Mixed anomalies: Chiral vortical effect and the Sommerfeld expansion

Michael Stone*
University of Illinois, Department of Physics 1110 West Green Street Urbana, Illinois 61801, USA

JiYoung Kim†
University of Illinois, Department of Physics 1110 West Green Street Urbana, Illinois 61801, USA

(Received 24 May 2018; published 16 July 2018)

We discuss the connection between the integer moments of the Fermi distribution function that occur in the Sommerfeld expansion and the coefficients that occur in anomalous conservation laws for chiral fermions. For an illustration, we extract the chiral magnetothermal energy current from the mixed gauge-gravity anomaly in the $3+1$-dimensional energy-momentum conservation law. We then use a similar method to confirm the conjecture that the $T^2/12$ thermal contribution to the chiral vortical effect current arises from the gravitational Pontryagin term in the $3+1$-dimensional chiral anomaly.

DOI: 10.1103/PhysRevD.98.025012

I. INTRODUCTION

A recent experiment [1] and its widespread coverage in the media [2] have focused attention on the idea that the physics of a system containing chiral fermions can be influenced by effects of gravitational origin even in flat space-time [3]. These effects occur because the coefficients in certain constitutive relations for transport currents are related to the coefficients in corresponding anomalous conservation laws. As anomalies are not renormalized by interactions, these anomaly-induced, nondissipative, contributions to transport currents should take the same values in both strongly coupled and free theories. In the free case, the currents can be computed without reference to any anomaly, and the free-theory computations reduce to the evaluation of integer moments of the Fermi function that turn out to be polynomials in the temperature and chemical potential. One is left with a sense that these Sommerfeld-expansion integrals somehow know about anomalies. This impression was made concrete by Loganayagam and Surówka [4], who observed that a generating function for the integer moments of the Fermi function bears a close resemblance to the product of the A-roof genus and the total Chern character that occurs in the general-dimension Dirac index theorem—and which, via the Bardeen-Zumino descent equations [5], is the ultimate source of the anomalies. Their observation led them to a "replacement rule" that allowed them to compute anomaly-induced contributions to transport and fluid dynamics in N space-time dimensions directly from the anomaly polynomial in $N+2$ dimensions [4,6–12].

In this paper, we illustrate some of the ideas by computing two of these currents—the thermomagnetic current that plays a central role in Ref. [1] and the thermal contribution to the chiral vortical effect (CVE) current that arises when a chiral fermion is in thermal equilibrium in a rotating frame—both from the free theory and from the corresponding anomaly. The first example is merely a repackaging of the gravitational-anomaly derivation of Hawking radiation [13–15], but it serves to set the stage for an explicit confirmation of the conjecture [3] that the thermal component of the CVE is related to the gravitational anomaly. These two derivations also help explicate the geometric origin of the replacement-rule mapping that takes the first Pontryagin class of the Riemann curvature tensor to minus the square of the temperature.

In Sec. II, we introduce the specific currents of which the anomaly-driven origin we seek to illustrate. In Sec. III, we will construct *gedanken* space-times in which the currents are created *ex nihilo* by tidal forces in the vicinity of black-hole event horizons. In Sec. IV, we use the similarity of the their generating functions and the observation that often only one of the formal eigenvalues of the curvature tensor will be nonzero to link the anomalies with the Sommerfeld integrals. A final section, Sec. V, will put these ideas into context.

II. ANOMALIES AND ANOMALY-INDUCED CURRENTS

The "mixed axial-gravitational anomalies" that are invoked in the condensed-matter context in Ref. [1] and also Ref. [16] are the $(3+1)$-dimensional anomalous conservation equation

*m-stone5@illinois.edu
†jkim623@illinois.edu

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_{N,\lambda} - \frac{1}{384\pi^2} \frac{\epsilon^{\rho\sigma\alpha\beta}}{\sqrt{-g}} \nabla_\mu [F_{\rho\sigma} R^{\nu\mu}{}_{\alpha\beta}] \tag{1}$$

for the energy-momentum tensor $T^{\mu\nu}$ of a unit-charge right-handed Weyl fermion and the anomalous conservation equation

$$\nabla_\mu J^\mu_N = -\frac{1}{32\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{768\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\sigma} \tag{2}$$

for the particle-number current J^μ_N. The right-hand side of Eq. (1) shows that energy-momentum is delivered to the fermion from two sources: the first is the expected Lorentz force $F^{\nu\lambda} J_{N,\lambda}$, and the second is the gravitational anomaly that requires a cooperation between the gauge field $F_{\mu\nu}$ and the tidal forces encoded in the Riemann tensor $R^{\nu\mu}{}_{\alpha\beta}$ of the background space-time geometry. The right-hand side of the gauge-current conservation law (2) contains *two* anomalous source terms: a gauge field Chern-character density and a geometric Pontryagin-class density. The two equations, (1) and (2), display *mixed anomalies* because the anomalous sources for both the geometry-related energy-momentum tensor $T^{\mu\nu}$ and the gauge field–related particle-number current J^μ_N contain expressions involving the background field that couples to the other.

Anomaly-induced currents appear in solid-state systems [1,16] and also in relativistic fluid dynamics [17] in which (in the $[-,+,+,+]$ metric convention) we have [18]

$$T^{\mu\nu} = p g^{\mu\nu} + (\epsilon + p) u^\mu u^\nu + \xi_{TB} (B^\mu u^\nu + B^\nu u^\mu) + \xi_{T\omega} (\omega^\mu u^\nu + \omega^\nu u^\mu), \tag{3}$$

$$J^\mu_N = n u^\mu + \xi_{JB} B^\mu + \xi_{J\omega} \omega^\mu, \tag{4}$$

$$J^\mu_S = s u^\mu + \xi_{SB} B^\mu + \xi_{S\omega} \omega^\mu. \tag{5}$$

Here, $T^{\mu\nu}$ and J^μ_N are the energy-momentum tensor and number current that we have already met, while J^μ_S is the entropy current. The first terms on the rhs of each of these expressions are the usual expressions for a relativistic fluid in which u^μ denotes the 4-velocity of the fluid, and ϵ, p, n, and s are, respectively, the energy density, pressure, particle-number density, and entropy density. The remaining anomaly-induced terms involve the angular-velocity 4-vector defined by

$$\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} u_\nu \partial_\sigma u_\tau. \tag{6}$$

With $\epsilon^{0123} = +1$ and $\epsilon_{0123} = -1$, and in the $u^\mu = (1,0,0,0)$ rest frame, we have $\omega^\mu = (0, \mathbf{\Omega})$, where $\mathbf{\Omega} = \frac{1}{2} \nabla \times \mathbf{u}$ is the local 3-vector angular velocity. The extra currents also involve the magnetic field $\mathbf{B}$ as it appears to an observer moving at velocity u^μ. We have

$$E^\mu = F^{\mu\nu} u_\nu, \qquad B^\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} u_\nu F_{\sigma\tau}, \tag{7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A^\mu = (\phi, \mathbf{A})$. Again, in the $u^\mu = (1,0,0,0)$ rest frame, we have $E^\mu = (0, \mathbf{E})$, $B^\mu = (0, \mathbf{B})$ and

$$\frac{1}{4\pi^2} \mathbf{E} \cdot \mathbf{B} = \frac{1}{4\pi^2} E_\mu B^\mu = -\frac{1}{32\pi^2} \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau}. \tag{8}$$

In relativistic fluid dynamics, the notion of the "velocity of the fluid" requires further specification. We will take u^μ to be the 4-velocity of the *no-drag* frame introduced in Refs. [19,20]. This is the frame in which the ξ coefficients take their simplest form and is usually the frame in which the fluid is in local thermodynamic equilibrium.

Demanding that no entropy production be associated with the anomaly-induced currents requires [20] that the six coefficients ξ_{TB}, $\xi_{T\omega}$, ξ_{JB}, $\xi_{J\omega}$, ξ_{SB}, and $\xi_{S\omega}$ depend at most on three underlying parameters through

$$\begin{aligned}
\xi_{JB} &= C\mu, \\
\xi_{J\omega} &= C\mu^2 + X_B T^2, \\
\xi_{SB} &= X_B T, \\
\xi_{S\omega} &= 2\mu T X_B + X_\omega T^2, \\
\xi_{TB} &= \frac{1}{2}(C\mu^2 + X_B T^2), \\
\xi_{T\omega} &= \frac{2}{3}(C\mu^3 + 3X_B \mu T^2 + X_\omega T^3).
\end{aligned} \tag{9}$$

Here, T is the temperature, and μ is the chemical potential associated with the U(1) particle-number current J^μ_N. For the ideal Weyl gas, the three parameters C, X_B, and X_ω take the values

$$C = \frac{1}{4\pi^2}, \qquad X_B = \frac{1}{12}, \qquad X_\omega = 0. \tag{10}$$

It is clear from the derivation in Ref. [20] that C is the coefficient of the term (8) in the chiral anomaly (2). It was conjectured in Ref. [3] that the parameter X_B is the coefficient appearing before the Pontryagin density in the same equation. This conjecture was originally motivated by the simple observation that both X_B and the Pontryagin coefficient depend on the same physical data (spin and chirality, but not charge), but it has gained support from consideration of global anomalies [21,22] and from calculations using AdS/CFT formalism [23]. It is not, however, straightforward to confirm the conjecture by extending the flat-space considerations in Ref. [20] to curved space.

For an ideal gas of right-handed Weyl fermions at rest in flat space [so, $u^\mu = (1,0,0,0)$], the term with ξ_{TB} in (3) leads to an anomaly-induced magnetothermal energy flux,

$$\mathbf{J}_\epsilon = \mathbf{B}\left(\frac{\mu^2}{8\pi^2} + \frac{1}{24}T^2\right), \tag{11}$$

which plays a central role in Ref. [1].

A similar gas in thermal equilibrium in a frame rotating at angular velocity $\mathbf{\Omega}$ [so that $u^\mu = (1, 0, 0, 0)$ on the rotation axis] acquires from the $\xi_{J\omega}$ term in (4) a CVE number current that (again on the rotation axis) is given by [24]

$$\mathbf{J}_N = \mathbf{\Omega}\left(\frac{\mu^2}{4\pi^2} + \frac{|\Omega|^2}{48\pi^2} + \frac{1}{12}T^2\right). \tag{12}$$

We do not need the gravitational anomaly to understand the origin of the magnetothermal current in (11). It is well known that solving the for the eigenvalues of the Weyl Hamiltonian in the presence of a magnetic field $\mathbf{B}$ yields a set of energy levels,

$$\epsilon_l(k) = \pm\sqrt{2|B|l + k^2}, \tag{13}$$

where k is the component of momentum parallel to $\mathbf{B}$. The levels have degeneracy $|B|/2\pi$ per unit area in a plane transverse to $\mathbf{B}$, and all levels are gapped except for $l = 0$. The $l = 0$ level is special in that only one sign of the square root is allowed, and we effectively have an array of gapless one-dimensional chiral fermions with

$$\epsilon(k) = +k. \tag{14}$$

Each one-dimensional chiral fermion contributes a current of

$$J_\epsilon = \int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\epsilon\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}} - \theta(-\epsilon)\right\} = 2\pi\left(\frac{\mu^2}{8\pi^2} + \frac{1}{24}T^2\right), \tag{15}$$

where $\beta = 1/T$ and the $-\theta(-\epsilon)$ counterterm affects a normal-ordering vacuum subtraction of the contribution of the Dirac/Fermi sea, ensuring that there is no current when $\mu = T = 0$. Combining (15) with the $|B|/2\pi$ areal degeneracy leads immediately to (11).

The free fermion computation of the CVE current is rather lengthier [24], but it also reduces to a Sommerfeld integral, in this case

$$\begin{aligned} J_N &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\epsilon^2 d\epsilon\left(\frac{1}{1+e^{\beta(\epsilon-(\mu+\Omega/2))}} - \frac{1}{1+e^{\beta(\epsilon-(\mu-\Omega/2))}}\right). \\ &= \frac{\mu^2\Omega}{4\pi^2} + \frac{\Omega^3}{48\pi^2} + \frac{1}{12}\Omega T^2. \end{aligned}$$

Although we do not *need* the mixed anomalies to obtain these currents, we *can* use them to do so and in doing so gain insight into the physical origin of the anomalies. We will devote the next section to the anomaly derivation of (11) and (12). We will see that a number of deep ideas are combined in these derivations.

III. CURRENTS FROM ANOMALIES

In this section, we will construct space-times in which the thermal contributions to the anomaly-induced currents arise from the gravitational source terms in the associated anomalous conservations laws.

A. Magnetothermal current

To derive the thermal part of (11) from the anomaly, we will take for granted the $4 \to 2$-dimensional reduction provided by the magnetic field and consider the current as that of our array of $1+1$-dimensional right-going fermions. We imagine a gedanken experiment in which we heat each right-going Fermi field to temperature T by terminating its space-time on the left by a $1+1$-dimensional black hole of which the Hawking temperature is T. The T^2 contribution to (11) is then the Fermi field's contribution to the outgoing Hawking radiation. To relate this interpretation to the anomaly, we review how [13–15] Hawking radiation arises from the $1+1$-dimensional version of the energy-momentum anomaly

$$\nabla_\mu T^{\mu\nu} = -\frac{c}{96\pi}\frac{\epsilon^{\nu\sigma}}{\sqrt{|g|}}\partial_\sigma R, \tag{16}$$

to which (1) reduces in a uniform $\mathbf{B}$ field. Here, $\epsilon^{01} = 1$, $R = R^{\alpha\beta}{}_{\alpha\beta} = 2R^{01}{}_{01} = 2R^{\mathrm{tr}}{}_{\mathrm{tr}}$ is the Ricci scalar, and $c = c_R - c_L$ is the difference between the conformal central charges of the right-going and left-going massless fields. As our magnetic field leaves us with only right-going fields, we have $c = 1$.

As the black hole is an externally imposed background space-time, we do not need its metric to satisfy the Einstein equations, and a suitable metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2, \tag{17}$$

where all that is required of $f(r)$ is that it tends to unity at large r and vanishes linearly as r approaches the event horizon $r = r_H$. In this metric, the Ricci scalar is given by

$$R = -f''. \tag{18}$$

A covariant energy-momentum conservation equation does not, on its own, lead to conserved energy and momentum. For that, we need a space-time symmetry, i.e., a Killing-vector field η^μ that obeys the isometry condition

$$\nabla_\mu\eta_\nu + \nabla_\nu\eta_\mu = 0. \tag{19}$$

Combining the isometry equation with (16) then gives us

$$\nabla_\mu (T^{\mu\nu}\eta_\nu) = -\frac{c}{96\pi}\frac{\epsilon^{\nu\sigma}\eta_\nu}{\sqrt{|g|}}\partial_\sigma R, \tag{20}$$

in which

$$\sqrt{|g|}\nabla_\mu (T^{\mu\nu}\eta_\nu) = \partial_\mu (\sqrt{|g|}T^{\mu\nu}\eta_\nu) \tag{21}$$

involves a conventional total divergence.

Our Schwarzschild metric possesses a Killing vector $\eta = \partial_t$ of which the covariant components are $(\eta_t, \eta_r) = (-f(r), 0)$. From this, we find that

$$\nabla_\mu T^{\mu\nu}\eta_\nu = (\partial_r \sqrt{|g|}T^r{}_t)/\sqrt{-g}. \tag{22}$$

We then have

$$\frac{\partial}{\partial r}(\sqrt{|g|}T^r{}_t) = -\frac{c}{96\pi}f\partial_r f'' = -\frac{c}{96\pi}\frac{\partial}{\partial r}\left(ff'' - \frac{1}{2}(f')^2\right), \tag{23}$$

and integrating from r_H to $r = \infty$ gives

$$\sqrt{|g|}T^r{}_t|_{r_H}^{\infty} = -\frac{c}{96\pi}\left(ff'' - \frac{1}{2}(f')^2\right)\Big|_{r_H}^{\infty}. \tag{24}$$

According to Ref. [15], the appropriate boundary condition is that $T^r{}_t$ is zero at the horizon. The rhs of (24) by contrast is zero at infinity and contributes $(c/96\pi)(f')^2/2$ at the horizon. As $\sqrt{|g|} \to 1$ at large r, we see that an energy current of magnitude

$$T^{rt}(z \to \infty) = -T^r{}_t(z \to \infty) = \frac{c\kappa^2}{48\pi}, \qquad \kappa = f'(r_H)/2, \tag{25}$$

has been built up by the anomaly as we move away from the horizon. The quantity κ is the *surface gravity* of the black hole.

To complete the derivation of (11), we recall the argument [25,26] that the geometry of the Euclidean section of our black-hole metric shows that the Hawking temperature is given by $T_H = \kappa/2\pi$. We begin by setting $t = -i\tau$ and see that in imaginary time our Schwarzschild space metric becomes

$$d\sigma^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2. \tag{26}$$

If we introduce a new radial coordinate,

$$\rho = \int_{r_H}^{r}\frac{dr'}{\sqrt{f(r')}} \approx \frac{2}{\sqrt{f'(r_H)}}\sqrt{r - r_H}, \tag{27}$$

where the approximation holds for r just above r_H. Then, in this same region,

$$\begin{aligned} d\sigma^2 &= f(r)d\tau^2 + \frac{1}{f(r)}dr^2 \\ &= f(r)d\tau^2 + d\rho^2, \\ &\approx f'(r_H)(r - r_H)d\tau^2 + d\rho^2 \\ &= \kappa^2\rho^2 d\tau^2 + d\rho^2. \end{aligned} \tag{28}$$

Comparison with the metric of plane polar coordinates now shows that if there is to be no conical singularity at r_H we must identify $\kappa\tau$ with the polar angle θ. Thus, the smooth Euclidean manifold described by (26) looks like the skin of a salami sausage (see Fig. 1) in which the circumferential coordinate θ is identified with $\theta + 2\pi$, or equivalently τ is identified with $\tau + \beta$ where $\beta = 2\pi/\kappa$. Green functions $G(r, t)$ in Minkowski signature space-time will be periodic in imaginary time with period β and are therefore [25,26] thermal Green functions with temperature $T_H = \beta^{-1} = \kappa/2\pi$ (or $k_B T_H = \hbar\kappa/2\pi c$ if we include dimensionful constants).

This derivation seems quite straightforward, but there are two subtleties that need to be discussed. First, anomalies are usually presented as being of two types: *consistent* and *covariant* [5]. Following Ref. [15], we have exclusively used the covariant form of the anomaly. Second, it is well known that Hawking radiation is observer dependent. These two issues are not unrelated. To illuminate this point, we will repeat the Hawking radiation calculation using the two-dimensional version of Kruskal-Szekeres coordinates.

We begin by defining a *tortoise coordinate* r_* by solving

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \tag{29}$$

and taking as a boundary condition that r_* coincides with r at large positive distance. In (t, r_*) coordinates, the metric becomes

$$ds^2 = e^{\phi}(-dt^2 + dr_*^2), \tag{30}$$

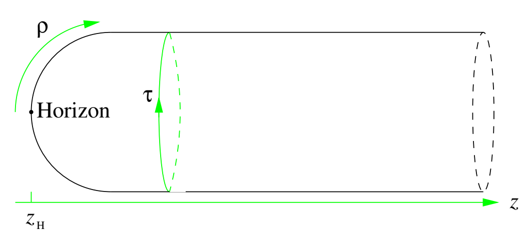

FIG. 1. The Euclidean, imaginary time, section of the two-dimensional black hole is asymptotically a cylinder of circumference $2\pi/\kappa$. The horizon is a single point at $\rho = 0$, or equivalently $z = z_H$.

where

$$\phi(r_*) = \ln f(r(r_*)) \tag{31}$$

and the event horizon lies at $r_* = -\infty$. On setting $z = r_* + i\tau$ and $\bar{z} = r_* - i\tau$, the Euclidean version of this metric takes the isothermal (conformal) form

$$d\sigma^2 = e^{\phi} d\bar{z}dz. \tag{32}$$

In $\bar{z}$, z, coordinates, the only nonvanishing Christoffel symbols are

$$\Gamma^z_{zz} = \partial_z\phi, \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}}\phi, \tag{33}$$

and the Ricci scalar is

$$R = -4e^{-\phi}\partial^2_{z\bar{z}}\phi. \tag{34}$$

In two-dimensional conformal field theory, we are used to defining energy-momentum operators $\hat{T}(z)$ and $\hat{\bar{T}}(\bar{z})$, where, for a free $c = 1$ boson field $\hat{\varphi}(z,\bar{z}) = \hat{\varphi}(z) + \hat{\varphi}(\bar{z})$, for example, we have

$$\begin{aligned}\hat{T}(z) &=: \partial_z\hat{\varphi}(z)\partial_z\hat{\varphi}(z): \\ &= \lim_{\delta\to 0}\left(\partial_z\hat{\varphi}(z+\delta/2)\partial_z\hat{\varphi}(z-\delta/2) + \frac{1}{4\pi\delta^2}\right).\end{aligned} \tag{35}$$

[Note that conformal field theory papers often define $\hat{T}(z)$ to be -2π times (35) so as to simplify the operator product expansion.] The operator $\hat{T}(z)$ has been constructed to be explicitly holomorphic in z, but at a price of tying its definition to the z, $\bar{z}$ coordinate system—both in the normal-ordered expression in the first line and by the explicit counterterm in the second line. It is not surprising, therefore, that under a change of coordinates the operator $\hat{T}(z)$ does not transform as a tensor but instead acquires an inhomogeneous Schwarzian-derivative c-number part [27]. If we want a genuine energy-momentum *tensor*, we must define

$$\begin{aligned}\hat{T}_{zz} &= \hat{T}(z) + \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right), \\ \hat{T}_{\bar{z}\bar{z}} &= \hat{\bar{T}}(\bar{z}) + \frac{c}{24\pi}\left(\partial^2_{\bar{z}\bar{z}}\phi - \frac{1}{2}(\partial_{\bar{z}}\phi)^2\right), \\ \hat{T}_{\bar{z}z} &= -\frac{c}{24\pi}\partial^2_{z\bar{z}}\phi,\end{aligned} \tag{36}$$

in which the c-number Schwarzians in the operator transformation are canceled by Schwarzians from the transformation of the c-number additions.

A direct computation, using the holomorphicity and antiholomorphicity of the operators $\hat{T}(z)$ and $\hat{\bar{T}}(\bar{z})$ together with the formulas for the Christoffel symbols, shows that

$$\nabla^z\hat{T}_{zz} + \nabla^{\bar{z}}\hat{T}_{\bar{z}z} = 0. \tag{37}$$

If, however, we keep only the right-going field, the chiral energy-momentum tensor becomes

$$\begin{aligned}\hat{T}_{zz} &= \hat{T}(z) + \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right), \\ \hat{T}_{\bar{z}\bar{z}} &= 0, \\ \hat{T}_{\bar{z}z} &= -\frac{c}{48\pi}\partial^2_{z\bar{z}}\phi.\end{aligned} \tag{38}$$

A similar computation shows that the chiral tensor obeys

$$\begin{aligned}\nabla^z\hat{T}_{zz} + \nabla^{\bar{z}}\hat{T}_{\bar{z}z} &= -\frac{c}{96\pi}\partial_z R, \\ \nabla^z\hat{T}_{z\bar{z}} + \nabla^{\bar{z}}\hat{T}_{\bar{z}\bar{z}} &= +\frac{c}{96\pi}\partial_{\bar{z}} R,\end{aligned} \tag{39}$$

where the second term on the left-hand side of the second equation is identically zero. In our z, $\bar{z}$ coordinates system, we have $\sqrt{g} = \sqrt{-g_{\bar{z}z}g_{z\bar{z}}} = -ie^{\phi}/2$ [perhaps more clearly, we can express this as $e^{\phi}dt\wedge dr_* = (e^{\phi}/2i)dz\wedge d\bar{z}$], and we can write these last two equations in a covariant manner as

$$\begin{aligned}\nabla^z\hat{T}_{zz} + \nabla^{\bar{z}}\hat{T}_{\bar{z}z} &= i\frac{c}{96\pi}\sqrt{g}\epsilon_{z\bar{z}}\partial^{\bar{z}}R, \\ \nabla^z\hat{T}_{z\bar{z}} + \nabla^{\bar{z}}\hat{T}_{\bar{z}\bar{z}} &= i\frac{c}{96\pi}\sqrt{g}\epsilon_{\bar{z}z}\partial^{z}R.\end{aligned} \tag{40}$$

In general Euclidean coordinates, we therefore have [28]

$$\nabla^{\mu}\hat{T}_{\mu\nu} = i\frac{c}{96\pi}\sqrt{g}\epsilon_{\nu\sigma}\partial^{\sigma}R. \tag{41}$$

The factor i appears in (42) because it is only the *imaginary* part of the Euclidean effective action that can be anomalous [29,30]. It is absent when we write the equation in Minkowksi signature space-time in which it becomes

$$\nabla_{\mu}\hat{T}^{\mu\nu} = -\frac{c}{96\pi}\frac{1}{\sqrt{|g|}}\epsilon^{\nu\sigma}\partial_{\sigma}R. \tag{42}$$

Because we have used a covariantly transforming energy momentum tensor, we find the covariant form of the anomaly. In this calculation, we also see that the anomaly arises solely from the c-number terms.

Now, define Euclidean Kruskal coordinates U and V by setting

$$\begin{aligned}Z &= U + iV = \exp\{\kappa(r_* + i\tau)\} = \exp\{\kappa z\}, \\ \bar{Z} &= U - iV = \exp\{\kappa(r_* - i\tau)\} = \exp\{\kappa\bar{z}\}\end{aligned} \tag{43}$$

so that

$$|Z|^2 = U^2 + V^2 = \exp\{2\kappa r_*\}. \tag{44}$$

In terms of these coordinates, we have

$$d\sigma^2 = f(r)\kappa^{-2} e^{-2\kappa r_*}(dU^2 + dV^2). \tag{45}$$

With κ being the surface gravity, this goes to the non-singular metric

$$d\sigma^2 = \mathrm{const}(dU^2 + dV^2) \tag{46}$$

[where the constant is determined by the exact form of $f(r)$] near the horizon point at $U^2 + V^2 = 0$) and to

$$d\sigma^2 = \kappa^{-2}(U^2 + V^2)^{-1}(dU^2 + dV^2) \tag{47}$$

at large distance. This last expression is the metric of a cylinder of circumference $2\pi/\kappa$, confirming the time periodicity.

For a general conformal "salami sausage" metric, we need $ds^2 = e^{\phi} dZ d\bar{Z}$ with $\phi = 0$ at $|Z| = 0$ and $\phi \approx -2\ln\kappa|Z|$ at large $|Z|$ where the circumference of the sausage becomes constant. The coefficient -2 is required by the Gauss-Bonnet theorem as the end cap is topologically a hemisphere. At short distance, the sausage looks like a spherical cap, and we have

$$e^{\phi} = 1 - \frac{1}{4}\bar{z}zR + O(|z|^4), \qquad \phi = -\frac{1}{4}\bar{z}zR + O(|z|^4), \tag{48}$$

where R is the Ricci scalar (twice the Gaussian curvature) at the horizon.

As Kruskal Z and $\bar{Z}$ coordinates are again isothermal, the chiral energy-momentum tensor is of the form

$$\hat{T}_{ZZ} = \hat{T}(Z) + \frac{c}{24\pi}\left(\partial^2_{ZZ}\phi - \frac{1}{2}(\partial_Z\phi)^2\right), \tag{49}$$

where $\hat{T}(Z)$ is the normal-ordered operator part that transforms inhomogeneously under conformal maps. The second term is the c-number counterterm of which the transformation cancels that of $\hat{T}(Z)$ so as to make $\hat{T}_{ZZ}$ transform as a tensor.

At short distance, the c-number part in $\hat{T}_{ZZ}$ vanishes—in fact, it vanishes identically on a sphere with conformal coordinates. Consequently, as $\hat{T}_{ZZ}$ is zero at the horizon, the expectation value of $\hat{T}(Z)$ is zero there, and hence everywhere. At large distance, however, we will have

$$\phi(Z, \bar{Z}) \sim -\ln\kappa Z - \ln\kappa\bar{Z}, \tag{50}$$

and so the c-number part gives us

$$\begin{aligned} T_{ZZ} &\sim \frac{c}{24\pi}\left(\frac{1}{Z^2} - \frac{1}{2}\frac{1}{Z^2}\right) \\ &= \frac{c}{48\pi}\frac{1}{Z^2}. \end{aligned} \tag{51}$$

Now, let us shift to the tortoise light-cone coordinates $z = r_* + i\tau$, $\bar{z} = r_* - i\tau$. Then,

$$\begin{aligned} \hat{T}_{zz} &= \left(\frac{\partial Z}{\partial z}\right)^2 \hat{T}_{ZZ} \\ &= \kappa^2 Z^2 \hat{T}_{ZZ} \\ &\to \frac{c\kappa^2}{48\pi}, \quad \text{as } r_* \to \infty. \end{aligned} \tag{52}$$

In the asymptotic Minkowski space $r_* = r$, and with the speed of light equal to 1 and $\pm$ denoting $r \pm t$ components, we have

$$\begin{aligned} \hat{T}_{++} &= \frac{1}{4}(\hat{T}_{rr} + \hat{T}_{tt} - 2\hat{T}_{rt}), \\ \hat{T}_{--} &= \frac{1}{4}(\hat{T}_{rr} + \hat{T}_{tt} + 2\hat{T}_{rt}), \\ \hat{T}_{+-} &= \frac{1}{4}(\hat{T}_{rr} - \hat{T}_{tt}), \end{aligned} \tag{53}$$

with $\hat{T}_{+-} = \hat{T}_{--} = 0$. Consequently, $\hat{T}_{++} = \hat{T}_{tt} = \hat{T}_{rt}$, and the Kruskal coordinate energy density and flux coincide with those from the Schwarzschild coordinate calculation. The breakup between the operator and c number is different, however. The c-number part in $\hat{T}_{zz}$ vanishes at large r, so the large-distance contribution to the Schwarzschild energy flux comes entirely from the expectation value of the operator $\hat{T}(z)$. In other words, the Schwarzschild observer sees the asymptotic energy being carried by actual particles. In Kruskal coordinates, the $\hat{T}_{ZZ}$ operator part has a vanishing expectation value everywhere, and the asymptotic energy flux comes entirely from the c-number term. Thus, $\hat{T}(Z)$ and $\hat{T}(z)$ record very different particle content, and the Schwarzschild r, t coordinate observer's zero-particle state is not the same as the Kruskal observer's zero-particle state.

The physical interpretation should now be clear: both the Schwarzschild time and the Kruskal time coordinate lines correspond to the flows of timelike Killing vectors. In each coordinate system, the field's mode expansions have well-defined yet different positive-frequency modes of which the coefficients are annihilation operators. The operators $\hat{T}_{ZZ}$ and $\hat{T}_{zz}$ are normal ordered so that the annihilators are all to the right. It is the positive frequencies that can excite a detector from its ground state, and the normal-ordered operators keep track of what a detector at fixed spatial coordinate would record in each coordinate system. Close to the horizon, a detector at fixed Schwarzschild coordinate

r sees a high-temperature thermal distribution of outgoing particles. However, at the horizon, their contribution to the expectation value of $\hat{T}_z$ is exactly canceled by the c-number term. As we move away from the horizon, this c-number term decreases and allows the total covariantly defined energy current to grow to its asymptotic value. On the other hand, the Kruskal observer never sees any particles, and all their energy flux comes from the c-number contribution that grows from zero at the horizon to the *same* asymptotic value as the Schwarzschild flux. Presumably, the source term in the Einstein equations will be the expectation value of a covariantly defined energy-momentum tensor. Therefore, it is independent of the observer's motion—but as we are not investigating the backreaction of the emitted radiation on the geometry, this is not our present concern.

B. Chiral vortical current

We now seek an analogous derivation of (12) from the Pontryagin term in (2). To do this, we need to modify our toy black hole to acquire a nonzero Pontryagin density. A suggestion of how to proceed comes from the Kerr metric of a rotating black hole. In Boyer-Lindquist coordinates (t, r, ϕ, θ), and with $\cos\theta = \chi$, this metric is

$$ds^2 = -\left(1 - \frac{2mr}{r^2 + a^2\chi^2}\right)dt^2 + \left(\frac{r^2 + a^2\chi^2}{r^2 + a^2 - 2mr}\right)dr^2 - \frac{4amr(1-\chi^2)}{r^2 + a^2\chi^2}dtd\phi \\ \times (1-\chi^2)\left(a^2 + r^2 + \frac{2a^2mr(1-\chi^2)}{r^2 + a^2\chi^2}\right)d\phi^2 + \frac{r^2 + a^2\chi^2}{1-\chi^2}d\chi^2, \tag{54}$$

where m is the mass and $J = ma$ is the angular momentum of the black hole.

This metric has two special horizon surfaces at the roots

$$r_\pm = m \pm \sqrt{m^2 - a^2} \tag{55}$$

of $r^2 + a^2 - 2mr = 0$. The outer horizon $r = r_+$ is the causal event horizon on which trapped photons are forced to orbit at fixed r, θ and angular velocity

$$\Omega_+ \stackrel{\text{def}}{=} \frac{d\phi}{dt} = \frac{a}{r_+^2}. \tag{56}$$

Both Ω_+ and the surface gravity

$$\kappa_+ = \frac{1}{4m} - m\Omega_+^2 \tag{57}$$

are constant over the horizon. As in the Schwarzschild case, the absence of a conical singularity in the Euclidean section requires $\tau \sim \tau + \beta_H$, where [31]

$$\beta_H = \frac{1}{T_H} = \frac{2\pi}{\kappa_+}. \tag{58}$$

The Kerr black hole is therefore both rotating and hot.

What is important for us is that the numerical coefficient

$$\frac{1}{4}\epsilon^{\lambda\mu\rho\sigma}R^a{}_{b\lambda\mu}R^b{}_{a\rho\sigma} = -\frac{48am^2r\chi(r^2 - 3a^2\chi^2)(3r^2 - a^2\chi^2)}{(r^2 + a^2\chi^2)^5} \tag{59}$$

of $dt \wedge dr \wedge d\phi \wedge d\chi$ in the Pontryagin-density 4-form $\mathrm{tr}\{R^2\}$ is nonzero. For small a/m, the coefficient is largest near the poles of rotation at $\chi = \pm 1$.

The Kerr metric can be conveniently written [31] in terms of the functions

$$\Delta = r^2 + a^2 - 2mr, \tag{60}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{61}$$

and two mutually orthogonal 1-forms

$$\omega = \frac{r^2 + a^2}{\rho^2}(dt - a\sin^2\theta d\phi), \tag{62}$$

$$\tilde{\omega} = \frac{r^2 + a^2}{\rho^2}\left(d\phi - \frac{a}{r^2 + a^2}dt\right), \tag{63}$$

as

$$ds^2 = -\frac{\Delta\rho^2}{(r^2 + a^2)^2}\omega^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta\tilde{\omega}^2. \tag{64}$$

Motivated by this rewriting, we consider a $3 + 1$ space with metric

$$ds^2 = -f(z)\frac{(dt - \Omega r^2 d\phi)^2}{(1 - \Omega^2 r^2)} + \frac{1}{f(z)}dz^2 + dr^2 \\ + \frac{r^2(d\phi - \Omega dt)^2}{(1 - \Omega^2 r^2)}. \tag{65}$$

The metric (65) has been constructed so that at large z, where $f(z) = 1$, it reduces to

$$ds^2 \to -dt^2 + dz^2 + dr^2 + r^2 d\phi^2, \tag{66}$$

where r, z, and ϕ are the cylindrical-coordinate radial, axial, and azimuthal coordinates of an asymptotically flat space-time. Thus, in contrast to previous usage, z is a real coordinate that provides a measure of the distance from the horizon at $f(z_H)=0$, and r is a measure of distance from the rotation axis. Replacing the complicated Kerr-metric coefficients by the function $f(z)$ allows us to ensure that the space-time curvature is concentrated near the horizon, which now appears to be planar and rotating at angular velocity Ω. We anticipate that outgoing Fermi fields in this space will be in asymptotic thermal equilibrium at temperature $T_H = f'(z_H)/4\pi$ in a frame rotating about the z axis with the horizon angular velocity Ω. They should therefore acquire a CVE current as they move though the curved and twisted near-horizon geometry.

The numerical coefficient of the Pontryagin density in this space-time is

$$\begin{aligned}
&\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\rho\sigma}\\
&\quad=\frac{2r\Omega f'(z)(8\Omega^2(1-f(z))+(1-\Omega^2r^2)^2f''(z))}{(1-\Omega^2r^2)^3}\\
&\quad\sim 2r\Omega f'(z)f''(z)\\
&\quad=\frac{\partial}{\partial z}(\Omega r[f'(z)]^2). \qquad (67)
\end{aligned}$$

In the last two lines, we have kept only the leading term in Ω. The error is $O[\Omega^3]$.

When we divide by $\sqrt{|g|}=r$ to get the Pontryagin-density scalar, we see that we have created in the region abutting the horizon an r-independent source term for the axial current of our anomalous relativistic fluid. We assume that this planar source drives a current only in the z direction. In that case, we find that to leading order in Ω the anomalous conservation law (2) becomes

$$\frac{\partial}{\partial z}(\sqrt{|g|}J_N^z) = -\frac{\Omega}{192\pi^2}\frac{\partial}{\partial z}(\sqrt{|g|}[f'(z)]^2). \qquad (68)$$

With boundary condition $J^z(z_H)=0$, we can again integrate up with respect to z to find

$$J_N^z(z\to\infty)=\frac{1}{12}\Omega T_H^2. \qquad (69)$$

This is the expected thermal contribution to the CVE current (12).

If we retain terms of order Ω^3, we do get a contribution to the on-axis current similar to that in (12), but with coefficient $1/24\pi^2$ rather than $1/48\pi^2$. Trying slightly modified metrics indicates that this correction to the current is sensitive to how the metric varies away from the axis of rotation. For example, omitting the $(1-\Omega^2r^2)$ factors in the denominators in (65) does not alter the on-axis asymptotic metric and does not affect the coefficient of the T^2 term in the current. It does, however, lead to the coefficient of Ω^3 becoming zero. Perhaps we should not be surprised by this as the notion of a rigidly rotating coordinate system such as that used by Ref. [24] is bound to be problematic away from the rotation axis.

Note that our CVE current (69) is not, as suggested in Ref. [32], simply proportional to the Chern-Simons current associated with the Pontryagin class. The latter current

$$J^{\lambda}_{\rm CS}=\frac{1}{2}\frac{\epsilon^{\lambda\mu\rho\sigma}}{\sqrt{|g|}}\left(\Gamma^{\alpha}{}_{\beta\mu}\partial_\rho\Gamma^{\beta}{}_{\alpha\sigma}+\frac{2}{3}\Gamma^{\alpha}{}_{\beta\mu}\Gamma^{\beta}{}_{\gamma\rho}\Gamma^{\gamma}{}_{\alpha\sigma}\right) \qquad (70)$$

has (to leading order in Ω) two nonzero components:

$$J^z_{\rm CS}=\Omega[f'(z)]^2, \qquad J^r_{\rm CS}=2\Omega f'(z)/r. \qquad (71)$$

It does satisfy

$$\partial_\lambda\sqrt{|g|}J^{\lambda}_{\rm CS}=\frac{1}{4}\epsilon^{\lambda\mu\rho\sigma}R^{\alpha}{}_{\beta\lambda\mu}R^{\beta}{}_{\alpha\rho\sigma} \qquad (72)$$

but obeys different boundary conditions in that $J^{\mu}_{\rm CS}(z)$ vanishes at $z=\infty$ rather than at the horizon. Our derivation in this section was, however, motivated by the discussion in Ref. [32].

IV. SOMMERFELD INTEGRALS AND ANOMALIES

In the Introduction, we made the claim that the Fermi-distribution moment integrals that appear in the higher-order terms of the Sommerfeld expansion somehow know about anomalies. In this section, we try to explain how this knowledge comes about by combining the ideas in Ref. [4] with the geometry behind our gedanken trick of using the Hawking effect as our heat source.

The first few such moment integrals are

$$\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\}=\left(\frac{\mu}{2\pi}\right),$$

$$\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\left(\frac{\epsilon}{2\pi}\right)\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\}=\frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2+\frac{T^2}{4!},$$

$$\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\frac{1}{2!}\left(\frac{\epsilon}{2\pi}\right)^2\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\}=\frac{1}{3!}\left(\frac{\mu}{2\pi}\right)^3+\left(\frac{\mu}{2\pi}\right)\frac{T^2}{4!},$$

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\frac{1}{3!}\left(\frac{\epsilon}{2\pi}\right)^3\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\} &= \frac{1}{4!}\left(\frac{\mu}{2\pi}\right)^4+\frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2\frac{T^2}{4!}+\frac{7}{8}\frac{T^4}{6!},\\
\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\frac{1}{4!}\left(\frac{\epsilon}{2\pi}\right)^4\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\} &= \frac{1}{5!}\left(\frac{\mu}{2\pi}\right)^5+\frac{1}{3!}\left(\frac{\mu}{2\pi}\right)^3\frac{T^2}{4!}+\left(\frac{\mu}{2\pi}\right)\frac{7}{8}\frac{T^4}{6!},\\
\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\frac{1}{5!}\left(\frac{\epsilon}{2\pi}\right)^5\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\} &= \frac{1}{6!}\left(\frac{\mu}{2\pi}\right)^6+\frac{1}{4!}\left(\frac{\mu}{2\pi}\right)^4\frac{T^2}{4!}+\frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2\frac{7}{8}\frac{T^4}{6!}+\frac{31}{24}\frac{T^6}{8!}.
\end{aligned}
\tag{73}
$$

These are all polynomials in the temperature and the chemical potential. It is essential for the simplicity of these results that the ϵ integral runs from $-\infty$ to $+\infty$. If we had kept only the positive energy part of the integrals, we would have instead

$$
\int_0^{\infty}\frac{d\epsilon}{2\pi}\frac{1}{k!}\left(\frac{\epsilon}{2\pi}\right)^k\frac{1}{1+e^{\beta(\epsilon-\mu)}} = -\frac{1}{(2\pi\beta)^{k+1}}\mathrm{Li}_{k+1}(-e^{\beta\mu}),
\tag{74}
$$

where the polylogarithm function $\mathrm{Li}_k(x)$ is defined by analytic continuation from the series

$$
\mathrm{Li}_k(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^k}, \qquad |z|<1.
\tag{75}
$$

The polynomial form of the full-range integral arises from the identity

$$
\mathrm{Li}_k(-e^{\beta\mu})+(-1)^k\mathrm{Li}_k(-e^{-\beta\mu}) = -\frac{(2\pi i)^k}{k!}\mathrm{B}_k\left(\frac{1}{2}+\frac{\beta\mu}{2\pi i}\right),
\tag{76}
$$

which holds for integer k and where $\mathrm{B}_k(x)$ are the Bernoulli polynomials defined by

$$
\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x).
\tag{77}
$$

The identity (76) is a special case of a general identity for the polylogarithm due to Hurwitz. A compact generating function

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}e^{\tau\epsilon/2\pi}\left\{\frac{1}{1+e^{\beta(\epsilon-\mu)}}-\theta(-\epsilon)\right\}\\
&\quad=\frac{1}{\tau}\left\{\frac{(\frac{\tau T}{2})}{\sin(\frac{\tau T}{2})}e^{\tau\mu/2\pi}-1\right\}, \qquad 0<\tau T/2\pi<1,
\end{aligned}
\tag{78}
$$

for the Fermi-distribution moments encapsulates these facts. Expanding both sides of (78) in powers of τ and comparing coefficients reveals the equalities in (73) and also explains the reason for the inclusion of the factors of $1/n!(2\pi)^n$ in the left-hand side integrals of (73). The generating function identity (78) is easily established by substituting $x=\exp\{\beta(\epsilon-\mu)\}$ and then using the standard integral

$$
\int_0^{\infty}dx\frac{x^{\alpha-1}}{1+x} = \frac{\pi}{\sin\pi\alpha}, \qquad 0<\alpha<1.
\tag{79}
$$

The authors of Ref. [4] point out that the generating function (78) is strongly reminiscent of the general formula

$$
\mathrm{Index}[\not{D}] = \int_{\mathcal{M}}\hat{A}[R]\mathrm{ch}[F]
\tag{80}
$$

for the index of the Dirac operator on a euclidean manifold $\mathcal{M}$. Here,

$$
\begin{aligned}
\mathrm{ch}[\tau F] &= \exp\{\tau F/2\pi i\}\\
&= 1+\tau\{F/2\pi i\}+\frac{\tau^2}{2}\{F/2\pi i\}^2+\ldots
\end{aligned}
\tag{81}
$$

is the total Chern character involving the gauge-field curvature $F=\frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$, and

$$
\begin{aligned}
\hat{A}[\tau R] &\overset{\mathrm{def}}{=} \sqrt{\det\left(\frac{\tau R/4\pi i}{\sinh\tau R/4\pi i}\right)}\\
&= 1+\frac{\tau^2}{(4\pi)^2}2\mathrm{tr}\{R^2\}\\
&\quad+\frac{\tau^4}{(4\pi)^4}\left[\frac{1}{288}(\mathrm{tr}\{R^2\})^2+\frac{1}{360}\mathrm{tr}\{R^4\}\right]+\cdots\\
&= 1-\frac{\tau^2}{24}\mathfrak{p}_1+\frac{\tau^4}{5760}(7\mathfrak{p}_1^2-4\mathfrak{p}_2)+\ldots.
\end{aligned}
\tag{82}
$$

is the A-roof genus involving the Riemann curvature matrix-valued 2-form

$$
R_{ij} = \frac{1}{2}R_{ij\mu\nu}dx^\mu dx^\nu.
\tag{83}
$$

In the last line of (82), the 4N-forms $\mathfrak{p}_n(R)$ are the Pontryagin classes normalized as is customary in the mathematics literature. It is tacitly understood that in the product of $\mathrm{ch}[F]$ and $\hat{A}(R)$ in (80) we only retain those terms of which the total form degree matches that of the manifold $\mathcal{M}$.

To derive the equalities in (82) and see the connection with (78), we make use of the algebraic trick that underlies the *splitting principle* from the general theory of characteristic classes. We regard the curvature 2-form of the N-dimensional manifold $\mathcal{M}$ as an n-by-n skew-symmetric matrix that can be reduced to the canonical form

$$\frac{1}{2\pi}R_{ij} \equiv \frac{1}{4\pi}R_{ij\mu\nu}dx^\mu dx^\nu = \begin{bmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -x_{N/2} \\ & & & x_{N/2} & 0 \end{bmatrix}_{ij}. \tag{84}$$

Here, the x_i are formal objects (Chern roots) that become real numbers when we evaluate the curvature 2-form at a point on some chosen vectors and only then perform the canonical-form reduction. In terms of the x_i, the A-roof genus and the total Pontryagin class are given by

$$\hat{A}[\tau R] = \sqrt{\det\left(\frac{\tau R/4\pi i}{\sinh \tau R/4\pi i}\right)} = \prod_{i=1}^{n/2} \frac{\tau x_i/2}{\sinh \tau x_i/2}, \tag{85}$$

$$\mathfrak{p}(\tau R) = \det(1 - \tau R/2\pi) = \prod_i (1 + \tau^2 x_i^2), \tag{86}$$

and

$$\mathfrak{p}(\tau R) = 1 + \tau^2 \mathfrak{p}_1(R) + \tau^4 \mathfrak{p}_2(R) + \ldots. \tag{87}$$

For the Pontryagin classes, the expressions

$$\mathfrak{p}_1(R) = \sum_i x_1^2 = -\frac{1}{(2\pi)^2}\left[\frac{1}{2}\mathrm{tr}\{R^2\}\right],$$
$$\mathfrak{p}_2(R) = \sum_{i<j} x_i^2 x_j^2 = \frac{1}{(2\pi)^4}\left[\frac{1}{8}(\mathrm{tr}\{R^2\})^2 - \frac{1}{4}\mathrm{tr}\{R^4\}\right] \tag{88}$$

account for the equality of the last two lines in (82). A similar expansion of (85) leads to the equality of the first two lines.

The discussion in Ref. [4] combines a general solution [33] to the constraints imposed by demanding the absence of entropy creation by the anomaly-induced currents with the generating function (78) to obtain an effective action for the ideal Weyl gas from the anomaly polynomial $\mathfrak{P}[R,F] \overset{\mathrm{def}}{=} \hat{A}[R]\mathrm{ch}[F]$. A key ingredient is the replacement rule [4,6–10,12]

$$F \to \mu$$
$$\mathfrak{p}_1(R) = -\frac{1}{8\pi^2}\mathrm{tr}(R^2) \to -T^2$$
$$\mathfrak{p}_n(R) = 0, \qquad n > 1. \tag{89}$$

The replacement-rule result is very striking, but one is left wondering whether the similarity of the Sommerfeld-integral generating function's factor

$$\frac{\tau T/2}{\sin(\tau T/2)} \tag{90}$$

to the anomaly polynomial's factor

$$\prod_{i=1}^{n/2} \frac{\tau x_i/2}{\sinh(\tau x_i/2)} \tag{91}$$

is anything more than a mere coincidence. The question of how the T^2 contributions to the currents are linked to the gravitational anomaly is also raised in Ref. [4] but was left unanswered because they work only in flat space. We believe that the illustrative examples in our Sec. III go some of the way toward explaining that the similarity is not a coincidence. The essential idea is that when we generate our temperature from the $1+1$-dimensional Schwarzschild sausage we need only to curve together the radial and time dimensions. As a consequence, only one x_i is nonzero, so only one nontrivial factor appears in the A-roof generating function. This also means that when expressed in terms of the Pontryagin classes only one of the $\mathfrak{p}_n$ can be nonzero. This will be $\mathfrak{p}_1(R) = x_1^2 = -T^2$, where the minus sign accounts for the difference between $\sinh \tau x/2$ and $\sin \tau T/2$.

V. DISCUSSION

In 1967, Sutherland [34] and Veltman [35] argued that partially conserved axial current (PCAC) and current algebra require the decay $\pi_0 \to \gamma\gamma$ to be strongly suppressed—a result contrary both to the experimental fact that this is the principal decay mode of the neutral pion and to the fact that the observed decay rate had been accurately calculated by Steinberger in 1949 [36] from a Pauli-Villars regulated Axial-Axial-Vector (AVV) triangle diagram. Two years later, the contradiction was resolved by Adler [37] and by Bell and Jackiw [38], who showed that the Sutherland-Veltman argument fails because it requires an illegitimate shift of the integration variable in the triangle diagram, which is only conditionally convergent. As a consequence, they found that, even for massless fermions, the axial current is not conserved.

The early understanding of such an anomalous failure of conservation laws was mostly of a formal mathematical character. The subtle issue of conditionally convergent Feynman integrals was followed by Kiskis, Nielsen and Schroer, and others making a connection with the mathematically deep Atiyah-Singer index theorem [39,40]. Fujikawa [41] then showed that the index theorem mandated the difference between the number of left- and right-handed Dirac eigenmodes led to the path-integral measure failing to be invariant under chiral transformations. It was only around 1982 that Peskin [42] and others realized that in the massless case the *physical* source of the $\mathbf{E}\cdot\mathbf{B}/4\pi^2$ chiral anomaly is that the $|\mathbf{B}|/2\pi$ density of gapless modes in the $\mathbf{B}$ field allows a steady $\dot{N}=\dot{k}_\parallel/2\pi=E_\parallel/2\pi$ flow of eigenstates out of the infinitely deep Dirac sea, which is acting as a Hilbert hotel. At about the same time, Nielsen and Ninomiya [43] showed that in crystals, where there are necessarily equal numbers of left- and right-handed Weyl nodes, the Hilbert-hotel picture is not needed because the Dirac seas of left- and right-handed fermions pass eigenstates to one another at their common seabed. This ambichiral traffic is the basis for our present understanding of Weyl semimetals. Later, Callan and Harvey [44] showed that, in the case of an uncanceled net anomaly, charge is supplied to the bottom of the Dirac sea via inflow from higher dimensions. Their bulk-edge and bulk-surface connections are central to the physics of the quantum Hall effect and topological insulators. In the latter, the picture is particularly clear because the top and bottom of the branches of gapless boundary modes merge with, and emerge from, the lower and upper edges of the higher-dimensional bulk states' energy gap. It is now also understood [45,46] that the spectral flow of eigenstates can be computed by including a Berry phase–induced anomalous velocity in semiclassical dynamics.

Today, we have a good *mathematical* understanding of gravitational anomalies [29,30], but a comparable physical explanation, analogous to the $\mathbf{E}\cdot\mathbf{B}$ spectral flow mechanism, does not seem to exist. In Ref. [47], an attempt was made to generalize the semiclassical Berry-phase picture to motion in curved space, but the generalization was frustrated by the unusual Lorentz transformation properties of massless particles with spin [48–51]. While the main result of the present work is the explicit derivation of the T^2 contribution to the CVE from the anomaly, we hope the simple gedanken space-time that we have constructed to do this will be useful for developing a physical understanding in the gravitational case also.

ACKNOWLEDGMENTS

M. S. would like to thank Andrzej Woszczyna of Kraków Jagiellonian University for his e-mail assistance in learning how to use the ccgrg package for *Mathematica*. We also benefited from useful comments on the text by Karl Landsteiner and Vatsal Dwivedi. This work was not directly supported by any funding agency, but it would not have been possible without resources provided by the Department of Physics at the University of Illinois at Urbana-Champaign.

[1] J. Gooth, A. C. Niemann, T. Meng, A. G. Grushin, K. Landsteiner, B. Gotsmann, F. Menges, M. Schmidt, C. Shekhar, V. Süß, R. Hühne, B. Rellinghaus, C. Felser, B. Yan, and K. Nielsch, Experimental signatures of the mixed axial-gravitational anomaly in the Weyl semimetal NbP, Nature (London) **547**, 324 (2017).

[2] For example, K. Chang, An Experiment in Zurich Brings Us Nearer to a Black Hole's Mysteries, on Page D2 of the July 25th edition of the New York Times, 2017, https://www.nytimes.com/2017/07/19/science/mixed-axial-gravitational-anomaly-weyl-semimetals-ibm.html.

[3] K. Landsteiner, E. Megías, and F. Peña-Benitez, Gravitational Anomaly and Transport, Phys. Rev. Lett. **107**, 021601 (2011).

[4] R. Loganayagam and P. Surówka, Anomaly/transport in an ideal Weyl gas, J. High Energy Phys. 04 (2012) 097.

[5] W. A. Bardeen and B. Zumino, Consistent and covariant anomalies in gauge and gravitational theories, Nucl. Phys. **B244**, 421 (1984).

[6] T. Azeyanagi, R. Loganayagam, G. S. Ng, and M. J. Rodriguez, Holographic thermal helicity, J. High Energy Phys. 08 (2014) 40.

[7] N. Banerjee, S. Dutta, S. Jain, R. Loganayagam, and T. Sharma, Constraints on anomalous fluid in arbitrary dimensions, J. High Energy Phys. 03 (2013) 48.

[8] K. Jensen, R. Loganayagam, and A. Yarom, Thermodynamics, gravitational anomalies and cones, J. High Energy Phys. 02 (2013) 88.

[9] K. Jensen, R. Loganayagam, and A. Yarom, Anomaly inflow and thermal equilibrium, J. High Energy Phys. 05 (2014) 134.

[10] K. Jensen, R. Loganayagam, and A. Yarom, Chern-Simons terms from thermal circles and anomalies, J. High Energy Phys. 05 (2014) 110.

[11] S. D. Chowdhury and J. R. David, Anomalous transport at weak coupling, J. High Energy Phys. 11 (2015) 048.

[12] V. Dwivedi and M. Stone, Chiral kinetic theory and anomalous hydrodynamics in even spacetime dimensions, J. Phys. A **50**, 155202 (2017).

[13] S. P. Robinson and F. Wilczek, Relationship between Hawking Radiation and Gravitational Anomalies, Phys. Rev. Lett. **95**, 011303 (2005); S. Iso, H. Umetsu, and F. Wilczek, Hawking Radiation from Charged Black Holes via

Gauge and Gravitational Anomalies, Phys. Rev. Lett. **96**, 151302 (2006).
[14] S. Iso, H. Umetsu, and F. Wilczek, Anomalies, Hawking radiations and regularity in rotating black holes, Phys. Rev. D **74**, 044017 (2006).
[15] R. Banerjee and S. Kulkarni, Hawking radiation and covariant anomalies, Phys. Rev. D **77**, 024018 (2008); Hawking radiation, effective actions and covariant boundary conditions, Phys. Lett. B **659**, 827 (2008); Kulkarni, Hawking radiation, covariant boundary conditions, and vacuum states, Phys. Rev. D **79**, 084045 (2009).
[16] A. Lucas, R. A. Davison, and S. Sachdev, Hydrodynamic theory of thermoelectric transport and negative magneto-resistance in Weyl semimetals, Proc. Natl. Acad. Sci. U.S.A. **113**, 9463 (2016).
[17] Y. Neiman and Y. Oz, Relativistic hydrodynamics with general anomalous charges, J. High Energy Phys. 03 (2011) 023.
[18] We are ignoring dissipative effects such as viscosity and diffusion.
[19] K. Rajagopal and A. V. Sadofyev, Chiral drag force, J. High Energy Phys. 10 (2015) 018.
[20] M. Stephanov and H.-U. Yee, The No-Drag Frame for Anomalous Chiral Fluid, Phys. Rev. Lett. **116**, 122302 (2016).
[21] S. Golkar and S. Sethi, Global anomalies and effective field theory, J. High Energy Phys. 05 (2016) 105.
[22] S. D. Chowdhury and J. R. David, Global gravitational anomalies and transport, J. High Energy Phys. 12 (2016) 116.
[23] K. Landsteiner, E. Megías, and F. Peña-Benitez, Holographic gravitational anomaly and chiral vortical effect, J. High Energy Phys. 09 (2011) 121.
[24] A. Vilenkin, Macroscopic parity violating effects: Neutrino fluxes from rotating black holes and in rotating thermal radiation, Phys. Rev. D **20**, 1807 (1979); Parity violating currents in thermal radiation, Phys. Lett. B **80**, 150 (1978).
[25] J. B. Hartle and S. W. Hawking, Path integral derivation of black-hole radiation, Phys. Rev. D **13**, 2188 (1976).
[26] G. W. Gibbons and M. J. Perry, Black Holes in Thermal Equilibrium, Phys. Rev. Lett. **36**, 985 (1976).
[27] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, Berlin, 1996).
[28] S. A Fulling, The gravitational anomaly: An elementary coordinate-space approach, Gen. Relativ. Gravit. **18**, 609 (1986).
[29] L. Alvarez-Gaume and E. Wittem, Gravitational anomalies, Nucl. Phys. **B234**, 269 (1984).
[30] L. Alvarez-Gaume and P. Ginsparg, The structure of gauge and gravitational anomalies, Ann. Phys. (N.Y.) **161**, 423 (1985).
[31] R. B. Mann and S. N. Solodukhin, Conical geometry and quantum entropy of a charged Kerr black hole, Phys. Rev. D **54**, 3932 (1996).
[32] A. Flachi and K. Fukushima, Chiral vortical effect in curved space and the Chern-Simons current, arXiv:1702.04753v2.
[33] R. Loganayagam, Anomaly induced transport in arbitrary dimensions, arXiv:1106.0277.
[34] D. G. Sutherland, Current algebra and some non-strong mesonic decays, Nucl. Phys. **B2**, 433 (1967).
[35] M. Veltman, Theoretical aspects of high energy neutrino interactions, Proc. R. Soc. A **301**, 107 (1967).
[36] J. Steinberger, On the use of subtraction fields and the lifetimes of some types of meson decay, Phys. Rev. **76**, 1180 (1949).
[37] S. L Adler, Axial-vector vertex in spinor electrodynamics, Phys. Rev. **177**, 2426 (1969).
[38] J. S. Bell and R. Jackiw, A PCAC puzzle: $\pi_0 \to \gamma\gamma$ in the σ-model, Nuovo Cimento A **60**, 47 (1969).
[39] J. Kiskis, Fermion zero modes and level crossing, Phys. Rev. D **18**, 3690 (1978).
[40] N. K. Nielsen and B. Schroer, Axial anomaly and Atiyah-Singer theorem, Nucl. Phys. **B127**, 493 (1977).
[41] K. Fujikawa, Path integral for gauge theories with fermions, Phys. Rev. D **21**, 2848 (1980); Erratum, Phys. Rev. D**22**, 1499(E) (1980).
[42] Personal communication from M. Peskin, during Recent advances in field theory and statistical mechanics, Les Houches Summer School, Session XXXIX, 1982; also mentioned as a private communication in J. Ambjørn, J. Greensite, and C. Peterson, The axial anomaly and the lattice Dirac sea, Nucl. Phys. **B221**, 381 (1983).
[43] H. B Nielsen and M. Ninomiya, The Adler-Bell-Jackiw anomaly and Weyl fermions in a crystal, Phys. Lett. **130B**, 389 (1983).
[44] C. G. Callan and J. A. Harvey, Anomalies and Fermion zero modes on strings and boundaries, Nucl. Phys. **B250**, 427 (1985).
[45] D. T. Son and N. Yamamoto, Berry Curvature, Triangle Anomalies, and the Chiral Magnetic Effect in Fermi Liquids, Phys. Rev. Lett. **109**, 181602 (2012).
[46] M. A. Stephanov and Y. Yin, Chiral Kinetic Theory, Phys. Rev. Lett. **109**, 162001 (2012).
[47] M. Stone, V. Dwivedi, and T. Zhou, Berry phase, Lorentz covariance, and anomalous velocity for Dirac and Weyl particles, Phys. Rev. D **91**, 025004 (2015).
[48] M. Stone, V. Dwivedi, and T. Zhou, Wigner Translations and the Observer Dependence of the Position of Massless Spinning Particles, Phys. Rev. Lett. **114**, 210402 (2015).
[49] J.-Y. Chen, D. T. Son, M. A. Stephanov, H.-U. Yee, and Y. Yin, Lorentz Invariance in Chiral Kinetic Theory, Phys. Rev. Lett. **113**, 182302 (2014).
[50] C. Duval and P. A. Horvathy, Chiral fermions as classical massless spinning particles, Phys. Rev. D **91**, 045013 (2015).
[51] C. Duval, M. Elbistan, P. A. Horvathy, and P.-M. Zhang, Wigner-Souriau translations and Lorentz symmetry of chiral fermions, Phys. Lett. B **742**, 322 (2015).