

Dynamical obstruction to perpetual motion from Lorentz-violating black holes

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Black holes in Lorentz-violating theories have been claimed to violate the second law of thermodynamics by perpetual motion energy extraction. We revisit this question for a Penrose splitting process in a spherically symmetric setting with two species of particles that move on radial geodesics that extend to infinity. We show that energy extraction by this process cannot happen in any theory in which gravity is attractive, in the sense of a geometric inequality that we describe. This inequality is satisfied by all known Einstein-æther and Hořava black hole solutions.

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I. INTRODUCTION

The defining property of a black hole is that nothing can escape from its interior which, in general relativity, is separated from the exterior by a null hypersurface called the event horizon. Although the event horizon is a global concept, its nullness implies that local Lorentz symmetry plays a crucial role in the definition of black holes. For some field excitations propagate superluminally, then these excitations are not confined by light cones and can penetrate null event horizons.

There are at least two distinct ways to introduce superluminality. First, one could have higher-order dispersion relations, such as $\omega^2 \propto k^2 + a^2 k^4 + \dots$, where ω is the frequency and k the wave number, and an unbounded maximum propagation speed. This scenario occurs in preferred foliation theories (cf. [1–5]). Remarkably, the concept of a black hole survives due to a new type of horizon, called the *universal horizon* [6,7]. See Ref. [8] for a detailed discussion. Second, each excitation could keep a linear dispersion relation but with differing propagation speeds. In this case different massless excitations propagate along the null cones of distinct effective metrics. This second scenario should still capture some aspects of the low-momentum behaviour present in the first scenario.

In the second scenario, a stationary black hole can have multiple horizons that cloak the interior; each is the Killing

horizon of a different effective metric, with the innermost one corresponding to the fastest excitation [6,9–15]. It has been argued that if such stacks of multiple horizons exist, they can allow a perpetuum mobile of the second kind, extracting unlimited energy from the black hole with no change in the hole’s entropy [16–18]. This conclusion is troubling as a point of principle, as it violates the generalized second law of thermodynamics, which has been a cornerstone of our understanding of the quantum properties of black holes [19] (for modern reviews see [20,21]). The conclusion would also be experimentally intriguing. While observations have provided strong constraints on Lorentz violations in both matter and gravity, no observations, including the recent gravitational wave observations [22,23], significantly constrain the speed of superluminal gravity polarisations that correspond to new degrees of freedom (d.o.f.) inherent in most models [24,25]. There is an underlying reason—in the limit from a Lorentz-violating theory to general relativity, the new d.o.f. do not necessarily decouple, the limit need not be smooth (see e.g., [26–28]), and the speeds of the new d.o.f. need not approach the speed of light.

The perpetual motion construction in [16] employs Hawking radiation, arguing that the temperature difference between two horizons allows heat transfer from a cold to a hot reservoir. The constructions in [17,18] are classical, arguing in terms of the energy of a particle that escapes to infinity from a splitting event between two horizons. All require that the region between the two horizons acts like an ergoregion for the slowest excitations. Moreover, moving from perpetual motion to a violation of the generalized second law of thermodynamics assumes that the process does not increase the black hole’s entropy and has no

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natural endpoint. The classical constructions of [17,18] also rely crucially on an external agent who sets the initial conditions for the process, arguing that the agent's net contribution to the energy budget can be made negligible. Finally, the arguments are purely kinematical, with the presumed local geometric structure of the ergoregion unconstrained by the requirement that it must exist as a solution to any otherwise well-behaved dynamical theory.

Here we reanalyze the perpetual motion question for two primary reasons. First, subtleties about external agents' contributions to energy budgets have generated unresolved debates of thirty years and counting [29,30]. These subtleties are a strong reason to bypass the debates and ask what is possible without external agents. Second, not all Lorentz violating spacetimes may be solutions of a viable gravitational theory, and we should not consider those spacetimes as representative of Lorentz violating gravity any more than we consider the negative mass Schwarzschild solution to represent general relativity.

Concretely, we consider a Penrose “one-to-two” splitting process involving two different particle species. The two particles move along radial geodesics of two different (effective) metrics. We assume both metrics are static, spherically symmetric and asymptotically flat, and that the geodesics extend to infinity. The energy budget is therefore unambiguously defined by the Killing energy at infinity. We show that energy extraction by this process cannot happen in any spacetime in which gravity is attractive, in the sense of a geometric inequality that we state. This inequality is equivalently a restriction on the configuration of the Lorentz-symmetry violating field that defines the preferred frame, and is satisfied in all Einstein-æther and Hořava gravity solutions that are analytically or numerically known.

Our results do not directly contest the perpetual mobile constructions of [17,18]. Nonetheless, they do indicate that removing the external agents from the construction and considering dynamics introduces a significant new obstruction. We view this result as a strong and qualitatively new argument against violation of the generalized second law in Lorentz-violating gravitational theories.

The structure of the paper is as follows: Sec. II recalls the Lorentz-violating geometric setting of the perpetual motion scenarios in [17,18], and Sec. III contrasts the thermodynamic paradoxes in these scenarios with the thermodynamically nonparadoxical Penrose processes in Einstein gravity. Our main result, the perpetual motion exclusion criterion, is presented in Sec. IV. Section V presents a partial converse, which demonstrates that the exclusion criterion is kinematically nontrivial. Section VI gives a brief summary and discussion. Technical material is deferred to three Appendices.

II. SCENARIO FOR PERPETUAL MOTION

We first recall the geometric setting of the perpetual motion scenarios in [17,18]. We consider Lorentz-noninvariant

theories of gravity that contain the metric g_{ab} and the æther [31] u^a , a dynamical unit timelike vector field that defines at each point a preferred timelike direction. Solutions therefore break local Lorentz symmetry.

Given g_{ab} and u^a , we construct new metrics

$$g_{ab}^{(i)} := -u_a u_b + c_i^{-2}(u_a u_b + g_{ab}), \quad (1)$$

where we have adopted the mostly plus sign convention, $u_a := g_{ab}u^b$, and c_i are positive constants. From the unit timelike condition on u^a it follows that $g_{ab}^{(i)}u^a u^b = -1$ and $u_a = g_{ab}^{(i)}u^b$ for all i . Examination of the light cones of $g_{ab}^{(i)}$ in the local Lorentz frame in which $u^a = (1, 0, 0, 0)$ shows that $g_{ab}^{(i)}$ has Lorentzian signature and c_i is its speed of light.

Fields obeying hyperbolic field equations in $g_{ab}^{(i)}$ provide a covariant description of excitations that propagate at different speeds in preferred frame theories [9]. In the point particle limit, the fields are replaced by causal geodesics in the respective metrics, and local interactions between the fields are described as collisions that preserve the four-momentum covector [17], as we outline in Appendix A.

We consider two particle species, a slower one denoted by A , propagating along geodesics of $g_{ab}^A := g_{ab}$, and a faster one, denoted by B , propagating along geodesics of g_{ab}^B given by (1) with $c_B := c > 1$. The absolute maximum speed of propagation is then c .

We assume that g_{ab}^A is a static, spherically symmetric and asymptotically flat black hole spacetime, such that the staticity Killing vector χ^a asymptotes to the standard Minkowski time translation Killing vector at infinity, satisfying $g_{ab}^A \chi^a \chi^b \rightarrow -1$ there, and that $g_{ab}^A \chi^a \chi^b$ changes sign at the Killing horizon. We call the future branch of this Killing horizon the A -horizon. We further assume that u^a is spherically symmetric and commutes with χ^a , and that it asymptotes to χ^a at the infinity. Then g_{ab}^B is spherically symmetric and asymptotically flat, χ^a is a hypersurface-orthogonal Killing vector of g_{ab}^B , and χ^a asymptotes to the standard Minkowski time translation Killing vector of g_{ab}^B at infinity, satisfying $g_{ab}^B \chi^a \chi^b \rightarrow -1$ there. Note that while both g_{ab}^A and g_{ab}^B are static in at least some neighborhood of infinity, the pairs (g_{ab}^A, u^a) and (g_{ab}^B, u^a) are only stationary since u^a need not be parallel to χ^a .

We assume that g_{ab}^A and u^a are sufficiently regular (say, smooth), and in particular that u^a is sufficiently regular across the A -horizon. As $c > 1$, $g_{ab}^B \chi^a \chi^b$ is negative outside and on the A -horizon. Immediately behind the A -horizon there is hence the *ergoregion*: a region in which $g_{ab}^A \chi^a \chi^b > 0$ but $g_{ab}^B \chi^a \chi^b < 0$. While g_{ab}^A is static only outside the A -horizon, g_{ab}^B is static in the union of the ergoregion, the outside of the A -horizon, and their joint boundary at the A -horizon, with χ^a providing the timelike hypersurface-orthogonal Killing vector. The causal structures of g_{ab}^A and g_{ab}^B hence differ: there exist no causal A -curves from the

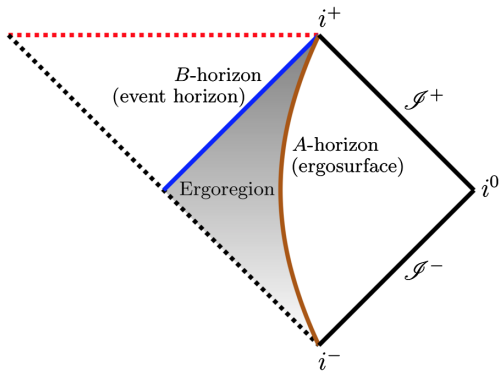


FIG. 1. Spacetime diagram of the static, spherically symmetric and asymptotically flat spacetime corresponding to the effective metric g_{ab}^B of the fastest particle, in the case where g_{ab}^B describes a black hole with a single Killing horizon and a spacelike singularity. The solid brown line marks the horizon for the A particles, which can be thought of as an ergosurface. The solid blue line marks the horizon for the B particles, which can be thought of as the event horizon. The shaded part in between is the ergoregion. The dotted red line marks the singularity. The spacetime outside the A -horizon will be referred to as the outside region and the spacetime inside the B -horizon as the black hole.

ergoregion to the \mathcal{S}^+ of g_{ab}^A , but there exist causal B -curves, and even null B -geodesics, from the ergoregion to the \mathcal{S}^+ of g_{ab}^B . Figure 1 shows the conformal diagram of g_{ab}^B in the case where the ergoregion terminates at a B -horizon, where $g_{ab}^B \chi^a \chi^b = 0 = u_a \chi^a$, and is followed by a region where $g_{ab}^B \chi^a \chi^b > 0$ and finally by a spacelike singularity. It is worth mentioning that the setup for (g_{ab}^A, u^a) and (g_{ab}^B, u^a) described above, though not tied to some specific dynamics, matches exactly that of known black hole solutions in Einstein-æther theory and Hořava gravity e.g., [6,9,10].

Now, as χ^a is A -spacelike within the ergoregion, A -particles can carry negative Killing energy there. This raises the possibility that a system of particles could enter the ergoregion, undergo interactions that create an A -particle with negative Killing energy and a B -particle that exits the ergoregion, and, by conservation of total Killing energy, give the exiting B -particle more Killing energy than originally entered the ergoregion. This process would thus extract energy from the black hole.

Versions of this process were considered in [17,18]. The process of [17] introduces an external agent who releases the initial system from close to the A -horizon, with an initial velocity that is sufficiently right-pointing in Fig. 1. The process of [18] introduces an external agent who uses a tether to lower the initial system from infinity past the A -horizon. In each case there is energy extraction if one can conclusively argue that the agents and their equipment make a negligible net contribution to the energy balance. Here we ask whether the processes have counterparts that involve no external agents.

III. ENERGY EXTRACTION VERSUS THERMODYNAMICS

Before proceeding, we pause to emphasise that energy extraction is not on its own a threat to thermodynamics. For example, for Kerr black holes in Einstein gravity, energy can be extracted by superradiance whenever the hole has nonzero angular momentum [32]; energy can also be extracted by the usual Penrose processes at least for sufficiently rapidly rotating Kerr black holes [32–36]. The key point is that the extracted energy does not come for free: the black hole loses mass by spinning down, while its horizon and entropy increase. Moreover, the process has a natural endpoint when black hole has lost all of its angular momentum [37]. There are no thermodynamical paradoxes.

Could the situation be similar in our case? Suppose for the moment that g_{ab}^A and g_{ab}^B were not related to each other through (1), but g_{ab}^B is just some composite metric defined in terms of g_{ab}^A and some other field Ψ . It is *a priori* conceivable that energy extraction then takes place as in [17,18], and as a result Ψ reconfigures itself such that eventually g_{ab}^B tends to g_{ab}^A , the two horizons merge, the ergoregion disappears and the process halts. Moreover, the entropy never decreases.

However, when g_{ab}^A and g_{ab}^B are related by (1), the unit timelike property of u^a shows that the ergoregion cannot disappear in a regular manner: driving the B -horizon towards the A -horizon makes the æther configuration singular. More broadly, whenever two different excitations propagate at different speeds, which is the hallmark of local Lorentz violation, it is quite difficult to construct a theory where the corresponding horizons merge smoothly.

IV. PERPETUAL MOTION EXCLUSION CRITERION

We now come to the main question of the paper: when both the initial and final energies are defined at asymptotically flat infinity, without external agents, does energy extraction still occur by some counterpart of the processes described in [17,18]?

We consider a pointlike object Σ that is dropped into the ergoregion from the infinity. Σ can be either an A -particle, following a causal A -geodesic, or a B -particle, following a causal B -geodesic. In the ergoregion Σ splits into two ejecta, an A -particle and a B -particle, each moving on a causal geodesic in their respective metric. The B -ejectum exits the ergoregion and escapes to infinity. All the geodesics are assumed to be radial, in the sense of having vanishing angular momenta with respect to the Killing vectors of spherical symmetry. All the particles are assumed to have positive energy in a local rest frame in the metric whose geodesic they follow.

The back-reaction of the particles on the metric and the æther is neglected throughout. The æther, since it has a tensor vacuum expectation value, generates modified light

cones for coupled standard model particles in its ground state, in the same way the background tensors in the standard model extension do [38]. Therefore, the components of the process need not be æther excitations (which take the form of æther-metric waves [39] at the linearized level), though such excitation could also be used. For Einstein gravity, the corresponding approximation would mean neglecting the production of gravitational waves. We stress that this approximation is typical when deriving Lorentz violation constraints from astrophysical observations [40–43].

Consider now the asymptotic observers who are static at infinity, following orbits of the Killing vector χ^a . For these observers, the energy of a particle with the momentum covector k_a is the Killing energy $-k_a\chi^a$. The observers see the above process as energy extraction if and only if the B -ejectum has a larger Killing energy than Σ . Under what conditions on the metrics and u^a can the asymptotic observers then see energy extraction?

Our main result is the following criterion:

Perpetual motion exclusion criterion: Suppose that the inequality

$$-g_{ab}^B\chi^a\chi^b < 1 \quad (2)$$

holds everywhere. Then energy extraction by the process described above cannot happen.

We sketch the proof here and detail it in Appendix B. By spherical symmetry of the geometry and the particle motion, we may drop the angles and work in the 1 + 1 spacetime dimensions shown in Fig. 1. We raise and lower indices with g_{ab}^B , and all dot products and normalisations are with respect to g_{ab}^B .

First, we introduce the æther frame (u^a, s^a) , where the spacelike unit vector s^a is orthogonal to u^a and tends to the usual outward-pointing radial vector at asymptotically flat infinity. This frame makes certain geometric properties manifest. In particular, a covector k_a can be decomposed in the æther frame as

$$k_a = E u_a + k_s s_a, \quad (3)$$

and similarly for the corresponding vector k^a . A (co)vector with $k_s > 0$ is called right-pointing and a (co)vector with $k_s < 0$ is called left-pointing. Also, since the Killing vector χ^a is B -timelike and future-pointing, we can parametrize it as

$$\chi^a = -(u \cdot \chi)(u^a + \tanh \eta s^a), \quad (4)$$

where η is the rapidity by which χ^a is boosted relative to u^a .

Now, within the ergoregion, the relative configuration of u^a , χ^a and the two light cones is as shown in Fig. 2. χ^a is right-pointing: this follows by continuity because the A -horizon is a future horizon and outside the A -horizon χ^a is inside the future A -light cone. It follows that

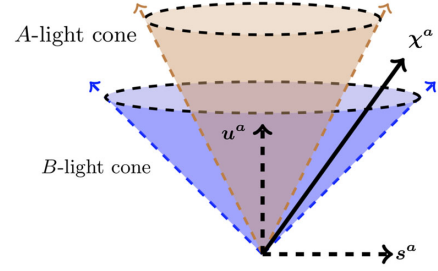


FIG. 2. The configuration of the two light cones and the Killing vector χ^a in the ergoregion, in the frame (u^a, s^a) .

$$0 < s \cdot \chi < -u \cdot \chi. \quad (5)$$

At the splitting event $\Sigma \rightarrow A$ -ejectum + B -ejectum in the ergoregion, conservation of the four-momentum reads

$$k_a^\Sigma = k_a^A + k_a^B, \quad (6)$$

where k_a^Σ , k_a^A and k_a^B are the respective momentum covectors of the three particles. By conservation of the Killing energy along geodesics in the two metrics, the energy extraction condition is $-k_a^A\chi^a < 0$. It follows that k_a^A is right-pointing, as can be seen by decomposing k_a^A as in (3), where $E > 0$ by the assumption of locally positive energy, and using (5).

It further follows that $(k^B)^a$ is more right-pointing than χ^a in Fig. 2. To see this, we introduce a function r that is constant on orbits of χ^a but strictly monotonic from orbit to orbit, increasing towards infinity. Recall that by assumption the B -ejectum makes it to infinity, where $-\chi \cdot \chi \rightarrow 1$, and by assumption $-\chi \cdot \chi < 1$ everywhere. The geodesic equation hence implies that the motion of the B -ejectum is strictly monotonic in r , and by the definition of r this means $(k^B)^a$ must be pointing more to the right than χ^a . In particular, k_a^B is right-pointing.

We can now turn to Σ . By (6) and the properties established about k_a^A and k_a^B , straightforward geometric considerations show that $(k^\Sigma)^a$ must be more right-pointing than $(k^B)^a$ in Fig. 2, and hence in particular more right-pointing than χ^a . Since Σ by assumption comes from infinity, and the motion of a B -particle that comes from infinity is strictly monotonic in r , this shows that Σ cannot be a B -particle, and it similarly shows that Σ cannot be a massless A -particle. Σ must hence be a massive A -particle.

Since Σ comes from infinity, and Killing energy is conserved along the trajectory of Σ , we have $-\chi^a k_a^\Sigma \geq M$, where $M > 0$ is the mass. Energy extraction requires $-k_a^A\chi^a < 0$. The remainder of the proof consist of showing that these two inequalities cannot be simultaneously satisfied; the details are in Appendix B.

Note that any theory in which gravity as described in the B -metric remains attractive would be expected to satisfy

inequality (2). The most obvious cases of theories to consider are Einstein-æther theory [31,44] and Hořava gravity [1,2,5] (the latter can be written covariantly in terms of a metric and a hypersurface orthogonal æther [45]). Under the assumptions of staticity, spherical symmetry, and asymptotic flatness, these two theories have the same black hole solutions [11,12]. For the two families of analytic solutions given in [10], inequality (2) does hold: by Eqs. (51c), (52), (61) and (63) in [10], these solutions satisfy $-u_a \chi^a < 1$, from which (2) follows by (4). We have also examined the nine numerical solutions given in [6], compatible with binary pulsar observational constraints [46] (although not compatible by the more recent observational constraints on the speed of gravitational waves [25]), and a selection of further numerical solutions generated by the code of [6]. In all cases we find that $-u_a \chi^a < 1$ holds, implying inequality (2).

V. PERPETUAL MOTION ADMISSION CRITERION

While we have provided evidence that inequality (2) is a well motivated assumption in our exclusion criterion, we now address briefly the converse question: what would it take to find a configuration in which energy extraction by our process *does* occur? We show that for any given A -metric, it is kinematically possible to write down æther configurations that allow energy extraction.

If the process involves massive B -particles, relaxing (2) becomes cumbersome because the qualitative behaviour of massive B -geodesics from and to the infinity is then sensitive to the locations and heights of the local maxima of $-g_{ab}^B \chi^a \chi^b$. However, if the process involves no massive B -particles, we find the following sharp criterion:

Perpetual motion admission criterion: Suppose the process involves no massive B -particles. Then, energy extraction occurs for some values of the A -ejectum mass if and only if Σ is a massive A -particle and the ergoregion has points at which $K \sqrt{\frac{c+1}{c-1}} < -u_a \chi^a$ and the vector $V^a := \chi^a - K \sqrt{1 - c^{-2}} u^a$ is B -timelike, where $K := \sqrt{\sup(-g_{ab}^A \chi^a \chi^b)} \geq 1$.

We give the proof of this criterion in Appendix C. We also show there that while the above statement of the criterion is in terms of algebraic properties that are transparently related to the B -metric and whose validity can be readily examined for any given Einstein-æther configuration, these algebraic properties encode a geometric property that is transparent in terms of the A -metric: energy extraction occurs if and only if, at the splitting event, the æther vector points to the *left* of the four-velocity of Σ in Fig. 1, by an A -boost whose velocity is greater than c^{-1} . The constant K encodes the initial speed with which Σ needs to be released from the infinity in order to reach the ergoregion; in particular, if $-g_{ab}^A \chi^a \chi^b < 1$ everywhere, we

have $K = 1$, and the optimal case for energy extraction is to release Σ with vanishing initial speed.

This geometric view makes it plain that for any given A -metric, it is possible to write down an æther configuration for which energy extraction occurs: you just need to make the æther sufficiently left-pointing somewhere in the ergoregion in Fig. 1. These æther configurations will however necessarily violate (2), and we show in Appendix C that the violation occurs with a wide margin.

Related geometric observations were used in [17] to instruct an external agent how to release an incoming particle from near the A -horizon.

VI. DISCUSSION

We have analysed classical extraction of energy from Lorentz-violating black holes in a Penrose splitting process which, if dynamically possible, would challenge the status of Lorentz-violating black holes as thermodynamical objects as implied by the Hawking effect. The main outcome was that such Penrose processes do not happen in any black hole configurations in which gravity remains attractive, in the kinematical sense of inequality (2), and we verified that they indeed do not happen in known analytical or numerical Lorentz-violating black hole solutions.

A central piece of input was that we considered energy extraction as seen by observers at an asymptotically flat infinity, without external agents operating somewhere in the spacetime. Our notion of energy is hence directly the Killing energy at infinity, which should be related to the conserved charges determined by the dynamics of the theory. Despite the limitations of spherical symmetry, the test particle approximation, and the limited number of particles and particle species considered, we view our results as a strong and qualitatively new argument against violation of the second law of black hole thermodynamics in theories of gravity without local Lorentz symmetry. Suggestions that the second law might be violated will now face the burden to explain how our obstruction is avoided.

That being said, suppose that there did exist a Lorentz-violating theory with a black hole that allows energy extraction by our process. Where would the energy be coming from? While our test particle approximation cannot address this question directly, we saw that energy extraction requires a bump in $-g_{ab}^B \chi^a \chi^b$. This suggests that the extraction will affect the bump in some way, and the ultimate fate of the bump will be related to the stability properties of the theory. We leave this question as a subject of future work.

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APPENDIX A: MOMENTUM COVECTOR CONSERVATION AT THE SPLITTING

In this Appendix we justify the momentum covector conservation at the collision event.

Suppose first that g_{ab}^A is Minkowski and u^a is a constant vector. By (1), g_{ab}^B is then also Minkowski, albeit with wider light cones, and all the translational Killing vectors of g_{ab}^A are also Killing vectors of g_{ab}^B . By Noether’s theorem, an action built from fields $\phi^{(j)}$ and their derivatives using g_{ab}^A and g_{ab}^B has hence four conserved quantities, corresponding to the four translational Killing vectors. When the fields are localized into relativistic point particles, these four conserved quantities reduce on every hyperplane spacelike in g_{ab}^B (and hence also spacelike in g_{ab}^A) to the sum of the four-momentum covectors of all the particles. Conservation of the total four-momentum covector at particle collisions, of any types of particles, hence follows from Noether’s theorem whenever the collisions are a localized limit of the underlying translationally invariant field theory.

When g_{ab}^A and u^a are arbitrary but still smooth, an action built from fields $\phi^{(j)}$ and their derivatives using g_{ab}^A and g_{ab}^B need no longer have conserved quantities that would correspond to translations. However, a particle collision process localised to a single spacetime event still conserves the four-momentum covector, as is seen by applying Noether’s theorem in a local inertial frame in a neighborhood of the collision event.

APPENDIX B: PROOF OF THE PERPETUAL MOTION EXCLUSION CRITERION

In this Appendix we give the detailed proof of the perpetual motion exclusion criterion stated in Sec. IV of the main text.

1. Notation

Recall from Eq. (1) of the main text that the A -metric and the B -metric are related by

$$g_{ab}^B = -u_a u_b + c^{-2}(u_a u_b + g_{ab}^A), \quad (\text{B1})$$

where u_a is unit timelike both in g_{ab}^A and g_{ab}^B , and $c > 1$. We use a notation adapted to the B -metric: unless otherwise specified, indices are raised and lowered with g_{ab}^B , and dot products and normalizations are with respect to g_{ab}^B . Given the spherical symmetry of the geometry and of the particle

motion, we may throughout drop the angles and work as if in $1 + 1$ dimensions.

2. Geometric preliminaries

We need four preliminaries.

First, we introduce the æther-adapted frame (u^a, s^a) , where the spacelike unit vector s^a is orthogonal to u^a and tends to the usual outward-pointing radial vector at the asymptotically flat infinity. We call this frame the æther frame. A covector k_a can be decomposed in the æther frame as

$$k_a = E u_a + k_s s_a, \quad (\text{B2})$$

and similarly for the corresponding vector k^a . A (co)vector with $k_s > 0$ is called right-pointing and a (co)vector with $k_s < 0$ is called left-pointing.

Second, since the Killing vector χ^a is B -timelike and future-pointing, we can parametrize it as

$$\chi^a = -(u \cdot \chi)(u^a + \tanh \eta s^a), \quad (\text{B3})$$

where η is the rapidity by which χ^a is boosted relative to u^a . The normalization factor in (B3) can be verified by contracting both sides with u_a .

Third, within the ergoregion, the relative configuration of u^a, χ^a and the two light cones is as shown in Fig. 2, in the frame (u^a, s^a) . u^a is straight up, within the future A -light cone, while χ^a is sandwiched between the two future light cones. χ^a is right-pointing: this follows by continuity because the A -horizon is a future horizon and outside the A -horizon χ^a is inside the future A -light cone. It follows that

$$0 < s \cdot \chi < -u \cdot \chi, \quad (\text{B4})$$

$$1 < c \tanh \eta. \quad (\text{B5})$$

Introducing now the assumption $-\chi \cdot \chi < 1$, (B3) implies $-u \cdot \chi < \cosh \eta$, and combining this with (B4) gives

$$0 < -u \cdot \chi < \cosh \eta. \quad (\text{B6})$$

Fourth, we introduce a function r that is constant on orbits of χ^a but strictly monotonic from orbit to orbit, increasing towards the infinity. (For some Einstein-æther solutions r may be chosen to be the area-radius, but we wish to proceed here without assumptions about the field equations.)

3. Conservation laws

Consider now the splitting event $\Sigma \rightarrow A$ -ejectum+ B -ejectum in the process described in the main text. Conservation of the four-momentum at the splitting event reads

$$k_a^\Sigma = k_a^A + k_a^B, \quad (\text{B7})$$

where k_a^Σ , k_a^A and k_a^B are the respective momentum covectors of the three particles. By conservation of the Killing energy along geodesics in the two metrics, the energy extraction condition is $-k_a^A \chi^a < 0$. It follows that k_a^A is right-pointing, as can be seen by decomposing k_a^A as in (B2), where $E > 0$ by the assumption of locally positive energy, and using (B4).

It further follows that $(k^B)^a$ is more right-pointing than χ^a in Fig. 2. To see this, recall that by assumption the B -ejectum makes it to the infinity, where $-\chi \cdot \chi \rightarrow 1$, and by assumption $-\chi \cdot \chi < 1$ everywhere. The geodesic equation hence implies that the motion of the B -ejectum is strictly monotonic in r , and by the definition of r this means $(k^B)^a$ must be pointing more to the right than χ^a . In particular, k_a^B is right-pointing.

We can now turn to Σ . By (B7) and the properties established about k_a^A and k_a^B , straightforward geometric considerations show that $(k^\Sigma)^a$ must be more right-pointing than $(k^B)^a$ in Fig. 2, and hence in particular more right-pointing than χ^a . Since Σ by assumption comes from the infinity, and the motion of a B -particle that comes from the infinity is strictly monotonic in r , this shows that Σ cannot be a B -particle, and it similarly shows that Σ cannot be a massless A -particle. Σ must hence be a massive A -particle. We may parametrize k_a^Σ as

$$k_a^\Sigma = M(\cosh \beta u_a + c \sinh \beta s_a), \quad (\text{B8})$$

where $M > 0$ is the mass. Note that $(g^A)^{ab} k_a^\Sigma k_b^\Sigma = -M^2$, where $(g^A)^{ab}$ is the usual inverse of g_{ab}^A . Since Σ comes from the infinity, we have $-\chi^a k_a^\Sigma \geq M$, which by (B3) gives

$$1 \leq -(u \cdot \chi)(\cosh \beta - c \tanh \eta \sinh \beta). \quad (\text{B9})$$

Finally, consider the ejecta. We parametrize k_a^A and k_a^B as

$$k_a^A = \sqrt{m^2 + p^2} u_a + c p s_a, \quad (\text{B10a})$$

$$k_a^B = \sqrt{\mu^2 + q^2} u_a + q s_a, \quad (\text{B10b})$$

where $m \geq 0$ and $\mu \geq 0$ are the respective masses. Note that $(g^A)^{ab} k_a^A k_b^A = -m^2$ and $k^B \cdot k^B = -\mu^2$. As $(k^B)^a$ is more right-pointing than χ^a , comparison of (B10b) and (B3) gives

$$\tanh \eta < \frac{q}{\sqrt{\mu^2 + q^2}}, \quad (\text{B11})$$

and since the B -ejectum reaches infinity, we have $-\chi^a k_a^B \geq \mu$, or

$$\mu \leq -(u \cdot \chi) \left(\sqrt{\mu^2 + q^2} - q \tanh \eta \right). \quad (\text{B12})$$

The energy extraction inequality, $-k_a^A \chi^a < 0$, reads

$$\sqrt{m^2 + p^2} < c p \tanh \eta. \quad (\text{B13})$$

Momentum conservation (B7) takes the form

$$M \cosh \beta = \sqrt{m^2 + p^2} + \sqrt{\mu^2 + q^2}, \quad (\text{B14a})$$

$$M \sinh \beta = p + q/c, \quad (\text{B14b})$$

and solving this pair for β gives

$$\tanh \beta = \frac{p + q/c}{\sqrt{m^2 + p^2} + \sqrt{\mu^2 + q^2}}. \quad (\text{B15})$$

Note that since $q > 0$ by (B11) and $p > 0$ by (B13), (B15) implies $\beta > 0$.

4. Contradiction

We shall now show that the above set of inequalities has no solutions.

To begin, we note by (B5) that we may introduce the positive number ϵ by

$$\tanh \epsilon = \frac{1}{c \tanh \eta}. \quad (\text{B16})$$

In terms of ϵ , (B13) becomes

$$\tanh \epsilon < \frac{p}{\sqrt{m^2 + p^2}} \quad (\text{B17})$$

and (B9) becomes

$$N \leq \frac{\sinh(\epsilon - \beta)}{\sinh \epsilon}, \quad (\text{B18})$$

where we have written $N := -1/(u \cdot \chi)$. Since $\beta > 0$, (B18) shows that $N < 1$. By (B6) we then have

$$\frac{1}{\cosh \eta} < N < 1. \quad (\text{B19})$$

Solving (B18) for β gives $0 < \beta \leq \beta_c$, where

$$e^{-\beta_c} = e^{-\epsilon} \left(N \sinh \epsilon + \sqrt{(N \sinh \epsilon)^2 + 1} \right). \quad (\text{B20})$$

For fixed ϵ , β_c is strictly decreasing in N . From (B19) we then have $\beta_c < \beta_e$, where

$$e^{-\beta_e} = e^{-\epsilon} \left(\frac{\sinh \epsilon}{\cosh \eta} + \sqrt{\left(\frac{\sinh \epsilon}{\cosh \eta} \right)^2 + 1} \right). \quad (\text{B21})$$

Collecting, we have $0 < \beta < \beta_e$.

From here on we consider the cases $\mu > 0$ and $\mu = 0$ in turn.

a. $\mu > 0$

Suppose $\mu > 0$. We write $q = \mu \sinh \psi$, where $\psi > \eta$ by (B11).

If $m > 0$, we write $p = m \sinh \theta$. (B15) then becomes

$$\tanh \beta = \frac{(m/\mu) \sinh \theta + c^{-1} \sinh \psi}{(m/\mu) \cosh \theta + \cosh \psi}. \quad (\text{B22})$$

From (B13) it follows that $\tanh \theta > c^{-1}$, and using this, an elementary analysis of (B22) shows that

$$\tanh \beta > c^{-1} \tanh \psi. \quad (\text{B23})$$

If instead $m = 0$, (B15) becomes

$$\tanh \beta = \frac{(p/\mu) + c^{-1} \sinh \psi}{(p/\mu) + \cosh \psi}, \quad (\text{B24})$$

leading again to (B23). We conclude that (B23) holds for all $m \geq 0$.

Starting from (B23), we have

$$c > \frac{\tanh \psi}{\tanh \beta} > \frac{\tanh \eta}{\tanh \beta_e}, \quad (\text{B25})$$

where the first inequality is (B23) while the second inequality uses $\psi > \eta$ and $0 < \beta < \beta_e$. Using (B16) and (B21), it can be shown that the rightmost expression in (B25) is equal to

$$c + \frac{\sqrt{c^2 - 1}}{\cosh \eta}, \quad (\text{B26})$$

which is strictly greater than c , in contradiction with (B25). This completes the proof for $\mu > 0$.

b. $\mu = 0$

Suppose $\mu = 0$.

If $m > 0$, we write again $p = m \sinh \theta$. (B15) then becomes

$$\tanh \beta = \frac{(m/q) \sinh \theta + c^{-1}}{(m/q) \cosh \theta + 1}. \quad (\text{B27})$$

As $\tanh \theta > c^{-1}$, an elementary analysis of (B27) shows that

$$\tanh \beta > c^{-1}. \quad (\text{B28})$$

If instead $m = 0$, (B15) becomes

$$\tanh \beta = \frac{(p/q) + c^{-1}}{(p/q) + 1}, \quad (\text{B29})$$

leading again to (B28). We conclude that (B28) holds for all $m \geq 0$.

It follows that (B25) is replaced by

$$c > \frac{1}{\tanh \beta} > \frac{1}{\tanh \beta_e}. \quad (\text{B30})$$

As the rightmost expression in (B30) is strictly greater than the rightmost expression in (B25), a contradiction again follows. This completes the proof for $\mu = 0$. \square

APPENDIX C: PROOF OF THE PERPETUAL MOTION ADMISSION CRITERION

In this Appendix we give the proof of the perpetual motion admission criterion stated in Sec. V of the main text. The notation is as in Appendix B. As stated in the assumptions of the admission criterion, any B -particles involved in the process are assumed to be massless.

1. Proof

The geometric preliminaries of Appendix B hold with one exception: since the $-\chi \cdot \chi < 1$ assumption has been dropped, the inequality $-u \cdot \chi < \cosh \eta$ in (B6) need no longer hold.

Consider the splitting event. k_a^A and k_a^B are given by (B10), where now $\mu = 0$ by assumption. The B -ejectum reaches infinity iff $q > 0$, and the energy extraction inequality is (B13), implying in particular that $p > 0$.

If Σ is a massless B -particle, k_a^Σ is a positive multiple of $u_a - s_a$, where the coefficient of s_a is negative since Σ by assumption comes from the infinity. Similarly, if Σ is a massless A -particle, k_a^Σ is a positive multiple of $u_a - cs_a$. Both of these cases are however inconsistent with the s_a -projection of the momentum conservation equation (B7). Hence Σ is a massive A -particle, and k_a^Σ is given by (B8) with $M > 0$.

The momentum conservation equation (B7) becomes (B14) with $\mu = 0$. For $m > 0$, momentum conservation is hence equivalent to (B14b) and (B27), and the energy extraction condition is $\tanh \theta > c^{-1}$; similarly, for $m = 0$, momentum conservation is equivalent to (B14b) and (B29), and the energy extraction condition is $p > 0$. An elementary analysis shows that a solution with some $m \geq 0$ exists iff

$$\tanh \beta > c^{-1}. \quad (\text{C1})$$

For given β satisfying (C1), the range of m for which solutions exist always includes $m = 0$.

To see the geometric meaning of (C1), recall that the four-velocity vector of Σ at the splitting event is $v_{\Sigma}^a := \cosh \beta u^a + c^{-1} \sinh \beta s^a$, as can be seen by raising the index of k_b^{Σ} with $(g^A)^{ab}$ and normalizing. As $(u^a, c^{-1}s^a)$ is a normalised frame in the A -metric, (C1) says that v_{Σ}^a points to the right of u^a by an A -metric boost whose velocity is greater than c^{-1} .

To summarize: energy extraction occurs for some values of the A -ejectum mass if and only if Σ is a massive A -particle and (C1) holds at the splitting event. Geometrically, (C1) says that the four-velocity vector of Σ at the splitting event points to the right of u^a by an A -metric boost whose velocity is greater than c^{-1} .

We next need to examine under what conditions Σ makes it to the ergoregion. Recall that $-g_{ab}^A \chi^a \chi^b \rightarrow 1$ at the infinity, and $-g_{ab}^A \chi^a \chi^b$ changes sign at the ergosurface. It follows that $K := \sqrt{\sup(-g_{ab}^A \chi^a \chi^b)}$ exists and satisfies $1 \leq K < \infty$. The A -metric geodesic equation shows that Σ makes it to the ergoregion iff its initial speed $v \geq 0$ at the infinity satisfies $1/\sqrt{1-v^2} > K$, except that when $-g_{ab}^A \chi^a \chi^b < 1$ everywhere, Σ makes it to the ergoregion also for $v = 0$. The geodesic equation further shows that increasing v makes the four-velocity vector of Σ more left-pointing in the ergoregion. It follows that the threshold case for energy extraction is when the momentum covector of the Σ -trajectory approaches $M(Ku_a - \sqrt{K^2 - 1}c s_a)$ at the infinity. Killing energy conservation on the threshold case trajectory gives $-\chi^a k_a^A = MK$, which by (B3) and (B8) takes the form

$$\begin{aligned} K &= -(u \cdot \chi)(\cosh \beta - c \tanh \eta \sinh \beta) \\ &= -(u \cdot \chi) \frac{\sinh(\epsilon - \beta)}{\sinh \epsilon}, \end{aligned} \quad (\text{C2})$$

where in the second equality ϵ is given by (B16) and we have used the same rearrangement as in (B18). As the last expression in (C2) is strictly decreasing in β , the condition (C1) shows that energy extraction occurs iff

$$K < -(u \cdot \chi) \frac{(1 - \tanh \eta)}{\sqrt{1 - c^{-2}}}, \quad (\text{C3})$$

where we have noted that $\cosh \beta \rightarrow 1/\sqrt{1 - c^{-2}}$ and $\sinh \beta \rightarrow c^{-1}/\sqrt{1 - c^{-2}}$ as $\tanh \beta \rightarrow c^{-1}$.

To express (C3) in terms of invariant quantities, we note from (B3) that

$$-\chi \cdot \chi = (u \cdot \chi)^2 (1 - \tanh^2 \eta). \quad (\text{C4})$$

Suppose that (C3) holds. As $\tanh \eta > c^{-1}$, (C3) implies

$$K \sqrt{\frac{c+1}{c-1}} < -u \cdot \chi, \quad (\text{C5})$$

and eliminating η with the help of (C4) and using (C5) shows that the vector

$$V^a := \chi^a - K \sqrt{1 - c^{-2}} u^a \quad (\text{C6})$$

is B -timelike. Conversely, if (C5) holds and V^a is B -timelike, direct algebra using (C4) shows that (C3) holds.

This completes the proof. \square

2. Comments

We end with two comments.

First, the B -timelikeness of V^a says geometrically that the four-velocity vector of Σ and the æther are related by an A -boost whose speed exceeds c^{-1} , but the B -timelikeness of V^a on its own does not specify the direction of the boost. The condition (C5) implies $K < -u \cdot \chi$, which by (C2) implies that $\beta > 0$, which says that the four-velocity vector of Σ points to the right of the æther. The only role of (C5) is hence to establish the relative orientation of the two vectors. (C5) could therefore be replaced by any weaker condition that does the same, such as $K < -u \cdot \chi$.

Second, an elementary analysis of (C3) and (C4) shows that there exist configurations in which $-\chi \cdot \chi > 1$ somewhere in the ergoregion but (C3) nevertheless does not hold, not even with $K = 1$. This shows that the perpetual motion exclusion criterion in the main text cannot be sharp at least when the processes are assumed to involve no massive B -particles.

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