Revisiting the Birkhoff theorem from a dual null point of view

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The Birkhoff theorem is a well-known result in general relativity, and it is used in many applications. However, its most general version, due to Bona, is almost unknown and presented in a form less accessible to the relativist and cosmologist community. Moreover, many wield it mistakenly as a simple transposition of Newton's iron sphere theorem. In the present work, we propose a modern, dual null, presentation— useful in many explorations, including black holes—of the theorem that renders accessible most of the results of Bona's version. In addition, we discuss the fluid contents admissible for the application of the theorem, beyond a vacuum, and we demonstrate how the formalism greatly simplifies solving the dynamical equations and allows one to express the solution as a power expansion in r. We present a family of solutions that share the properties predicted by the Birkhoff theorem and discuss the existence of trapped and antitrapped regions. The formalism manifestly shows how the type of region—trapped or untrapped—determines the character of the Killing vector.

I. INTRODUCTION

The Birkhoff theorem [1–4] states that the vacuum spherically symmetric solutions are static and independent of changes in the matter distribution sourcing the gravitational field, provided the latter changes preserve the spherical symmetry. It is often referred as the general relativistic counterpart of Newton's iron sphere theorem [5,6], yet one should be wary that this is justified only when one is dealing with the gravitational field in a vacuum (where the case of a cosmological constant is included). Indeed, as pointed out in Ref. [7], Birkhoff's theorem is commonly misinterpreted as determining only the gravitational field inside a spherically symmetric matter distribution by its enclosed mass, while the static thin spherical shell surrounding a spherical central object initially proposed by Ref. [8] demonstrates that the intermediate vacuum region's gravity depends also on the outer shell's mass. We may speculate that the similarity between the field equations of general relativity (GR) in the case of spherical symmetry and the Newtonian equations for a central field induces this misunderstanding. However, this equivocated procedure oversees the nonlinearity of GR, which distinguishes it markedly from Newtonian gravity.

In popular textbooks such as Hobson and Lasenby [9], the presentation of the theorem is, for short, that the only vacuum solution with spherical symmetry is Schwarzschild's (although the original formulation was that the only vacuum solution with spherical symmetry is static [10]). Physically, the Birkhoff theorem implies that if a spherically symmetric star undergoes strictly radial pulsations, then it cannot propagate any disturbance into the surrounding space [11]. This is obviously related to the fact that the lowest multipolar radiation that propagates in general relativity is quadrupole radiation.

In the present work, we consider the Birkhoff theorem and discuss it from a formulation particularly fruitful for the exploration of causal structures, in particular, of dynamical black holes, based on the behavior of the expansion of null congruences. This so-called dual null formalism is of great interest, as it is particularly adequate and useful to deal with dynamical black holes, and underlies many recent results

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regarding both the thermodynamics of black holes and more general cosmological settings [13–18].

From an observational point of view, all the information we get from the Universe reaches us through null paths [19–21]. In fact, both electromagnetic radiation and the recently detected gravitational waves [22,23] travel on null congruences, and hence the dual null formalism, being developed from the consideration of null vectors, presents itself as a particularly appropriate tool to connect theoretical discussions and an understanding of the observable Universe. Although a known result, the understanding of the Birkhoff theorem can benefit from those modern tools.

We show in this work that the dual null formalism allows us to extend the Birkhoff theorem to more general geometric frameworks, such as planar or cylindrically symmetric spacetimes, as well as ADS/CFT settings [24–26]. Although the generalization of the Birkhoff theorem to the latter geometrical cases has been previously obtained in the literature—see Stephani *et al.* [27] and references therein—this fact is widely ignored and was derived in a different way in the present work. Moreover, besides characterizing naturally the admissible matter models that are compatible with the theorem, the dual null formalism allows us to find all the solutions for sources that can be expressed as a power series on r in a simple way. Finally, the dual null formalism manifestly shows that the character of the theorem's additional Killing vector, time-like or spacelike, naturally follows from the type of region it applies, trapped or untrapped.

The outline of the present work is as follows. In Sec. II, we briefly review the literature in connection with the Birkhoff theorem. This will enable us to situate our work with regard to the alternative approaches to its derivation as well as to some of the efforts pursued in the literature to generalize it. In Sec. III, we present the dual null formalism used in this work and develop the new proof of the Birkhoff theorem. In Sec. IV, we discuss the symmetry requirements that are actually needed and obtain the most general admissible matter models which are compatible with the theorem. Finally, we give a brief discussion of our results in Sec. VI.

A quick remark on the notation: In most instances, we use the abstract index notation as in Wald's textbook [28]. However, we swap to the intrinsic mathematical notation (without indices) when it is convenient. The translation from the two notations can be readily made by the use of the base vectors and 1-forms. For a vector V^a we write $V^a = V^\mu \partial_\mu = \partial_V$ and for a 1-form $\omega_a = \omega_\mu dx^\mu$. If $\omega_a = \partial_a f$, for a scalar function f, then $\omega = df = \partial_\mu f\, dx^\mu$.

II. GENERAL FORMULATIONS OF THE BIRKHOFF THEOREM

A generalized and geometrically minded version of the Birkhoff theorem was put forward by Goenner [29], pointing out that the theorem relies on the existence of a three-parameter group of (global) isometries with two-dimensional non-null orbits and of an additional Killing vector associated with a G_4 group of motions [30]. The Birkhoff theorem for spherically symmetric vacuum solutions and the Taub theorem for plane-symmetric vacuum solutions were both generalized to vacuum solutions with conformal symmetries. In particular, it was proved that any conformally spherically (respectively, plane-) symmetric vacuum solution to the Einstein equations must be the Schwarzschild (respectively, either Taub-Kasner or flat) solution.

Upgraded versions of the theorem can be found in Refs. [27,29,31,32], and the most evolved phrasing for this geometric approach is due to Bona [33], for metrics that are conformally reducible, that is, $g = Y^2 \hat{g}$, where $\hat{g}$ is reducible as the metric of a direct product spacetime. Let

$$ds^2 = Y^2(x^C)(\gamma_{AB} dx^A dx^B + h_{\alpha\beta} dy^\alpha dy^\beta), \quad (1)$$

where $h_{\alpha\beta}$ and y^α are a two-dimensional metric and a coordinate system, respectively, on the two-dimensional orbits O_2 of G_3. Analogously, γ_{AB} and x^A the corresponding metric and coordinates on V_2, which is the orthogonal submanifold to O_2 according to g. Bona's statement of the Birkhoff theorem is

(i) *Theorem* [33].—Metrics with a group G_3 of motions on non-null orbits O_2 and with Ricci tensors of type [(11)(1,1)] and [(111,1)] admit a group G_4 provided that $dY \neq 0$,

emphasizing the requirement $dY \neq 0$ and the appropriate Segré types [33].

Other attempts at generalizing the Birkhoff theorem can be found in the literature. Generalization to higher dimensions was achieved by Bronnikov and Melnikov [34], and a thorough discussion on the relationship between manifold dimensionality and the existence of Birkhoff-like theorems was made by Schmidt [35]. Goswami and Ellis [36–38] have investigated the possibility of extending it by analyzing whether the theorem remains approximately true both for an approximately spherical vacuum solution [36] and also for an approximately vacuum configuration [37]. They resort to the analysis of perturbations with the $1 + 1 + 2$ formalism developed by Clarkson [39]. The difficulties associated with this pragmatic line of research stem from the need to remain in the neighborhood of the vacuum spherically symmetric models and, thus, of defining the conditions that guarantee the existence of such a neighborhood.

Following a diverse path, Hernández-Pastora [40] pursued an attempt to get a relationship between the spherical symmetry and the multipole structure of the so-called monopole solution.

The Birkhoff theorem was also investigated in connection with conformal rescaling [41], with the possibility of extending it to modified theories of gravity [42–44], with

different hypotheses, as in Ref. [45]—where the key condition (for Bona) $dY \neq 0$ is abandoned and the theorem still applies under some additional conditions on the matter sources—and with regard to many other features [46–49].

III. BIRKHOFF THEOREM IN DUAL NULL FORMALISM

In this section, we briefly present the main tools of the dual null formalism and apply them to prove and discuss the necessary conditions for the validity of the Birkhoff theorem. It is worth pointing out that the dual null formalism is distinct from operating in null coordinates, as it deals with optical scalars related to null congruences. Such quantities are independent of coordinate choice and can be analyzed in any coordinate set. Null coordinates are useful in order to represent and compute more simply the relevant quantities of the dual null formalism, and we take advantage of this in the following.

A. Spherically symmetric spacetimes and dual null formalism

In dual null coordinates, any spherically symmetric metric can be parametrized as

$$ds^2 = -e^f(du\,dv + dv\,du) + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (2)$$

where $f = f(u, v)$, $r = r(u, v)$, and we omit the tensor product symbol $\otimes$ for short. Metric (2) is of the form (1) for $Y = r(u, v)$, $\gamma_{AB}dx^A\,dx^B = -\frac{e^f}{r^2}du\,dv$, and $h_{\alpha\beta}dy^\alpha\,dy^\beta = d\theta^2 + \sin^2\theta\,d\phi^2$.

The coordinates in Eq. (2) are also a codimension-two foliation of the spacetime. The orbits of the G_3 group, here the group of rotations in three dimensions, are two-dimensional spheres corresponding to O_2. Each two-dimensional sphere is characterized by the pair (u, v) that are the coordinates on V_2.

The null coordinates on V_2 are not unique. Hence, by making a coordinate change of the form $(u, v) \rightarrow (U, V)$:

$$u \rightarrow U(u), \qquad v \rightarrow V(v), \quad (3)$$

with $U'(u) > 0$ and $V'(v) > 0$ for all u, v, in order to not reverse the orientation of the new coordinates. We obtain a new pair of dual null coordinates:

$$ds^2 = -e^{F(U,V)}(dU\,dV + dV\,dU) + r^2(U, V)d\Omega^2, \quad (4)$$

with

$$F(U, V) = f(u(U), v(V)) - \ln U'(u(U)) - \ln V'(v(V)). \quad (5)$$

Let k^a be a null vector field orthogonal to the orbits of the coordinates θ and ϕ everywhere in the spacetime.

We define its expansion $\Theta_{(k)}$ as the relative variation of the area form on the orthogonal spheres when transported along the integral curves of k^a:

$$\Theta_{(k)} = \frac{\mathcal{L}_k(r^2\sqrt{\det h})}{r^2\sqrt{\det h}} = \frac{2}{r}k^a\partial_a r, \quad (6)$$

where $\mathcal{L}_k$ is the Lie derivative with respect to k^a. Using the coordinate base vectors ∂_u and ∂_v, we define the two null expansions related to our coordinates in Eq. (2):

$$\Theta_{(u)} = \frac{2}{r}\partial_u r, \qquad \Theta_{(v)} = \frac{2}{r}\partial_v r. \quad (7)$$

The null expansions transform under (3) as

$$\Theta_{(u)} \rightarrow U'(u)\Theta_{(u)}, \qquad \Theta_{(v)} \rightarrow V'(v)\Theta_{(v)}. \quad (8)$$

We see that the value of the null expansions depends on the coordinate choice, but their sign and the locus where they vanish are not. Based on this, we may classify each sphere in the spacetime as

  (i) regular, normal or untrapped, if $\Theta_u\Theta_v < 0$;
  (ii) trapped or future trapped, if $\Theta_u\Theta_v > 0$ and $\Theta_u < 0$;
  (iii) antitrapped or past trapped, if $\Theta_u\Theta_v > 0$, and $\Theta_u > 0$; or
  (iv) marginal, if $\Theta_u\Theta_v = 0$.

This classification has been an important tool in the study of black hole physics, especially in the case of dynamical solutions (see, for example, Refs. [13,50,51], and references therein, for motivation and applications of this formalism).

Let the basis forms related to the coordinates (u, v) be denoted du and dv. Since $\Theta_{(u)}du$ is invariant under a change of coordinates [52], we may build a 1-form $\mathcal{K}_a$ called the mean curvature form as

$$\mathcal{K}_a = \Theta_{(u)}\partial_a u + \Theta_{(v)}\partial_a v, \quad (9)$$

where $\partial_a u$ and $\partial_a v$ are the abstract index notation version of du and dv, respectively.

With the aid of the mean curvature form, we are able to express simply the null expansion respective to any null vector field by just contracting it to $\mathcal{K}_a$:

$$\Theta_{(k)} = k^a\mathcal{K}_a, \quad (10)$$

for any k^a null and orthogonal to O_2. We may also generalize the definition for any vectors in V_2, be it time- or spacelike, by defining what we call the 2-expansion, in order to distinguish it from the usual expansion defined as the divergence of timelike vector fields, as was made in Ref. [18] in order to deal with the separation between a collapse and a cosmological expansion (see [53–63]) and as a tool to define dynamical universal horizons in Ref. [64]

(see [65–74]). Let be X^a any vector orthogonal to the orbits O_2; then its 2-expansion, denoted $\Theta_{(X)}$, is defined as

$$\Theta_{(X)} = X^a \mathcal{K}_a. \tag{11}$$

For the expansions of the null coordinate basis, the Raychaudhuri equations are written as

$$\mathcal{L}_u \Theta_{(u)} - \Theta_{(u)} \partial_u f + \frac{\Theta_{(u)}^2}{2} + R_{uu} = 0, \tag{12a}$$

$$\mathcal{L}_v \Theta_{(v)} - \Theta_{(v)} \partial_v f + \frac{\Theta_{(v)}^2}{2} + R_{vv} = 0. \tag{12b}$$

The uv component of the Einstein tensor may be written in terms of the null expansions as

$$G_{uv} = \mathcal{L}_v \Theta_{(u)} + \Theta_{(u)} \Theta_{(v)} + \frac{e^f}{r^2}. \tag{13}$$

Note that, since ∂_u and ∂_v are coordinate base vectors, they commute, and then $\mathcal{L}_u \Theta_{(v)} = \mathcal{L}_v \Theta_{(u)}$.

Equations (12) and (13) together with Einstein's equation

$$G_{ab} = T_{ab} \tag{14}$$

for a given energy-momentum tensor T_{ab} capture the full dynamics of the problem and completely determine a spherically symmetric solution.

B. Properties of vacuum spacetimes

Until this point, the only hypothesis made on the spacetime was spherical symmetry. In this section, we also assume that it satisfies Einstein's equation in a vacuum in an open domain $\mathcal{D}$, of the form $\mathcal{D}_2 \times O_2$, where $\mathcal{D}$ is the image under the coordinate map (u, v) of an open domain of $\mathbb{R}^2$. This domain $\mathcal{D}$ can be described as a spherical shell with finite thickness that lasts for some finite time interval.

On $\mathcal{D}$, $R_{ab} = G_{ab} = 0$. The full dynamics are determined in terms of the null expansions by the three equations below:

$$\mathcal{L}_u \Theta_{(u)} - \Theta_{(u)} \partial_u f + \frac{\Theta_{(u)}^2}{2} = 0, \tag{15a}$$

$$\mathcal{L}_v \Theta_{(v)} - \Theta_{(v)} \partial_v f + \frac{\Theta_{(v)}^2}{2} = 0, \tag{15b}$$

$$\mathcal{L}_v \Theta_{(u)} + \Theta_{(u)} \Theta_{(v)} + \frac{e^f}{r^2} = 0. \tag{15c}$$

Using Eqs. (15), we can deduce several general results valid for vacuum solutions that we present in the following.

Proposition III.1.—Given the hypotheses above, there exists a pair of dual null coordinates (U, V) such that $|\Theta_{(U)}(U, V)| = |\Theta_{(V)}(U, V)|$ in $\mathcal{D}$.

Proof.—We can rewrite Eq. (15a) as

$$\partial_u \Theta_{(u)} - \Theta_{(u)} \partial_u f + \frac{\Theta_{(u)}^2}{2} = 0 \Rightarrow$$

$$r^{-1} \partial_u (\Theta_{(u)} r) - \Theta_{(u)} \partial_u f = 0 \Rightarrow$$

$$\frac{\partial_u (\Theta_{(u)} r)}{\Theta_{(u)} r} = \partial_u f \Rightarrow$$

$$r\Theta_{(u)} = C_1(v) e^f, \tag{16}$$

where $C_1(v)$ is an arbitrary nonvanishing function that comes from the integration in u. Repeating the same procedure with Eq. (15b), we obtain

$$r\Theta_{(v)} = C_2(u) e^f, \tag{17}$$

where $C_2(u)$ is also an arbitrary function. For any functions C_1 and C_2, we can make a gauge transformation $(u, v) \rightarrow (U, V)$ of the form Eq. (3) with the choice

$$U(u) = \int^u |C_2(s)| ds,$$

$$V(v) = \int^v |C_1(s)| ds, \tag{18}$$

noting that $U'(u)$, $V'(v) > 0$, for all u, v, in order to have a well-behaved coordinate transformation. We obtain

$$r\Theta_{(U)} = |C_2(u)| C_1(v) e^f,$$

$$r\Theta_{(V)} = |C_1(v)| C_2(u) e^f. \tag{19}$$

Dividing Eqs. (19) by each other, we obtain the wished result.   □

This result shows that there exists one special set (U, V) of dual null coordinates in vacuum spherically symmetric spacetimes for which the two null expansions have the same absolute value at each event on $\mathcal{D}$. Note that this special set of dual null coordinates is unique up to a constant rescaling.

The next proposition shows that this special pair of null coordinates is useful to reduce the dynamical equations to equations on only one independent coordinate.

Proposition III.2.—Let $\Theta_{(U)} + \Theta_{(V)} = 0$ $(\Theta_{(U)} - \Theta_{(V)} = 0)$ and the new coordinates $\chi_\pm = \frac{1}{2}(U \pm V)$. We denote with $\partial_\pm$ the derivatives with respect to $\chi_\pm$.

Then

(i) $\partial_+ \Theta_{(U)} = \partial_+ \Theta_{(V)} = 0$ $(\partial_- \Theta_{(U)} = \partial_- \Theta_{(V)} = 0)$.

(ii) $\partial_+ r(U, V) = 0$ $[\partial_- r(U, V) = 0]$.

(iii) If $\Theta_{(U)} \neq 0$, then $\partial_+ f = 0$ $(\partial_- f = 0)$.

(iv) If $\Theta_{(U)} + \Theta_{(V)} = 0$ $(\Theta_{(U)} - \Theta_{(V)} = 0)$, then ∂_+ (∂_-) is a Killing vector.

Proof.—First, we remark that $\partial_\pm = \partial_U \pm \partial_V$.

(i) Let $\mathcal{O} = \Theta_{(U)} = \mp \Theta_{(V)}$. Since ∂_U and ∂_V commute, we have

$$\partial_V \Theta_{(U)} = \partial_U \Theta_{(V)} \Rightarrow$$
$$\partial_V \mathcal{O} = \mp \partial_U \mathcal{O} \Rightarrow (\partial_U \pm \partial_V) \mathcal{O} = 0, \quad (20)$$

where the sign choice depends directly on the choice in $\Theta_V \pm \Theta_U = 0$.

(ii) Consider the case $\Theta_{(U)} + \Theta_{(V)} = 0$, the other case being similar. By Eq. (7), $\mathcal{O} = \partial_U \ln r^2 = -\partial_V \ln r^2$. Considering that ∂_+ commute with both ∂_U and ∂_V and item (i), we have

$$0 = \partial_+ \partial_U \ln r^2 = \partial_U \partial_+ \ln r^2 \Rightarrow \partial_+ \ln r^2 = C_1(V),$$
$$(21)$$

$$0 = \partial_+ \partial_V \ln r^2 = \partial_V \partial_+ \ln r^2 \Rightarrow \partial_+ \ln r^2 = C_2(U).$$
$$(22)$$

Since $C_1(V) = C_2(U)$, they are constant. We can then write

$$\ln r^2 = C\chi_+ + H(\chi_-), \quad (23)$$

with H an arbitrary function and C an arbitrary constant. Computing the expansions for U and V using Eq. (23), we obtain

$$\Theta_{(U)} = \frac{C + H'}{2} = -\frac{C - H'}{2} = -\Theta_{(V)} \Rightarrow C = 0,$$
$$(24)$$

which implies that the null expansions do not depend on χ_+.

(iii) Adding Eq. (15a) and minus (15b) and writing the expansion in terms of $\mathcal{O}$, we obtain

$$(\partial_U \pm \partial_V)\mathcal{O} - \mathcal{O}(\partial_U \pm \partial_V)f = 0. \quad (25)$$

If $\Theta_U \pm \Theta_V$, the first term vanishes by item (i). Therefore, if $\mathcal{O} \neq 0$, then

$$(\partial_U \pm \partial_V)f = 0. \quad (26)$$

(iv) We denote by $\chi_\pm^a = \frac{\partial x^a}{\partial \chi^\pm}$ the components of $\partial_\pm$ in the coordinates system x^a. Then

$$\mathcal{L}_{\chi_\pm} g_{ab} = \chi_\pm^c \partial_c g_{ab} + g_{ac} \partial_b \chi_\pm^c + g_{cb} \partial_a \chi_\pm^c$$
$$= \partial_\pm g_{ab} = 0, \quad (27)$$

as the functions in the metric components, namely, $f(U, V)$ and $r(U, V)$, do not depend on $\chi_\pm$. □

In the next proposition, we relate the classification of the spacetime region with the character of the Killing field.

Proposition III.3.—If

(i) $\Theta_{(U)} = -\Theta_{(V)} \neq 0$, ∂_+ is a timelike Killing vector field;

(ii) $\Theta_{(U)} = \Theta_{(V)} \neq 0$, ∂_- is a spacelike Killing vector field.

Proof.—

(i) From Proposition III.2, in this case $\partial_+ = \partial_U + \partial_V$ is a Killing vector field. Then

$$g_{ab}\chi_+^a \chi_+^b = -2e^f < 0. \quad (28)$$

(ii) Analogously,

$$g_{ab}\chi_-^a \chi_-^b = 2e^f > 0. \quad (29)$$

□

Propositions III.2 and III.3 imply that spherically symmetric vacuum spacetimes are also static [75], provided the region is *regular* or *untrapped*, that is, the null expansions have opposite sign. Case (ii) shows that in *trapped regions*, where both null expansions have the same sign, an additional Killing vector field still exists, but it is spacelike and the region is not static but spatially homogeneous.

Our construction also shows that the Killing field is always orthogonal to the orbits O_2 and the isometry it generates commutes with the O_2 rotations.

IV. BEYOND VACUUM AND SPHERICAL SYMMETRY

We used the dual formalism under the hypothesis that the spacetime is spherically symmetric such that we used a codimension-two foliation of the spacetime using the spheres corresponding to the orbits of the action of SO(3). We also assumed that it was a vacuum solution.

Since Bona proved the Birkhoff theorem under weaker hypotheses [33], in this section we explore better the conditions necessary in order to prove it under our formalism.

In the proofs above, the only relevant equations were the Raychaudhuri equation for the null congruences. If we can weaken the hypotheses while keeping Eqs. (15a) and (15b) unchanged, we will obtain a stronger version of our result.

A. Discussing the symmetry condition

The consequence of spherical symmetry in Raychaudhuri equations is the fact that the shear and vorticity of the null congruences must vanish, which implies that the evolution of the expansions depends only on themselves.

In order to guarantee the vanishing of the shear and vorticity of null congruences, we can replace spherical symmetry by any maximal symmetry for two-dimensional manifolds. Therefore, we may replace the hypothesis of spherical symmetry with the statement that $h_{\alpha\beta}$ must have constant curvature, which includes planar and hyperbolic symmetries on $h_{\alpha\beta}$. The most general line element that preserves our equations is

$$ds^2 = -e^{f(u,v)}(du\,dv + dv\,du) + r^2(u,v)h_{\alpha\beta}dy^\alpha\,dy^\beta, \quad (30)$$

where

$$h_{\alpha\beta}dy^\alpha\,dy^\beta = d\theta^2 + S_\epsilon^2(\theta)d\phi^2, \quad (31)$$

where $\theta \in (0, \infty)$ and
  (i) $S_1(\theta) = \sin\theta$, for spherical symmetry;
  (ii) $S_0(\theta) = \theta$, for planar symmetry; and
  (iii) $S_{-1}(\theta) = \sinh\theta$, for hyperbolic symmetry.
  This change leaves Eqs. (12a) and (12b) invariant, while Eq. (13) becomes

$$G_{uv} = \mathcal{L}_v\Theta_u + \Theta_u\Theta_v + \frac{\epsilon e^f}{r^2}, \quad (32)$$

where ϵ take the values 1, 0, or -1, corresponding to the spherical, planar, or hyperbolic symmetry, respectively.

This is equivalent to Bona's wording in terms of the G_3 group of symmetry with two-dimensional orbits O_2, with the difference that in Bona's paper [33] the only exigence on O_2 is that it is non-null. In our case, since we use a dual null basis on V_2, O_2 must be spacelike. If the orbits O_2 are Lorentzian, the orthogonal vector space to the orbits is spacelike; therefore, the optical focusing equations we are using cannot be applied. In this sense, our formalism is less general than Bona's.

We could use a formalism similar to prove the Birkhoff theorem in that case, by using the corresponding focusing equations for spacelike geodesics. However, as spacelike geodesics are much less interesting under the physical point of view than the null cones, and one of the objectives of this work is to discuss the physical meaning of the hypotheses of the Birkhoff theorem, we will not pursue in this direction.

B. Discussing the vacuum condition

The vacuum condition has the only effect of making Raychaudhuri equations homogeneous, since $R_{uu} = R_{vv} = 0$.

Therefore, we should determine the broadest class of energy-momentum tensors—or, equivalently, Ricci tensors—that produces the same result.

Since $\partial_u \equiv u^a\partial_a$ and $\partial_v \equiv v^a\partial_a$ are null, the vanishing of the uu component of the Ricci tensor is equivalent to $R_a^b u^a = \lambda_u(u,v)u^b$ and analogously to ∂_v. Therefore, ∂_u

and ∂_v are two null eigenvectors of the Ricci tensor. Since $g_{uv} = u^a v_a = -e^f \neq 0$, their respective eigenvalues λ_u and λ_v coincide:

$$\lambda_v(v^b u_b) = (R_a^b v^a)u_b$$
$$= (R_a^b u^a)v_b = \lambda_u u^b v_b. \quad (33)$$

If the two null basis vectors are eigenvectors with the same eigenvalue, we have that $\partial_u \pm \partial_v$ also are eigenvectors with λ_u as their eigenvalue. This shows that the condition of vanishing R_{uu} and R_{vv} is equivalent to imposing that the Ricci tensor have a timelike and a spacelike eigenvector in V_2, with the same eigenvalue.

As the induced metric in the symmetric orbits is of constant curvature, this implies that the restriction of R_{ab} to the subspace tangent to the orbits is proportional to the metric itself, that is,

$$R_{AB} = R(u,v)g_{AB}, \quad (34)$$

for $A, B \in \{\theta, \phi\}$. This implies that the Ricci tensor has two linearly independent eigenvectors w^a and z^a, with the same eigenvalue. The space spanned by w^a and z^a is orthogonal to ∂_u and ∂_v, therefore tangent to the orbits of the angular coordinates. This means we have $R_{uu} = R_{vv}$ for Ricci tensors of the Segre type [(1,1)(11)] (two pairs of double eigenvalues) or [(111,1)] (one quadruple eigenvalue). This is the same hypothesis for the Ricci tensor used in the generalized version by Bona.

C. Admissible matter models

By Einstein's equations, the Segre type of the Ricci tensor corresponds to the Segre type of the energy-momentum tensor. Therefore, it is worth determining the most general matter model that satisfies the requirements for the application of the Birkhoff theorem.

The most general T_{ab} with two pairs of double eigenvectors and presenting the symmetry requirements may be written as

$$T_{ab} = \lambda_1 r^2\gamma_{ab} + \lambda_2 r^2 h_{ab} \Rightarrow$$
$$T_{ab} = \lambda_1(-2e^f\partial_{(a}u\partial_{b)}v) + \lambda_2(r^2 h_{ab}). \quad (35)$$

Defining a new basis

$$n^a = \frac{e^{-f/2}}{2}[u^a + v^a],$$

$$e^a = \frac{e^{-f/2}}{2}[u^a - v^a], \quad (36)$$

which satisfy

$$n^a n_a = -1, \qquad e^a e_a = 1, \quad (37)$$

we have

$$T_{ab}n^a n^b = -\lambda_1, \qquad T_{ab}e^a e^b = \lambda_1, \qquad (38)$$

which leads to

$$T_{ab} = -\lambda_1 n_a n_b + \lambda_1 e_a e_b + \lambda_2 r^2 h_{ab}. \qquad (39)$$

We may interpret Eq. (39) as the energy-momentum tensor of a fluid with energy density $-\lambda_1$ and anisotropic pressure, with value λ_1 in the direction orthogonal to the orbits O_2 and value λ_2 tangent to it. If we apply the weak energy condition, then $\lambda_1 < 0$, which means that the fluid must have *negative pressure* in the e^a direction.

An important feature of the energy-momentum tensor in (39) is that the flow velocity n^a is not uniquely defined, as a boost transformation of the form

$$n'^a = \cosh \omega n^a + \sinh \omega e^a,$$
$$e'^a = \cosh \omega e^a + \sinh \omega n^a \qquad (40)$$

for arbitrary ω preserves its form. This means that there exists a one-parameter family of observers, with different velocities, that are "comoving" to the fluid. This is a vacuumlike property, and in Ref. [45] this kind of fluid is called a Dminikova vacuum, or D vacuum.

A particularly simple realization of matter of this form corresponds to $\lambda_1 = \lambda_2 = -\Lambda$, where the energy-momentum tensor correspond to a cosmological constant (in this case, the Segre type is [(1,111)]).

Another case of interest is the presence of a non-null electromagnetic field F_{ab}. In the absence of charges and radiation [76], the energy-momentum tensor may be written as

$$T_{ab} = \frac{1}{2}(E^2 + B^2)[n_a n_b - e_a e_b + r^2 h_{ab}], \qquad (41)$$

which corresponds to a Segre type [(1,1)(11)], for $\lambda_2 = -\lambda_1 = \frac{E^2 + B^2}{2}$.

We see that the most known cases where the Birkhoff theorem is usually applied in the literature are quite particular, as they correspond to $\lambda_1 = \pm\lambda_2$. In the next section, we will solve Einstein's equation for a general matter model satisfying the above conditions.

In general, we may represent an energy-momentum tensor of the Segre types required as

$$T^{ab} = \lambda_1 g^{ab} + (\lambda_2 - \lambda_1)r^{-2}h^{ab}. \qquad (42)$$

The energy-momentum conservation is written

$$0 = \nabla_a T^{ab} = g^{ab}\partial_a \lambda_1 + r^{-2}h^{ab}\partial_a(\lambda_2 - \lambda_1)$$
$$+ (\lambda_2 - \lambda_1)\nabla_a(r^{-2}h^{ab}). \qquad (43)$$

If $\lambda_2 = \lambda_1$, this implies $\partial_a \lambda_1 = 0$, which means that the eigenvalue must be constant, as is well known for the

cosmological constant. Therefore, $\lambda_2 \neq \lambda_1$ is necessary in order to obtain models with varying λ_1. We will see in the next section that only λ_1 appears directly in Einstein's equations, but λ_2 affects implicitly the solution, as it is related to λ_1 according to Eq. (43).

A thorough presentation of the field Lagrangians that produce this type of energy-momentum tensors can be found in Ref. [32]. Another important remark concerns the interpretation of the results in terms of matter models: This analysis is equally valid for extra terms in Einstein equations provided by modified gravity theories.

D. Summarizing our results

We conclude this section by stating the generalized version of the Birkhoff theorem in our language.

Theorem IV.1.—Let g be a metric tensor of a spacetime that admits a codimension-two foliation of the form Eq. (30) where h is two-dimensional Riemannian metric tensor, induced on the two-dimensional spacelike leaves of the foliation.

If h has constant curvature, $\Theta_u \neq 0$, as defined in Eq. (7), and the energy-momentum tensor has the form given in Eq. (39), then g has an additional isometry generated by a Killing vector χ orthogonal to the leaves of the foliation.

In addition, if the spacetime region considered is regular, then χ is timelike and the metric is static. If the spacetime region is trapped or antitrapped, then χ is spacelike and the metric is homogeneous.

V. SOLVING THE EQUATIONS

In this section, we aim to determine the solutions that satisfy the theorem, by using the tools we have already prepared. We have to consider two types of solutions: those for regular or untrapped regions, where $\Theta_U + \Theta_V = 0$, and those for trapped regions, corresponding to $\Theta_U - \Theta_V = 0$.

A. Regular regions

With no loss of generality, we assume $\mathcal{O} = \Theta_U > 0$ and $\Theta_V = -\mathcal{O} < 0$.

It is useful to remark that in this case $\partial_+ r = 0$, which means that the vectors ∂_+ and ∂_r are orthogonal, which implies that ∂_- is proportional to ∂_r. Indeed, it is straightforward to verify that

$$\mathcal{O} = \frac{1}{r}\partial_- r \Rightarrow \partial_- = r\mathcal{O}\partial_r, \qquad (44)$$

wherever $\partial_- r \neq 0$.

We must revisit Eq. (19) and note that, by redefining f using the transformation in Eq. (5), all the functions on the right-hand side are in the exponential term. Since the coordinates U and V are unique up to a rescaling transformation, we can set the proportionality constant as 2, and then

$$rO = 2e^f, \tag{45}$$

which allows us to write the line element as

$$ds^2 = -\frac{rO}{2}(dU\,dV + dV\,dU) + r^2 h_{\alpha\beta}dx^\alpha\,dx^\beta. \tag{46}$$

Also, according to Proposition III.2, no metric component depends on χ^+, and its basis vector is orthogonal to ∂_r, which makes the pair of coordinates (χ^+, r) a natural choice to describe the solution. Using Eq. (6) and the definition of χ^+, we obtain

$$dr = \frac{rO}{2}(dU - dV),$$
$$d\chi^+ = \frac{1}{2}(dU + dV), \tag{47}$$

which leads to

$$ds^2 = -rO d\chi^{+2} + \frac{dr^2}{rO} + r^2 h_{\alpha\beta}dx^\alpha\,dx^\beta. \tag{48}$$

Now, we have only to solve the equations for $O(r)$. Considering a matter model of the form Eq. (39), applying the Einstein equations Eq. (14) to Eq. (13), and considering Eq. (45), we have

$$\frac{1}{2}(\partial_V\Theta_U + \partial_U\Theta_V) + \Theta_U\Theta_V + \epsilon\frac{e^f}{r^2} = -\lambda_1 e^f \Rightarrow$$
$$\frac{1}{2}(\partial_V - \partial_U)O - O^2 + \frac{O}{2}\left(\frac{\epsilon}{r} + \lambda_1 r\right) = 0 \Rightarrow$$
$$-\frac{1}{2}\partial_- O - O^2 + \frac{O}{2}\left(\frac{\epsilon}{r} + \lambda_1 r\right) = 0. \tag{49}$$

Using Eq. (44), we are able to write a differential equation with respect to r:

$$r\partial_r O + 2O = \left(\frac{\epsilon}{r} + \lambda_1(r)r\right) \Rightarrow$$
$$\partial_r(r^2 O) = (\epsilon + \lambda_1(r)r^2) \Rightarrow$$
$$rO = \epsilon + \frac{b}{r} + \frac{1}{r}\int^r \lambda_1(s)s^2\,ds. \tag{50}$$

In order to gain a better insight on our family of solutions, we consider the case where $\lambda_1(r)$ admits a representation as a sum or a series of powers of r, provided it is uniformly convergent on $\mathcal{D}$:

$$\lambda_1(r) = -\sum_i c_i r^i. \tag{51}$$

Then, we can integrate it term by term and find

$$rO = \epsilon + \frac{b}{r} - \frac{c_{-3}\ln r}{r} - \sum_{i\neq-3}\frac{c_i r^{i+2}}{i+3}. \tag{52}$$

The most common sources studied in black hole physics are particular cases of our model. The cosmological constant is equivalent to $c_0 = \Lambda$, the electrostatic central field to $c_{-4} = q^2$. We also see that in this case the solutions behave "linearly": The addition of sources, provided they satisfy the requirements of the Birkhoff theorem, corresponds to the addition of a respective term in the solution. It is worth noticing that, in the Newtonian limit, $rO \sim 1 + 2\Phi$, where Φ is the potential and Eq. (50) is the relativistic analog of the Poisson equation. As in the Newtonian case, the solution (52) must satisfy boundary conditions at the innermost and outermost radius of the domain $\mathcal{D}$, including $r \to 0$ and $r \to \infty$ as possible cases. However, as explicitly shown in Ref. [7], the boundary conditions at the outermost radius may affect physics in $\mathcal{D}$, in opposition to the result of Newtonian gravity.

Another fact of interest is that all those solutions are of Petrov type D, meaning that the geometry corresponds only to the Coulombian part of the gravitational field. This is expected, since the high degree of symmetry of those solutions eliminates any form of gravitational radiation term.

For the spherical solutions, we are able to compute the Misner-Sharp mass [50] of the general solution as

$$M = \frac{r}{2}(1 - g^{ab}\partial_a r\partial_b r). \tag{53}$$

From the line element in Eq. (48), we have $g^{ab}\partial_a r\partial_b r = rO$. Using the general solution Eq. (52), we have

$$M = \frac{1}{2}\left(-b + c_{-3}\ln r + \sum_{i\neq-3}\frac{c_i r^{i+3}}{i+3}\right), \tag{54}$$

which allows us to identify that the integration constant $b = -2m$, where m is the central mass, as it is the component of the total Misner-Sharp mass which is independent of the radius and corresponds to the Schwarzschild mass in the absence of sources. The other terms give the energy contribution of each kind of source.

B. Trapped and antitrapped regions

Those regions correspond to $\Theta_U = \Theta_V = O$. Trapped regions present $O < 0$, and antitrapped regions have $O > 0$. We follow the changes in the equations we presented for regular regions. In this case $\partial_- r = 0$, and then we replace Eq. (44) by

$$\frac{1}{r}\partial_+ r = O \Rightarrow \partial_+ = rO\partial_r. \tag{55}$$

Equation (45) becomes

$$rO = \pm 2e^f, \tag{56}$$

because we need to include the case where $\mathcal{O} < 0$. The line element in Eq. (46) becomes

$$ds^2 = -\frac{r|\mathcal{O}|}{2}(dU\,dV + dV\,dU) + r^2 h_{\alpha\beta}dy^\alpha\,dy^\beta. \quad (57)$$

We replace Eqs. (47) by

$$dr = \frac{r\mathcal{O}}{2}(dU + dV),$$
$$d\chi^+ = \frac{1}{2}(dU - dV). \quad (58)$$

The line element in the coordinates (χ^-, r) is given by

$$ds^2 = r|\mathcal{O}|d\chi^{-2} - \frac{dr^2}{r|\mathcal{O}|} + r^2 h_{\alpha\beta}dy^\alpha\,dy^\beta. \quad (59)$$

We notice that, in trapped regions, the coordinate r is timelike and the corresponding metric element is negative, as expected from Proposition III.3 along with the fact that $\partial_- r = 0$.

Equation (49) becomes

$$\frac{1}{2}\partial_+\mathcal{O} + \mathcal{O}^2 \pm \frac{\mathcal{O}}{2}\left(\frac{\epsilon}{r} + \lambda_1 r\right) = 0, \quad (60)$$

where the $+$ sign correspond to antitrapped regions and the $-$ sign to trapped regions. Changing the χ^+ coordinate to r, using Eq. (55), we obtain

$$\frac{1}{2}r\partial_r\mathcal{O} + \mathcal{O} \pm \left(\frac{\epsilon}{r} + \lambda_1 r\right) = 0, \Rightarrow$$
$$r\mathcal{O} = \mp\left(\epsilon + \frac{b}{r} + \frac{1}{r}\int^r \lambda_1(s)s^2\,ds\right). \quad (61)$$

Therefore, the only difference in the case of trapped and antitrapped regions lies in the character of the Killing vector and of the r coordinate. The absolute value of the metric components coincide.

Notice that the metric solutions we found fail to cover the marginal surfaces that correspond to $\mathcal{O} = 0$. The three-dimensional locus defined by the marginal surfaces is an apparent (or trapping) horizon [13], which is the boundary between trapped and untrapped regions. While our choice of coordinates $(\chi^\pm, r)$ makes use of the symmetry of the problem in order to simplify its resolution, the metrics in Eqs. (48) and (59) have the Schwarzschild form in usual coordinates, and the marginal surfaces correspond to coordinate singularities. A coordinate system that covers both sides of those marginal surfaces is easily built by

known methods as, for instance, the definition of an Eddington-Finkelstein-like system of coordinates. This analysis leads to the known fact that the Killing field is null on a marginal surface.

VI. CONCLUSION

We have shown how to obtain the Birkhoff theorem from the dual null formalism, naturally relating the result with the type of region considered, if regular or trapped. Only in a regular region does the theorem lead to static solutions.

The formalism has also enabled us to prove a very general version of the Birkhoff theorem, coming short of being completely general in that we did not consider symmetries with timelike orbits as done by Bona [33]. However, we have obtained general matter sources for which the theorem is valid, and thus, with the aid of dual null formalism, we found all the solutions for sources that can be expressed as a power series on r.

The Birkhoff theorem is much invoked in the literature in relation to the idea that, given a spherically symmetric distribution of matter, the gravitational physics at some given value of the radial coordinate depends only on the overall mass of the distribution inside that radius. This is, of course, not true, in general, and a clear counterexample is provided by the well-known Lemaître-Tolman-Bondi dust solution [77], in which the gravitational physics, at some spherical shell, depends not only on the integrated Misner-Sharp mass but also on an energy parameter that weights the spatial curvatures and the initial energy conditions. Other misuses have been discussed in Ref. [7]. We thus believe that the present work is transparent and useful in making it absolutely clear what is the scope of applicability of the Birkhoff theorem in general relativity and also as a guide for the investigation of analogous results in modified gravity theories.

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