\phi})^{\nu\mu}$. The Hamiltonian density (7) will be useful in Sec. IV for the derivation of a Schrödinger-like equation. In the following sections, we will set $A_\mu = 0$ to avoid an electromagnetic interaction, as even a very tiny electromagnetic interaction spoils the test of WEP, and we included it here only for completeness. Strictly speaking, only a neutral particle is immune to electromagnetic interaction, and in that case, the scalar field must be real. In flat spacetime, we can discard the $(k_\phi)^\mu$ term as it only contributes a total derivative for a real scalar. Similarly, $(\tilde{k}_{\phi\phi})^{\mu\nu}$ can only take the real symmetric and traceless part and can be shifted to the fermion sector with $c_{\mu\nu} \to c_{\mu\nu} - \frac{1}{2}(\tilde{k}_{\phi\phi})^{\mu\nu}$ through a coordinate transformation [27]. However, all of the above issues are not very relevant here when coupled with gravity. For a gravity-coupled neutral scalar, we only need to ignore the $S^{\mu\nu}$ and $k_{\phi A}^{\mu\nu}$ terms. For completeness, we will still use the complex scalar to demonstrate all of the results.

III. STATIC METRIC AND TRADITIONAL ROUTE TO THE NONRELATIVISTIC EQUATION

In curved spacetime, LV coefficients can also contribute to the energy momentum tensor [8] and, through the Einstein equation, affect the corresponding metric solutions. Here, because the statement of WEP involves a "free-moving" test particle and we are only interested in matter–gravity couplings, for simplicity, we can adopt a test particle assumption [29], where the spacetime metric is untouched by the LV coefficients associated with the matter sector. So we can still make use of the conventional metric from GR, and "free motion" implies we have to take $A_\mu = 0$ in (2), which gives

$$\{g^{\mu\nu}\nabla_\mu\nabla_\nu - \tilde{k}_{\phi\phi}^{\mu\nu}[\partial_\mu\partial_\nu + \Gamma^\lambda{}_{\mu\lambda}\partial_\nu] + ik_\phi^{\mu*}[\partial_\mu + \Gamma^\lambda{}_{\mu\lambda}] + ik_\phi{}^\mu\partial_\mu - (m^2 + \xi R)\}\Phi = 0, \tag{8}$$

where, for simplicity, we also assumed Riemann spacetime instead of Riemann-Cartan spacetime; otherwise, $\frac{1}{e}\partial_\mu[eg^{\mu\nu}\partial_\nu]\Phi = g^{\mu\nu}[\nabla_\mu\nabla_\nu - T_{(\mu\nu)}{}^\lambda\nabla_\lambda]\Phi$, where $T^\lambda{}_{\mu\nu}$ is the torsion tensor. For simplicity, we can take the isotropic static metric [35],

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = V^2dt^2 - \delta_{\hat{i}\hat{j}}W^2dx^idx^j, \tag{9}$$

as an example. Then the only nonzero Christoffel symbols are given by

$$\Gamma^i_{jk} = [\delta^i_j\partial_kW + \delta^i_k\partial_jW - \delta_{jk}\partial_iW]/W, \qquad \Gamma^0{}_{0j} = \frac{\partial_jV}{V},$$
$$\Gamma^j{}_{00} = \frac{1}{2}\frac{\partial_jV^2}{W^2}, \qquad \Gamma^\lambda_{i\lambda} = \partial_iV/V + 3\partial_iW/W. \tag{10}$$

Defining $\mathcal{F} \equiv \frac{V}{W}$ and substituting (10), $g^{00} = -1/V^2$ and $g^{ij} = \delta^{ij}/W^2$ into (8), we get

$$\begin{aligned}\{-\partial_0^2 + \mathcal{F}^2[\Delta + \vec{\nabla}\ln(VW)\cdot\vec{\nabla}] - V^2(m^2+\xi R)\}\Phi \\ = V^2\{\tilde{k}_{\phi\phi}^{\mu\nu}[\partial_\mu\partial_\nu + \delta^i_\mu\partial_i\ln(VW^3)\partial_\nu] \\ - [2ia^\mu\partial_\mu + ik_\phi^{j*}\partial_j\ln(VW^3)]\}\Phi.\end{aligned} \tag{11}$$

The Ricci scalar for the metric (9) is given by

$$R = \frac{2}{VW^4}[W^2\nabla^2V + 2WV\nabla^2W + W\vec{\nabla}V\cdot\vec{\nabla}W - V(\vec{\nabla}W)^2]. \tag{12}$$

Note that R differs by a minus sign if using convention $\mathrm{diag}(\eta_{\mu\nu}) = (1,-1,-1,-1)$.

Now substituting the ansatz $\Phi(t,\vec{r}) = e^{-imt}\psi(t,\vec{r})$ into (11), we can get

$$\begin{aligned}&[m^2\psi + 2im\dot{\psi} - \ddot{\psi}] \\ &\quad + \frac{\mathcal{F}^2}{1+\tilde{k}_{\phi\phi}^{00}V^2}[\Delta + \vec{\nabla}\ln(VW)\cdot\vec{\nabla} - W^2(m^2+\xi R)]\psi \\ &= \frac{V^2}{1+\tilde{k}_{\phi\phi}^{00}V^2}\{[\tilde{k}_{\phi\phi}^{i0}\partial_i\ln(VW^3) + \tilde{k}_{\phi\phi}^{(0i)}\partial_i](\partial_0 - im) \\ &\quad + \tilde{k}_{\phi\phi}^{ij}[\partial_i + \partial_i\ln(VW^3)]\partial_j - i[2a^0(\partial_0 - im) \\ &\quad + 2\vec{a}\cdot\vec{\nabla} + (\vec{a} - i\vec{b})\cdot\vec{\nabla}\ln(VW^3)]\}\psi,\end{aligned} \tag{13}$$

where $\tilde{k}_{\phi\phi}^{(0i)} \equiv (\tilde{k}_{\phi\phi}^{0i} + \tilde{k}_{\phi\phi}^{i0})$. Because most of the tests of EP and LLS until now have been done near the Earth's surface, where the metric functions are asymptotically flat, *i.e.*, $g_{\mu\nu} \simeq \eta_{\mu\nu}$, we can resort to the approximation scheme in [27], where terms proportional to the product of LV coefficients and metric perturbation of powers of l and n, respectively, are denoted by $\mathcal{O}(l,n)$. Next, we proceed our calculations with the Schwarzschild metric $V = (1+\frac{1}{2}\chi)(1-\frac{1}{2}\chi)^{-1}$, $W = (1-\frac{1}{2}\chi)^2$, where $\chi \equiv -\frac{GM}{c^2r}$. Now $R = 0$, when $r \neq 0$, even $R^\kappa{}_{\lambda\mu\nu} \neq 0$ in general. Below, we will expand $g_{\mu\nu}$ in powers of χ and keep only terms up to $\mathcal{O}(0,2)$ and $\mathcal{O}(1,1)$. In doing so, we also take advantage of the Virial theorem in which $\chi \sim \frac{\bar{v}^2}{c^2}$. In essence, that means we can also take $\bar{v}$ (assuming in natural units that $c = 1$) as an expansion parameter. Also note that in laboratory experiments, $|\partial_i\chi| \ll |\chi/L|$ [27], where L is the typical experimental scale, so we can treat $\nabla_i\chi$ as higher order compared to χ and ignore its product with LV coefficients. Under these assumptions, we can rearrange (13) as below:

$$
\begin{aligned}
i\dot{\psi} = &\left\{\mathcal{F}^2\left[1-\left(\tilde{k}_{\phi\phi}^{00}+\frac{a^0}{m}\right)V^2\right]\frac{\hat{\vec{p}}^2}{2m}-\frac{\mathcal{F}^2}{2m}\vec{\nabla}\ln(VW)\cdot\vec{\nabla}\right.\\
&+\frac{m}{2}[(V^2-1)-\tilde{k}_{\phi\phi}^{00}V^4]\\
&\left.+V^2\left[\frac{\tilde{k}_{\phi\phi}^{(0i)}}{2}\hat{p}_i-\frac{\tilde{k}_{\phi\phi}^{ij}}{2m}\hat{p}_i\hat{p}_j+\frac{\vec{a}\cdot\hat{\vec{p}}}{m}-\frac{a^0}{2}(V^2+1)\right]\right\}\psi\\
&+V^2\frac{\tilde{k}_{\phi\phi}^{(0i)}}{2m}\nabla_i\dot{\psi}+\left(1-\frac{a^0}{m}V^2\right)\frac{\ddot{\psi}}{2m}. \qquad (14)
\end{aligned}
$$

At order $\mathcal{O}(0,1)$, we have $i\dot{\psi}=[-\frac{\vec{\nabla}^2}{2m}+m\chi]\psi$, which is roughly the order of $m\bar{v}^2$. So we know $\frac{\ddot{\psi}}{2m}\sim m(\bar{v}^2)^2\sim m\chi^2$, and then we can temporarily ignore the last two terms proportional to $\nabla_i\dot{\psi}$ and $\frac{\ddot{\psi}}{2m}$ in (14) and get

$$
\begin{aligned}
i\dot{\psi} = &\left\{\left[(1+4\chi)-\left(\tilde{k}_{\phi\phi}^{00}+\frac{a^0}{m}\right)(1+6\chi)\right]\frac{\hat{\vec{p}}^2}{2m}+\frac{\chi}{4m}\vec{\nabla}\chi\cdot\vec{\nabla}\right.\\
&+\left[m\chi(1+\chi)-m\tilde{k}_{\phi\phi}^{00}\left(\frac{1}{2}+2\chi\right)\right]-(1+3\chi)a^0\\
&\left.+(1+2\chi)\left[\frac{\tilde{k}_{\phi\phi}^{(0i)}}{2}\hat{p}_i-\frac{\tilde{k}_{\phi\phi}^{ij}}{2m}\hat{p}_i\hat{p}_j+\frac{\vec{a}\cdot\hat{\vec{p}}}{m}\right]\right\}\psi, \qquad (15)
\end{aligned}
$$

up to $\mathcal{O}(\chi^2)$, except for the LI term $\frac{\chi}{4m}\vec{\nabla}\chi\cdot\vec{\nabla}$. Now defining the terms in the large braces in (15) as $\hat{H}_0$, and adding the correction $\frac{\ddot{\psi}}{2m}=-\frac{1}{2m}(\hat{H}_0)^2\psi$ and $\nabla_i\dot{\psi}=-i\nabla_i(\hat{H}_0\psi)$ back into (15) to replace the last two terms in (14), we can obtain the desired order:

$$
\begin{aligned}
i\dot{\psi} = &\left\{\left[\left(1+3\chi-\frac{\tilde{k}_{\phi\phi}^{00}}{2}\right)\frac{\hat{\vec{p}}^2}{2m}+m\chi\left(1+\frac{\chi}{2}\right)-\frac{(\hat{\vec{p}}^2)^2}{8m^3}\right]\right.\\
&+(1+\chi)\left[\frac{\vec{a}\cdot\hat{\vec{p}}}{m}-\frac{\tilde{k}_{\phi\phi}^{ij}}{2m}\hat{p}_i\hat{p}_j\right]+(1+2\chi)\left[\frac{\tilde{k}_{\phi\phi}^{(0i)}}{2}\hat{p}_i-a^0\right]\\
&\left.-\frac{m}{2}\tilde{k}_{\phi\phi}^{00}(1+3\chi)\right\}\psi+\left\{\frac{i}{2m}\left(1+\frac{13\chi}{2}\right)\vec{\nabla}\chi\cdot\hat{\vec{p}}\right.\\
&+\left[2\frac{a^0}{m}-\tilde{k}_{\phi\phi}^{00}\right]\chi\frac{\hat{\vec{p}}^2}{2m}+\frac{1}{4m}(\Delta\chi+2(\vec{\nabla}\chi)^2)\\
&\left.+\tilde{k}_{\phi\phi}^{(0i)}\frac{\hat{\vec{p}}^2}{4m^2}\hat{p}_i\right\}\psi. \qquad (16)
\end{aligned}
$$

Note that, as the procedure implies, the above equation will be valid only up to $\mathcal{O}(0,2)$ and $\mathcal{O}(1,1)$. We divide the right-hand side of (16) into two parts. In fact, comparing with the NR Hamiltonian (26) obtained by a quite different method, we find that, except for the $\vec{\nabla}\chi\cdot\hat{\vec{p}}$ term (belonging to the latter brace), the part enclosed by the former brace is consistent with (26) up to the desired orders, whereas those in the latter brace may be classified as divergent higher-order terms. Indeed, we can even verify this coincidence (of the NR results obtained with different methods) by choosing another metric, e.g., the uniform accelerating metric. So it is interesting to explore whether the above NR procedure can be improved to yield completely consistent results with the FWT or even extended to higher orders systematically. This question is beyond the scope of this paper. In the next section, we will utilize the FWT [32–34,36,37] to show that the NR approximation can indeed be obtained systematically.

IV. SCHRÖDINGER-LIKE EQUATION FOR SCALAR FIELD AND FWT

The Foldy-Wouthuysen transformation for a scalar field was first introduced in [34] and later refined by [32,36,37]. In order to perform FWT for a scalar field, first we have to obtain a Schrödinger-like Hamiltonian from the scalar Lagrangian (1), and then we can do a pseudounitary transformation parallel to the case of the fermion. Next, with a series expansion in terms of $\frac{1}{m}$, we can obtain the NR approximation to any desired order we prefer. Below, we will show the FWT up to $\mathcal{O}(1,1)$, $\mathcal{O}(0,2)$ in a static Schwarzschild metric, and we will perform the FWT both directly [34,36] and indirectly with a unitary transformation [32] performed first. We will show that these two procedures give the same result, and the result is consistent with the part enclosed by the first brace in (16).

To formally remove the second-order time derivatives, first we can obtain the Hamiltonian equation of motion with a canonical formalism. From Hamiltonian $H_\Phi=\int d^3\vec{x}\mathcal{H}$, where $\mathcal{H}$ is given by (7), we get

$$
\begin{aligned}
\dot{\Phi} &= \frac{\delta H_\Phi}{\delta\pi_\Phi}=-\frac{\pi_{\Phi^\dagger}}{e\tilde{G}^{00}}-\frac{\tilde{G}^{0j}}{\tilde{G}^{00}}D_j\Phi-i\left[\frac{k^{0*}_{\phi}}{\tilde{G}^{00}}-qA_0\right]\Phi,\\
\dot{\pi}_{\Phi^\dagger} &= -\frac{\delta H_\Phi}{\delta\Phi^\dagger}=-D_j\left[\frac{\tilde{G}^{j0}}{\tilde{G}^{00}}\pi_{\Phi^\dagger}\right]-i\left[\frac{k^0{}_\phi}{\tilde{G}^{00}}-qA_0\right]\pi_{\Phi^\dagger}\\
&\quad+D_i[e\bar{G}^{ij}D_j\Phi]-e\left[m^2+\xi R+\frac{1}{2}k_{\phi A}{}^{\mu\nu}F_{\mu\nu}\right.\\
&\quad\left.-\frac{|k^0_\phi|^2}{\tilde{G}^{00}}\right]\Phi+ie\left\{\left[k^j_\phi D_j\Phi+\frac{1}{e}D_j(ek^j_\phi{}^*\Phi)\right]\right.\\
&\quad\left.-\left[k^0{}_\phi\frac{\tilde{G}^{0j}}{\tilde{G}^{00}}D_j\Phi+\frac{1}{e}D_j\left(e\frac{\tilde{G}^{j0}}{\tilde{G}^{00}}k^{0*}_\phi\Phi\right)\right]\right\}. \qquad (17)
\end{aligned}
$$

Then we can define $\Theta=+\frac{i}{m}\pi_{\Phi^\dagger}$ and symmetrize fields Φ, Θ by the definition $\Psi\equiv\binom{\eta}{\zeta}\equiv\frac{1}{\sqrt{2}}\binom{\Phi+\Theta}{\Phi-\Theta}$. For notational convenience, we can also define $\bar{g}^{\mu\nu}\equiv g^{\mu\nu}-K^{\mu\nu}$ and hence $\tilde{G}^{00}=\bar{g}^{00}$. With these definitions, Eq. (17) can be cast into the Schrödinger form $i\dot{\Psi}=\hat{H}_\Psi\Psi$, where

$$\hat{H}_\Psi = \left[\frac{a^0}{\bar{g}^{00}} - qA_0 + \frac{1}{2}\nabla_j\left(\frac{S^{0j}}{\bar{g}^{00}}\right) + \frac{1}{2}\left\{\hat{\pi}_j, \frac{\bar{g}^{0j}}{\bar{g}^{00}}\right\}\right]\hat{1} + \left[\frac{1}{2}\nabla_j\left(\frac{\bar{g}^{0j}}{\bar{g}^{00}}\right) - \frac{b^0}{\bar{g}^{00}} - \frac{1}{2}\left\{\hat{\pi}_j, \frac{S^{0j}}{\bar{g}^{00}}\right\}\right]i\sigma_1 + \left\{\frac{e}{2m}\bar{M}^2 + \frac{m}{2e\bar{g}^{00}} + \frac{1}{2m}\hat{\pi}_i(e\bar{G}^{ij}\hat{\pi}_j) - \hat{O}_k\right\}i\sigma_2 + \left\{\frac{e}{2m}M^2 - \frac{m}{2e\bar{g}^{00}} + \frac{1}{2m}\hat{\pi}_i(e\bar{G}^{ij}\hat{\pi}_j) - \hat{O}_k\right\}\sigma_3. \quad (18)$$

In (18), $\hat{\pi} \equiv [\hat{p}_i - qA_i]$, $\hat{O}_k \equiv \frac{1}{2m}\{[\vec{\nabla}\cdot(e\vec{b}) - \{\hat{\pi}_j, ea^j\}] + [\nabla_j[e(a^0S^{0j} - b^0\bar{g}^{0j})/\bar{g}^{00}] + \{\hat{\pi}_j, \frac{e}{\bar{g}^{00}}(a^0\bar{g}^{0j} + b^0S^{0j})\}]\}$, and σ_i, $i = 1, 2, 3$, are the Pauli matrices. For completeness, until now, we have not assumed the isotropic metric and $A_\mu = 0$ or done any approximation. From the definition of pseudohermiticity $\hat{\sigma}_3\hat{O}^\dagger\hat{\sigma}_3 = \hat{O}$ [36], it is straightforward to verify that $\hat{H}_\Psi$ in (18) is pseudohermitian. The pseudohermiticity requirement is necessary to ensure that all of the eigenenergies of $\hat{H}_\Psi$ are real values. We also note the formal similarity of pseudohermiticity defined by σ_3 and that defined by γ^0 in spinor space, *i.e.*, $\gamma^0\mathcal{M}^\dagger\gamma^0 = \mathcal{M}$. This indicates that σ_3 plays a role very similar to that of γ^0, as can be seen from the process of dividing operators into even and odd parts in FWT [33,34]. In the following, we will take $A_\mu = 0$ and the isotropic metric (9), so (18) becomes

$$\hat{H}_\Psi = \left[\frac{a^0}{g^{00}} - \frac{K^{0j}}{g^{00}}\hat{p}_j - \frac{1}{2}\hat{p}_j\left(\frac{\tilde{k}^{j0}_{\phi\phi}}{g^{00}}\right)\right]\hat{1} - \left[\frac{i}{2}\hat{p}_j\left(\frac{\tilde{k}^{j0}_{\phi\phi}}{g^{00}}\right) + \frac{b^0}{g^{00}} - \frac{S^{j0}}{g^{00}}\hat{p}_j\right]i\sigma_1 + \left\{\frac{e}{2m}M^2 + \frac{m}{2e\bar{g}^{00}} + \frac{1}{2m}\hat{p}_i(e\bar{G}^{ij}\hat{p}_j) - \hat{O}_k\right\}i\sigma_2 + \left\{\frac{e}{2m}M^2 - \frac{m}{2e\bar{g}^{00}} + \frac{1}{2m}\hat{p}_i(e\bar{G}^{ij}\hat{p}_j) - \hat{O}_k\right\}\sigma_3, \quad (19)$$

where $M^2 \equiv [m^2 + \xi R]$, $\hat{O}_k \equiv \frac{1}{2m}[\vec{\nabla}\cdot(e\vec{b}) - \{\hat{p}_j, ea^j\}]$ and $\bar{g}^{00} = -(\frac{1}{V^2} + K^{00})$, $\bar{G}^{ij} = \frac{\delta_{ij}}{W^2} - \tilde{k}^{ij}_{\phi\phi}$, $e = VW^3$. Note by replacing $\bar{g}^{00}$ with g^{00} in the denominators, we have already ignored terms with second-order LV couplings.

A. Pseudounitary transformation

With the relativistic Hamiltonian (19), we can perform FWT directly to obtain the NR approximation. However, we wish to perform a pseudo-unitary transformation first, which will make the Hamiltonian more suitable for FWT, and then we will do the FWT afterward. We call this procedure the Cognola-Vanzo-Zerbini (CVZ) method, which was first introduced in [32]. For a similarity transformation to be defined as pseudounitary, its associated operator $\hat{U}$ must satisfy $\hat{\sigma}_3\hat{U}^\dagger\hat{\sigma}_3 = \hat{U}^{-1}$ [32,34,36]. The goal of the desired pseudounitary transformation is to make the term proportional to m, $\frac{e}{2m}M^2 + \frac{m}{2e\bar{g}^{00}}$, associated with σ_2, vanish. Because the square brackets in (19) associated with $\hat{1}$ and $i\sigma_1$ do not contain any term proportional to m, we can perform a "rotation" only in the space spanned by σ_2 and σ_3; *i.e.*, define $\hat{U} \equiv f + g\sigma_1$ to eliminate the mass proportional term in the large brace multiplied by $i\sigma_2$. Assuming $f, g \in \mathbb{R}^\infty$, the pseudounitary condition of $\hat{U}$ indicates $\hat{U}^{-1} = f - g\sigma_1$ and $f^2 - g^2 = 1$. With a little algebra, the mass-eliminating requirement gives $\frac{f-g}{f+g} = e\sqrt{-\bar{g}^{00}} = W^3[1 + \tilde{k}^{00}_{\phi\phi}V^2]^{\frac{1}{2}}$. Combined with $f^2 - g^2 = 1$, we get

$$\hat{U} = \frac{1}{2}(e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}}\left[1 + e\sqrt{-\bar{g}^{00}} + (1 - e\sqrt{-\bar{g}^{00}})\sigma_1\right]. \quad (20)$$

Then we can use (20) to perform a pseudounitary transformation $\hat{H}'_\Psi \equiv \hat{U}^{-1}\hat{H}_\Psi\hat{U}$ on (19), which gives

$$\hat{H}'_\Psi = \left\{\frac{m}{\sqrt{-\bar{g}^{00}}} + \frac{\xi R}{2m\sqrt{-\bar{g}^{00}}} + (e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}}\left[\frac{1}{2m}\hat{p}_i(e\bar{G}^{ij}\hat{p}_j) - \hat{O}_k\right](e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}}\right\}\sigma_3 + \left\{\frac{\xi R}{2m\sqrt{-\bar{g}^{00}}} + (e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}}\left[\frac{1}{2m}\hat{p}_i(e\bar{G}^{ij}\hat{p}_j) - \hat{O}_k\right](e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}}\right\}i\sigma_2 - \left[\frac{b^0}{g^{00}} + \frac{i}{2}\hat{p}_j\left(\frac{\tilde{k}^{j0}_{\phi\phi}}{g^{00}}\right) + \frac{S^{0j}}{g^{00}}\hat{p}_j + \frac{3}{2}\frac{K^{0j}}{g^{00}}\nabla_j\ln W\right]i\sigma_1 + \left\{\frac{a^0}{g^{00}} - \frac{K^{0j}}{g^{00}}\hat{p}_j - \frac{1}{2}\hat{p}_j\left(\frac{\tilde{k}^{j0}_{\phi\phi}}{g^{00}}\right) + \frac{3}{2}\frac{S^{0j}}{g^{00}}\nabla_j\ln W\right\}\hat{1}. \quad (21)$$

Following the spirit of FWT [33,34], we can separate $\hat{H}'_\Psi$ into even and odd parts according to whether they commutate or anticommutate with σ_3, where σ_3 plays the role of γ^0 in the fermion case, as mentioned before. In other words, we can write $\hat{H}'_\Psi = m\sigma_3 + \mathcal{E} + \mathcal{O}$, where $[\mathcal{E}, \sigma_3] = 0$ and $\{\mathcal{O}, \sigma_3\} = 0$. Ignoring the nonminimal coupling term ξR and those which are products of the derivatives of χ and LV coefficients, the even and odd operators are

$$\mathcal{E} = \left\{ (e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}} \left[\frac{1}{2m} \hat{p}_i (e\bar{G}^{ij}\hat{p}_j) - \hat{O}_k \right] (e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}} + m\left(\frac{1}{\sqrt{-\bar{g}^{00}}} - 1 \right) \right\} \sigma_3 + \left\{ \frac{a^0}{g^{00}} - \frac{K^{0j}}{g^{00}} \hat{p}_j \right\} \hat{1}, \tag{22}$$

$$\mathcal{O} = \left\{ (e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}} \left[\frac{1}{2m} \hat{p}_i (e\bar{G}^{ij}\hat{p}_j) - \hat{O}_k \right] (e\sqrt{-\bar{g}^{00}})^{-\frac{1}{2}} \right\} i\sigma_2 - \left[\frac{b^0}{g^{00}} + \frac{S^{0j}}{g^{00}} \hat{p}_j \right] i\sigma_1. \tag{23}$$

Now, clearly, $\mathcal{E}$ is already diagonal and hence decouples the two-component field Ψ, whereas $\mathcal{O}$ is off-diagonal and still needs to be diagonalized. In order to make the off-diagonal part smaller and smaller, we can perform a further unitary transformation,

$$\Psi' \to A^{-1}\Psi', \qquad \hat{H}'_\Psi \to A^{-1}\hat{H}'_\Psi A - iA^{-1}(\partial_t A), \tag{24}$$

where $A = \exp[-\frac{1}{2m}\sigma_3\mathcal{O}]$ [34]. For a static metric, this transformation leads to

$$\begin{aligned} \hat{\hat{H}}_\Psi &= e^{\frac{1}{2m}\sigma_3\mathcal{O}} \hat{H}'_\Psi e^{-\frac{1}{2m}\sigma_3\mathcal{O}} \\ &= \hat{H}'_\Psi + \frac{1}{2m}[\sigma_3\mathcal{O}, \hat{H}'_\Psi] + \frac{1}{8m^2}[\sigma_3\mathcal{O}, [\sigma_3\mathcal{O}, \hat{H}'_\Psi]] \\ &\quad + \frac{1}{3!(2m)^3}[\sigma_3\mathcal{O}, [\sigma_3\mathcal{O}, [\sigma_3\mathcal{O}, \hat{H}'_\Psi]]] + \cdots \\ &= \sigma_3 m + \left\{ \mathcal{E} + \frac{1}{2m}\sigma_3\mathcal{O}^2 - \frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] + \cdots \right\} \\ &\quad + \left\{ \frac{1}{2m}\sigma_3[\mathcal{O}, \mathcal{E}] - \frac{1}{3m^2}\mathcal{O}^3 + \cdots \right\}. \end{aligned} \tag{25}$$

Note that, compared to m, all terms in $\mathcal{O}$, $\mathcal{E}$ are either proportional to various powers of the metric perturbation χ and its derivatives or powers of tiny LV coefficients or some products between the two, which are all small parameters (as mentioned before, in a weak gravitation field, $\hat{\vec{p}}^2/2m \sim m\chi \ll m$ can also be regarded as small). Therefore, products of $\mathcal{O}$, $\mathcal{E}$ must be much smaller, which legitimizes the approximation procedure of the expansion in (25) [35]. Substituting the Schwarzschild metric $V=(1+\frac{1}{2}\chi)(1-\frac{1}{2}\chi)^{-1}$ and $W = (1-\frac{1}{2}\chi)^2$ into (22) and (23) and preserving only terms up to $\mathcal{O}(0,2)$, $\mathcal{O}(1,1)$, from (25), we get

$$\begin{aligned} \hat{H}_{\mathrm{CVZ}} = &\left\{ m + \left[m\chi\left(1+\frac{\chi}{2}+\frac{\chi^2}{4}\right) - m\frac{\tilde{k}^{00}_{\phi\phi}}{2}(1+3\chi) \right] \right. \\ &+ \left[(1+3\chi+5\chi^2) - \frac{\tilde{k}^{00}_{\phi\phi}}{2}(1+5\chi) \right] \frac{\hat{\vec{p}}^2}{2m} - \frac{(\hat{\vec{p}}^2)^2}{8m^3} \\ &+ \frac{(1+\chi)}{2m}[2\vec{a}\cdot\hat{\vec{p}} - \tilde{k}^{ij}_{\phi\phi}\hat{\vec{p}}_i\hat{\vec{p}}_j] - \frac{3}{4m}[2(\vec{\nabla}\chi)^2 + \Delta\chi] \\ &\left. - \frac{i}{2m}(3+10\chi)\vec{\nabla}\chi\cdot\hat{\vec{p}} \right\} \sigma_3 + (1+2\chi)[K^{0j}\hat{p}_j - a^0]\hat{1}. \end{aligned} \tag{26}$$

Note that $-\frac{(\hat{\vec{p}}^2)^2}{8m^3}$ comes from the lowest-order LI contribution of $\frac{1}{2m}\sigma_3\mathcal{O}^2$, and all other terms except for m come from $\mathcal{E}$. Up to $\mathcal{O}(1,1)$, $\mathcal{O}(0,2)$, we have not calculated $-\frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]]$. Compared to direct FWT, which will be shown below, we see that the pseudounitary transformation saves the work of calculating commutators in (25) if the NR approximation is only required to proceed to the next leading order. As mentioned before, except for the last two terms in the large brace, Eq. (26) agrees well with the terms in the first brace of (16), indicating that it is still possible to improve the NR procedure using the conventional method.

B. Foldy-Wouthuysen transformation

In this subsection, we show that direct FWT on (19) can also lead to the same result in (26). For calculational convenience, we can separate both $\mathcal{E}$ and $\mathcal{O}$ into LI and LV parts, *i.e.*, $\mathcal{E} = \mathcal{E}_{LI} + \mathcal{E}_{LV}$ and $\mathcal{O} = \mathcal{O}_{LI} + \mathcal{O}_{LV}$. In detail,

$$\mathcal{E}_{LI} = \left\{ \frac{em}{2} - \frac{m}{2eg^{00}} - m + \frac{1}{2m}\hat{p}_i(VW\hat{p}_i) \right\} \sigma_3, \tag{27}$$

$$\begin{aligned} \mathcal{E}_{LV} = &\left\{ -\frac{m}{2}\tilde{k}^{00}_{\phi\phi}\mathcal{F}^3 + \frac{VW^3}{2m}[2\vec{a}\cdot\hat{\vec{p}} - \tilde{k}^{ij}_{\phi\phi}\hat{p}_i\hat{p}_j] \right\} \sigma_3 \\ &+ \{V^2[K^{0j}\hat{p}_j - a^0]\}\hat{1}, \end{aligned} \tag{28}$$

$$\mathcal{O}_{LI} = \left\{ \frac{em}{2} + \frac{m}{2eg^{00}} + \frac{1}{2m}\hat{p}_i(VW\hat{p}_i) \right\} i\sigma_2, \tag{29}$$

$$\begin{aligned} \mathcal{O}_{LV} = &\left\{ \frac{m}{2}\tilde{k}^{00}_{\phi\phi}\mathcal{F}^3 + \frac{VW^3}{2m}[2\vec{a}\cdot\hat{\vec{p}} - \tilde{k}^{ij}_{\phi\phi}\hat{p}_i\hat{p}_j] \right\} i\sigma_2 \\ &+ \{V^2[b^0 + S^{0j}\hat{p}_j]\}i\sigma_1. \end{aligned} \tag{30}$$

So expanded in terms of χ and its derivatives, we have up to linear order of LV coefficients,

$$\begin{aligned}\mathcal{O}^2 &= \mathcal{O}_{LI}^2 + \{\mathcal{O}_{LI}, \mathcal{O}_{LV}\}\\ &= -\hat{1}\Bigg\{9m^2\chi^2\left(1+\frac{5}{2}\chi\right) + \left(\frac{\hat{\vec{p}}^2}{2m}\right)^2 - 3\chi\left(1+\frac{5}{4}\chi\right)\hat{\vec{p}}^2 + 3i\left(1+\frac{5}{2}\chi\right)\vec{\nabla}\chi\cdot\hat{\vec{p}} + \frac{3}{2}\left[\Delta\chi + \frac{5}{2}(\vec{\nabla}\chi)^2\right]\\ &\quad + \frac{\tilde{k}_{\phi\phi}^{00}}{2}(1+6\chi)\hat{\vec{p}}^2 - 3m^2\tilde{k}_{\phi\phi}^{00}\chi + \left[(1-2\chi)\frac{\hat{\vec{p}}^2}{m^2} - 6\chi\right]\left[\vec{a}\cdot\hat{\vec{p}} - \frac{\tilde{k}_{\phi\phi}^{ij}}{2}\hat{p}_i\hat{p}_j\right]\Bigg\},\end{aligned} \tag{31}$$

and

$$[\mathcal{O},[\mathcal{O},\mathcal{E}]] = [\mathcal{O}_{LI},[\mathcal{O}_{LI},\mathcal{E}_{LI}]] + [\mathcal{O}_{LI},[\mathcal{O}_{LI},\mathcal{E}_{LV}]] + [\mathcal{O}_{LI},[\mathcal{O}_{LV},\mathcal{E}_{LI}]] + [\mathcal{O}_{LV},[\mathcal{O}_{LI},\mathcal{E}_{LI}]], \tag{32}$$

where

$$[\mathcal{O}_{LI},[\mathcal{O}_{LI},\mathcal{E}_{LI}]] = \left\{[(5\chi\hat{\vec{p}}^2 - 8i\vec{\nabla}\chi\cdot\hat{\vec{p}} - \Delta\chi)]\frac{\hat{\vec{p}}^2}{m} - \frac{1}{2}\left(\frac{\hat{\vec{p}}^2}{m}\right)^3 + 6m\chi[2(i\vec{\nabla}\chi\cdot\hat{\vec{p}} + \Delta\chi) - 6m^2\chi^2 - \chi\hat{\vec{p}}^2]\right\}\sigma_3, \tag{33}$$

$$[\mathcal{O}_{LI},[\mathcal{O}_{LI},\mathcal{E}_{LV}]] + [\mathcal{O}_{LI},[\mathcal{O}_{LV},\mathcal{E}_{LI}]] + [\mathcal{O}_{LV},[\mathcal{O}_{LI},\mathcal{E}_{LI}]] = \left\{\frac{\tilde{k}_{\phi\phi}^{00}}{4}\chi\frac{\hat{\vec{p}}^2}{m} + \frac{\tilde{k}_{\phi\phi}^{00}}{16}\left(\frac{\hat{\vec{p}}^2}{m}\right)^2 - \frac{17\chi}{8m}\left[\frac{\vec{a}\cdot\hat{\vec{p}}}{m} - \frac{\tilde{k}_{\phi\phi}^{ij}}{2m}\hat{p}_i\hat{p}_j\right]\frac{\hat{\vec{p}}^2}{m}\right\}\sigma_3. \tag{34}$$

Substituting all of the above Eqs. (27)–(34) back into (25), we get the NR scalar Hamiltonian (up to second-order commutators of FWT) as

$$\begin{aligned}\hat{H}_{\mathrm{FWT}} = \Bigg\{&m + m\chi\left(1+\frac{\chi}{2}+\frac{\chi^2}{4}\right) - \frac{i}{2m}(3+10\chi)\vec{\nabla}\chi\cdot\hat{\vec{p}} + \left[(1+3\chi+5\chi^2) - \frac{\tilde{k}_{\phi\phi}^{00}}{2}(1+5\chi)\right]\frac{\hat{\vec{p}}^2}{2m} - \frac{(\hat{\vec{p}}^2)^2}{8m^3}\\ &-m\frac{\tilde{k}_{\phi\phi}^{00}}{2}(1+3\chi) + \frac{(1+\chi)}{2m}[2\vec{a}\cdot\hat{\vec{p}} - \tilde{k}_{\phi\phi}^{ij}\hat{\vec{p}}_i\hat{\vec{p}}_j] - \frac{3}{4m}\Delta\chi - \left(1+\frac{9}{4}\chi\right)[2\vec{a}\cdot\hat{\vec{p}} - \tilde{k}_{\phi\phi}^{ij}\hat{\vec{p}}_i\hat{\vec{p}}_j]\frac{\hat{\vec{p}}^2}{4m^3}\Bigg\}\sigma_3\\ &+ (1+2\chi)[K^{0j}\hat{p}_j - a^0]\hat{1}.\end{aligned} \tag{35}$$

Compared with (26), we see that, except for the LV term proportional to $\frac{\hat{\vec{p}}^2}{4m^3}$, the NR Hamiltonian obtained by direct FWT is completely the same as that obtained with the CVZ method; though to the next lowest order, the latter can be obtained without substantially calculating any commutators. At first glance, this is a little surprising because the results are expected to differ by a pseudounitary transformation; however, inspecting the CVZ method, we see that it is exactly the pseudounitary transformation, which ensures that the NR Hamiltonian is the same as that obtained with direct FWT [38]. The pseudounitary transformation preserves both the charge and matrix elements of the Hamiltonian after transformation [36].

C. Consistency check and partial support

Another confirmation can be seen by applying the different methods mentioned above to the linear accelerating metric $g_{00} = -[1+\frac{\vec{a}\cdot\vec{x}}{c^2}]^2$, $g_{ij} = \delta_{ij}$. With either direct FWT, the CVZ method, or even the unsystematically traditional method in Sec. III, we can obtain an NR Hamiltonian:

$$\begin{aligned}\hat{H}_{\mathrm{NRL}} = m(1+\phi) &+ \left[(1+\phi) - \frac{\tilde{k}_{\phi\phi}^{00}}{2}(1+3\phi)\right]\frac{\hat{\vec{p}}^2}{2m}\\ &- \frac{\tilde{k}_{\phi\phi}^{00}}{2}m(1+3\phi) + (1+\phi)\left[\frac{\vec{a}\cdot\hat{\vec{p}}}{m} - \frac{\tilde{k}_{\phi\phi}^{ij}}{2m}\hat{\vec{p}}_i\hat{\vec{p}}_j\right]\\ &- \frac{i}{2m}\vec{\nabla}\phi\cdot\hat{\vec{p}} + (1+2\phi)[K^{0j}\hat{p}_j - a^0] - \frac{(\hat{\vec{p}}^2)^2}{8m^3},\end{aligned} \tag{36}$$

where $\phi \equiv \frac{\vec{a}\cdot\vec{x}}{c^2}$. The correctness of (35) and (36) can be partially supported by comparing the LI part of these Hamiltonians with the Eqs. (20) and (21) in [37]. We can even compare the LI part of (36) with the fermion Hamiltonian obtained in [29,35,39]. The consistency of this comparison lies in the fact that each spinor component satisfies the Klein-Gordon equation as dictated by the relativistic dispersion relation. In other words, the NR Hamiltonian for a scalar field is equivalent to that of a fermion field by ignoring its spin contribution. In the same spirit, we can also expect an equivalence between the LV contribution to the NR scalar Hamiltonian with the fermion counterpart; see [27,29]. Comparing Eq. (36) with Eqs. (21), (23), and (27) in [29] by ignoring the spin interactions, we find

$$(k_\phi)^0 \sim [\tilde{a}^0 - me^0], \qquad (k_\phi)^j \sim [\tilde{a}^j - me^j],$$
$$\frac{\tilde{k}^{00}_{\phi\phi}}{2} \sim c_{00}, \qquad \frac{\tilde{k}^{ij}_{\phi\phi}}{2} \sim c_{(ij)}, \qquad K^{0i} \sim 2c_{(0i)}. \tag{37}$$

Note that to avoid notational confusion of the real part of $(k_\phi)^\mu$, *i.e.*, a^μ, with the fermionic LV coefficient "a^μ", we instead use $\tilde{a}^\mu$ to represent the latter in (37). Relation (37) is also consistent with the CPT properties of the corresponding LV coefficients. The incompleteness of the formal similarity between the spin-independent NR LV Hamiltonian (36) and the one in [29] can be attributed to the fact that we only preserve LV perturbations to $\mathcal{O}(1,1)$ for simplicity, whereas in [29], these perturbations are preserved to much higher orders. An interesting scenario is that if we start with a Lagrangian describing spin-1 boson (such as meson) and carry out the above procedure again, we may also establish a relationship between the spin-dependent LV coefficients for an effective boson with the more fundamental fermionic LV couplings, like b^μ, $H^{\mu\nu}$, $g^{\lambda\mu\nu}$, etc.

Finally, we mention again the advantage of the CVZ method over direct FWTs, at least to the next lowest order of the NR approximation, is that the CVZ method can largely save work in calculating various commutators, such as $[\mathcal{O}, [\mathcal{O}, \mathcal{E}]]$ in FWT.

V. RELATION TO THE TEST OF THE EQUIVALENCE PRINCIPLE

Next, we discuss the relevance of the scalar Hamiltonian to the test of the EP. Actually, there are various inequivalent definitions on EP in the extensive discussions found in the literature [40]. Thus there is no doubt that discussions of inequivalent subjects necessarily cause conflicting conclusions on the validity of EP [19,20]. As mentioned at the very beginning, we constrain ourselves to the WEP. Speaking more precisely, we mean the equivalence between the law of mechanics for any free-moving test body with negligible self-gravity in a sufficiently small local region of spacetime (in a gravitational field) with that in a uniform accelerating frame (with proper acceleration) in the absence of gravity [40,41]. Note that in this statement, universal free fall (UFF, the world line of any free-moving test body with given initial conditions is independent of its mass and internal properties) cannot be equivalent to WEP [42] and ceases to be valid in a quantum domain. More importantly, UFF is even meaningless in quantum mechanics as the world line (or trajectory for an object) is purely a classical concept. In this sense, it is better to view UFF as a classical manifestation of WEP. On the contrary, WEP can still be safely guaranteed in a quantum realm, particularly constrained to the NR region reduced in [20,21,43] from GR. Actually, WEP provides a key to "gauge away" the gravitational analogy of gauge potential, the first derivatives of metric tensor, *i.e.*, $\partial_\rho g_{\mu\nu} \sim \Gamma_{\rho\mu\nu}$, and thus is an essential ingredient to attach quantum matter (neglecting spin-gravity couplings) to the classical gravitational background. In relativistic quantum field theory, WEP may not be valid due to the nonlocal nature of the radiative corrections even in a classical GR background [44].

In this respect, we think it is more meaningful to test WEP in an extended theory of GR, especially in the quantum domain. Many alternative theories fit into this category, such as Einstein-Cartan theory [45], metric-affine theory [46], etc. In a much broader context, it is valuable to incorporate Lorentz and CPT violation together with the test of WEP in a single framework, especially considering the intimate relationship between LLI and WEP, as indicated by Schiff's conjecture [15,18]. SME provides such an ideal test ground. In fact, testing WEP in SME allows more exotic signals, such as the distinctive nature between gravitational force and acceleration in the presence of LV [27]. Discussion of EP in the context of SME is abundant [27–30,47]; however, it seems that two important points have been overlooked or not been taken seriously, which we further address below.

First, it is logically more consistent to start with an intrinsically curved metric instead of a uniform accelerating metric, though the latter is an excellent approximation in most circumstances (up to an irrelevant constant), *e.g.*, $g_{00} \simeq -(1+2\chi) = -[1+2\vec{g}\cdot\Delta\vec{r}/c^2 + 2\frac{GM}{c^2R}] \sim -(1+2\phi)$ (R is Earth's radius). However, this approximation cannot be reliable to higher orders. In essence, the metric $g_{00} = -(1+\vec{a}\cdot\vec{x}/c^2)^2$, $g_{ij} = \delta_{ij}$ is only a general relativistic description of uniform acceleration, which is essentially flat and contains no information of gravity. Comparing (35) with (36), we see, even staying at the metric level and in the absence of LV, the two Hamiltonians cannot be equivalent at orders other than $\mathcal{O}(\chi)$, including the $\vec{\nabla}\chi \cdot \hat{\vec{p}}$ or $\vec{\nabla}\phi \cdot \hat{\vec{p}}$ term. Viewed in another way, the failure of this match may precisely reflect the realm of validity in the statement of WEP, "a sufficiently small local region of spacetime". Going beyond this "local" patch of spacetime necessarily means going out of the domain of WEP, where "violation" is naturally expected even in GR.

Second, determining how small should be considered as local enough, which depends on experimental capabilities. For an experimental apparatus capable of achieving the precision of $\delta s\mu$gal in a gravitational acceleration measurement near Earth's surface, the length scale is roughly about $L = L(\delta s) \sim \frac{\delta|\vec{g}|_{\max}R}{2g} \sim 10^{-8} \times \frac{\delta s R}{2g}$ to ensure the local requirement of the WEP test; otherwise, even the conventional tidal gravity can have a non-null effect. For example, if the gravimeter precision is on the order of 1 mgal, the length scale involved in the gravimeter measurement must be less than 3.24 m, which is easy to satisfy. For a 1μgal precision measurement, the length scale is smaller by a factor of 1000, which excludes many conventional macroscopic gravitational experiments.

On the other hand, if the flavor-dependent LV coefficients $\tilde{k}^{\mu\nu}_{\phi\phi}$, $\tilde{k}^{\mu}_{\phi}$ are nonzero, WEP is apparently violated. To see this, we collect the LV Hamiltonian up to $\mathcal{O}(\chi)$ from (35) as shown below:

$$\hat{H}_{\mathrm{FWT}} = \left[m\chi\left(1-\frac{3\tilde{k}^{00}_{\phi\phi}}{2}-2\frac{a^0}{m}\right)-\frac{m\tilde{k}^{00}_{\phi\phi}}{2}-a^0\right] + \left(1-\frac{\tilde{k}^{00}_{\phi\phi}}{2}\right)\frac{\hat{\vec{p}}^2}{2m}+(1+2\chi)K^{0j}\hat{p}_j + \frac{(1+\chi)}{2m}[2\vec{a}\cdot\hat{\vec{p}}-\tilde{k}^{ij}_{\phi\phi}\hat{\vec{p}}_i\hat{\vec{p}}_j]. \tag{38}$$

The first term in the large square bracket can be regarded as potential energy, which depends not only on the LV corrected mass term $m(1-\frac{3\tilde{k}^{00}_{\phi\phi}}{2}-2\frac{a^0}{m})$ but also directly on the combination of LV coefficients, $-[m\tilde{k}^{00}_{\phi\phi}/2+a^0]$. In general, the LV coefficients are directionally dependent and hence necessarily lead to breaking of UFF even in the context of classical mechanics. We can see this more transparently from the classical Lagrangian (40) derived below. In fact, even when the usual coordinate transformation $z\to z'=z+\frac{g}{2}t^2,\ t\to t'=t$ is performed on the Schrödinger equation [21] associated with the Hamiltonian (38), it cannot be reduced to the free motion case even locally ($\chi\to\vec{g}\cdot\Delta\vec{r}/c^2$) due to the presence of LV coefficients. So LV necessarily violates WEP by definition. Inspection of (35) also reveals that the gravitational redshift associated with $\tilde{k}^{\mu\nu}_{\phi\phi}$ depends on the number of its zero indices, so this can be utilized to discriminate different LV coefficients, as already been noticed in [29]. This also prevents us from using a coordinate transformation to the local patch of a uniform acceleration frame to transform Hamiltonian (38) to the flat space one with LV couplings.

To see the violation of WEP in another way, from the quadratic dispersion relation,

$$\left(\frac{1}{V^2}+\tilde{k}^{00}_{\phi\phi}\right){p_0}^2+\left(\tilde{k}^{ij}_{\phi\phi}-\frac{\delta^{ij}}{W^2}\right)p_ip_j+\tilde{k}^{(0i)}_{\phi\phi}p_ip_0 + \frac{i}{W^2}\nabla_i\ln(VW)p_i-2(a^0p_0+a^jp_j)=m^2, \tag{39}$$

derived from (2), we can construct a classical relativistic Lagrangian [48]:

$$L=-\mu[V^2(1-\tilde{k}^{00}_{\phi\phi}V^2)u^{02}-W^2(\delta_{ij}+K_{ij}W^2)u^iu^j +2V^2W^2K_{0i}u^0u^i]^{\frac{1}{2}}+\left[W^2a_j-\frac{i}{2}\nabla_j\ln(VW)\right]u^j - a_0V^2u^0, \tag{40}$$

where $\mu\equiv\{m^2+\frac{1}{4W^2}[\vec{\nabla}\ln(VW)]^2\}$, $K^{\mu\nu}\equiv\mathrm{Re}[\tilde{k}^{\mu\nu}_{\phi\phi}]=K^{\nu\mu}$, $u^\mu\equiv\frac{dx^\mu}{d\tau}$. As a simple approximation, we have only retained the LV coefficients in the above calculation to linear order. It can be readily verified that the particle trajectory obtained from (40) deviates from geodesic equation $\frac{du^0}{d\tau}+2(\vec{u}\cdot\vec{\nabla}\ln V)u^0=0$, $\frac{d\vec{u}}{d\tau}+\frac{1}{2}\frac{\vec{\nabla}V^2}{W^2}u^{02}+2(\vec{u}\cdot\vec{\nabla}\ln W)\vec{u}-\vec{u}^2\vec{\nabla}\ln W=0$ without LV and hence apparently violates WEP classically (*i.e.*, UFF). Note that (40) is only an example illustration to show that the inclusion of LV necessarily indicates deviation from geodesic for a classical particle trajectory. Because the equation of motion derived from (40) automatically includes various products of LV coefficients with ∂_ig_{00} or ∂_ig_{jk}, to be self-consistent, we have to include higher-order LV contributions, as well, which is beyond the scope of this paper.

At the end of this section, we note that there are several subtleties in the discussion of WEP. One issue is that the nonlocal nature of vacuum polarization may induce nonminimal couplings even starting with a minimal coupled action [44], as mentioned before. This effect can introduce a very tiny length scale, the Compton wavelength λ_C of a massive particle, say, the electron, and this will definitely violate WEP due to the tidal effects. The other issue is particular for the presence of LV, the so-called vacuum Cherenkov radiation [49–52]. For an energetically charged particle whose velocity exceeds the phase velocity of the LV photon, the charge is expected to radiate [49,50]. Similarly, a Cherenkov-type process can occur for modified electroweak and gravity sectors, as well, leading to the emission of W, Z bosons and gravitons, respectively [52]. The back-reaction due to this radiation can lead to a deviation from geodesic motion [49]; however, except for the electromagnetic Maxwell-Chern-Simons theory [50], due to the existence of threshold energy, this scenario will be nonrelevant for a NR particle in general. For the LV-charged fermion, the situation is slightly complicated. Certain spin-flip LV coefficients such as H, d, and g can also lead to threshold-free vacuum Cherenkov radiation [51], and this will drive even a NR-charged particle away from its geodesic. For an effective neutral particle composed of charged fermions, it is still unclear whether the composite-charged fermions in the bound state can radiate or not. If they can, the back-reaction may lead to WEP violation, as well, though this could be a higher-order LV effect.

VI. SUMMARY

In this work, we have derived a NR gravitationally coupled scalar Hamiltonian from the scalar Lagrangian of minimal SME. Using the test particle assumption, we derive it from two different methods in a static isotropic metric. One derivation utilizes the usual ansatz $\Phi(t,\vec{r})=e^{-imt}\Psi(t,\vec{r})$. The other is the FWT transformation with a pseudounitary transformation developed by Cognola *et al.* [32], and we call it the CVZ method. At least to $\mathcal{O}(1,1)$, the results (16) and (26), obtained from the two different methods, match. In the former method, we used an iteration procedure to perturbatively eliminate additional time derivative terms such as $\frac{\ddot{\psi}}{2m}$, which proves to be crucial for correct approximation. This method is a bit loose, though we think it is much more straightforward, and it will be interesting to explore whether this method can be further developed to obtain higher-order corrections systematically. We also check the CVZ method with a direct FWT, and the result (35) confirms (26) very well. However, at least for the next-leading-order approximation, the CVZ method appears more economical, as it largely saves the work in calculating various commutators.

In the context of SME, various NR Hamiltonians stemming from fermion Lagrangian have been developed in the literature [9,53]. It is natural because matter is composed of fermions. However, in an effective point of view, it is complementary to start directly with a bosonic action as many quantum tests of WEP use bosonic atoms [11,54,55] as test particles. Our result provides such an example for the spin-0 boson, which may be useful to the analysis of the ^{88}Sr atom [54]. Generalization to the spin-1 case will be straightforward and may be more interesting because spin interaction allows experimental testing in a more general framework, such as the metric-affine theory with torsion and nonmetricity [56,57], so more broad test schemes [24,58] are involved. As a bonus, comparison of the NR Hamiltonian for scalar and fermion fields enables us to bridge a relation between the corresponding LV coefficients; see (37). Accordingly, we may also be able to establish a relation between the LV coefficients of the spin-1 boson field and those of the fermion field in a future work. Then the spin-dependent LV coefficients, such as $H^{\mu\nu}$, $d^{\mu\nu}$, and $g^{\lambda\mu\nu}$ may be able to match the counterparts of the spin-1 boson, which is not attainable in the scalar case.

Finally, we also discuss the relevance of the scalar Hamiltonian with the test of WEP, which in our conservative point of view is still valid in the semiclassical context in the nonrelativistic regime reduced from GR. Therefore, tests of WEP are more natural in an extended theory of GR. With both a classical Lagrangian and a NR Hamiltonian, we show that, classically, the presence of LV indeed leads to deviation of the geodesic, which is apparently a signal of UFF violation. Furthermore, as the LV coefficients are directionally dependent and receive the gravitational redshift differently, we argue that this also leads to breaking of WEP even when transformed to a uniform accelerating frame with $\vec{a} = -\vec{g}$. Specifically, if LV leads to vacuum Cherenkov radiation, due to the back-reaction of the emitted quanta to test particle, more subtle WEP violation effects are expected for a composite neutral scalar.

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[1] CMS Collaboration, Nat. Phys. **10**, 557 (2014).
[2] B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016); **116**, 241103 (2016); **118**, 221101 (2017); **119**, 161101 (2017).
[3] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, J. High Energy Phys. 02 (2013) 062; P. Chen, Y. C. Ong, D. N. Page, M. Sasaki, and D.-h. Yeom, Phys. Rev. Lett. **116**, 161304 (2016).
[4] S. Liberati, Classical Quantum Gravity **30**, 133001 (2013); J. Phys. Conf. Ser. **631**, 012011 (2015); J. D. Tasson, Rep. Prog. Phys. **77**, 062901 (2014).
[5] V. A. Kostelecký and N. Russell, Rev. Mod. Phys. **83**, 11 (2011).
[6] V. A. Kostelecký and S. Samuel, Phys. Rev. Lett. **63**, 224 (1989); **66**, 1811 (1991); Phys. Rev. D **39**, 683 (1989); R. Gambini and J. Pullin, Phys. Rev. D **59**, 124021 (1999); J. Alfaro, H. A. Morales-Tecotl, and L. F. Urrutia, Phys. Rev. Lett. **84**, 2318 (2000); G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature **393**, 763 (1998).
[7] D. Colladay and V. A. Kostelecký, Phys. Rev. D **55**, 6760 (1997); **58**, 116002 (1998).
[8] V. A. Kostelecký, Phys. Rev. D **69**, 105009 (2004).
[9] V. A. Kostelecký and M. Mewes, Phys. Rev. D **80**, 015020 (2009); **85**, 096005 (2012); **88**, 096006 (2013).
[10] V. A. Kostelecký, CPT and Lorentz Symmetry, *Proceedings of the Fifth-Seventh Meeting on CPT and Lorentz Symmetry* (World Scientific, Singapore, 2010).
[11] T. Damour, Classical Quantum Gravity **13**, A33 (1996); M. Zych and C. Brukner, arXiv:1502.00971; G. Rosi, G. D. Amico, L. Cacciapuoti, F. Sorrentino, M. Prevedelli, M. Zych, C. Brukner, and G. M. Tino, Nat. Commun. **8**, 15529 (2017).
[12] V. V. Flambaum, Phys. Rev. Lett. **117**, 072501 (2016).

[13] T. A. Wagner, S. Schlamminger, J. H. Gundlach, and E. G. Adelberger, Classical Quantum Gravity **29**, 184002 (2012).

[14] L. Shao and N. Wex, Sci. China Phys. Mech. Astron. **59**, 699501 (2016).

[15] C. M. Will, Living Rev. Relativity **17**, 4 (2014).

[16] S. Weinberg, Phys. Lett. **9**, 357 (1964); S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1966), p. 537.

[17] A. Halprin and H. B. Kim, Phys. Lett. B **469**, 78 (1999).

[18] L. Schiff, Am. J. Phys. **28**, 340 (1960).

[19] G. Z. Adunas, E. Rodriguez-Milla, and D. V. Ahluwalia, Gen. Relativ. Gravit. **33**, 183 (2001); P. J. Orlando, R. B. Mann, K. Modi, and F. A. Pollock, Classical Quantum Gravity **33**, 19LT01 (2016).

[20] E. Okon and C. Callender, Eur. J. Phil. Sci. **1**, 133 (2011).

[21] M. Nauenberg, Am. J. Phys. **84**, 879 (2016); H. Padmanabhan and T. Padmanabhan, Phys. Rev. D **84**, 085018 (2011).

[22] V. Bargmann, Ann. Math. **59**, 1 (1954); S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1966), p. 62.

[23] P. Touboul *et al.*, Phys. Rev. Lett. **119**, 231101 (2017).

[24] X.-C. Duan, X.-B. Deng, M.-K. Zhou, K. Zhang, W.-J. Xu, F. Xiong, Y.-Y. Xu, C.-G. Shao, J. Luo, and Z.-K. Hu, Phys. Rev. Lett. **117**, 023001 (2016).

[25] K. S. Throne, D. L. Lee, and A. P. Lightman, Phys. Rev. D **7**, 3563 (1973).

[26] W.-T. Ni, Int. J. Mod. Phys. D **25**, 1630002 (2016).

[27] V. A. Kostelecký and J. D. Tasson, Phys. Rev. D **83**, 016013 (2011).

[28] Y. Bonder, E. Fischbach, H. Hernandez-Coronado, D. E. Krause, Z. Rohrbach, and D. Sudarsky, Phys. Rev. D **87**, 125021 (2013).

[29] Y. Bonder, Phys. Rev. D **88**, 105011 (2013).

[30] M. A. Hohensee, N. Leefer, D. Budker, C. Harabati, V. A. Dzuba, and V. V. Flambaum, Phys. Rev. Lett. **111**, 050401 (2013); V. V. Flambaum, Phys. Rev. Lett. **117**, 072501 (2016).

[31] R. Geiger and M. Trupke, Phys. Rev. Lett. **120**, 043602 (2018).

[32] G. Cognola, L. Vanzo, and S. Zerbini, Gen. Relativ. Gravit. **18**, 971 (1986).

[33] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[34] K. M. Case, Phys. Rev. **95**, 1323 (1954).

[35] Y. N. Obukhov, Phys. Rev. Lett. **86**, 192 (2001).

[36] H. Feshbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).

[37] A. Accioly and H. Blas, Phys. Rev. D **66**, 067501 (2002).

[38] A. J. Silenko and O. V. Teryaev, Phys. Rev. D **71**, 064016 (2005).

[39] F. W. Hehl and W.-T. Ni, Phys. Rev. D **42**, 2045 (1990).

[40] E. D. Casola and S. Liberati, Am. J. Phys. **83**, 39 (2015).

[41] E. Poisson and C. M. Will, *Gravity, Newotonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014).

[42] H. Hernandez-Coronado and E. Okon, Phys. Lett. A **377**, 2293 (2013).

[43] L. Seveso and M. G. A. Paris, Ann. Phys. (Amsterdam) **380**, 213 (2017).

[44] I. T. Drummond and S. J. Hathrell, Phys. Rev. D **22**, 343 (1980).

[45] E. Cartan, C. R. Acad. Sci. (Paris) **174**, 593 (1922).

[46] Y. Ne'eman and F. W. Hehl, Classical Quantum Gravity **14**, A251 (1997); M. Blagojević and F. W. Hehl, *Gauge Theories of Gravitation* (Imperial College Press, London, 2013).

[47] Q. G. Bailey, V. A. Kostelecký, and R. Xu, Phys. Rev. D **91**, 022006 (2015); B. Altschul *et al.*, Adv. Space Res. **55**, 501 (2015).

[48] V. A. Kostelecký and N. Russell, Phys. Lett. B **693**, 443 (2010).

[49] R. Lehnert and R. Potting, Phys. Rev. Lett. **93**, 110402 (2004); Phys. Rev. D **70**, 125010 (2004); B. Altschul, Phys. Rev. Lett. **98**, 041603 (2007).

[50] C. Kaufhold and F. R. Klinkhamer, Nucl. Phys. **B734**, 1 (2006); Phys. Rev. D **76**, 025024 (2007).

[51] M. Schreck, Phys. Rev. D **96**, 095026 (2017).

[52] D. Colladay, J. P. Noordmans, and R. Potting, Phys. Rev. D **96**, 035034 (2017); Symmetry **9**, 248 (2017); V. A. Kostelecký and J. D. Tasson, Phys. Lett. B **749**, 551 (2015).

[53] V. A. Kostelecký and C. D. Lane, J. Math. Phys. (N.Y.) **40**, 6245 (1999).

[54] M. G. Tarallo, T. Mazzoni, N. Poli, D. V. Sutyrin, X. Zhang, and G. M. Tino, Phys. Rev. Lett. **113**, 023005 (2014).

[55] S. Herrmann, H. Dittus, and C. Lämmerzahl, Classical Quantum Gravity **29**, 184003 (2012).

[56] S. Capozziello and M. De Laurentis, Phys. Rept. **509**, 167 (2011); A. Delhom-Latorre, G. J. Olmo, and M. Ronco, Phys. Lett. B **780**, 294 (2018).

[57] V. A. Kostelecký, N. Russell, and J. D. Tasson, Phys. Rev. Lett. **100**, 111102 (2008); J. Foster, V. A. Kostelecký, and R. Xu, Phys. Rev. D **95**, 084033 (2017); R. Lehnert, W. M. Snow, Z. Xiao, and R. Xu, Phys. Lett. B **772**, 865 (2017).

[58] S. A. Hojman and F. A. Asenjo, Classical Quantum Gravity **34**, 115011 (2017).