

General Poincaré gauge theory: Hamiltonian structure and particle spectrum

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Basic aspects of the Hamiltonian structure of the parity-violating Poincaré gauge theory are studied. We found all possible primary constraints, identified the corresponding critical parameters, and constructed the generic form of the canonical Hamiltonian. In addition to being important in their own right, these results offer dynamical information that is essential for a proper understanding of the particle spectrum of the theory, calculated in the weak field approximation around the Minkowski background.

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I. INTRODUCTION

Weyl's idea of gauge invariance [1] turned out to be a key principle underlying the dynamical structure of all the fundamental physical interactions. Following this idea and the subsequent works of Yang, Mills, and Utiyama [2], Kibble and Sciama [3] formulated a new theory of gravity, the Poincaré gauge theory (PG, aka PGT), based on gauging the Poincaré group of spacetime symmetries. In PG, spacetime is characterized by a Riemann-Cartan geometry, in which the torsion and curvature are the field strengths associated with the translation and Lorentz subgroups of the Poincaré group; for more details, see [4–10].

Earlier investigations of PG were mostly focused on the class of *parity preserving* Lagrangians quadratic in the field strengths; see, for instance, Hayashi and Shirafuji [5] or Obukhov [11]. We denote this class of models as PG^+ . Sezgin and Niuwenhuizen [12] analyzed the particle spectrum of PG^+ in the weak field approximation around the Minkowski background M_4 . Using the absence of ghosts and tachyons as physical requirements, they found a number of restrictions on the PG^+ parameters that ensure the propagating torsion modes to be well behaved.

General dynamical aspects of PG^+ , including the identification of its physical degrees of freedom (d.o.f.), are most naturally understood in Dirac's Hamiltonian approach for constrained dynamical systems [13]. Blagojević and Nikolić [14,15] started a systematic Hamiltonian analysis of PG^+ , focusing on its generic aspects. They identified a subset of the primary constraints that are always present (“sure” constraints, associated to the local Poincaré symmetry). Moreover, if certain critical parameters vanished, they found additional primary constraints (“if-constraints”),

constructed the total Hamiltonian, and discussed certain aspects of the consistency procedure. Further advances in this direction were made by Cheng *et al.* [16] and Chen *et al.* [17], who found that the nonlinear nature of constraints may drastically change the number of propagating modes obtained in the linearized analysis. Yo and Nester [18] made a detailed study of this phenomenon in PG^+ , concluding that there are apparently only two good propagating torsion modes. For an interesting application of this result to cosmology, see Shie *et al.* [19].

There are no physical arguments that favor the conservation of parity in the gravitational interaction. *Parity violating models* based on the general PG, with all possible quadratic invariants in the Lagrangian, were considered already in the 1980s [20], but the subject remained without wider response. Recently, there has been increased interest in a better understanding of both the basic structure and various dynamical aspects of these models, including cosmological applications and wave solutions [21–27]. In particular, one should mention the analysis of the particle spectrum carried out by Karananas [25], who made a suitable extension of the weak field approximation method used earlier in PG^+ [12] and applied it to the general PG. According to his results, it seems that the set of good modes that can coexist is significantly enlarged in comparison to PG^+ .

The objective of the present work is to examine the Hamiltonian structure of the general PG, based on the if-constraint formalism [14,15,18], and use it to clarify the physical content of its particle spectrum, calculated in the weak field approximation around M_4 . In this regard, a particularly important role is played by both the critical parameters appearing in the analysis of the primary constraints, and the structure of the canonical Hamiltonian. By comparing the properties of the particle spectrum to those found in Ref. [25], we noted certain differences. On the other side, elements of the Hamiltonian structure developed

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here can be a good starting point for studying the nonlinear dynamics of PG.

The paper is organized as follows. In Sec. II, we give a short account of the Lagrangian formalism for the general PG. In Secs. III and IV, we find the canonical critical parameters, identify the related if-constraints, and construct the generic, “most dynamical” canonical Hamiltonian, determined by the nonvanishing critical parameters. Then, in Secs. V and VI, we derive the linearized gravitational field equations and use them to identify the mass eigenvalues of the torsion modes. The conditions for the absence of ghosts and tachyons, as well as the reality conditions of the mass eigenvalues, are examined in Sec. VII. Essential features of the particle spectrum are either tested by or derived from the Hamiltonian structure of PG. In contrast to the results obtained in [25], we show that the two spin-2 torsion modes cannot propagate simultaneously. In Sec. VIII, we give a short summary of our results, and six appendices contain useful technical details, including an outline of the Hamiltonian formalism describing the case of vanishing critical parameters.

Our conventions are as follows. The Latin indices (i, j, \dots) are the local Lorentz indices, the Greek indices (μ, ν, \dots) are the coordinate indices, and both run over 0,1,2,3; the orthonormal frame (tetrad) is $b^i{}_\mu$, the inverse tetrad is $h_i{}^\mu$, the Lorentz connection is $\omega^{ij}{}_\mu$, and $\eta_{ij} = (1, -1, -1, -1)$ and $g_{\mu\nu} = \eta_{ij}b^i{}_\mu b^j{}_\nu$ are the metric components in the local Lorentz and coordinate frame, respectively; a totally antisymmetric tensor ε_{ijkl} is normalized to $\varepsilon_{0123} = +1$, and the dual of an antisymmetric tensor X_{ij} is $\star X_{ij} = (1/2)\varepsilon_{ij}{}^{mn}X_{mn}$.

II. LAGRANGIAN FORMALISM

In this section, we give a short account of the Lagrangian formalism for the general parity-violating PG. Basic dynamical variables are the tetrad field $b^i = b^i{}_\mu dx^\mu$ and the antisymmetric spin connection $\omega^{ij} = \omega^{ij}{}_\mu dx^\mu = -\omega^{ji}$ (1-forms), which represent the gauge potentials associated with translations and Lorentz transformations, respectively. The corresponding field strengths are the torsion and the curvature (2-forms),

$$\begin{aligned} T^i &:= db^i + \omega^i{}_k \wedge b^k = \frac{1}{2} T^i{}_{\mu\nu} dx^\mu \wedge dx^\nu, \\ R^{ij} &:= d\omega^{ij} + \omega^i{}_k \wedge \omega^{kj} = \frac{1}{2} R^{ij}{}_{\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \quad (2.1)$$

which satisfy the Bianchi identities

$$\nabla T^i = R^i{}_k \wedge b^k, \quad \nabla R^{ij} = 0. \quad (2.2)$$

The underlying spacetime continuum is described by Riemann-Cartan geometry [7–9].

A. Field equations

The PG dynamics is determined by a Lagrangian $L = L_M + L_G$, where L_M describes matter and its interaction with gravity, and L_G is the pure gravitational part. In the framework of tensor calculus, the gravitational field equations in vacuum are obtained by varying the action $I_G = \int d^4x L_G(b^i{}_\mu, T_{ijk}, R_{ijkl})$ with respect to $b^i{}_\mu$ and $\omega^{ij}{}_\mu$. After introducing the covariant gravitational momenta

$$H_i{}^{\mu\nu} := \frac{\partial L_G}{\partial T^i{}_{\mu\nu}}, \quad H_{ij}{}^{\mu\nu} := \frac{\partial L_G}{\partial R^{ij}{}_{\mu\nu}}, \quad (2.3a)$$

and the associated energy-momentum and spin currents

$$E_i{}^\nu := \frac{\partial L_G}{\partial b^i{}_\nu}, \quad E_{ij}{}^\mu := \frac{\partial L_G}{\partial \omega^{ij}{}_\mu}, \quad (2.3b)$$

the gravitational field equations take a compact form:

$$(1\text{ST}) \quad \mathcal{E}_i{}^\nu := -\frac{\delta L_G}{\delta b^i{}_\nu} = \nabla_\mu H_i{}^{\mu\nu} - E_i{}^\nu = 0, \quad (2.4a)$$

$$(2\text{ND}) \quad \mathcal{E}_{ij}{}^\nu := -\frac{\delta L_G}{\delta \omega^{ij}{}_\nu} = \nabla_\mu H_{ij}{}^{\mu\nu} - E_{ij}{}^\nu = 0. \quad (2.4b)$$

The explicit expressions for the energy-momentum and spin currents are given by

$$\begin{aligned} E_i{}^\nu &= h_i{}^\nu L_G - T^m{}_{ki} H_m{}^{k\nu} - \frac{1}{2} R^{mn}{}_{ki} H_{mn}{}^{k\nu}, \\ E_{ij}{}^\mu &= -2H_{[ij]}{}^\mu. \end{aligned} \quad (2.5)$$

In the presence of matter, the right-hand sides of (2.4a) and (2.4b) contain the corresponding matter currents.

B. Quadratic PG models

We assume the Lagrangian density L_G to contain all possible quadratic invariants, constructed out of the three irreducible components of the torsion and the six irreducible components of the curvature (Appendix A). Relying on the Lagrangian 4-form given in Ref. [27], one finds that the corresponding Lagrangian density has the form $L_G = b\mathcal{L}_G$, where $b := \det(b^i{}_\mu)$ and

$$\begin{aligned} \mathcal{L}_G &= -(a_0 R + 2\Lambda_0) - \bar{a}_0 X \\ &+ \frac{1}{2} T^{ijk} \sum_{n=1}^3 (a_n {}^{(n)}T_{ijk} - \bar{a}_n \star {}^{(n)}T_{ijk}), \\ &+ \frac{1}{4} R^{ijkl} \sum_{n=1}^6 (b_n {}^{(n)}R_{ijkl} - \bar{b}_n \star {}^{(n)}R_{ijkl}). \end{aligned} \quad (2.6)$$

Here, the irreducible components of the field strengths are defined in Appendix A, the parity even and parity odd

sectors are described by the parameters (a_n, b_n, Λ_0) and (\bar{a}_n, \bar{b}_n) , respectively, and the star symbol denotes the duality operation with respect to the frame indices of the field strengths. Another form of \mathcal{L}_G , useful for comparison with the literature, is given in Appendix B. Knowing \mathcal{L}_G , one finds that the covariant momentum densities (2.3a) can be written in the form $H_{imn} = b\mathcal{H}_{imn}$ and $H_{ijmn} = b\mathcal{H}_{ijmn}$, where

$$\begin{aligned}\mathcal{H}_{imn} &= 2 \sum_{n=1}^3 (a_n {}^{(n)}T_{imn} - \bar{a}_n {}^*T_{imn}), \\ \mathcal{H}_{ijmn} &= {}^L\mathcal{H}_{ijmn} + \mathcal{H}'_{ijmn},\end{aligned}\quad (2.7a)$$

and

$$\begin{aligned}{}^L\mathcal{H}_{ijmn} &= -2a_0(\eta_{im}\eta_{jn} - \eta_{jm}\eta_{in}) + 2\bar{a}_0\epsilon_{ijmn}, \\ \mathcal{H}'_{ijmn} &= 2 \sum_{n=1}^6 (b_n {}^{(n)}R_{ijmn} - \bar{b}_n {}^*{}^{(n)}R_{ijmn}).\end{aligned}\quad (2.7b)$$

C. On the choice of Lagrangian parameters

In the Lagrangian (2.6), the two parity sectors are presented in a very symmetric way, but the set of three identities (A3a) implies that not all of the parameters (\bar{a}_n, \bar{b}_n) are independent. To resolve this issue, we choose the conditions

$$\bar{a}_2 = \bar{a}_3, \quad \bar{b}_2 = \bar{b}_4, \quad \bar{b}_3 = \bar{b}_6, \quad (2.8)$$

which reduce the number of Lagrangian parameters to $21 - 3 = 18$. Note that the above conditions are not unique.

Further freedom in the choice of parameters follows from the existence of three topological invariants. The Euler and Pontryagin invariants are defined by the 4-forms

$$I_E := R^{ij} \wedge R^{mn} \epsilon_{mnij}, \quad I_P := R^{ij} \wedge R_{ij}, \quad (2.9a)$$

respectively, whereas the third invariant is based on the Nieh-Yan identity,

$$I_{NY} := T^i \wedge T_i - R_{ij} \wedge b^i \wedge b^j \equiv d(b^i \wedge T_i). \quad (2.9b)$$

These invariants produce vanishing contributions to the field equations, which implies that not all of the Lagrangian parameters are dynamically independent. Indeed, they can be used to eliminate three more terms from the Lagrangian, leaving us with the final number of $18 - 3 = 15$ independent parameters; see Ref. [23]₃ for more details. In this paper, we use only the conditions in Eq. (2.8), allowing thereby for an easier comparison to the literature.

For a clear understanding of the physical content of PG, it is convenient to use dimensionless parameters (coupling constants). The Lagrangian parameters in Eq. (2.6) are not

dimensionless, but the transition to their dimensionless counterparts can be easily realized by suitable rescalings; see, for instance, Ref. [27]. However, to make the general exposition simpler and more compact, we find it useful to keep the Lagrangian parameters in the form Eq. (2.6), which corresponds to using the units $c = \hbar = 2\kappa = 1$. The true dimensionless parameters can be reintroduced later whenever needed.

III. PRIMARY CONSTRAINTS

Hamiltonian structure is by itself a particularly important aspect of PG as a gauge theory [13]. Moreover, it also offers dynamical information that is essential for a proper understanding of the particle spectrum of PG.

We begin the subject by analyzing the primary constraints (PC) of PG. The canonical momenta $(\pi_i^\mu, \Pi_{ij}^\mu)$ associated to the basic Lagrangian variables $(b^i_\mu, \omega^{ij}_\mu)$ are

$$\pi_i^\mu := \frac{\partial L_G}{\partial(\partial_0 b^i_\mu)} = b\mathcal{H}_i^{0\mu}, \quad \Pi_{ij}^\mu := \frac{\partial L_G}{\partial(\partial_0 \omega^{ij}_\mu)} = b\mathcal{H}_{ij}^{0\mu}. \quad (3.1)$$

Since the field strengths do not depend on the velocities $\partial_0 b^i_0$ and $\partial_0 \omega^{ij}_0$, the above relations define ten constraints that are always present in the theory (“sure” PCs), regardless of the values of the coupling constants. They read

$$\pi_i^0 \approx 0, \quad \Pi_{ij}^0 \approx 0, \quad (3.2)$$

and their existence is directly related to the local Poincaré symmetry of PG.

Before we proceed, let us note that at each point of a spatial hypersurface $\Sigma_0: x^0 = \text{const}$, one can define the unit timelike vector \mathbf{n} , normal to Σ_0 . Then, any spacetime vector V_k can be decomposed into a component V_\perp along \mathbf{n} and a component $V_{\bar{k}}$ in the tangent space of (“parallel” to) Σ_0 ; that is, $V_k = n_k V_\perp + V_{\bar{k}}$, where $V_\perp = n^k V_k$ and $n^k V_{\bar{k}} = 0$ (Appendix C).

To find additional constraints that may appear in Eq. (3.1), it is useful to define the parallel gravitational momenta

$$\hat{\pi}_{i\bar{k}} := \pi_i^\alpha b_{k\alpha} = J\mathcal{H}_{i\perp\bar{k}}, \quad (3.3a)$$

$$\hat{\Pi}_{ij\bar{k}} := \Pi_{ij}^\alpha b_{k\alpha} = J\mathcal{H}_{ij\perp\bar{k}}, \quad (3.3b)$$

such that $\hat{\pi}_{i\bar{k}} n^k = 0$, $\hat{\Pi}_{ij\bar{k}} n^k = 0$, and J is defined by $b = NJ$, with $N = n_k b^k_0$. Depending on the values of the coupling constants, these relations may produce additional constraints (primary “if-constraints”). In analogy to the above orthogonal-parallel decomposition of a vector V_k , one can introduce a similar decomposition of the field strengths,

$$T_{ikm} = T_{i\bar{k}\bar{m}} + (n_k T_{i\perp\bar{m}} + n_m T_{i\bar{k}\perp}) = \bar{T}_{ikm} + \mathcal{T}_{ikm}, \quad (3.4a)$$

$$R_{ijkm} = R_{ij\bar{k}\bar{m}} + (n_k R_{ij\perp\bar{m}} + n_m R_{ij\bar{k}\perp}) = \bar{R}_{ijkm} + \mathcal{R}_{ijkm}. \quad (3.4b)$$

It is very useful for further analysis to know that the parallel components $\bar{T}_{ikm} := T_{i\bar{k}\bar{m}}$ and $\bar{R}_{ijkm} := R_{ij\bar{k}\bar{m}}$ are independent not only of the “velocities” $T_{i\perp\bar{m}}, R_{ij\perp\bar{m}}$, but also of the unphysical variables (b^i_0, ω^{ij}_0) ; for more details, see Refs. [7,14,15].

A. Torsion sector

The torsion piece of the Lagrangian in Eq. (2.6) depends on the velocities $\partial_0 b^i_\alpha$ only through $T_{i\perp\bar{k}}$. The linearity of $\mathcal{H}_{i\perp\bar{k}}$ in \bar{T} and \mathcal{T} allows us to rewrite Eq. (3.3a) in the form

$$\phi_{i\bar{k}} := \frac{\hat{\pi}_{i\bar{k}}}{J} - \mathcal{H}_{i\perp\bar{k}}(\bar{T}) = \mathcal{H}_{i\perp\bar{k}}(\mathcal{T}), \quad (3.5)$$

where all possible velocity terms are moved to the right-hand side. Now, we decompose this equation into irreducible parts with respect to the group of rotations in Σ_0 (Appendix C):

$${}^S\phi := \frac{\hat{\pi}_{\bar{k}\bar{k}}}{J} + \bar{a}_2 \varepsilon^{\bar{k}\bar{m}\bar{n}} T_{\bar{k}\bar{m}\bar{n}} = -2a_2 T_{\bar{k}\perp}, \quad (3.6a)$$

$$\begin{aligned} \phi_{\perp\bar{k}} &:= \frac{\hat{\pi}_{\perp\bar{k}}}{J} + \frac{2}{3}(a_1 - a_2) T_{\bar{m}\bar{k}}^{\bar{m}} + \frac{1}{3}(2\bar{a}_1 + \bar{a}_2) \varepsilon_{\bar{k}}^{\bar{m}\bar{n}} T_{\perp\bar{m}\bar{n}} \\ &= \frac{2}{3}(2a_1 + a_2) T_{\perp\perp\bar{k}} + \frac{2}{3}(\bar{a}_1 - \bar{a}_2) \varepsilon_{\bar{k}}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp}, \end{aligned} \quad (3.6b)$$

$$\begin{aligned} {}^A\phi_{i\bar{k}} &:= \frac{\hat{\pi}_{i\bar{k}}}{J} - \frac{2}{3}(a_1 - a_3) T_{\perp i\bar{k}} - \frac{1}{3}(\bar{a}_1 + 2\bar{a}_3) \varepsilon_{i\bar{k}}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}}^{\bar{m}} \\ &= -\frac{2}{3}(a_1 + 2a_3) T_{[\bar{k}\perp]} - \frac{2}{3}(\bar{a}_1 - \bar{a}_2) \varepsilon_{i\bar{k}}^{\bar{m}\bar{n}} T_{\perp\perp\bar{n}}, \end{aligned} \quad (3.6c)$$

$$\begin{aligned} {}^T\phi_{i\bar{k}} &:= \frac{\hat{\pi}_{i\bar{k}}}{J} + \bar{a}_1 \left[\varepsilon_{i\bar{k}}^{\bar{m}\bar{n}} T_{\bar{k}\bar{m}\bar{n}} - \frac{1}{3} \eta_{i\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}} T_{\bar{k}\bar{m}\bar{n}} \right] \\ &= -2a_1 T_{i\bar{k}\perp}. \end{aligned} \quad (3.6d)$$

Here, the set $({}^S\phi, \phi_{\perp\bar{k}}, {}^A\phi_{i\bar{k}}, {}^T\phi_{i\bar{k}})$, defined by the scalar, vector, antisymmetric, and traceless-symmetric parts of $\phi_{i\bar{k}}$, represents the set of all possible new constraints. The mechanism by which these if-constraints become true constraints is simply explained in the parity even case, characterized by four critical parameters: $a_2, (2a_1 + a_2), (a_1 + 2a_3)$, and a_1 . When some of these parameters vanish, the corresponding velocity terms on the right-hand sides of

(3.6) also vanish, and consequently, the associated if-constraints become true constraints. However, if none of the critical parameters vanishes, there are no new constraints.

The same mechanism works also in the general PG. Whereas the critical parameters for ${}^S\phi$ and ${}^T\phi_{i\bar{k}}$ remain the same as in PG⁺, a_2 and a_1 , the structure of the if-constraints $\phi_{\perp\bar{k}}$ and ${}^A\phi_{i\bar{k}}$ is more complicated, as the right-hand sides of Eqs. (3.6b) and (3.6c) depend on two velocities, $T_{[\bar{m}\bar{n}]\perp}$ and $T_{\perp\perp\bar{k}}$. To find the related critical parameters, we first transform ${}^A\phi_{i\bar{k}}$ into the axial 3-vector ${}^A\phi_{\bar{k}} := \varepsilon_{\bar{k}}^{\bar{m}\bar{n}A} \phi_{\bar{m}\bar{n}}$, so that Eq. (3.6c) goes over into

$${}^A\phi_{\bar{k}} = \frac{4}{3}(\bar{a}_1 - \bar{a}_2) T_{\perp\perp\bar{k}} - \frac{2}{3}(a_1 + 2a_3) \varepsilon_{\bar{k}}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp}. \quad (3.7)$$

Then, the set of equations involving $(\phi_{\perp\bar{k}}, {}^A\phi_{\bar{k}})$ can be written in the matrix form as

$$\begin{pmatrix} \phi_{\perp\bar{k}} \\ {}^A\phi_{\bar{k}} \end{pmatrix} = \frac{2}{3} A \begin{pmatrix} T_{\perp\perp\bar{k}} \\ \varepsilon_{\bar{k}}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp} \end{pmatrix}, \quad (3.8a)$$

where

$$A := \begin{pmatrix} 2a_1 + a_2 & \bar{a}_1 - \bar{a}_2 \\ 2(\bar{a}_1 - \bar{a}_2) & -(a_1 + 2a_3) \end{pmatrix},$$

$$\det A = -[(2a_1 + a_2)(a_1 + 2a_3) + 2(\bar{a}_1 - \bar{a}_2)^2]. \quad (3.8b)$$

If the matrix A has two distinct eigenvalues, one can construct the invertible matrix P that transforms A into a diagonal form, $D_A := P^{-1}AP$. Then, Eq. (3.8a) implies

$$\phi_{\bar{k}} := P^{-1} \begin{pmatrix} \phi_{\perp\bar{k}} \\ {}^A\phi_{\bar{k}} \end{pmatrix} = \frac{2}{3} D_A P^{-1} \begin{pmatrix} T_{\perp\perp\bar{k}} \\ \varepsilon_{\bar{k}}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp} \end{pmatrix}, \quad (3.9)$$

where the column $\phi_{\bar{k}}$ represents two diagonalized if-constraints, and the diagonal elements of D_A are the critical parameters,

$$c_{\pm}(A) = \frac{1}{2} \left(\text{tr} A \pm \sqrt{(\text{tr} A)^2 - 4 \det A} \right). \quad (3.10)$$

More details on this construction can be found in Appendix D. In general, the number of true constraints in Eq. (3.9) is equal to the number of vanishing critical parameters.

- (i) The critical parameters of the torsion sector are $a_2, c_{\pm}(A)$, and a_1 .

B. Curvature sector

For the curvature sector, we use the linearity of $\mathcal{H}'_{ij\perp k}$ in \bar{R} and \mathcal{R} to rewrite Eq. (3.3b) in the form

$$\Phi_{ij\bar{k}} := \frac{\Pi_{ij\bar{k}}}{J} - \mathcal{H}_{ij\perp\bar{k}}(\bar{R}) = \mathcal{H}'_{ij\perp\bar{k}}(\mathcal{R}). \quad (3.11)$$

The content of the object $\Phi_{ij\bar{k}}$ is described by two three-dimensional (3d) tensors, $\Phi_{ij\bar{k}} = (\Phi_{\perp j\bar{k}}, \Phi_{\bar{i} j\bar{k}})$. The irreducible decomposition of $\Phi_{\perp j\bar{k}}$ takes the form defined in (C4):

$$\begin{aligned} {}^S\Phi &\equiv \frac{\Pi_{\perp\bar{k}}}{J} + 6a_0 + \frac{1}{2}(b_4 - b_6)\underline{R} + \frac{1}{2}(\bar{b}_2 + \bar{b}_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}} \\ &= (b_4 + b_6)R_{\perp\perp} - \frac{1}{2}(\bar{b}_2 - \bar{b}_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp}, \end{aligned} \quad (3.12a)$$

$$\begin{aligned} {}^A\Phi_{\perp j\bar{k}} &\equiv \frac{{}^A\Pi_{\perp j\bar{k}}}{J} + (b_2 - b_5)\underline{R}_{j\bar{k}} - \frac{1}{2}(\bar{b}_2 + \bar{b}_5)({}^A\varepsilon_j^{\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}}) \\ &= (b_2 + b_5)R_{\perp j\perp\bar{k}} - \frac{1}{2}(\bar{b}_2 - \bar{b}_5)({}^A\varepsilon_j^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\bar{k}\perp}), \end{aligned} \quad (3.12b)$$

$$\begin{aligned} {}^T\Phi_{\perp j\bar{k}} &\equiv \frac{{}^T\Pi_{\perp j\bar{k}}}{J} + (b_1 - b_4)R_{j\bar{k}} + \frac{1}{2}(\bar{b}_1 + \bar{b}_2)({}^T\varepsilon_j^{\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}}) \\ &= (b_1 + b_4)R_{\perp j\perp\bar{k}} - \frac{1}{2}(\bar{b}_1 - \bar{b}_2)({}^T\varepsilon_j^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}). \end{aligned} \quad (3.12c)$$

The irreducible parts of $\Phi_{\bar{i} j\bar{k}} = -\Phi_{j\bar{i}\bar{k}}$ are the pseudoscalar, the vector, and the traceless symmetric part; see (C5):

$$\begin{aligned} {}^P\Phi &\equiv \frac{{}^P\Pi}{J} + 12\bar{a}_0 + (b_2 - b_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}} - (\bar{b}_1 + 2\bar{b}_2 + \bar{b}_3)\underline{R} \\ &= -(b_2 + b_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp} - 2(\bar{b}_2 - \bar{b}_3)R_{\perp\perp}, \end{aligned} \quad (3.13a)$$

$$\begin{aligned} {}^V\Phi_{\bar{i}} &\equiv \frac{{}^V\Pi_{\bar{i}}}{J} - (b_4 - b_5)R_{\perp\bar{i}} + \frac{1}{2}(\bar{b}_2 + \bar{b}_5)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{i}\bar{k}\bar{m}\bar{n}} \\ &= (b_4 + b_5)R_{\bar{i}\perp} - (\bar{b}_2 - \bar{b}_5)\varepsilon_{\bar{i}}^{\bar{k}\bar{n}}R_{\perp\bar{k}\perp\bar{n}}, \end{aligned} \quad (3.13b)$$

$$\begin{aligned} {}^T\Phi_{\bar{i} j\bar{k}} &\equiv \frac{{}^T\Pi_{\bar{i} j\bar{k}}}{J} + (b_1 - b_2)R_{\perp j\bar{k}\bar{i}} - {}^T\mathcal{H}'_{\bar{i} j\perp\bar{k}}(\bar{R}) \\ &= (b_1 + b_2)R_{\bar{i} j\perp\bar{k}} - (\bar{b}_1 - \bar{b}_2)({}^T\varepsilon_{\bar{i} j}^{\bar{n}\Sigma}R_{\perp\bar{n}\perp\bar{k}}). \end{aligned} \quad (3.13c)$$

In Eqs. (3.12) and (3.13), the underlined objects do not contain velocities, $\underline{R} := R^{\bar{m}\bar{n}}_{\bar{m}\bar{n}}$ and $\underline{R}_{\bar{i} j} := R_{\bar{i}\bar{n}j}^{\bar{n}}$, the superscript Σ denotes symmetrization, and $\mathcal{H}'_{\bar{i} j\perp\bar{k}}$ in Eq. (3.13c) denotes the parity odd part of the covariant momentum,

$$\begin{aligned} {}^T\mathcal{H}'_{\bar{i} j\perp\bar{k}}(\bar{R}) &= - \left\{ \frac{1}{4}\varepsilon_{\bar{i}}^{\bar{m}\bar{n}}[(\bar{b}_1 + 2\bar{b}_2 + \bar{b}_5)R_{\bar{i} j\bar{m}\bar{n}} \right. \\ &\quad \left. + (\bar{b}_1 - \bar{b}_5)R_{\bar{m}\bar{n}\bar{i} j}] \right\}. \end{aligned}$$

Looking at the type of velocities appearing in the above equations, one can see that the critical parameters can be found by grouping these equations into suitably chosen pairs.

1. Spin-0 pair

Consider first Eqs. (3.12a) and (3.13a), which contain the same set of velocities, $R_{\perp\perp}$ and $\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp}$. They can be written in the matrix form as

$$\begin{pmatrix} {}^S\Phi \\ {}^P\Phi \end{pmatrix} = B_0 \begin{pmatrix} R_{\perp\perp} \\ \varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp} \end{pmatrix}, \quad (3.14a)$$

where

$$\begin{aligned} B_0 &:= \begin{pmatrix} b_4 + b_6 & -\frac{1}{2}(\bar{b}_2 - \bar{b}_3) \\ -2(\bar{b}_2 - \bar{b}_3) & -(b_2 + b_3) \end{pmatrix}, \\ \det B_0 &= -(b_4 + b_6)(b_2 + b_3) - (\bar{b}_2 - \bar{b}_3)^2. \end{aligned} \quad (3.14b)$$

In analogy to what we found in the previous subsection, the critical parameters are the eigenvalues of B_0 , $c_{\pm}(B_0)$, and the related column of the if-constraints reads

$${}^0\Phi := P_0^{-1} \begin{pmatrix} {}^S\Phi \\ {}^P\Phi \end{pmatrix} = D_0 P_0^{-1} \begin{pmatrix} R_{\perp\perp} \\ \varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp} \end{pmatrix}, \quad (3.15)$$

where P_0 is the matrix that diagonalizes B_0 , $D_0 = P_0^{-1}B_0P_0$.

2. Spin-1 pair

Similarly, after transforming ${}^A\Phi_{\perp\bar{i} j}$ into ${}^A\Phi_{\bar{k}} := \varepsilon_{\bar{k}}^{\bar{m}\bar{n}}\Phi_{\perp\bar{m}\bar{n}}$, Eq. (3.12b) becomes

$${}^A\Phi_{\bar{i}} = (b_2 + b_5)\varepsilon_{\bar{i}}^{\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} + (\bar{b}_2 - \bar{b}_5)R_{\bar{i}\perp}, \quad (3.16)$$

and the matrix form of Eqs. (3.16) and (3.13b) reads

$$\begin{pmatrix} {}^A\Phi_{\bar{i}} \\ {}^V\Phi_{\bar{i}} \end{pmatrix} = B_1 \begin{pmatrix} \varepsilon_{\bar{i}}^{\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} \\ R_{\bar{i}\perp} \end{pmatrix}, \quad (3.17a)$$

where

$$\begin{aligned} B_1 &:= \begin{pmatrix} b_2 + b_5 & \bar{b}_2 - \bar{b}_5 \\ -(\bar{b}_2 - \bar{b}_5) & b_4 + b_5 \end{pmatrix}, \\ \det B_1 &= (b_4 + b_5)(b_2 + b_5) + (\bar{b}_2 - \bar{b}_5)^2. \end{aligned} \quad (3.17b)$$

As before, the critical parameters are $c_{\pm}(B_1)$, and the if-constraints are determined by the matrix P_1 that diagonalizes B_1 ,

$${}^1\Phi_{\bar{i}} := P_1^{-1} \begin{pmatrix} {}^A\Phi_{\bar{i}} \\ {}^V\Phi_{\bar{i}} \end{pmatrix} = D_1 P_1^{-1} \begin{pmatrix} \varepsilon_{\bar{i}}^{\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} \\ R_{\bar{i}\perp} \end{pmatrix}. \quad (3.18)$$

3. Spin-2 pair

To find the critical parameters in the spin-2 sector, it is convenient to replace ${}^T\Phi_{i\bar{j}\bar{k}}$ with the expression ${}^T\Phi_{i\bar{k}} := T(\varepsilon_i^{\bar{m}\bar{n}}\Phi_{\bar{m}\bar{n}\bar{k}})$. Indeed, ${}^T\Phi_{i\bar{k}}$ refers to the same set of velocities that appears in Eq. (3.12c),

$${}^T\Phi_{i\bar{k}} = (b_1 + b_2)T(\varepsilon_i^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}) + 2(\bar{b}_1 - \bar{b}_2)T R_{\perp\bar{i}\perp\bar{k}}, \quad (3.19)$$

which allows Eqs. (3.12c) and (3.19) to be written in the matrix form

$$\begin{pmatrix} {}^T\Phi_{\perp\bar{j}\bar{k}} \\ {}^T\Phi_{j\bar{k}} \end{pmatrix} = B_2 \begin{pmatrix} {}^T R_{\perp\bar{j}\perp\bar{k}} \\ T(\varepsilon_j^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}) \end{pmatrix}, \quad (3.20a)$$

where

$$B_2 := \begin{pmatrix} b_1 + b_4 & -\frac{1}{2}(\bar{b}_1 - \bar{b}_2) \\ 2(\bar{b}_1 - \bar{b}_2) & b_1 + b_2 \end{pmatrix}, \quad \det B_2 = (b_1 + b_2)(b_1 + b_4) + (\bar{b}_1 - \bar{b}_2)^2. \quad (3.20b)$$

Hence, the critical parameters are $c_{\pm}(B_2)$, and the column of if-constraints has the form

$${}^2\Phi_{j\bar{k}} := P_2^{-1} \begin{pmatrix} {}^T\Phi_{\perp\bar{j}\bar{k}} \\ {}^T\Phi_{j\bar{k}} \end{pmatrix} = D_2 P_2^{-1} \begin{pmatrix} {}^T R_{\perp\bar{j}\perp\bar{k}} \\ T(\varepsilon_j^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}) \end{pmatrix}. \quad (3.21)$$

- (ii) The critical parameters in the curvature sector are $c_{\pm}(B_0)$, $c_{\pm}(B_1)$, and $c_{\pm}(B_2)$.

C. Critical parameters and if-constraints

Since the if-constraints belong to irreducible representations of 3d rotations, they are characterized by a specific spin content. Their structure is best understood by grouping them into pairs with definite spin, as shown in Table I. In this classification, the parity eigenvalues are absent since parity is not conserved.

The generic set of the critical parameters $c_{\pm}(F)$, $F = A, B_0, B_1, B_2$, is defined provided the parity odd parameters in F do not vanish; see Appendix D. Hence, the limit of the final expressions $c_{\pm}(F)$ when these parameters tend to zero is not well defined. However, since in that case F is already diagonal, one can identify c_{\pm} directly from F .

TABLE I. Critical parameters and if-constraints.

Spin	Critical parameters	If-constraints
0	$a_2, c_{\pm}(B_0)$	${}^s\phi, ({}^0\Phi)_{\pm}$
1	$c_{\pm}(A), c_{\pm}(B_1)$	$(\phi_{\bar{k}})_{\pm}, ({}^1\Phi_{\bar{k}})_{\pm}$
2	$a_1, c_{\pm}(B_2)$	${}^T\phi_{i\bar{k}}, ({}^2\Phi_{i\bar{k}})_{\pm}$

The total number of the primary if-constraints is $10 \times 3 = 30$, the same as the number of the parallel canonical momenta (3.3). The if-constraints and the associated critical parameters have a decisive influence on the structure of the canonical Hamiltonian.

IV. HAMILTONIAN

The procedure for constructing the canonical (and total) Hamiltonian in PG^+ is well known [7,14,15,18], but its extension to PG, although in principle straightforward, is technically rather complicated.

Starting with the standard definition of the canonical Hamiltonian density,

$$\mathcal{H}_c = \pi_i^{\alpha} \partial_0 b^i_{\alpha} + \frac{1}{2} \Pi_{ij}^{\alpha} \partial_0 \omega^{ij}_{\alpha} - b\mathcal{L}, \quad (4.1)$$

one can rewrite it in the Dirac-ADM form,

$$\mathcal{H}_c = N\mathcal{H}_{\perp} + N^{\alpha}\mathcal{H}_{\alpha} - \frac{1}{2} \omega^{ij}_0 \mathcal{H}_{ij} + \partial_{\alpha} D^{\alpha}, \quad (4.2)$$

where N and N^{α} are the lapse and shift functions (see Appendix C), and

$$\begin{aligned} \mathcal{H}_{ij} &= 2\pi_{[i}^{\alpha} b_{j]\alpha} + \nabla_{\alpha} \Pi_{ij}^{\alpha}, \\ \mathcal{H}_{\alpha} &= \pi_i^{\beta} T^i_{\alpha\beta} + \frac{1}{2} \pi_{ij}^{\beta} R^{ij}_{\alpha\beta} - b^k_{\alpha} \nabla_{\beta} \pi_k^{\beta}, \\ \mathcal{H}_{\perp} &= \pi_i^{\bar{k}} T^i_{\perp\bar{k}} + \frac{1}{2} \Pi_{ij}^{\bar{k}} R^{ij}_{\perp\bar{k}} - J\mathcal{L} - n^k \nabla_{\beta} \pi_k^{\beta}, \\ D^{\alpha} &= b^i_0 \pi_i^{\alpha} + \frac{1}{2} \omega^{ij}_0 \Pi_{ij}^{\alpha}. \end{aligned} \quad (4.3)$$

Since \mathcal{H}_{\perp} is the only term that depends on the form of the Lagrangian, explicit construction of the whole \mathcal{H}_c reduces just to the construction of its dynamical piece \mathcal{H}_{\perp} . In this process, we focus our attention on the ‘‘most dynamical’’ case when all the critical parameters are nonvanishing (that is, when none of the if-constraints becomes a true constraint). Such an assumption is sufficient for our study of the particle spectrum of PG. Extension of the formalism to include vanishing critical parameters is outlined in Appendix D.

A. Torsion sector

Isolating the torsion contribution to \mathcal{L}_G , one finds the corresponding part of \mathcal{H}_{\perp} ,

$$\mathcal{H}_{\perp}^T = \frac{1}{2} \phi_{i\perp\bar{k}} T^{i\perp\bar{k}} - J\bar{\mathcal{L}}_{T^2} - n^k \nabla_{\beta} \pi_k^{\beta}, \quad (4.4a)$$

where $\bar{\mathcal{L}}_{T^2} = \mathcal{L}_{T^2}(\bar{T})$ does not contain velocities. In order to express the velocities in terms of the phase-space variables, we decompose the first term into four irreducible parts:

$$\begin{aligned} \phi_{i\perp\bar{k}} T^{i\perp\bar{k}} &= \phi_{\perp\bar{k}} T^{\perp\perp\bar{k}} + \frac{1}{2} {}^A\phi_{\bar{i}} \varepsilon^{\bar{i}\bar{m}\bar{n}} T^{\bar{m}\bar{n}\perp} \\ &+ {}^T\phi_{\bar{i}\bar{k}} {}^T T^{\bar{i}\perp\bar{k}} + \frac{1}{3} {}^S\phi T_{\bar{k}\perp}^{\bar{k}}. \end{aligned} \quad (4.4b)$$

If $a_1, a_2 \neq 0$, the velocities from the last two terms can be directly eliminated using Eqs. (3.6a) and (3.6d),

$$\frac{1}{3} {}^S\phi T_{\bar{k}\perp}^{\bar{k}} + {}^T\phi_{\bar{i}\bar{k}} {}^T T^{\bar{i}\perp\bar{k}} = \frac{1}{6a_2} {}^S\phi {}^S\phi + \frac{1}{2a_1} {}^T\phi_{\bar{i}\bar{k}} {}^T\phi^{\bar{i}\bar{k}}. \quad (4.5a)$$

Continuing with the first two terms in (4.4b), we note that for $\det A \neq 0$, one can use the relation $A^{-1} \times (3.8a)$ to eliminate the velocities. Introducing the notation $\varphi_{\bar{k}} := (\phi_{\perp\bar{k}}, {}^A\phi_{\bar{k}})^T$, the result takes a compact matrix form,

$$\begin{aligned} \left(\phi_{\perp\bar{k}}, \frac{1}{2} {}^A\phi_{\bar{k}} \right) \begin{pmatrix} T^{\perp\perp\bar{k}} \\ \varepsilon^{\bar{k}\bar{m}\bar{n}} T^{\bar{m}\bar{n}\perp} \end{pmatrix} &= \frac{3}{2 \det A} \varphi_{\bar{k}}^T T \varphi^{\bar{k}}, \\ T &:= \begin{pmatrix} 2a_1 + a_2 & \bar{a}_1 - \bar{a}_2 \\ \bar{a}_1 - \bar{a}_2 & -(a_1 + 2a_3)/2 \end{pmatrix}, \quad \det T = \frac{1}{2} \det A. \end{aligned} \quad (4.5b)$$

Hence, the resulting form of \mathcal{H}_{\perp}^T reads

$$\begin{aligned} \mathcal{H}_{\perp}^T &= \frac{1}{2} J \phi_T^2 - J \bar{\mathcal{L}}_{T^2} - n^k \nabla_{\beta} \pi_{k\beta}, \\ \phi_T^2 &:= \frac{1}{6a_2} {}^S\phi {}^S\phi + \frac{1}{2a_1} {}^T\phi_{\bar{i}\bar{k}} {}^T\phi^{\bar{i}\bar{k}} + \frac{3}{2 \det A} \varphi_{\bar{k}}^T T \varphi^{\bar{k}}. \end{aligned} \quad (4.6)$$

B. Curvature sector

In a similar manner, one finds the curvature contribution to \mathcal{H}_{\perp} :

$$\mathcal{H}_{\perp}^R = \frac{1}{4} \Phi_{ij\bar{k}} R^{ij\perp\bar{k}} - J \bar{\mathcal{L}}_{R^2} - a_0 R^{\bar{m}\bar{n}}{}_{\bar{m}\bar{n}} + \bar{a}_0 \varepsilon^{\bar{m}\bar{n}\bar{k}} R_{\perp\bar{m}\bar{n}\bar{k}}, \quad (4.7a)$$

where $\bar{\mathcal{L}}_{R^2} = \mathcal{L}_{R^2}(\bar{R})$ does not contain velocities, and

$$\begin{aligned} \Phi_{ij\bar{k}} R^{ij\perp\bar{k}} &= \frac{2}{3} {}^S\Phi R_{\perp\perp} - {}^A\Phi_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}} R_{\perp\bar{m}\perp\bar{n}} + 2 {}^T\Phi_{\perp\bar{j}\bar{k}} {}^T R^{\perp\bar{j}\perp\bar{k}}, \\ &- \frac{1}{6} {}^P\Phi \varepsilon_{\bar{i}\bar{j}\bar{k}} R^{\bar{i}\bar{j}\perp\bar{k}} + {}^V\Phi^{\bar{i}} R_{\bar{i}\bar{k}\perp}^{\bar{k}} \\ &- \frac{1}{2} {}^T\Phi^{\bar{i}\bar{k}} T(\varepsilon_{\bar{i}\bar{m}\bar{n}} R_{\bar{m}\bar{n}\perp\bar{k}}). \end{aligned} \quad (4.7b)$$

Summing up the scalar and pseudoscalar term from the expression (4.7b) and using the relation $B_0^{-1} \times (3.14a)$ to eliminate the velocities, one obtains

$$\begin{aligned} \frac{2}{3} {}^S\Phi R_{\perp\perp} + \frac{1}{6} {}^P\Phi \varepsilon_{\bar{i}\bar{j}\bar{k}} R^{\bar{i}\bar{j}\perp\bar{k}} &= J \frac{1}{6 \det B_0} {}^{(0)}\Phi^T R_0 {}^{(0)}\Phi, \\ R_0 &= \begin{pmatrix} -4(b_2 + b_3) & 2(\bar{b}_2 - \bar{b}_3) \\ 2(\bar{b}_2 - \bar{b}_3) & b_4 + b_6 \end{pmatrix}, \quad \det R_0 = 4 \det B_0, \end{aligned} \quad (4.8)$$

where ${}^{(0)}\Phi^T := ({}^S\Phi, {}^P\Phi)$.

Similarly, the sum of the axial vector and vector term, combined with $B_1^{-1} \times (3.17a)$, yields

$$\begin{aligned} -{}^A\Phi_{\bar{k}} \varepsilon^{\bar{k}\bar{m}\bar{n}} R_{\perp\bar{m}\perp\bar{n}} + {}^V\Phi^{\bar{i}} R_{\bar{i}\perp} &= -J \frac{1}{\det B_1} {}^{(1)}\Phi_i^T R_1 {}^{(1)}\Phi^{\bar{i}}, \\ R_1 &= \begin{pmatrix} b_4 + b_5 & -(\bar{b}_2 - \bar{b}_5) \\ -(\bar{b}_2 - \bar{b}_5) & -(b_2 + b_5) \end{pmatrix}, \quad \det R_1 = -\det B_1, \end{aligned} \quad (4.9)$$

where ${}^{(1)}\Phi_i^T = ({}^A\Phi_i, {}^V\Phi_i)$.

Finally, using $B_2^{-1} \times (3.20a)$, the sum of the two tensor terms is given by

$$\begin{aligned} 2 {}^T\Phi_{\perp\bar{j}\bar{k}} {}^T R^{\perp\bar{j}\perp\bar{k}} - \frac{1}{2} {}^T\Phi^{\bar{i}\bar{k}} T(\varepsilon_{\bar{i}\bar{m}\bar{n}} R_{\bar{m}\bar{n}\perp\bar{k}}) \\ &= J \frac{1}{4 \det B_2} {}^{(2)}\Phi_{\bar{i}\bar{k}}^T R_2 {}^{(2)}\Phi^{\bar{i}\bar{k}}, \\ R_2 &= \begin{pmatrix} 4(b_1 + b_2) & 2(\bar{b}_2 - \bar{b}_1) \\ 2(\bar{b}_2 - \bar{b}_1) & -(b_1 + b_4) \end{pmatrix}, \quad \det R_2 = -4 \det B_2, \end{aligned} \quad (4.10)$$

where ${}^{(2)}\Phi_{\bar{i}\bar{k}}^T := ({}^T\Phi_{\perp\bar{i}\bar{k}}, {}^T\Phi_{\bar{i}\bar{k}})$.

Summing up the above three contributions, one obtains the expression for \mathcal{H}_{\perp}^R as

$$\begin{aligned} \mathcal{H}_{\perp}^R &= \frac{1}{4} J \Phi_R^2 - J \mathcal{L}_{R^2}(\bar{R}) - a_0 R^{\bar{m}\bar{n}}{}_{\bar{m}\bar{n}} + \bar{a}_0 \varepsilon^{\bar{m}\bar{n}\bar{k}} R_{\perp\bar{m}\bar{n}\bar{k}}, \\ \Phi_R^2 &:= \frac{1}{6 \det B_0} {}^{(0)}\Phi^T R_0 {}^{(0)}\Phi - \frac{1}{\det B_1} {}^{(1)}\Phi_i^T R_1 {}^{(1)}\Phi^{\bar{i}} \\ &+ \frac{1}{4 \det B_2} {}^{(2)}\Phi_{\bar{i}\bar{k}}^T R_2 {}^{(2)}\Phi^{\bar{i}\bar{k}}. \end{aligned} \quad (4.11)$$

The complete expression $\mathcal{H}_{\perp} = \mathcal{H}_{\perp}^T + \mathcal{H}_{\perp}^R$ will be used in Sec. VII to formulate the conditions for the positivity of energy of the isolated spin modes.

C. Consistency conditions

The complete canonical Hamiltonian of PG, with $\mathcal{H}_{\perp} = \mathcal{H}_{\perp}^T + \mathcal{H}_{\perp}^R$, is calculated by assuming that none of the critical parameters is vanishing. In the next step, one can construct the total Hamiltonian that generates the temporal evolution of dynamical variables. Since the only primary constraints are the sure constraints (3.2), the total Hamiltonian is given by

$$\mathcal{H}_T = \mathcal{H}_c + u^i \pi_i^0 + \frac{1}{2} u^{ij} \Pi_{ij}^0, \quad (4.12)$$

where u^i and u^{ij} are canonical multipliers.

By construction, the components \mathcal{H}_{ij} , \mathcal{H}_α , and \mathcal{H}_\perp of the canonical Hamiltonian do not depend on the unphysical variables b^i_0 and ω^{ij}_0 . Hence, by demanding that the primary constraints be preserved during the time evolution, one finds the set of secondary constraints,

$$\mathcal{H}_\perp \approx 0, \quad \mathcal{H}_\alpha \approx 0, \quad \mathcal{H}_{ij} \approx 0. \quad (4.13)$$

General arguments, based on the existence of local Poincaré invariance, show that these constraints are first class [13]; see also [28]. Hence, the Dirac consistency algorithm is completed at the level of the secondary constraints (4.13).

The present PG model has $N_1 = 20$ first-class constraints and $N_2 = 0$ second-class constraints. Since the number of the Lagrangian variables is $N = 40$ (16 tetrad, plus 24 connection components), the number of the Lagrangian d.o.f. is $N^* = (2N - 2N_1 - N_2)/2 = 20$. They are the same as those found in the weak field approximation of PG: 2 massless spin-2 modes and 18 massive torsion modes (two spin-0, six spin-1, and ten spin-2 modes). However, we shall show that not all of these d.o.f. are physically acceptable, in contrast to earlier expectations [25]. To do that, we will first calculate the mass eigenvalues $m_\pm^2(J)$ for the torsion modes with spin $J = 0, 1, 2$.

V. LINEARIZED FIELD EQUATIONS

In this section, we start our analysis of the particle spectrum of PG by deriving the weak field approximation of the gravitational field equations (2.4) around the Minkowski background M_4 ; for consistency, we assume $\Lambda_0 = 0$. Such an approximation is based on the following weak field expansion of the basic dynamical variables,

$$b^i_\mu = \delta^i_\mu + \tilde{b}^i_\mu + O_2, \quad \omega^{ij}_\mu = \tilde{\omega}^{ij}_\mu + O_2.$$

To simplify the notation, we omit writing the tilde sign and the symbol O_2 , with an implicit understanding of their effects. Furthermore, we find it technically convenient to use the following abbreviations:

$$\begin{aligned} A_n &= a_n - a_1, & B_n &= b_n - b_1, \\ \bar{A}_n &= \bar{a}_n - \bar{a}_1, & \bar{B}_n &= \bar{b}_n - \bar{b}_1. \end{aligned} \quad (5.1)$$

A. First field equation

In the first field equation (2.4a), the covariant momentum associated to torsion has the form

$$\begin{aligned} \mathcal{H}_{imn} &= 2a_1 T_{imn} + \frac{2}{3} A_2 (\eta_{im} \mathcal{V}_n - \eta_{in} \mathcal{V}_m) + 2A_3 \varepsilon_{imnl} \mathcal{A}^l, \\ &\quad - \bar{a}_1 T_{irs} \varepsilon^{rs}_{mn} - \frac{2}{3} \bar{A}_2 \varepsilon_{imns} \mathcal{V}^s \\ &\quad + 2\bar{A}_3 (\eta_{im} \mathcal{A}_n - \eta_{in} \mathcal{A}_m), \end{aligned} \quad (5.2)$$

where $\bar{a}_2 = \bar{a}_3$ yields $\bar{A}_2 = \bar{A}_3$. Then, after calculating the linearized form of E_i^ν ,

$$\begin{aligned} E_i^\nu &= 2a_0 G^\nu_i - \bar{a}_0 (R_{mnki} \varepsilon^{mnk\nu} + h_i^\nu X) \\ &= 2a_0 G^\nu_i - 2\bar{a}_0 X_i^\nu, \end{aligned}$$

the linearized (1ST) takes the form

$$\begin{aligned} \mathcal{E}_{in} &= \partial^m \mathcal{H}_{imn} - 2a_0 G_{ni} + 2\bar{a}_0 X_{in} \\ &= -2a_1 \partial^m T_{imn} + \frac{2}{3} A_2 (\partial_i \mathcal{V}_n - \eta_{in} \partial \mathcal{V}) \\ &\quad - 2A_3 \varepsilon_{imnk} \partial^m \mathcal{A}^k + \frac{2}{3} \bar{A}_2 \varepsilon_{imnk} \partial^m \mathcal{V}^k \\ &\quad + 2\bar{A}_2 (\partial_i \mathcal{A}_n - \eta_{in} \partial \mathcal{A}) - 2a_0 G_{ni} \\ &\quad + 2(\bar{a}_0 - \bar{a}_1) X_{in} = 0, \end{aligned} \quad (5.3)$$

where we used (E3), and $\partial \mathcal{V} := \partial_i \mathcal{V}^i$, $\partial \mathcal{A} := \partial_i \mathcal{A}^i$.

B. Second field equation

Using the formulas obtained in the weak field approximation,

$$\begin{aligned} \nabla_\mu^L H_{ij}{}^{\mu n} &= 2a_0 (T^n_{ij} - \delta^n_i \mathcal{V}_j + \delta^n_j \mathcal{V}_i) \\ &\quad - \bar{a}_0 \varepsilon_{ij}{}^{rs} (T^n_{rs} - \delta^n_r \mathcal{V}_s + \delta^n_s \mathcal{V}_r), \\ 2\mathcal{H}_{[ij]n} &= -\frac{4}{3} (2a_1 + a_2) \eta_{n[i} \mathcal{V}_{j]} + 2(a_1 + 2a_3) \varepsilon_{ijnk} \mathcal{A}^k \\ &\quad - \frac{4}{3} \bar{A}_2 \varepsilon_{ijnk} \mathcal{V}^k - 4\bar{A}_3 \eta_{n[i} \mathcal{V}_{j]}, \end{aligned} \quad (5.4)$$

the linearized form of (2ND) reads

$$\begin{aligned} \mathcal{E}_{ijn} &= \partial^m \mathcal{H}'_{ijmn} + 2a_0 (T^n_{ij} - \delta^n_i \mathcal{V}_j + \delta^n_j \mathcal{V}_i) \\ &\quad - \bar{a}_0 \varepsilon_{ij}{}^{rs} (T^n_{rs} - \delta^n_r \mathcal{V}_s + \delta^n_s \mathcal{V}_r) + 2\mathcal{H}_{[ij]n} = 0. \end{aligned} \quad (5.5a)$$

Using the double duality relations for the curvature, see Appendix C in Ref. [29], the term $\partial^m \mathcal{H}'_{ijmn}$ is found to have the form

$$\begin{aligned}
\partial^m \mathcal{H}'_{ijmn} &= (b_2 + b_1) \partial^m (\eta_{ir} \Psi_{js} - \eta_{jr} \Psi_{is}) \varepsilon^{rs}{}_{mn} + \frac{1}{6} B_3 \varepsilon_{ijmn} \partial^m X \\
&+ (b_4 + b_1) [(\partial_i \Phi_{jn} - \eta_{in} \partial^m \Phi_{jm}) - (i \leftrightarrow j)] + \frac{1}{6} B_6 (\eta_{jn} \partial_i - \eta_{in} \partial_j) R \\
&+ B_5 [(\partial_i \hat{R}_{[jn]} - \eta_{in} \partial^m \hat{R}_{[jm]}) - (i \leftrightarrow j)] - \bar{B}_5 \partial^m (\eta_{ir} \hat{R}_{[js]} - \eta_{jr} \hat{R}_{[is]}) \varepsilon^{rs}{}_{mn} \\
&+ (\bar{b}_2 - \bar{b}_1) [(\partial_i \Psi_{jn} - \eta_{in} \partial^m \Psi_{jm}) - (i \leftrightarrow j)] + \frac{1}{6} \bar{B}_3 (\eta_{jn} \partial_i - \eta_{in} \partial_j) X \\
&- (\bar{b}_4 - \bar{b}_1) \partial^m (\eta_{ir} \Phi_{js} - \eta_{jr} \Phi_{is}) \varepsilon^{rs}{}_{mn} - \frac{1}{6} \bar{B}_6 \varepsilon_{ijmn} \partial^m R.
\end{aligned} \tag{5.5b}$$

VI. PARTICLE SPECTRUM

The particle spectrum of PG contains important information of its physical content. Recently, Karananas [25] made a detailed analysis of this problem by extending the spin-projection operator formalism, used earlier in the context of PG⁺ [12], and applying it to study the PG field excitations around the Minkowski background. His work resulted in the mass formulas for the spin-0, spin-1, and spin-2 massive torsion modes, together with the related restrictions on the parameter space, stemming from the requirements for the absence of ghosts and tachyons.

In this section, we study the same problem by analyzing the linearized field equations along the lines presented in [5]. The obtained results are tested by verifying their compatibility with the expressions for the critical parameters found in the canonical analysis, whereas the absence of ghosts and tachyons is studied in the next section.

A. Spin-0 modes

The spin-0 sector is determined by the traces of the field equations \mathcal{E}_{in} , $\partial^i \mathcal{E}_{ijn}$, and $\partial^k (*\mathcal{E})_{kln}$, where $*\mathcal{E}_{kln} := (1/2) \varepsilon_{kl}{}^{ij} \mathcal{E}_{ijn}$ is the dual of \mathcal{E}_{ijn} :

$$\begin{aligned}
&-a_2 \partial V - 3\bar{a}_2 \partial A + a_0 R + \bar{a}_0 X = 0, \\
&(b_4 + b_6) \square R + (\bar{b}_3 - \bar{b}_2) \square X + 4(2a_0 + a_2) \partial V \\
&\quad + 12(\bar{a}_2 - \bar{a}_0) \partial A = 0, \\
&(b_2 + b_3) \square X - (\bar{b}_3 - \bar{b}_2) \square R - 12(a_0 + 2a_3) \partial A \\
&\quad + 8(\bar{a}_2 - \bar{a}_0) \partial V = 0.
\end{aligned} \tag{6.1}$$

With $X = 3\partial A$, the first equation can be used to express R in terms of ∂V and ∂A , whereupon the remaining two equations are written in the matrix form as

$$(K_0 \square + 4a_0 N_0) U = 0, \tag{6.2a}$$

$$\begin{aligned}
K_0 &= \begin{pmatrix} a_2(b_4 + b_6) & -3a_0(\bar{b}_2 - \bar{b}_3) - 3(\bar{a}_0 - \bar{a}_2)(b_4 + b_6) \\ a_2(\bar{b}_2 - \bar{b}_3) & 3a_0(b_2 + b_3) - 3(\bar{a}_0 - \bar{a}_2)(\bar{b}_2 - \bar{b}_3) \end{pmatrix}, \\
N_0 &= \begin{pmatrix} (2a_0 + a_2) & -3(\bar{a}_0 - \bar{a}_2) \\ -2(\bar{a}_0 - \bar{a}_2) & -3(a_0 + 2a_3) \end{pmatrix}, \quad U = \begin{pmatrix} \partial V \\ \partial A \end{pmatrix}.
\end{aligned} \tag{6.2b}$$

The determinants of K_0 and N_0 are given by

$$\begin{aligned}
\det K_0 &= 3a_0 a_2 [(b_4 + b_6)(b_2 + b_3) + (\bar{b}_2 - \bar{b}_3)^2], \\
\det N_0 &= -3[(2a_0 + a_2)(a_0 + 2a_3) + 2(\bar{a}_0 - \bar{a}_2)^2].
\end{aligned} \tag{6.2c}$$

For $\det K_0 \neq 0$, one can multiply (6.2a) by K_0^{-1} , and obtain the Klein-Gordon equation for the massive spin-0 torsion modes,

$$(\square + M_0) U = 0, \quad M_0 = 4a_0 K_0^{-1} N_0. \tag{6.3}$$

The masses of these modes are given by the eigenvalues of the mass matrix M_0 ,

$$\begin{aligned}
m_{\pm}^2(0) &= \frac{1}{2} \left(\text{tr} M_0 \pm \sqrt{(\text{tr} M_0)^2 - 4(\det M_0)} \right) \\
&= \frac{2a_0}{\det K_0} \left(\text{tr} f_0 + \sqrt{(\text{tr} f_0)^2 - 4 \det f_0} \right),
\end{aligned} \tag{6.4a}$$

where $f_0 := (\det K_0) K_0^{-1} N_0$, and

$$\begin{aligned}
\text{tr} f_0 &= 3a_0(2a_0 + a_2)(b_2 + b_3) - 12a_0(\bar{a}_0 - \bar{a}_2)(\bar{b}_2 - \bar{b}_3) \\
&\quad - 3[a_2(a_0 + 2a_3) + 2(\bar{a}_0 - \bar{a}_2)^2](b_4 + b_6), \\
\det f_0 &= (\det K_0)(\det N_0).
\end{aligned} \tag{6.4b}$$

It is interesting to note that $\det K_0$ is proportional to the product of two critical parameters, a_2 and $\det B_0$, characterizing the spin-0 sector of the set of if-constraints (see Table I). Hence, when the critical parameters vanish, we have $\det K_0 = 0$, the mass eigenvalues (6.4) become infinite, and consequently, the spin-0 modes do not propagate. In the linear regime, this mechanism provides a Lagrangian description of the dynamical role of if-constraints.

As a further test of our mass formula (6.4), we calculated its form in the parity-even sector $(\bar{a}_0, \bar{a}_n, \bar{b}_n) = 0$, and found the well-known result for the spin-0[±] torsion modes:

$$m_{+}^2(0) = \frac{4a_0(2a_0 + a_2)}{a_2(b_4 + b_6)}, \quad m_{-}^2(0) = -\frac{4(a_0 + 2a_3)}{(b_2 + b_3)}.$$

B. Spin-1 modes

To understand the linearized dynamics of the spin-1 sector, it is convenient to start with the antisymmetric part of (1ST), $\mathcal{E}_{[ij]}$, and its dual, ${}^*\mathcal{E}_{ij}$. Taking derivatives of these equations yields

$$\begin{aligned} \frac{1}{3}(2a_1 + a_2)(\square\mathcal{V}_j - \partial_j\partial\mathcal{V}) + \bar{A}_3(\square\mathcal{A}_j - \partial_j\partial\mathcal{A}) + 2A_0\partial^i\hat{R}_{[ij]} + 2\bar{A}_0\partial^i X_{[ij]} &= 0, \\ (a_1 + 2a_3)(\square\mathcal{A}_j - \partial_j\partial\mathcal{A}) - \frac{2}{3}\bar{A}_2(\square\mathcal{V}_j - \partial_j\partial\mathcal{V}) - 2A_0\partial^i X_{[ij]} + 2\bar{A}_0\partial^i\hat{R}_{[ij]} &= 0. \end{aligned}$$

Then, the solutions for $\partial^m\hat{R}_{[mi]}$ and $\partial^m X_{[mi]}$ are found to be given in the matrix form as

$$\begin{aligned} 2\begin{pmatrix} -\partial^m\hat{R}_{[mi]} \\ \partial^m X_{[mi]} \end{pmatrix} &= G(\square U_i - \partial_i(\partial U)), \quad U_i = \begin{pmatrix} \mathcal{V}_i \\ \mathcal{A}_i \end{pmatrix}, \quad g := A_0^2 + \bar{A}_0^2, \\ G &:= \frac{1}{g} \begin{pmatrix} \frac{1}{3}[A_0(2a_1 + a_2) - 2\bar{A}_0\bar{A}_2] & [A_0\bar{A}_2 + \bar{A}_0(a_1 + 2a_3)] \\ -\frac{1}{3}[\bar{A}_0(2a_1 + a_2) + 2A_0\bar{A}_2] & -[\bar{A}_0\bar{A}_2 - A_0(a_1 + 2a_3)] \end{pmatrix}. \end{aligned} \quad (6.5)$$

Next, consider the trace of (2ND), $\eta^{jk}\mathcal{E}_{ijk}$, and of its dual, $\eta^{jk}{}^*\mathcal{E}_{ijk}$. Using the identities (E4), these trace components take the form

$$\begin{aligned} -2(b_4 + b_5)\partial^m\hat{R}_{[mi]} + 2(\bar{b}_2 - \bar{b}_5)\partial^m X_{[mi]} + \frac{1}{2}(b_4 + b_6)\partial_i R - \frac{1}{2}(\bar{b}_2 - \bar{b}_3)\partial_i X \\ + 2(2a_0 + a_2)\mathcal{V}_i - 6(\bar{a}_0 - \bar{a}_2)\mathcal{A}_i &= 0, \end{aligned} \quad (6.6a)$$

$$\begin{aligned} -4(\bar{b}_2 - \bar{b}_5)\partial^m\hat{R}_{[mi]} - 4(b_2 + b_5)\partial^m X_{[mi]} + (\bar{b}_2 - \bar{b}_3)\partial_i R + (b_2 + b_3)\partial_i X \\ - 8(\bar{a}_0 - \bar{a}_2)\mathcal{V}_i - 12(a_0 + 2a_3)\mathcal{A}_i &= 0. \end{aligned} \quad (6.6b)$$

Using the expressions for $\partial^m\hat{R}_{[mi]}$ and $\partial^m X_{[mi]}$ found in (6.5), and the expression for R determined by the trace of (1ST), Eqs. (6.6) multiplied by $-2g$ can be written in the matrix form as

$$(K_1\square - 4gN_1)U_i + (L_1 - K_1)\partial_i(\partial U) = 0, \quad (6.7a)$$

where

$$\begin{aligned} K_1 &= B'_1(gG), \quad B'_1 := -2 \begin{pmatrix} b_4 + b_5 & \bar{b}_2 - \bar{b}_5 \\ 2(\bar{b}_2 - \bar{b}_5) & -2(b_2 + b_5) \end{pmatrix}, \\ N_1 &= \begin{pmatrix} (2a_0 + a_2) & -3(\bar{a}_0 - \bar{a}_2) \\ -4(\bar{a}_0 - \bar{a}_2) & -6(a_0 + 2a_3) \end{pmatrix}, \quad U_i = \begin{pmatrix} \mathcal{V}_i \\ \mathcal{A}_i \end{pmatrix}, \\ L_1 &= -\frac{g}{a_0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} K_0 = -4gN_1M_0^{-1}. \end{aligned} \quad (6.7b)$$

The determinants of K_1 and N_1 are given by

$$\begin{aligned} \det K_1 &= \frac{8}{3}g(\det A)(\det B_1), \\ \det N_1 &= -6[(a_0 + 2a_3)(2a_0 + a_2) + 2(\bar{a}_0 - \bar{a}_2)^2]. \end{aligned} \quad (6.7c)$$

When $\det K_1 \neq 0$, one can multiply Eq. (6.7a) by K_1^{-1} and obtain the matrix Klein-Gordon equation for the massive spin-1 torsion modes,

$$\begin{aligned}
(\square + M_1)\tilde{U}_i &= 0, & M_1 &:= -4gK_1^{-1}N_1, \\
\tilde{U}_i &:= U_i + M_0^{-1}\partial_i U, & \partial^i \tilde{U}_i &= 0.
\end{aligned} \tag{6.8}$$

The eigenvalues of the mass matrix M_1 are given by

$$m_{\pm}^2(1) = \frac{-2g}{\det K_1} \left(\text{tr} f_1 \pm \sqrt{(\text{tr} f_1)^2 - 4 \det f_1} \right), \tag{6.9a}$$

where $f_1 := (\det K_1)K_1^{-1}N_1$, and

$$\begin{aligned}
\det f_1 &= (\det N_1)(\det K_1), \\
\text{tr} f_1 &= 4(b_2 + b_5)[(2a_0 + a_2)[(a_0 - a_1)(a_1 + 2a_3) - (\bar{a}_0 - \bar{a}_1)^2] + 2(a_0 - a_1)(\bar{a}_0 - \bar{a}_2)^2 \\
&\quad + 4(b_4 + b_5)[(a_0 + 2a_3)[(a_0 - a_1)(2a_1 + a_2) - 2(\bar{a}_0 - \bar{a}_1)^2] + 2(a_0 - a_1)(\bar{a}_0 - \bar{a}_2)^2 \\
&\quad + 8(\bar{b}_2 - \bar{b}_5)[-(2a_0 + a_2)(a_0 + 2a_3)(\bar{a}_0 - \bar{a}_1) + 2[(a_0 - a_1)^2 + (\bar{a}_0 - \bar{a}_1)^2](\bar{a}_0 - \bar{a}_2) \\
&\quad - 2(\bar{a}_0 - \bar{a}_1)(\bar{a}_0 - \bar{a}_2)^2].
\end{aligned} \tag{6.9b}$$

The determinant of K_1 is the product of two critical parameters associated to the spin-1 sector (see Table I). A discussion of what happens when at least one of these parameters vanishes is given in Appendix D.

In the parity even sector, our mass formula (6.9) yields the familiar result for the spin-1 $^{\pm}$ torsion modes:

$$\begin{aligned}
m_+^2(1) &= \frac{6(a_0 - a_1)(a_0 + 2a_3)}{(a_1 + 2a_3)(b_2 + b_5)}, \\
m_-^2(1) &= \frac{6(a_0 - a_1)(2a_0 + a_2)}{(2a_1 + a_2)(b_4 + b_5)}.
\end{aligned}$$

C. Spin-2 modes

Although, in principle, the analysis of the spin-2 sector is not much more complicated than the one for the spin-1 case, the fact that there are lots of variables makes the general procedure rather complex and difficult to follow. In Ref. [27], the mass eigenvalues of the spin-2 torsion modes were found by studying a class of exact wave solutions, defined by an ansatz that creates only the tensorial irreducible part of the torsion, whereas the vector and axial vector parts vanish. This motivates us to simplify the present discussion by considering a dynamical system with vanishing spin-0 and spin-1 modes, $\mathcal{V}_i = 0$ and $\mathcal{A}_i = 0$. The physical content of such a system is described solely by the spin-2 tensor t_{ijk} (Appendix A). Such a technical simplification does not influence the final result for the spin-2 mass eigenvalues.

The adopted assumptions have two additional consequences: $X = 0$, which follows from $X = 3\partial\mathcal{A}$; and $R = 0$, which follows from the trace of (1ST). To analyse the spin-2 sector, we need the symmetrized version of (1ST),

$$-a_1\Theta_{ik} - a_0\Phi_{ik} + \bar{A}_0\Psi_{ik} = 0, \tag{6.10}$$

where $\Theta_{ik} := \partial^m t_{ikm} = \partial^m T_{(ik)m}$, as follows from the definition (A1) of t_{ikm} . Moreover, we also need two equations that follow from (2ND), $\partial^m \mathcal{E}_{m(ik)}$, and $\partial^m (*\mathcal{E})_{m(ik)}$:

$$(b_1 + b_4)[\square\Phi_{ik} - 2\partial_{(i}\partial^m\Phi_{k)m}] + \bar{B}_2[\square\Psi_{ik} - 2\partial_{(i}\partial^m\Psi_{k)m}] - 2A_0\Theta_{ik} - 2\bar{A}_0\Psi_{ik} = 0, \tag{6.11a}$$

$$(b_1 + b_2)[\square\Psi_{ik} - 2\partial_{(i}\partial^m\Psi_{k)m}] - \bar{B}_4[\square\Phi_{ik} - 2\partial_{(i}\partial^m\Phi_{k)m}] - 2\bar{A}_0\Theta_{ik} + 2A_0\Psi_{ik} = 0. \tag{6.11b}$$

Since Φ_{ik} has a nontrivial Riemannian part associated to the massless graviton, a proper description of the torsion spin-2 modes is obtained by using Eq. (6.10) to eliminate Φ_{ik} from Eqs. (6.11):

$$(b_1 + b_4)\square(-a_1\Theta_{ik} + \bar{A}_0\Psi_{ik}) + a_0\bar{B}_2\square\Psi_{ik} - 2a_0(A_0\Theta_{ik} + \bar{A}_0\Psi_{ik}) = 0, \tag{6.12a}$$

$$a_0(b_1 + b_2)\square\Psi_{ik} - \bar{B}_2\square(-a_1\Theta_{ik} + \bar{A}_0\Psi_{ik}) - 2a_0(\bar{A}_0\Theta_{ik} - A_0\Psi_{ik}) = 0. \tag{6.12b}$$

These equations can be compactly represented in the matrix form as

$$(K_2\square + 2a_0N_2)U_{ik} = 0, \tag{6.13}$$

where

$$\begin{aligned}
K_2 &:= \begin{pmatrix} a_1(b_1 + b_4) & -\bar{A}_0(b_1 + b_4) - a_0(\bar{b}_2 - \bar{b}_1) \\ -a_1(\bar{b}_2 - \bar{b}_1) & \bar{A}_0(\bar{b}_2 - \bar{b}_1) - a_0(b_2 + b_1) \end{pmatrix}, \\
N_2 &:= \begin{pmatrix} A_0 & \bar{A}_0 \\ \bar{A}_0 & -A_0 \end{pmatrix}, & U_{ik} &:= \begin{pmatrix} \Theta_{ik} \\ \Psi_{ik} \end{pmatrix}.
\end{aligned}$$

For $\det K_2 \neq 0$, Eq. (6.13) is equivalent to

$$(\square + M_2)U_{ik} = 0, \quad M_2 := 2a_0 K_2^{-1} N_2, \quad (6.14)$$

where M_2 is the mass matrix of the spin-2 torsion mode.

The matrices K_2 and N_2 are of the same form as those found in Ref. [27], Eq. (4.50), up to inessential differences in conventions. Hence, the mass eigenvalues are also the same. Expressed in terms of the matrix $f_2 = (\det K_2) K_2^{-1} N_2$, they are given by

$$m_{\pm}^2(2) = \frac{a_0}{\det K_2} \left(\text{tr} f_2 \pm \sqrt{(\text{tr} f_2)^2 - 4 \det f_2} \right), \quad (6.15a)$$

where

$$\begin{aligned} \det f_2 &= (\det K_2)(\det N_2), \\ \text{tr} f_2 &= -a_0(a_0 - a_1)(b_1 + b_2) - 2a_0(\bar{a}_0 - \bar{a}_1)(\bar{b}_2 - \bar{b}_1) \\ &\quad + [-a_1(a_0 - a_1) + (\bar{a}_0 - \bar{a}_1)^2](b_1 + b_4). \end{aligned} \quad (6.15b)$$

As expected, the determinant of K_2 is proportional to the product of the critical parameters given in the third line of Table I,

$$\det K_2 = -a_0 a_1 \det B_2, \quad \det N_2 = -(A_0^2 + \bar{A}_0^2). \quad (6.16)$$

In the parity-even sector, the above formulas produce the well-known result,

$$m_+^2(2) = \frac{2a_0(a_0 - a_1)}{a_1(b_1 + b_4)}, \quad m_-^2(2) = \frac{2(a_0 - a_1)}{b_1 + b_2}.$$

The above procedure can be extended to the case with nonvanishing spin-0 and spin-1 terms. After a straightforward but rather clumsy calculation, we found that the new terms do not influence the mass eigenvalues, they only modify the spin-2 state U_{ik} . A compact form of the result reads

$$\begin{aligned} U_{ik} &\rightarrow \tilde{U}_{ik} \\ &:= Z \left[\tilde{U}_{ik} - G \partial_{(i} U_{k)} + \frac{1}{3} H (M_0^{-1} \partial_i \partial_k U + \eta_{ik} U) \right], \end{aligned} \quad (6.17a)$$

where

$$\begin{aligned} Z &:= \frac{1}{a_1} \begin{pmatrix} -a_0 & \bar{A}_0 \\ 0 & a_1 \end{pmatrix}, \quad \tilde{U}_{ik} := \begin{pmatrix} \Phi_{ik} \\ \Psi_{ik} \end{pmatrix}, \\ H &:= \frac{1}{a_0} \begin{pmatrix} a_2 & 3(\bar{a}_2 - \bar{a}_0) \\ 0 & -2a_0 \end{pmatrix}. \end{aligned} \quad (6.17b)$$

The role of Z is to replace Φ_{ik} in \tilde{U}_{ik} by its form obtained from the symmetrized (1ST). The spin-2 nature of \tilde{U}_{ik} is ensured by the properties $\partial^i \tilde{U}_{ik} = 0$, $\eta^{ik} \tilde{U}_{ik} = 0$. In fact, these properties are sufficient to completely determine \tilde{U}_{ik} .

D. Comparison with Karananas' mass formulas

Our mass formulas are found to be consistent with the expressions for the canonical critical parameters, displayed in Table I. A more detailed test can be conducted by comparing them to the recent calculations of Karananas [25]. The first step in this direction is to compare the Lagrangian (5) in Ref. [25] with our expression (B1). Although the procedure is straightforward, a number of misprints found in Ref. [25] complicate the process. Nevertheless, we established the following correspondence between the related parameters:

$$\begin{aligned} a_0 &= \lambda, & \bar{a}_0 &= \Lambda = 0, \\ a_1 &= \lambda + t_1, & a_2 &= 2(-\lambda + t_3), \\ a_3 &= (-\lambda + t_2)/2, \\ \bar{a}_1 &= -2t_5, & \bar{a}_2 &= \bar{a}_3 = -t_4, \\ b_1 &= 4(r_1 - r_3), & b_2 &= 4r_3, \\ b_3 &= 4(r_2 - r_3), & b_4 &= 4(r_1 - r_3 + r_4), \\ b_5 &= 4(r_3 + r_5), & b_6 &= 4(r_1 - r_3 + 3r_4), \\ \bar{b}_1 &= r_7 - 3r_8, & \bar{b}_2 &= \bar{b}_4 = r_7 + r_8, \\ \bar{b}_3 &= \bar{b}_6 = -4r_6 + r_7 + r_8, & \bar{b}_5 &= -3r_7 + r_8. \end{aligned} \quad (6.18)$$

The remaining part of the comparison is rather simple. By substituting the above expressions into Eqs. (6.4) and (6.9), one finds that the resulting mass eigenvalues for the spin-0 and spin-1 torsion modes exactly reproduce the respective result that Karananas gives in his Appendix A. Moreover, we also found that, with the exception of minor differences, our mass formula (6.15) for the spin-2 modes is in agreement with his result (A.3.5); see also subsection IV.E in Ref. [27]. Although the difference is small, it might be responsible for more serious discrepancies in the physical properties of the spin-2 modes, found in the next section.

VII. PHYSICAL RESTRICTIONS ON THE SPACE OF PARAMETERS

In this section, we study the physical requirements of the absence of ghosts ($E > 0$), the absence of tachyons ($m^2 > 0$), and the reality (m^2 real) in the spectrum of torsion modes. Our approach is based on the Hamiltonian analysis developed in Secs. III and IV, subject to the assumption that all the critical parameters are nonvanishing, or equivalently, that all the torsion modes are propagating. In what follows, we shall examine whether such an

assumption is compatible with the adopted physical requirements.

Our general strategy is the following. The conditions of the *positivity of energy* can be read from the dynamical component \mathcal{H}_\perp^R of the canonical Hamiltonian; see Eq. (4.11). By introducing the matrices

$$F_J := \frac{1}{\det B_J} R_J, \quad J = 0, 1, 2,$$

these conditions can be expressed by demanding that the eigenvalues of F_J be positive. Using the general formula for the eigenvalues of a 2×2 matrix [see (3.9)], one can express these conditions in a more practical form as

$$E_J > 0: \det F_J > 0, \text{tr} F_J > 0. \quad (7.1)$$

The absence of tachyons is effectively described by the conditions of *positivity* of the eigenvalues $m_\pm^2(J)$ of the mass matrices M_J :

$$m_\pm^2(J) > 0: \det M_J > 0, \text{tr} M_J > 0. \quad (7.2)$$

Moreover, the presence of square roots in the mass eigenvalues requires to check their *reality*:

$$m_\pm^2(J) \text{ real: } (\text{tr} M_J)^2 - 4 \det M_J > 0. \quad (7.3)$$

By applying these general physical criteria to the specific spin- J sectors, one obtains a set of restrictions on the original Lagrangian parameters. An important goal of our analysis is to clarify the issue of their mutual (in)consistency. We shall always use $a_0 > 0$, the condition that ensures the correct limit to GR.

A. Spin-0 sector

1. Positivity of energy

The energy of the spin-0 modes is positive if the eigenvalues of the matrix $F_0 = R_0/\det B_0$ are positive. Since $\det R_0 = 4 \det B_0$, the first condition $\det F_0 > 0$ implies that $\det B_0 > 0$, or, equivalently,

$$(b_2 + b_3)(b_4 + b_6) + (\bar{b}_2 - \bar{b}_3)^2 < 0, \quad (7.4a)$$

$$\Rightarrow (b_2 + b_3)(b_4 + b_6) < 0. \quad (7.4b)$$

Then, the second condition takes the form $\text{tr} R_0 > 0$. In combination with Eq. (7.4b), it yields the relations

$$b_2 + b_3 < 0, \quad b_4 + b_6 > 0, \quad (7.5)$$

which coincide with those appearing in PG^+ . The independent conditions are the condition Eq. (7.4a) and, for instance, the first one in Eq. (7.5),

$$(b_2 + b_3)(b_4 + b_6) + (\bar{b}_2 - \bar{b}_3)^2 < 0, \quad b_2 + b_3 < 0. \quad (7.6)$$

These two conditions coincide with the first two relations found in Eq. (48) of Ref. [25] (the third relation is redundant).

2. Positivity of $m_\pm^2(0)$

The mass matrix M_0 of the spin-0 torsion modes has the form (6.3),

$$M_0/4a_0 = K_0^{-1} N_0 = \frac{1}{\det K_0} f_0, \quad \det K_0 = -3a_0 a_2 \det B_0. \quad (7.7)$$

The positivity of its eigenvalues is expressed by the conditions $\det M_0 > 0$ and $\text{tr} M_0 > 0$:

$$\frac{\det N_0}{\det K_0} > 0, \quad \frac{1}{\det K_0} \text{tr} f_0 > 0. \quad (7.8)$$

Since $\det B_0 > 0$, they take the form

$$a_2 \det N_0 < 0, \quad (7.9a)$$

$$a_2 \text{tr} f_0 < 0. \quad (7.9b)$$

As shown in Appendix F, these general conditions can be transformed into an unexpectedly simple form, in which the parameters $(b_n \bar{b}_n)$ are completely absent:

$$a_2[(2a_0 + a_2)(a_0 + 2a_3) + 2(\bar{a}_0 - \bar{a}_2)^2] > 0, \quad (7.10)$$

$$a_2(2a_0 + a_2) > 0.$$

Returning to the parameters introduced in Eq. (6.18), this result takes the form

$$(t_3 - \lambda)(t_2 t_3 + t_4^2) > 0, \quad (t_3 - \lambda)t_3 > 0.$$

The first formula is equivalent to Karananas's result [25], but the second one is different.

B. Spin-1 sector

1. Positivity of energy

Starting with $F_1 := R_1/\det B_1$ and using $\det R_1 = -\det B_1$, the first condition for the positivity of energy, $\det F_1 > 0$, reads

$$\det B_1 \equiv (b_2 + b_5)(b_4 + b_5) + (\bar{b}_2 - \bar{b}_5)^2 < 0, \quad (7.11a)$$

$$\Rightarrow (b_2 + b_5)(b_4 + b_5) < 0. \quad (7.11b)$$

The second condition, written as $\text{tr}R_1 < 0$ and combined with Eq. (7.11b), yields

$$b_2 + b_5 > 0, \quad b_4 + b_5 < 0, \quad (7.12)$$

which is the PG^+ result. As the two independent conditions, we choose

$$(b_2 + b_5)(b_4 + b_5) + (\bar{b}_2 - \bar{b}_5)^2 < 0, \quad b_4 + b_5 < 0. \quad (7.13)$$

Again, there is a complete agreement with the first two relations in Eq. (49) of [25], whereas the third relation is redundant.

2. Positivity of $m_{\pm}^2(1)$

To make the technical exposition more compact, we introduce the following notation:

$$\begin{aligned} \mu_2 &:= 2a_0 + a_2, & \mu_3 &:= a_0/2 + a_3, \\ k_2 &:= 2a_1 + a_2, & k_3 &:= a_1/2 + a_3. \\ \det A &= -2[k_2k_3 + (\bar{a}_1 - \bar{a}_2)^2], \\ \det N_1 &= -12[\mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2]. \end{aligned}$$

The mass matrix of the spin-1 torsion modes was found in subsection [VIB](#),

$$\begin{aligned} M_1 &= -4gK_1^{-1}N_1 = -\frac{4g}{\det K_1}f_1, \\ \det K_1 &= \frac{8}{3}g(\det A)(\det B_1), \end{aligned} \quad (7.14)$$

with $g \equiv A_0^2 + \bar{A}_0^2$. The positivity of the mass eigenvalues is expressed by the requirements

$$\frac{\det N_1}{\det K_1} > 0, \quad \frac{1}{\det K_1} \text{tr}f_1 < 0. \quad (7.15)$$

Since $\det B_1 < 0$, these conditions are equivalent to

$$(\det A)(\det N_1) < 0, \quad (7.16a)$$

$$(\det A)\text{tr}f_1 > 0. \quad (7.16b)$$

The expression for $\text{tr}f_1$ is given in subsection [VIB](#); see also Appendix [F](#).

A simple inspection of Eq. (7.16a) shows that it can be realized by $\det A < 0$, $\det N_1 > 0$, or vice versa, whereas, as shown in Appendix [F](#), Eq. (7.16b) can be replaced by a much simpler expression. The resulting conditions, equivalent to Eq. (7.16), are defined in (F7):

$$\begin{aligned} \text{(i)} \quad & k_2k_3 + (\bar{a}_1 - \bar{a}_2)^2 < 0, \quad \mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2 > 0, \\ & \mu_3k_2A_0 - 2\mu_3\bar{A}_0^2 + A_0(\bar{a}_0 - \bar{a}_2)^2 < 0, \\ \text{(ii)} \quad & k_2k_3 + (\bar{a}_1 - \bar{a}_2)^2 > 0, \quad \mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2 < 0, \\ & \mu_3k_2A_0 - 2\mu_3\bar{A}_0^2 + A_0(\bar{a}_0 - \bar{a}_2)^2 > 0. \end{aligned} \quad (7.17)$$

As before, they do not depend on the parameters (b_n, \bar{b}_n) . Going over to the parameters defined in Eq. (6.18), the relations (i) read

$$\begin{aligned} (t_1 + t_2)(t_1 + t_3) + (t_4 - 2t_5)^2 &< 0, \quad t_2t_3 + t_4^2 > 0, \\ t_2(t_1^2 + 4t_5^2) + t_1(t_2t_3 + t_4^2) &> 0. \end{aligned}$$

The first two inequalities in the set (i) coincide with those found in Ref. [25]; the third one is a bit different, but the whole complementary set (ii) is missing.

C. Spin-2 sector

1. Positivity of energy

The first condition for the positivity of the eigenvalues of $F_2 = R_2/\det B_2$, $\det F_2 > 0$, combined with $\det R_2 = -4\det B_2$, takes the form

$$\det B_2 \equiv (b_1 + b_2)(b_1 + b_4) + (\bar{b}_2 - \bar{b}_1)^2 < 0, \quad (7.18a)$$

$$\Rightarrow (b_1 + b_2)(b_1 + b_4) < 0. \quad (7.18b)$$

The second condition combined with Eq. (7.18b) yields relations that are also valid in PG^+ ,

$$b_1 + b_2 < 0, \quad b_1 + b_4 > 0. \quad (7.19)$$

The two independent conditions are

$$(b_1 + b_2)(b_1 + b_4) + (\bar{b}_2 - \bar{b}_1)^2 < 0, \quad b_1 + b_2 < 0. \quad (7.20)$$

Comparing these conditions to the first two relations in Eq. (50) of Ref. [25], one finds a complete agreement (the third relation is redundant).

2. Positivity of $m_{\pm}^2(2)$

The mass matrix for the spin-2 modes is found in subsection [VIC](#):

$$M_2 = 2a_0K_2^{-1}N_2 = \frac{2a_0}{\det K_2}f_2,$$

$$\det K_2 = -a_0a_1 \det B_2,$$

$$\det N_2 = -[(a_0 - a_1)^2 + (\bar{a}_0 - \bar{a}_1)^2].$$

The positivity of the mass eigenvalues is expressed by the requirements

$$\frac{\det N_2}{\det K_2} > 0, \quad \frac{2a_0}{\det K_2} \text{tr}f_2 > 0. \quad (7.21)$$

The condition $\det N_2 < 0$ implies $\det K_2 < 0$, whereupon, relying on $\det B_2 < 0$, one obtains

$$a_1 < 0, \quad (7.22a)$$

$$\text{tr}f_2 < 0, \quad (7.22b)$$

where $\text{tr}f_2$ is calculated in subsection VIC.

3. Is the spin-2 sector free of ghosts and tachyons?

Let us recall that in PG^+ , the conditions $a_1 < 0$ and $b_1 + b_2 < 0$ imply $\text{tr}f_2 > 0$, so that one of the two spin-2 $^\pm$ modes is always a tachyon, as is well known. In what follows, we will prove, somewhat unexpectedly, that the same conclusion also holds in the general PG.

To show this, we rewrite $\text{tr}f_2$ in a compact notation as

$$\begin{aligned} \text{tr}f_2 &= \alpha_2(b_1 + b_2) + \beta_2(\bar{b}_2 - \bar{b}_1) + \gamma_2(b_1 + b_4), \\ \alpha_2 &< 0, \end{aligned} \quad (7.23a)$$

where the coefficients α_2 , β_2 and γ_2 can be read from Eq. (6.15b),

$$\begin{aligned} \alpha_2 &= -a_0(a_0 - a_1), & \beta_2 &= -2a_0(\bar{a}_0 - \bar{a}_1)^2, \\ \gamma_2 &= -a_1(a_0 - a_1) + (\bar{a}_0 - \bar{a}_2)^2, \end{aligned}$$

and $\alpha_2 < 0$ follows from Eq. (7.22a). Since $b_1 + b_4 > 0$, one finds

$$\frac{\text{tr}f_2}{b_1 + b_4} = \alpha_2 \frac{b_1 + b_2}{b_1 + b_4} + \beta_2 \frac{\bar{b}_2 - \bar{b}_1}{b_1 + b_4} + \gamma_2. \quad (7.23b)$$

Having in mind the first relation in Eq. (7.20), written as

$$\frac{b_1 + b_2}{b_1 + b_4} + x^2 < 0, \quad x := \frac{\bar{b}_2 - \bar{b}_1}{b_1 + b_4},$$

we find it useful to rewrite Eq. (7.23b) in an equivalent form,

$$\begin{aligned} \frac{\text{tr}f_2}{b_1 + b_4} &= \alpha_2 \left(\frac{b_1 + b_2}{b_1 + b_4} + x^2 \right) + F_2(x), \\ F_2(x) &:= -\alpha_2 x^2 + \beta_2 x + \gamma_2. \end{aligned} \quad (7.23c)$$

A critical argument in our analysis comes from the observation that the discriminant of the quadratic function $F_2(x)$, $\Delta_2 = \beta_2^2 + 4\alpha_2\gamma_2$, is automatically negative,

$$\begin{aligned} \Delta_2 &= 4a_0a_1[(\bar{a}_0 - \bar{a}_1)^2 + (a_0 - a_1)^2] \\ &= -4a_0a_1 \det N_2 < 0. \end{aligned} \quad (7.24)$$

Next, since $\alpha_2 < 0$ (the parabola F_2 opens upward) and $\Delta_2/\alpha_2 > 0$ (minimum of F_2 is positive), it follows that $F_2(x) > 0$ for any x . Hence, using Eq. (7.20), one obtains the result

$$\text{tr}f_2 > (b_1 + b_4)F_2(x) > 0, \quad (7.25)$$

which contradicts to (7.22b). Thus,

S2. The two no-tachyon conditions in Eqs. (7.22a) and (7.22b) are mutually exclusive; hence, the two spin-2 torsion modes cannot be simultaneously physical.

Such a conclusion is not in agreement with the result found by Karananas [25].

4. No-ghost conditions: spin-2 versus spin-1 sector

The no-ghost conditions for spin-1 and spin-2 sectors are in contradiction to each other. Indeed, Eq. (7.12) implies that $b_2 > b_4$, whereas Eq. (7.19) implies that $b_4 > b_2$. Hence, only one of these two sectors can be physical. The result is in agreement with the Corrigendum in [25].

D. Reality conditions

The structure of the general reality conditions in Eq. (7.3) looks rather cumbersome. However, after replacing $|\text{tr}f_0|$, $|\text{tr}f_1|$, and $|\text{tr}f_2|$ with their minimal values, calculated from the inequalities (F3), (F6), and (7.25), respectively, the reality conditions in Eq. (7.3) transform into

$$\begin{aligned} \text{spin } 0: & (b_4 + b_6)^2 a_2 \det N_0 + 12a_0(2a_0 + a_2)^2 \det B_0 < 0, \\ \text{spin } 1: & g(b_2 + b_5)^2 (\det A)(\det N_1) - 24\alpha_1^2 \det B_1 < 0, \\ \text{spin } 2: & (b_1 + b_4)^2 a_1 \det N_2 + 4a_0(a_0 - a_1)^2 \det B_2 > 0; \end{aligned} \quad (7.26)$$

see Sec. VI and Appendix F. These formulas are much simpler than Eq. (7.3), but they represent only sufficient conditions for the reality of the corresponding mass eigenvalues.

VIII. SUMMARY AND CONCLUSIONS

In this paper, we investigated generic aspects of the Hamiltonian structure of the general parity-violating PG, and used them to study the torsion particle spectrum [30].

Making use of Dirac's Hamiltonian approach, we identified the set of all *if-constraints*, the expressions that become true constraints if the corresponding *critical parameters* c_n vanish. Both the *if-constraints* and the associated critical parameters have a crucial influence on the PG dynamics. Then, we constructed the generic form of the *canonical Hamiltonian* \mathcal{H}_c , determined by taking all the critical parameters to be nonvanishing. An extension of the procedure to allow for a proper treatment of the vanishing critical parameters is outlined in Appendix D.

Apart from being important by itself, the Hamiltonian structure introduced here turns out to be intrinsically related to the particle spectrum of PG. To examine that subject, we first calculated the *mass eigenvalues* $m_{\pm}^2(J)$ of the torsion modes with spin $J = 0, 1$, and 2 , relying on the weak field approximation of the gravitational field equations around M_4 . As a test of the results, we verified that $m_{\pm}^2(J)$ are proportional to the inverse critical parameters $1/c_n$. As a consequence, whenever some of c_n vanish, the corresponding values of $m_{\pm}^2(J)$ become infinite, thereby preventing the associated torsion modes from propagating. This is consistent with the canonical effects of the vanishing critical parameters in PG^+ (in the weak field approximation). A

comparison of our mass formulas to those found by Karananas [25] leads to the following conclusions:

- (k1) For the spin-0 and spin-1 torsion modes, our results confirm those of Karananas.
- (k2) For the spin-2 modes, there are certain differences, noted already in Ref. [27].

The *absence of ghosts* (positivity of energy) in the particle spectrum is ensured by demanding the positivity of the specific spin- J terms in the canonical Hamiltonian, whereas the conditions for the *absence of tachyons* are defined by the requirement $m_{\pm}^2(J) > 0$. A detailed analysis shows that these requirements can be formulated as follows:

$$\begin{aligned}
 \text{Spin 0: } & (b_2 + b_3)(b_4 + b_6) + (\bar{b}_2 - \bar{b}_3)^2 < 0, & b_2 + b_3 < 0, \\
 & a_2[(2a_0 + a_2)(a_0/2 + a_3) + (\bar{a}_0 - \bar{a}_2)^2] < 0, & a_2(2a_0 + a_2) > 0. \\
 \text{Spin 1: } & (b_2 + b_5)(b_4 + b_5) + (\bar{b}_2 - \bar{b}_5)^2 < 0, & b_4 + b_5 < 0, \\
 & \text{(i) } (2a_1 + a_2)(a_1/2 + a_3) + (\bar{a}_1 - \bar{a}_2)^2 < 0, & (2a_0 + a_2)(a_0/2 + a_3) + (\bar{a}_0 - \bar{a}_2)^2 > 0, \\
 & & (a_0 - a_1)[(a_0/2 + a_3)(2a_1 + a_2) + (\bar{a}_0 - \bar{a}_2)^2] - 2(a_0/2 + a_3)(\bar{a}_0 - \bar{a}_1)^2 < 0; \\
 & \text{(ii) an alternative set of conditions, obtained by (i) } \rightarrow (-1) \times (i). \\
 \text{Spin 2: } & (b_1 + b_2)(b_1 + b_4) + (\bar{b}_2 - \bar{b}_1)^2 < 0, & b_1 + b_2 < 0, \\
 & \text{the conditions for the absence of tachyons are } \textit{mutually exclusive}. & \tag{8.1}
 \end{aligned}$$

The results for the absence of ghosts (first line in each spin sector) are identical to those of Karananas, whereas the formulas describing the absence of tachyons show a number of less or more serious differences. In particular, the whole set of conditions (ii) in the spin-1 sector is missing in Karananas's analysis, but the most important difference is found in the spin-2 sector, where the two conditions for the absence of tachyons are in contradiction to each other, in contrast to Karananas's conclusion.

The presence of square roots in the expressions for the mass eigenvalues $m_{\pm}^2(J)$ requires us to verify their reality. A sufficient form of the *reality conditions*, compactly presented at the end of Sec. VII, is much simpler than their general form.

In conclusion, our analysis clarifies the structure of the particle spectrum of the general PG by improving the results found by Karananas, in particular the status of the spin-2 sector. On the other hand, elements of the Hamiltonian structure introduced here, including its extension to the case of vanishing critical parameters outlined in Appendix D, are a good starting point for further investigation of the full nonlinear dynamics of PG.

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APPENDIX A: IRREDUCIBLE DECOMPOSITION OF THE FIELD STRENGTHS

The torsion tensor has three irreducible pieces:

$$\begin{aligned}
 {}^{(2)}T_{imn} &= \frac{1}{3}(\eta_{im}\mathcal{V}_n - \eta_{in}\mathcal{V}_m), \\
 {}^{(3)}T_{imn} &= \varepsilon_{imnk}\mathcal{A}^k, \\
 {}^{(1)}T_{imn} &= T_{imn} - {}^{(2)}T_{imn} - {}^{(3)}T_{imn} = \frac{4}{3}t_{i[mn]}, \tag{A1a}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{V}_n &:= T^k{}_{kn}, & \mathcal{A}_k &:= \frac{1}{6}\varepsilon_{krst}T^{rst}, \\
 t_{imn} &:= T_{(im)n} + \frac{1}{3}\eta_{n(i}\mathcal{V}_{m)} - \frac{1}{3}\eta_{im}\mathcal{V}_n. \tag{A1b}
 \end{aligned}$$

The Riemann-Cartan curvature tensor can be decomposed into six irreducible pieces:

$$\begin{aligned}
(2)R_{ijmn} &= \frac{1}{2}(\eta_{ik}\Psi_{jl} - \eta_{jk}\Psi_{il})\varepsilon^{kl}_{mn}, \\
(3)R_{ijmn} &= \frac{1}{12}X\varepsilon_{ijmn}, \\
(4)R_{ijmn} &= \frac{1}{2}(\eta_{im}\Phi_{jn} - \eta_{jm}\Phi_{in}) - (m \leftrightarrow n), \\
(5)R^{ij} &= \frac{1}{2}(\eta_{im}\hat{R}_{[jn]} - \eta_{jm}\hat{R}_{[in]}) - (m \leftrightarrow n), \\
(6)R_{ijmn} &= \frac{1}{12}R(\eta_{im}\eta_{jn} - \eta_{jm}\eta_{in}), \\
(1)R_{ijmn} &= R_{ijmn} - \sum_{a=2}^6 (a)R_{ijmn}, \tag{A2a}
\end{aligned}$$

where

$$\begin{aligned}
\hat{R}_{im} &:= Ric_{im} = R_{inm}{}^n, & R &:= Ric^m{}_m, \\
X_{ij} &:= \frac{1}{2}R_{ikmn}\varepsilon^{kmn}{}_j, & X &:= X^n{}_n, \\
\Phi_{ij} &:= Ric_{(ij)} - \frac{1}{4}\eta_{ij}R, & \Psi_{ij} &:= X_{(ij)} - \frac{1}{4}\eta_{ij}X. \tag{A2b}
\end{aligned}$$

The above definitions are the tensor counterparts of the corresponding formulas given in terms of the differential forms; see [27,29]. They imply the following relations characterizing the parity-odd sector:

$$\begin{aligned}
T^{ijk\star(2)}T_{ijk} &= T^{ijk\star(3)}T_{ijk}, \\
R^{ijkl\star(2)}R_{ijkl} &= R^{ijkl\star(4)}R_{ijkl}, \\
R^{ijkl\star(3)}R_{ijkl} &= R^{ijkl\star(6)}R_{ijkl}, \tag{A3a}
\end{aligned}$$

and also

$$\begin{aligned}
T^{ijk\star(1)}T_{ijk} &= (1)T^{ijk\star(1)}T_{ijk}, \\
R^{ijkl\star(1)}R_{ijkl} &= (1)R^{ijkl\star(1)}R_{ijkl}, \\
R^{ijkl\star(5)}R_{ijkl} &= (5)R^{ijkl\star(5)}R_{ijkl}. \tag{A3b}
\end{aligned}$$

APPENDIX B: ALTERNATIVE FORM OF THE LAGRANGIAN

In this appendix, we rewrite our Lagrangian (2.6) in an equivalent form that allows an easier comparison to the literature [22,25]:

$$\begin{aligned}
\mathcal{L}_G &= -(a_0R + 2\Lambda_0 + \bar{a}_0X) + \mathcal{L}_{T^2} + \mathcal{L}_{R^2}, \\
\mathcal{L}_{T^2} &= h_1T^{ijk}T_{ijk} + h_2T^{imn}T_{nmi} + h_3V_mV^n \\
&\quad + \varepsilon^{mnkl}(\bar{h}_4T^i{}_{mn}T_{ikl} + \bar{h}_5T_{mn}{}^iT_{kli}), \tag{B1a}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{R^2} &= \frac{1}{2}(f_1R^{ijmn}R_{ijmn} + f_2R^{ijmn}R_{imjn} + f_3R^{ijmn}R_{mnij} \\
&\quad + f_4Ric^{im}Ric_{im} + f_5Ric^{im}Ric_{mi} + f_6R^2) \\
&\quad + \frac{1}{2}\varepsilon^{mnkl}(\bar{f}_7R_{mnkl}R + \bar{f}_8R_{ijmn}R^{ij}{}_{kl} \\
&\quad + \bar{f}_9R_{mnij}R_{kl}{}^{ij} + \bar{f}_{10}R_{mnij}R^{ij}{}_{kl}). \tag{B1b}
\end{aligned}$$

The parameters (h_n, \bar{h}_n) and (f_n, \bar{f}_n) can be expressed in terms of the ‘‘irreducible’’ parameters appearing in Eq. (2.6), as follows:

$$\begin{aligned}
h_1 &= \frac{1}{6}(2a_1 + a_3), & h_2 &= \frac{1}{3}(a_1 - a_3), \\
h_3 &= -\frac{1}{3}(a_1 - a_2), \\
\bar{h}_4 &= -\frac{1}{24}(4\bar{a}_1 + \bar{a}_2 + \bar{a}_3), & \bar{h}_5 &= -\frac{1}{6}(2\bar{a}_1 - \bar{a}_2 - \bar{a}_3), \tag{B2a}
\end{aligned}$$

and

$$\begin{aligned}
f_1 &:= \frac{1}{12}(2b_1 + 3b_2 + b_3), & f_2 &:= \frac{1}{3}(b_1 - b_3), \\
f_3 &:= \frac{1}{12}(2b_1 - 3b_2 + b_3), & f_4 &:= -\frac{1}{2}(b_1 + b_2 - b_4 - b_5), \\
f_5 &:= -\frac{1}{2}(b_1 - b_2 - b_4 + b_5), & f_6 &:= \frac{1}{12}(2b_1 - 3b_4 + b_6), \\
\bar{f}_7 &:= \frac{1}{24}(2\bar{b}_1 - \bar{b}_3 - \bar{b}_6), & \bar{f}_8 &:= -\frac{1}{16}(\bar{b}_1 + \bar{b}_2 + \bar{b}_4 + \bar{b}_5), \\
\bar{f}_9 &:= -\frac{1}{16}(\bar{b}_1 - \bar{b}_2 - \bar{b}_4 + \bar{b}_5), & \bar{f}_{10} &:= -\frac{1}{8}(\bar{b}_1 - \bar{b}_5). \tag{B2b}
\end{aligned}$$

Relying on the existence of three topological invariants (2.9), Karananas [25] imposed three conditions on the Lagrangian parameters in (B1): $\bar{a}_0, f_6, \bar{f}_8 = 0$.

APPENDIX C: (3+1) DECOMPOSITION OF SPACETIME

The dynamical content of canonical constraints is greatly clarified by using a decomposition of tensor fields with respect to the subgroup of 3d rotations in the spatial hypersurface Σ_0 : $x^0 = \text{const}$.

Let e_α be a basis of three coordinate tangent vectors in Σ_0 , $e_\alpha = \partial_\alpha$ ($\alpha = 1, 2, 3$), and \mathbf{n} the unit normal to Σ_0 , $n_k = h_k^0/\sqrt{g^{00}}$. The four vectors (\mathbf{n}, e_α) define the so-called ADM basis of tangent vectors in spacetime. The decomposition of the vector e_0 in the ADM basis is given by

$$\mathbf{e}_0 = N\mathbf{n} + N^\alpha \mathbf{e}_\alpha, \quad (\text{C1a})$$

where N and N^α , known as the lapse and shift functions, respectively, are linear in b^k_0 :

$$\begin{aligned} N &= \mathbf{e}_0 \mathbf{n} = n_k b^k_0 = 1/\sqrt{g^{00}}, \\ N^\alpha &= \mathbf{e}_0 \mathbf{e}_\beta^3 g^{\beta\alpha} = h_{\bar{k}}^\alpha b^k_0 = -g^{0\alpha}/g^{00}. \end{aligned} \quad (\text{C1b})$$

Introducing the projectors on \mathbf{n} and \mathbf{e}_α , given respectively by

$$(P_\perp)_k^l = n_k n^l, \quad (P_\parallel)_k^l = \delta_k^l - n_k n^l,$$

one can express a spacetime vector V_k in terms of its orthogonal (to Σ_0) and ‘‘parallel’’ (living in the tangent space of Σ_0) components:

$$V_k = n_k V_\perp + V_{\bar{k}}, \quad (\text{C2})$$

where $V_\perp := n^k V_k$ and $V_{\bar{k}} := V_k - n_k V_\perp$. Here, we use a convention that a bar over an index k denotes its parallel projection, so that $n^k V_{\bar{k}}$ vanishes. The objects V_\perp and $V_{\bar{k}}$ are respectively a scalar and a vector with respect to 3d rotations in Σ_0 .

Consider now a second-rank tensor, X_{ik} . Its orthogonal-parallel decomposition reads

$$X_{ik} = n_i X_{\perp\bar{k}} + n_i n_k X_{\perp\perp} + n_k X_{\bar{i}\perp} + X_{\bar{i}\bar{k}}. \quad (\text{C3})$$

Here, $X_{\perp\bar{k}}$ is a vector and $X_{\perp\perp}$ a scalar with respect to 3d rotations, whereas the irreducible parts of $X_{\bar{i}\bar{k}}$ are its trace, antisymmetric, and traceless symmetric parts:

$$\begin{aligned} {}^S X &:= X_{\bar{i}\bar{k}}, & {}^A X_{\bar{i}\bar{k}} &:= X_{[\bar{i}\bar{k}]}, & {}^T X_{\bar{i}\bar{k}} &:= X_{(\bar{i}\bar{k})} - \frac{1}{3} \eta_{\bar{i}\bar{k}} X^{\bar{m}\bar{m}}, \\ X_{\bar{i}\bar{k}} &= {}^A X_{\bar{i}\bar{k}} + {}^T X_{\bar{i}\bar{k}} + \frac{1}{3} \eta_{\bar{i}\bar{k}} {}^S X. \end{aligned} \quad (\text{C4a})$$

As a consequence,

$$X^{\bar{i}\bar{k}} Y_{\bar{i}\bar{k}} = {}^A X_{\bar{i}\bar{k}} {}^A Y_{\bar{i}\bar{k}} + {}^T X_{\bar{i}\bar{k}} {}^T Y_{\bar{i}\bar{k}} + \frac{1}{3} {}^S X {}^S Y. \quad (\text{C4b})$$

Now, it is straightforward to extend these considerations to any tensor. As a particularly interesting example, we consider the spacetime tensor $X_{ijk} = -X_{jik}$, which is decomposed into the set of spatial tensors ($X_{\perp\perp\perp}, X_{\perp\bar{j}\bar{k}}, X_{\bar{i}\bar{j}\perp}, X_{\bar{i}\bar{j}\bar{k}}$). The irreducible parts of $X_{\bar{i}\bar{j}\bar{k}} = -X_{\bar{j}\bar{i}\bar{k}}$ are the pseudoscalar, the vector, and the traceless symmetric tensor, respectively:

$$\begin{aligned} {}^P X &:= \epsilon^{\bar{i}\bar{j}\bar{k}} X_{\bar{i}\bar{j}\bar{k}}, & {}^V X_{\bar{i}} &:= X_{\bar{i}\bar{k}}^{\bar{k}}, \\ {}^T X_{\bar{i}\bar{j}\bar{k}} &:= X_{\bar{i}(\bar{j}\bar{k})} + \frac{1}{2} \eta_{\bar{i}(\bar{j}} {}^V X_{\bar{k})} - \frac{1}{2} \eta_{\bar{j}\bar{k}} {}^V X_{\bar{i}}. \end{aligned} \quad (\text{C5a})$$

The tensor part satisfies the cyclic identity ${}^T X_{\bar{i}\bar{j}\bar{k}} + {}^T X_{\bar{k}\bar{i}\bar{j}} + {}^T X_{\bar{j}\bar{k}\bar{i}} = 0$. The epsilon tensor $\epsilon_{\bar{i}\bar{j}\bar{k}}$ is defined by $\epsilon_{\bar{i}\bar{j}\bar{k}} := \epsilon_{\perp\bar{i}\bar{j}\bar{k}}$ and satisfies the identities

$$\begin{aligned} \epsilon_{\bar{i}\bar{j}\bar{k}} \epsilon^{\bar{m}\bar{n}\bar{k}} &= -(\delta_{\bar{i}}^{\bar{m}} \delta_{\bar{j}}^{\bar{n}} - \delta_{\bar{j}}^{\bar{m}} \delta_{\bar{i}}^{\bar{n}}), \\ \epsilon_{\bar{i}\bar{n}\bar{k}} \epsilon^{\bar{m}\bar{n}\bar{k}} &= -3\delta_{\bar{i}}^{\bar{m}}, & \epsilon_{\bar{m}\bar{n}\bar{k}} \epsilon^{\bar{m}\bar{n}\bar{k}} &= -6. \end{aligned}$$

The related decomposition formulas read:

$$\begin{aligned} X_{\bar{i}\bar{j}\bar{k}} &= \frac{4}{3} {}^T X_{[\bar{i}\bar{j}]\bar{k}} - \eta_{\bar{k}[\bar{i}} {}^V X_{\bar{j}]} - \frac{1}{6} \epsilon_{\bar{i}\bar{j}\bar{k}} {}^P X, \\ X^{\bar{i}\bar{j}\bar{k}} Y_{\bar{i}\bar{j}\bar{k}} &= \frac{4}{3} {}^T X^{\bar{i}\bar{j}\bar{k}} {}^T Y_{\bar{i}\bar{j}\bar{k}} + {}^V X^{\bar{i}V} Y_{\bar{i}} - \frac{1}{6} {}^P X {}^P Y. \end{aligned} \quad (\text{C5b})$$

APPENDIX D: GENERAL CONSTRUCTION OF \mathcal{H}_\perp

In this appendix, we discuss the general structure of \mathcal{H}_\perp , including the case when some of the critical parameters vanish. In a simplified but self-evident notation, the relations that define critical parameters have the following typical form (see Sec. III):

$$\varphi = FV, \quad (\text{D1})$$

where

$$\varphi := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad F := \begin{pmatrix} a & \bar{b} \\ \bar{c} & d \end{pmatrix}, \quad V := \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

Here, φ represents the if-constraints, V are the corresponding velocities, and F is the matrix with eigenvalues c_1, c_2 . Since F is chosen to represent A, B_0, B_1 or B_2 , the parameter \bar{c} is proportional to \bar{b} , $\bar{c} = \kappa \bar{b}$. If $\bar{b} = 0$, the matrix F is already diagonal, and the construction of \mathcal{H}_\perp is quite simple. When $\bar{b} \neq 0$, which is typical for the parity-violating PG, the matrix F needs first to be diagonalized. The diagonal form D of F is constructed as

$$\begin{aligned} D &= P^{-1} F P, & P &:= \begin{pmatrix} -\bar{b} & -\bar{b} \\ a - c_1 & a - c_2 \end{pmatrix}, \\ D &= \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \end{aligned} \quad (\text{D2})$$

where P is invertible provided $\det P = \bar{b}(c_2 - c_1) \neq 0$, and

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} a - c_2 & \bar{b} \\ -a + c_1 & -\bar{b} \end{pmatrix}.$$

Left multiplication of (D1) by P^{-1} yields

$$\varphi' = D V', \quad (\text{D3a})$$

where $\varphi' := P^{-1}\varphi$ and $V' := P^{-1}V$, or, equivalently,

$$\varphi'_1 = c_1 V'_1, \quad \varphi'_2 = c_2 V'_2. \quad (\text{D3b})$$

To construct the related F -part of \mathcal{H}_\perp , note that its typical form reads

$$\mathcal{H}_\perp^F = \varphi'^T Q V \equiv \varphi'^T (P^T Q P) V', \quad Q := \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}; \quad (\text{D4})$$

see Sec. IV. Further discussion depends on the specific values of c_1 and c_2 .

(1) When $c_1, c_2 \neq 0$, Eq. (D1) implies $V = F^{-1}\varphi$, and $\mathcal{H}_\perp^F = \varphi^T Q F^{-1}\varphi$ coincides with the result found in Sec. IV. (2) The case $c_1 = c_2 = 0$ is rather trivial: both if-constraints φ'_n become true constraints that appear in the total Hamiltonian, but $\mathcal{H}_\perp^F = 0$. (3) Finally, when only one critical parameter vanishes (which requires $\det F = 0$), say $c_2 = 0$, then $\varphi'_2 = 0$ (a new constraint), V'_2 remains undetermined, and $\varphi'_1 = c_1 V'_1$. Hence, Eq. (D4) implies that

$$\mathcal{H}_\perp^F = (\bar{b}^2 q_1 + d^2 q_2) \frac{1}{c_1} (\varphi'_1)^2 + \varphi'_1 (\bar{b}^2 q_1 - ad q_2) V'_2. \quad (\text{D5})$$

The result can be also expressed in terms of the original if-constraints φ_n by noting that $\varphi'_2 = 0$ implies $\varphi'_1 = -\varphi_1/\bar{b}$. The factor $1/c_1$ in the first term shows a typical dependence on the critical parameters, known from PG⁺, whereas the second term, linear in the undetermined velocity V'_2 , can be absorbed into the total Hamiltonian; see [14,15,23]. The presence of an extra constraint φ'_2 requires us to complete the whole consistency procedure.

In the context of the weak field approximation, the form of \mathcal{H}_\perp^F in Eq. (D5) determines the no-ghost conditions for the case (3):

$$\det F = ad - \bar{b}\bar{c} = 0, \quad \sigma c_1 > 0, \quad (\text{D6})$$

where σ is the sign of $(\bar{b}^2 q_1 + d^2 q_2)$ and $c_1 = a + d$.

Now, we have a comment on kind of “non-analyticity” of the above results. Since the assumption $\bar{b} \neq 0$ ensures the regularity of the matrix P , the diagonal matrix D in Eq. (D2) has no valid limit for $\bar{b} \rightarrow 0$. Hence, the expressions for c_n when $\bar{b} = 0$ cannot be obtained by taking the limit $\bar{b} \rightarrow 0$ of the generic result. However, since the matrix F for $\bar{b} = 0$ is already diagonal, the critical parameters c_n can be obtained directly from F . The same conclusion also holds for the form of \mathcal{H}_\perp^F .

APPENDIX E: LINEARIZED BIANCHI IDENTITIES

In Secs. V and VI, many technical simplifications were obtained with the help of the linearized Bianchi identities,

$$\varepsilon^{\mu\nu\lambda\rho} R^i{}_{j\nu\rho} = 0, \quad \varepsilon^{\mu\nu\lambda\rho} \partial_\nu T_{i\lambda\rho} = \varepsilon^{\mu\nu\lambda\rho} R_{i\nu\lambda\rho}, \quad (\text{E1})$$

and their consequences. In particular, the first identity implies that

$$\partial^k X_{ik} = 0, \quad \partial^i G_{ik} = 0, \quad (\text{E2})$$

where $G_{ik} := Ric_{ik} - (1/2)\eta_{ik}R$, and the second identity yields

$$\begin{aligned} X_i{}^j &= -\frac{1}{2}\varepsilon^{jkmn}\partial_k T_{imn}, & X &= 3\partial\mathcal{A}, \\ \varepsilon^{ijmn}R_{ijmk} &= 2X_k{}^n - \delta_k^n X, \\ 2Ric_{[mn]} &= -\partial^k T_{kmn} + 2\partial_{[m}\mathcal{V}_{n]}. \end{aligned} \quad (\text{E3})$$

As a consequence,

$$\partial^m \Phi_{im} = \partial^m \hat{R}_{[im]} + \frac{1}{4}\partial_i R, \quad \partial^m \Psi_{im} = \partial^m X_{[mi]} - \frac{1}{4}\partial_i X. \quad (\text{E4})$$

APPENDIX F: SIMPLIFIED CONDITIONS FOR THE ABSENCE OF TACHYONS

In this appendix, we derive a simplified form of the conditions (7.9) and (7.16), describing the absence of tachyons in the spin-0 and spin-1 sectors, respectively; the spin-2 sector is discussed in subsection VII C.

1. Spin-0 sector

The expression for $\text{tr}f_0$, found in subsection VI A, can be represented in a suitable form as

$$\frac{1}{3}\text{tr}f_0 = \alpha_0(b_2 + b_3) + \beta_0(\bar{b}_2 - \bar{b}_3) + \gamma_0(b_4 + b_6), \quad (\text{F1a})$$

where

$$\begin{aligned} \alpha_0 &= a_0(2a_0 + a_2), & \beta_0 &= 4a_0(\bar{a}_2 - \bar{a}_0), \\ \gamma_0 &= -[a_2(a_0 + 2a_3) + 2(\bar{a}_0 - \bar{a}_2)^2]. \end{aligned}$$

After dividing this equation by $(b_4 + b_6) > 0$, one obtains

$$\frac{\text{tr}f_0}{3(b_4 + b_6)} = \alpha_0 \frac{b_2 + b_3}{b_4 + b_6} + \beta_0 \frac{\bar{b}_2 - \bar{b}_3}{b_4 + b_6} + \gamma_0.$$

By noting that the first relation in Eq. (7.6) can be written as

$$\frac{b_2 + b_3}{b_4 + b_6} + x^2 < 0, \quad x := \frac{\bar{b}_2 - \bar{b}_3}{b_4 + b_6},$$

we find it useful to rewrite Eq. (F1a) in the form

$$\begin{aligned} \frac{\text{tr}f_0}{3(b_4 + b_6)} &= \alpha_0 \left(\frac{b_2 + b_3}{b_4 + b_6} + x^2 \right) + F_0(x), \\ F_0(x) &:= -\alpha_0 x^2 + \beta_0 x + \gamma_0. \end{aligned} \quad (\text{F1b})$$

Further analysis is based on an important property of the quadratic function $F_0(x)$, based on Eq. (7.9a); its discriminant, $\Delta_0 = \beta_0^2 + 4\alpha_0\gamma_0$, is always negative,

$$\begin{aligned} \Delta_0 &= -4a_0a_2[(2a_0 + a_2)(a_0 + 2a_3) + 2(\bar{a}_0 - \bar{a}_2)^2] \\ &\equiv (4/3)a_0a_2 \det N_0 < 0. \end{aligned} \quad (\text{F2})$$

Similar considerations applied to $a_2\text{tr}f_0$ modify Eq. (F1b) by an overall multiplicative factor a_2 . To simplify the discussion, we introduce a suitable notation: $\alpha'_0 := a_2\alpha_0$ and $F'_0(x) := a_2F_0(x)$. Note that the discriminant Δ'_0 of the new function $F'_0(x)$ remains negative. Now, we are ready to prove the following statement:

S0. Given $\Delta_0 < 0$, the condition $\alpha'_0 \equiv a_2\alpha_0 > 0$ is equivalent to $a_2\text{tr}f_0 < 0$.

To prove this equivalence, we start by assuming $\alpha'_0 > 0$, which implies

$$a_2\text{tr}f_0 < 3(b_4 + b_6)F'_0(x). \quad (\text{F3})$$

Moreover, the parabola $F'_0(x)$ opens downward, and $\Delta'_0/\alpha'_0 < 0$ (negative at vertex) ensures that $F'_0(x) < 0$ for any x . Hence, $a_2\text{tr}f_0 < 0$, what was to be shown.

The reverse statement $a_2\text{tr}f_0 < 0 \Rightarrow \alpha'_0 > 0$ can be easily proven by reductio ad absurdum, that is, by showing that $\alpha'_0 < 0$ implies $a_2\text{tr}f_0 > 0$, which is a contradiction.

The statement **S0** allows us to replace Eq. (7.9b) with the much simpler condition $a_2 > 0$.

2. Spin-1 sector

For the spin-1 sector, we first rewrite $\text{tr}f_1$ in the form

$$\frac{1}{4}\text{tr}f_1 = \alpha_1(b_4 + b_5) + \beta_1(\bar{b}_2 - \bar{b}_5) + \gamma_1(b_2 + b_5), \quad (\text{F4a})$$

where

$$\begin{aligned} \alpha_1 &:= 2\mu_3k_2A_0 - 4\mu_3\bar{A}_0^2 + 2A_0(\bar{a}_0 - \bar{a}_2)^2, \\ \beta_1 &:= -4\mu_2\mu_3\bar{A}_0 + 4(A_0^2 + \bar{A}_0^2)(\bar{a}_0 - \bar{a}_2) - 4\bar{A}_0(\bar{a}_0 - \bar{a}_2)^2, \\ \gamma_1 &:= 2A_0\mu_2k_3 - \mu_2\bar{A}_0^2 + 2A_0(\bar{a}_0 - \bar{a}_2)^2. \end{aligned}$$

After dividing by $(b_2 + b_5) > 0$, one can rewrite Eq. (F4a) in a suitable form

$$\begin{aligned} \frac{1}{4(b_2 + b_5)}\text{tr}f_1 &= \alpha_1 \left(\frac{b_4 + b_5}{b_2 + b_5} + x^2 \right) + F_1(x), \\ F_1(x) &:= -\alpha_1x^2 + \beta_1x + \gamma_1, \quad x := \frac{\bar{b}_2 - \bar{b}_5}{b_2 + b_5}. \end{aligned} \quad (\text{F4b})$$

As a consequence of Eq. (7.16a), the discriminant Δ_1 of the quadratic function $F_1(x)$ is automatically negative,

$$\begin{aligned} \Delta_1 &:= 16(A_0^2 + \bar{A}_0^2)[\mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2][k_2k_3 + (\bar{a}_1 - \bar{a}_2)^2] \\ &\equiv \frac{2}{3}(A_0^2 + \bar{A}_0^2)(\det N_1)(\det A) < 0. \end{aligned} \quad (\text{F5})$$

To relate our considerations to the properties of $(\det A)\text{tr}f_1$, we multiply Eq. (F4b) by $\det A$, and introduce a suitable notation $\alpha'_1 := (\det A)\alpha_1$ and $F'_1(x) := (\det A)F_1(x)$. The new discriminant Δ'_1 is also negative. Now, one can prove the following statement:

S1. For $\Delta_1 < 0$, the condition $\alpha'_1 \equiv (\det A)\alpha_1 < 0$ is equivalent to $(\det A)\text{tr}f_1 > 0$.

The proof goes as follows. Starting with $\alpha'_1 < 0$, one obtains

$$(\det A)\text{tr}f_1 > 4(b_2 + b_5)F'_1(x). \quad (\text{F6})$$

Then, by noting that the parabola opens upward ($\alpha'_1 < 0$) and $\Delta'_1/\alpha'_1 > 0$ (positive at vertex), one concludes that $F'_1(x) > 0$. Hence, $(\det A)\text{tr}f_1 > 0$.

As before, the reverse statement $(\det A)\text{tr}f_1 > 0 \Rightarrow \alpha'_1 < 0$ can be proven by showing that $\alpha'_1 > 0$ leads to $(\det A)\text{tr}f_1 < 0$, which is a contradiction.

The condition $\Delta_1 < 0$, combined with $(\det A)\alpha_1 < 0$, can be realized in two ways:

- (i) $\det A > 0, \det N_1 < 0, \alpha_1 < 0$.
- (ii) $\det A < 0, \det N_1 > 0, \alpha_1 > 0$. (F7)

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