Propagation of gravitational waves in strong magnetic fields

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The propagation of gravitational waves is explored in the cosmological context. It is explicitly demonstrated that the propagation of gravitational waves could be influenced by the medium. It is shown that in the thermal radiation, the propagation of gravitational waves in general relativity is different from that in the scalar-tensor theory. The propagation of gravitational waves is investigated in the uniform magnetic field. As a result, it is found that cosmic magnetic fields could influence the propagation of gravitational waves to a non-negligible extent. The corresponding estimation for the spiral galaxy NGC 6946 effect is made.

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I. INTRODUCTION

It has been proved by LIGO that a two-black-holes system emits strong gravitational waves in the coalescence phase [1]. The first detection was from black holes with about 30 solar masses, and the following ones were from the mergers of two black holes (black hole binary) [2–5]. Very recently, the so-called multimessenger astronomy has started with the discovery of strong gravitational waves from the collision of two neutron stars [6] and the electromagnetic radiation was detected in coincidence with the gravitational wave.

It is very difficult and complicated to analyze the processes of black hole mergers and scatterings because gravitational dynamics is too strong. In spite of the difficulties, there has been accurate numerical simulations, which reproduce the observational results [7,8] although various approximate approaches [9] and analytic ideas [10,11] to calculate the gravitational wave signatures in the strong gravitational field regimes have also been proposed.

On the other hand, the existence of cosmic magnetic fields have been known and those origins have also been explored. In particular, the origins of large-scale magnetic fields observed in clusters of galaxies can be primordial magnetic fields from inflation and the following cosmological phases in the early universe (for reviews on cosmic magnetic fields, see, e.g., [12–21]).

Moreover, various modified gravity theories including the scalar-tensor theory have especially been studied in the cosmological context recently in order to explain the late-time cosmic acceleration (for recent reviews on modified gravity theories as well as the dark energy problem, see, for example, [22–26]). The cosmological bounds from the Neutron Star Merger GW170817 [6] have been examined in the scalar-tensor and $F(R)$ gravity theories [27]. The constraints [28] on alternative theories of gravity have been calculated with GW150914 and GW151226 [2,29,30]. Various features of gravitational waves from modified gravity theories have also been studied [31–33].

In this paper, we clarify how the propagation of gravitational waves could be changed by the medium. Usually the radiation is made of the quanta or relativistic particles at the high temperature as in the early universe after the inflation. In the radiation dominated era, the universe expands as $a \propto t^{-\frac{1}{2}}$. Here a is the scale factor of the universe and t is the cosmological time. On the other hand, it is known that the power law behavior $a \propto t^{-\frac{1}{2}}$ in the radiation dominated universe can also be realized by the classical scalar-tensor theory.[1] In order to distinguish the above two kinds of the

[1]For example, it has been shown that any evolution of the universe expansion can be realized in the scalar-tensor theory in [34]. See also [35,36].

radiation dominated universe, we show that the propagation of gravitational waves in the thermal radiation in general relativity is different from that in the classical scalar-tensor theory. Usually, the radiation is made of photons, which are quanta of the electromagnetic field. The classical electromagnetic field is different from the photon. As an example of the classical electromagnetic field, we investigate the propagation of gravitational waves in the uniform magnetic field and we demonstrate that the effects from the magnetic field to the propagation could not be negligible.

The structure of the paper is the following. In Sec. II, we explore the propagation of gravitational waves in general matter. In Sec. III, we investigate the propagation in quanta and thermal matter with finite temperature. In Sec. IV, we analyze the propagation of gravitational waves under the existence of magnetic fields. In Sec. V, we consider the case of gravitational waves in $F(R)$ gravity. Finally, conclusions are given in Sec. VI.

II. PROPAGATION OF GRAVITATIONAL WAVE IN MATTERS

The gravitational wave is given by the perturbation from the background geometry,

$$g_{\mu\nu} \to g_{\mu\nu} + \kappa^2 h_{\mu\nu}, \tag{1}$$

where $|h_{\mu\nu}| \ll 1$ is the perturbation with respect to a given background $g_{\mu\nu}$. Then by imposing the gauge condition

$$\nabla^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0, \tag{2}$$

the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \tag{3}$$

take the perturbed form as follows:

$$\frac{1}{2}[-\nabla^2 h_{\mu\nu} - 2R^\lambda{}_\nu{}^\rho{}_\mu h_{\lambda\rho} + R^\rho{}_\mu h_{\rho\nu} + R^\rho{}_\nu h_{\rho\mu} - h_{\mu\nu} R + g_{\mu\nu} R^{\rho\lambda} h_{\rho\lambda}] = \kappa^2 \delta T_{\mu\nu}. \tag{4}$$

Let us denote the scale of $\kappa^2 T_{\mu\nu}$ by M^2, that is, $\kappa^2 T_{\mu\nu} \sim M^2$. If we assume that M^2 can be small enough, we can expand the left-hand side (LHS) and the right-hand side (RHS) of the Einstein equation with respect to M^2 as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = I_R^{(0)} + M^2 I_R^{(1)} + M^4 I_R^{(2)} + \mathcal{O}((M^2)^3),$$
$$\kappa^2 T_{\mu\nu} = M^2 I_T^{(1)} + M^4 I_T^{(2)} + \mathcal{O}((M^2)^3). \tag{5}$$

We should note that the RHS starts with the $\mathcal{O}(M^2)$ term, and therefore the $\mathcal{O}(1)$ term $I_R^{(0)}$ in the LHS should vanish, which gives the flat vacuum solution $g_{\mu\nu} = \eta_{\mu\nu}$. Then the $\mathcal{O}(M^2)$ term $M^2 I_T^{(1)}$, which expresses the matters in the flat background and equation $M^2 I_R^{(1)} = M^2 I_T^{(1)}$, gives the $\mathcal{O}(M^2)$ correction to the geometry. We should note that the energy-momentum tensor $T_{\mu\nu}$ in Eq. (3) depends on the metric; therefore the $\mathcal{O}(M^4)$ term $M^4 I_T^{(2)}$ in the RHS expresses the matter in the background with the $\mathcal{O}(M^2)$ correction. Then the equation $M^4 I_R^{(2)} = M^4 I_T^{(2)}$ gives the $\mathcal{O}(M^4)$ correction to the geometry. By iterating the above procedure, we can find the background geometry by the perturbation with respect to M^2. The corrections to the geometry, which includes the gravitational wave, appear as the perturbative series with respect to κ^2. Therefore we have two parameters M^2 and κ^2 for the perturbative expansions. The parameter M^2 is conceptually different from the parameter κ^2, and they are independent from each other. Then the LHS and the RHS of the Einstein equation can be expressed by the double expansion with respect to M^2 and κ^2,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = I_R^{(0)} + M^2 I_R^{(1,0)} + \kappa^2 I_R^{(0,1)} + M^2 \kappa^2 I_R^{(2)} + \mathcal{O}(M^4, \kappa^2, \kappa^2 M^2),$$
$$\kappa^2 T_{\mu\nu} = M^2 I_T^{(1,0)} + M^2 \kappa^2 I_T^{(1,1)} + \mathcal{O}(M^4, \kappa^4). \tag{6}$$

In Eq. (6), $\mathcal{O}(\kappa^2)$, $\kappa^2 I_R^{(0,1)}$, $M^2 \kappa^2 I_R^{(2)}$, $M^2 \kappa^2 I_T^{(1,1)}$, and $\mathcal{O}(\kappa^2 M^4)$ terms express the propagation of the gravitational wave because the energy-momentum tensor $T_{\mu\nu}$ in Eq. (3) depends on the metric, and if we consider the perturbation as (1), there is a variation of the energy-momentum tensor $T_{\mu\nu}$ in (4). On the other hand, the $\mathcal{O}(\kappa^4)$ terms include the nonlinear interaction between the gravitational wave. We now neglect the interactions between the gravitational wave, we omit the $\mathcal{O}(M^4, \kappa^4)$ terms, and we consider the κ^2 terms including the leading corrections with respect to M^2, that is, $\kappa^2 I_R^{(0,1)}$, $M^2 \kappa^2 I_R^{(2)}$, and $M^2 \kappa^2 I_T^{(1,1)}$. The effect of the term $M^2 \kappa^2 I_T^{(1,1)}$ is similar to the propagation of light in the medium (such as water). As we know by the Cerenkov radiation, the speed of the light decreases in the medium. Although the propagation of the light also follows the geodesics, the incident light makes the electric charge or electric or magnetic moment distributions fluctuate, and the fluctuations with the electric or magnetic dipole moments generate the light. The generated light interferes with the incident light, and the decrease of the propagation speed of the light occurs. These effects are known as a polarization and can be expressed as the changes of the permittivity and permeability. Even for the gravitational wave, there occur similar phenomena, which were also recently reported in the paper by Flaugher and Weinberg [37] in detail for the propagation of the gravitational wave in the cold dark matter. The incident gravitational wave makes the medium fluctuate and the fluctuation with a quadrupole moment generates an additional gravitational wave. The RHS in (4) or the term $M^2\kappa^2 I_T^{(1,1)}$ in (6) expresses such effects although our formulation is rather simplified compared with the paper [37].

III. QUANTA AND THERMAL MATTER

In this section, we consider the real scalar field as the matter. We treat the scalar field as the quantum field at the finite temperature. In case of the high temperature or in a massless case, the scalar field plays the role of the radiation. On the other hand, in the limit that the temperature vanishes but the density is finite, we obtain the dust, which can be a cold dark matter. After that, we compare the obtained results with those in the classical scalar-tensor model [27,38].

Even in the classical scalar-tensor model, we can realize a matter dominated (filled with dust) or radiation dominated universe. Then we find that in the case of the matter dominated universe, the result for the quantum field coincides with the result in the classical scalar-tensor theory, but in the other case, the tensor structure of $\delta T_{\mu\nu}$ is different in the two cases; therefore the propagation of the gravitational wave changes in general.

In curved space-time, the energy-momentum tensor of a free real scalar field ϕ with mass M is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \frac{1}{2}M^2\phi^2\right). \tag{7}$$

In the flat background, we find

$$T_{00} = \rho = \frac{1}{2}\left(\pi^2 + \sum_{n=1,2,3}(\partial_n\phi)^2 + M^2\phi^2\right),$$
$$T_{ij} = \partial_i\phi\partial_j\phi + \frac{1}{2}\delta_{ij}\left(\pi^2 - \sum_{n=1,2,3}(\partial_n\phi)^2 - M^2\phi^2\right). \tag{8}$$

Here $\pi = \dot{\phi}$ is the momentum conjugate to ϕ. We also obtain

$$\frac{\partial T_{\mu\nu}}{\partial g_{\rho\sigma}} = \frac{1}{2}(\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\sigma\delta_\nu{}^\rho)\left(-\frac{1}{2}g^{\eta\zeta}\partial_\eta\phi\partial_\zeta\phi - \frac{1}{2}M^2\phi^2\right) + \frac{1}{2}g_{\mu\nu}\partial^\rho\phi\partial^\sigma\phi, \tag{9}$$

which has the following form in the flat background:

$$\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{4}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\left(\pi^2 - \sum_{n=1,2,3}(\partial_n\phi)^2 - M^2\phi^2\right) + \frac{1}{2}\delta_{ij}\partial^k\phi\partial^l\phi. \tag{10}$$

We now evaluate $\frac{\partial T_{ij}}{\partial g_{kl}}$ in (10) at the finite temperature T. In order to make the situation definite, we assume that the three-dimensional space is the square box where the lengths of the edges are L, and we impose the periodic boundary condition on the scalar field ϕ. Then the momentum $\boldsymbol{k}$ is given by

$$\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{n}, \qquad \boldsymbol{n} = (n_x, n_y, n_z). \tag{11}$$

Here n_x, n_y, and n_z are integers. If we define

$$\phi(\boldsymbol{x}) \equiv \frac{1}{L^{\frac{3}{2}}}\sum_{\boldsymbol{n}} e^{i\frac{2\pi\boldsymbol{n}\cdot\boldsymbol{x}}{L}}\phi_{\boldsymbol{n}}, \qquad \pi(\boldsymbol{x}) \equiv \frac{1}{L^{\frac{3}{2}}}\sum_{\boldsymbol{n}} e^{i\frac{2\pi\boldsymbol{n}\cdot\boldsymbol{x}}{L}}\pi_{\boldsymbol{n}}, \tag{12}$$

we find

$$\int d^3x\phi(\boldsymbol{x})^2 = \sum_{\boldsymbol{n}}\phi_{-\boldsymbol{n}}\phi_{\boldsymbol{n}},$$
$$\int d^3x\pi(\boldsymbol{x})^2 = \sum_{\boldsymbol{n}}\pi_{-\boldsymbol{n}}\pi_{\boldsymbol{n}}. \tag{13}$$

The Hamiltonian is given by

$$H = \frac{1}{2}\sum_{\boldsymbol{n}}\left(\pi_{-\boldsymbol{n}}\pi_{\boldsymbol{n}} + \left(\frac{4\pi^2\boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2\right)\phi_{-\boldsymbol{n}}\phi_{\boldsymbol{n}}\right). \tag{14}$$

Here $\tilde{\pi}(\boldsymbol{k})$ and $\tilde{\phi}(\boldsymbol{l})$ satisfy the following commutation relation:

$$[\pi_{\boldsymbol{n}}, \phi_{\boldsymbol{n}'}] = -i\delta_{\boldsymbol{n}+\boldsymbol{n}',0}. \tag{15}$$

We now define the creation and annihilation operators $a_{\boldsymbol{n}}^\pm$ by

$$a_{\boldsymbol{n}}^\pm = \frac{1}{\sqrt{2}}\left(\frac{\pi_{\boldsymbol{n}}}{(\frac{4\pi^2\boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2)^{\frac{1}{4}}} \pm i\left(\frac{4\pi^2\boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2\right)^{\frac{1}{4}}\phi_{\boldsymbol{n}}\right). \tag{16}$$

We should note $(a_{\boldsymbol{n}}^\pm)^\dagger = a_{-\boldsymbol{n}}^\mp$ because $\pi_{\boldsymbol{n}}{}^\dagger = \pi_{-\boldsymbol{n}}$ and $\phi_{\boldsymbol{n}}{}^\dagger = \phi_{-\boldsymbol{n}}$. The operators $a_{\boldsymbol{n}}^\pm$ satisfy the following commutation relation:

$$[a_{\boldsymbol{n}}^-, a_{\boldsymbol{n}'}^+] = \delta_{\boldsymbol{n}+\boldsymbol{n}',0}, \qquad [a_{\boldsymbol{n}}^\pm, a_{\boldsymbol{n}'}^\pm] = 0. \tag{17}$$

Equation (16) can be solved with respect to $\pi_{\boldsymbol{n}}$ and $\phi_{\boldsymbol{n}}$ as follows:

$$\phi_{\boldsymbol{n}} = \frac{1}{i(\frac{4\pi^2\boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2)^{\frac{1}{4}}\sqrt{2}}(a_{\boldsymbol{n}}^+ - a_{\boldsymbol{n}}^-),$$
$$\pi_{\boldsymbol{n}} = \frac{(\frac{4\pi^2\boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2)^{\frac{1}{4}}}{\sqrt{2}}(a_{\boldsymbol{n}}^+ + a_{\boldsymbol{n}}^-). \tag{18}$$

The Hamiltonian (14) can be rewritten as

$$H = \sum_{\boldsymbol{n}} \sqrt{\frac{4\pi^2 \boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2} \left(a^+_{-\boldsymbol{n}} a^-_{\boldsymbol{n}} + \frac{1}{2} \right). \tag{19}$$

We now neglect the zero-point energy,

$$H \to \tilde{H} = \sum_{\boldsymbol{n}} \sqrt{\frac{4\pi^2 \boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2}\, a^+_{-\boldsymbol{n}} a^-_{\boldsymbol{n}}. \tag{20}$$

We define the number operator by

$$N \equiv \sum_{\boldsymbol{n}} a^+_{-\boldsymbol{n}} a^-_{\boldsymbol{n}}. \tag{21}$$

Then we find the following expression of the partition function:

$$Z(\beta,\mu) = \mathrm{tr}\, \mathrm{e}^{-\beta\tilde{H} - i\mu N} = \mathrm{e}^{-\sum_{\boldsymbol{n}} \ln\left(1-\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}\right)},$$
$$E_{\boldsymbol{n}} \equiv \sqrt{\frac{4\pi^2 \boldsymbol{n}\cdot\boldsymbol{n}}{L^2} + M^2}. \tag{22}$$

Here $\beta = \frac{1}{k_{\mathrm{B}}T}$ with the Boltzmann constant k_{B} and μ is the chemical potential. Then we find the thermal average of the operator $a^+_{\boldsymbol{m}} a^-_{\boldsymbol{n}}$ is given as follows:

$$\begin{aligned} \langle a^+_{\boldsymbol{m}} a^-_{\boldsymbol{n}} \rangle_{T,\mu} &= -\delta_{\boldsymbol{m}+\boldsymbol{n},0} \frac{1}{\beta} \frac{\partial \ln Z(\beta,\mu)}{\partial E_{\boldsymbol{n}}} \\ &= \delta_{\boldsymbol{m}+\boldsymbol{n},0} \frac{\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}{1-\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}. \end{aligned} \tag{23}$$

By normal ordering the operator $\frac{\partial T_{ij}}{\partial g_{kl}}$ in (10), we acquire

$$\begin{aligned} :\frac{\partial T_{ij}}{\partial g_{kl}}: &= \frac{1}{L^3} \sum_{\boldsymbol{m},\boldsymbol{n}} \mathrm{e}^{i\frac{2\pi(\boldsymbol{m}+\boldsymbol{n})\cdot\boldsymbol{x}}{L}} \left\{ \frac{1}{4} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \left(:\pi_{\boldsymbol{m}} \pi_{\boldsymbol{n}} + \left(\frac{(2\pi)^2}{L^2} \boldsymbol{m}\cdot\boldsymbol{n} - M^2 \right) :\phi_{\boldsymbol{m}}\phi_{\boldsymbol{n}}: \right) - \frac{(2\pi)^2}{2L^2} \delta_{ij} m^k n^l :\phi_{\boldsymbol{m}}\phi_{\boldsymbol{n}}: \right\} \\ &= \frac{1}{L^3} \sum_{\boldsymbol{m},\boldsymbol{n}} \mathrm{e}^{i\frac{2\pi(\boldsymbol{m}+\boldsymbol{n})\cdot\boldsymbol{x}}{L}} \left\{ \frac{1}{4} (\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) \left(\sqrt{E_{\boldsymbol{m}} E_{\boldsymbol{n}}} + \frac{1}{\sqrt{E_{\boldsymbol{m}} E_{\boldsymbol{n}}}} \left(\frac{(2\pi)^2}{L^2} \boldsymbol{m}\cdot\boldsymbol{n} - M^2 \right) \right) - \frac{(2\pi)^2}{2L^2\sqrt{E_{\boldsymbol{m}} E_{\boldsymbol{n}}}} \delta_{ij} m^k n^l \right\} a^+_{\boldsymbol{m}} a^-_{\boldsymbol{n}}. \end{aligned} \tag{24}$$

Therefore we obtain

$$\begin{aligned} \left\langle :\frac{\partial T_{ij}}{\partial g_{kl}}: \right\rangle_T &= \frac{1}{2L^3} \sum_{\boldsymbol{n}} \left\{ \frac{(2\pi)^2 n^k n^l}{L^2 E_{\boldsymbol{n}}} \delta_{ij} \right\} \frac{\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}{1-\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}} \\ &= \frac{1}{6L^3} \sum_{\boldsymbol{n}} \left\{ \frac{(2\pi)^2 \boldsymbol{n}\cdot\boldsymbol{n}}{L^2 E_{\boldsymbol{n}}} \delta_{ij}\delta^{kl} \right\} \frac{\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}{1-\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}. \end{aligned} \tag{25}$$

Particularly in the case of massless, $M = 0$, we find

$$\left\langle :\frac{\partial T_{ij}}{\partial g_{kl}}: \right\rangle_{T,M=0} = \frac{1}{6L^3} \delta_{ij}\delta^{kl} \sum_{\boldsymbol{n}} \frac{2\pi\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}}{L} \frac{\mathrm{e}^{-\frac{2\pi\beta\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}}{L} - i\mu}}{1-\mathrm{e}^{-\frac{2\pi\beta\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}}}{L} - i\mu}}. \tag{26}$$

The expectation value of the number operator in (21) is given by

$$\langle N \rangle_{T,M=0} = \sum_{\boldsymbol{n}} \frac{\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}{1-\mathrm{e}^{-\beta E_{\boldsymbol{n}} - i\mu}}. \tag{27}$$

In the limit of $L \to \infty$, we obtain

$$\begin{aligned} \left\langle :\frac{\partial T_{ij}}{\partial g_{kl}}: \right\rangle_T &= \frac{1}{6(2\pi)^3} \delta_{ij}\delta^{kl} \int d^3k \frac{k^2}{\sqrt{k^2+M^2}} \\ &\quad \times \frac{\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}} - i\mu}}{1-\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}} - i\mu}} \\ &= \frac{1}{12\pi^2} \delta_{ij}\delta^{kl} \int_0^\infty dk \frac{k^4}{\sqrt{k^2+M^2}} \\ &\quad \times \frac{\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}} - i\mu}}{1-\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}} - i\mu}}, \end{aligned} \tag{28}$$

and in massless case, $M = 0$,

$$\left\langle :\frac{\partial T_{ij}}{\partial g_{kl}}: \right\rangle_{T,M=0} = \frac{1}{12\pi^2} \delta_{ij}\delta^{kl} \int_0^\infty dk \frac{k^3 \mathrm{e}^{-\beta k - i\mu}}{1-\mathrm{e}^{-\beta k - i\mu}}. \tag{29}$$

The expectation value of the number density n is given by

$$\langle n \rangle_{T,M=0} \equiv \lim_{L\to\infty} \frac{\langle N \rangle_{T,M=0}}{L^3} = \frac{1}{2\pi^2} \int_0^\infty dk \frac{k^2 \mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}} - i\mu}}{1-\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}} - i\mu}}. \tag{30}$$

By using (8), we also find

$$\langle\rho\rangle_T = \frac{1}{2\pi^2}\int_0^\infty dk \frac{k^2(k^2+M^2)^{\frac{1}{2}}e^{-\beta(k^2+M^2)^{\frac{1}{2}}-i\mu}}{1-e^{-\beta(k^2+M^2)^{\frac{1}{2}}-i\mu}},$$

$$\langle T_{ij}\rangle_T = \delta_{ij}\langle p\rangle_T = \frac{\delta_{ij}}{6\pi^2}\int_0^\infty dk \frac{k^4 e^{-\beta(k^2+M^2)^{\frac{1}{2}}-i\mu}}{(k^2+M^2)^{\frac{1}{2}}(1-e^{-\beta(k^2+M^2)^{\frac{1}{2}}-i\mu})}. \tag{31}$$

In the massless limit $m \to 0$, we acquire

$$\langle\rho\rangle_T = 3\langle p\rangle_T = \frac{1}{4\pi^2}\int_0^\infty dk \frac{k^3 e^{-\beta k - i\mu}}{1-e^{-\beta k-i\mu}} = \frac{1}{4\pi^2\beta^4}\int_0^\infty ds \frac{s^3 e^{-s-i\mu}}{1-e^{-s-i\mu}}. \tag{32}$$

When we explore the dark matter, the number of the particles might be fixed. Let the number be N_0, and then the partition function in (22) is replaced by

$$Z_{N_0}(\beta) = \int_0^{2\pi} d\mu e^{i\mu N_0} Z(\beta,\mu) = \int_0^{2\pi} d\mu e^{i\mu N_0 - \sum_n \ln(1-e^{-\beta E_n - i\mu})}. \tag{33}$$

Especially if we consider the limit of $T \to 0$, only the ground state can contribute, and we find

$$\left\langle :\frac{\partial T_{ij}}{\partial g_{kl}}: \right\rangle_{T=0,N=N_0} = 0, \tag{34}$$

and

$$\langle\rho\rangle_{T=0,N=N_0} = \rho_0 \equiv \frac{N_0 M}{L^3}, \qquad \langle T_{ij}\rangle_{T=0,N=N_0} = 0. \tag{35}$$

Until now, we have treated the scalar field as a quantum field at finite temperature. Instead of this, we often take the real scalar field as a classical field. We now investigate whether there is any difference in the two treatments. The action of the general scalar field with potential has the following form:

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi,$$
$$\mathcal{L}_\phi = -\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{36}$$

Then we find

$$T_{\mu\nu} = -\omega(\phi)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi, \tag{37}$$

and instead of (9), we obtain

$$\frac{\partial T_{\mu\nu}}{\partial g_{\rho\sigma}} = \frac{1}{2}(\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\sigma\delta_\nu{}^\rho)\left(-\frac{1}{2}g^{\eta\zeta}\omega(\phi)\partial_\eta\phi\partial_\zeta\phi - V(\phi)\right) + \frac{1}{2}g_{\mu\nu}\omega(\phi)\partial^\rho\phi\partial^\sigma\phi. \tag{38}$$

When we assume the Friedmann-Robertson-Walker (FRW) universe with a flat spatial part,

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1,2,3}(dx^i)^2, \tag{39}$$

and $\phi = t$ in (36), a power-law behavior for the scale factor $a(t)$ of the universe,

$$a(t) = \left(\frac{t}{t_0}\right)^\alpha, \tag{40}$$

with t_0 and α real constants, can be realized by choosing

$$\omega(\phi) = \frac{2\alpha}{\kappa^2 t_0^2\phi^2}, \qquad V(\phi) = \frac{3\alpha^2-\alpha}{\kappa^2 t_0^2\phi^2}. \tag{41}$$

In case of the FRW universe (39) filled by the perfect fluid whose equation of state (EoS) parameter w is constant, α in (40) is given by

$$\alpha = \frac{2}{3(1+w)}. \tag{42}$$

For dust where $w = 0$, in (38), by using (41) and $\phi = t$, we find

$$-\frac{1}{2}g^{\eta\zeta}\omega(\phi)\partial_\eta\phi\partial_\zeta\phi - V(\phi) = \frac{1}{2}\omega(\phi) - V(\phi) = \frac{2}{3\kappa^2 t_0^2\phi^2} - \frac{2}{3\kappa^2 t_0^2\phi^2} = 0, \tag{43}$$

and therefore

$$\frac{\partial T_{ij}}{\partial g_{kl}} = 0, \tag{44}$$

which is consistent with (34). On the other hand, in case $w = \frac{1}{3}$, which corresponds to the radiation, we obtain $\alpha = \frac{1}{2}$ and

$$\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{2}(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k)\frac{1}{4\kappa^2 t_0^2\phi^2}, \tag{45}$$

whose tensor structure is different from that of the real radiation in (29).

In general, in the case of the quantum field at finite temperature, we find the tensor structure of $\frac{\partial T_{ij}}{\partial g_{kl}}$ as

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_T \propto \delta_{ij}\delta^{kl}, \tag{46}$$

but for the tensor-scalar theory, we find

$$\frac{\partial T_{ij}}{\partial g_{kl}} \propto \frac{1}{2}(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k). \tag{47}$$

Because of the difference of the tensor structure, the propagation of the gravitational wave is different in the case of the quantum field at finite temperature and the case of the classical scalar-tensor theory, in general. Especially in the case of the quantum field, because $\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \rangle_T$ always includes the factor δ^{kl}, by the condition $h_\mu{}^\mu = 0$ in (2), as long as we consider the gravitational wave with $h_{tt} = 0$, the term does not contribute. We may investigate the radiation as a comprehensible example. The usual radiation, for example in the early universe, is made of many quanta or particles at finite temperature as is known in the (quantum) statistical physics. The radiation is realized by the massless particles or in the limit of the high temperature. On the other hand, the FRW universe in the radiation dominated era can be realized by the classical scalar-tensor theory. The tensor structure of $\frac{\partial T_{ij}}{\partial g_{kl}}$ is different in the two cases, as shown in (46) and (47). The difference of the tensor structure generates the difference of the propagation of the gravitational wave [38]. In fact, the equation for the gravitational wave in the scalar-tensor theory is given by

$$0 = \left(2\dot{H} + 6H^2 + H\partial_t - \partial_t^2 + \frac{\Delta}{a^2}\right)h_{ij}, \tag{48}$$

but in the case of the quantum field with the finite temperature, we have

$$0 = \left(6\dot{H} + 12H^2 + H\partial_t - \partial_t^2 + \frac{\Delta}{a^2}\right)h_{ij}, \tag{49}$$

where Δ is the Laplacian.

IV. MAGNETIC FIELD

In this section, we analyze the propagation of the gravitational wave under the magnetic field. The energy-momentum tensor of the electromagnetic field in curved space-time is given by

$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}g^{\eta\zeta}F_{\rho\eta}F_{\sigma\zeta},$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{50}$$

which gives

$$\frac{\partial T_{\mu\nu}}{\partial g_{\rho\sigma}} = -g^{\rho\eta}g^{\sigma\zeta}F_{\mu\eta}F_{\nu\zeta} - \frac{1}{8}(\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\sigma\delta_\nu{}^\rho)g^{\xi\tau}g^{\eta\zeta}F_{\xi\eta}F_{\tau\zeta} + \frac{1}{2}g_{\mu\nu}g^{\rho\xi}g^{\sigma\tau}g^{\eta\zeta}F_{\xi\eta}F_{\tau\zeta}. \tag{51}$$

Equation (4) shows that there are mainly two kinds of effects in the magnetic field. The LHS in (4) receives the change of the geometry due to the existence of the magnetic field, and we obtain nontrivial connections and curvatures. The RHS tells that the gravitational wave gives some fluctuation of the distribution of the magnetic field, which becomes a new source of the gravitational field.

We should note that the contributions from the change of the geometry are the same order with the contributions from the fluctuation of the magnetic field.

A. Change of geometry by magnetic field

Because Eq. (50) gives the effects via the fluctuation of the distribution in the magnetic field, we now examine the change of the geometry. We assume that the background is almost flat but there is a constant magnetic field along the z direction, $F_{xy} = -F_{yx} = B$. Then we find

$$T_{xx} = T_{yy} = \frac{1}{2}B^2 + \mathcal{O}(\kappa^2),$$
$$T_{zz} = -T_{tt} = -\frac{1}{2}B^2 + \mathcal{O}(\kappa^2),$$
$$\text{other components} = 0. \tag{52}$$

The Einstein equation (3) leads to

$$R = 0, \qquad R_{\mu\nu} = \kappa^2 T_{\mu\nu}. \tag{53}$$

The parameter M^2 in (5) and (6) corresponds to κ^2B^2. Then before considering the gravitational wave, we need to consider the $\mathcal{O}(\kappa^2B^2)$ correction, corresponding to (5), from the flat background $g_{\mu\nu} = \eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa^2B^2\zeta_{\mu\nu}. \tag{54}$$

Then the Einstein equation (3) gives

$$\partial_\mu\partial^\rho\zeta_{\nu\rho} + \partial_\nu\partial^\rho\zeta_{\mu\rho} - \partial_\rho\partial^\rho\zeta_{\mu\nu} - \partial_\mu\partial_\nu(\eta^{\rho\lambda}\zeta_{\rho\lambda}) - \eta_{\mu\nu}(\partial^\rho\partial^\sigma\zeta_{\rho\sigma} - \partial^2(\eta^{\rho\sigma}\zeta_{\rho\sigma})) = \frac{2}{B^2}T_{\mu\nu}, \tag{55}$$

which corresponds to the equation $M^2I_R^{(1)} = M^2I_T^{(1)}$ in (5). A solution of (55) is given by

$$\zeta_{xx} = -\frac{1}{2}y^2, \qquad \zeta_{yy} = -\frac{1}{2}x^2, \qquad \zeta_{zz} = \frac{1}{2}y^2,$$
$$\zeta_{tt} = -\frac{1}{2}x^2, \qquad \text{other components} = 0. \tag{56}$$

We find that the connections read

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\Gamma^y_{xx} = -\frac{1}{2}y\kappa^2 B^2,$$
$$\Gamma^y_{xy} = \Gamma^y_{yx} = -\Gamma^x_{yy} = -\frac{1}{2}x\kappa^2 B^2,$$
$$\Gamma^z_{zy} = \Gamma^z_{yz} = -\Gamma^y_{zz} = \frac{1}{2}y\kappa^2 B^2,$$
$$\Gamma^t_{xt} = \Gamma^t_{tx} = \Gamma^x_{tt} = \frac{1}{2}x\kappa^2 B^2,$$
$$\text{other components} = 0. \tag{57}$$

Because

$$R^\lambda{}_{\mu\rho\nu} = -\Gamma^\lambda_{\mu\rho,\nu} + \Gamma^\lambda_{\mu\nu,\rho} + \mathcal{O}((\kappa^2 B^2)^2), \tag{58}$$

we obtain

$$R^x{}_{yxy} = -R^x{}_{yyx} = -R^y{}_{xxy} = R^y{}_{xyx} = \kappa^2 B^2,$$
$$R^x{}_{txt} = -R^x{}_{ttx} = -R^t{}_{xtx} = R^t{}_{xxt} = \frac{1}{2}\kappa^2 B^2,$$
$$R^y{}_{zyz} = -R^y{}_{zzy} = R^z{}_{yzy} = -R^z{}_{yyz} = -\frac{1}{2}\kappa^2 B^2,$$
$$\text{other components} = 0. \tag{59}$$

The above results are consistent with (53). The expressions in (56) with (54) show that we should require

$$\kappa^2 B^2 x^2, \qquad \kappa^2 B^2 y^2 \ll 1, \tag{60}$$

or we need to consider the higher order terms with respect to $\kappa^2 B^2$. The gauge conditions in (2) can be explicitly written as

$$\begin{aligned}
0 &= \nabla^\mu h_{\mu x} \\
&= \partial^\mu h_{\mu x} + \frac{1}{2}\kappa^2 B^2 (y^2(h_{xx} - h_{zx}) + x^2(h_{yx} + h_{tx}) \\
&\quad - y h_{xy} + x h_{yy} + x h_{tt}), \\
0 &= \nabla^\mu h_{\mu y} = \partial^\mu h_{\mu y} + \frac{1}{2}\kappa^2 B^2 (y^2(h_{xy} - h_{zy}) \\
&\quad + x^2(h_{yy} + h_{yx}) + y h_{xx} - x h_{yy} - y h_{zz}), \\
0 &= \nabla^\mu h_{\mu z} = \partial^\mu h_{\mu z} + \frac{1}{2}\kappa^2 B^2 (y^2(h_{xz} - h_{zz}) \\
&\quad + x^2(h_{yz} + h_{tz}) - y h_{yz}), \\
0 &= \nabla^\mu h_{\mu t} = \partial^\mu h_{\mu t} + \frac{1}{2}\kappa^2 B^2 (y^2(h_{xt} - h_{zt}) \\
&\quad + x^2(h_{yt} + h_{tt}) - x h_{tt}), \\
0 &= g^{\mu\nu} h_{\mu\nu} = h_{xx} + h_{yy} + h_{zz} - h_{tt} \\
&\quad + \frac{1}{2}\kappa^2 B^2 y^2 (h_{xx} - h_{zz}) + \frac{1}{2}\kappa^2 B^2 y^2 (h_{yy} + h_{tt}).
\end{aligned} \tag{61}$$

The above equations indicate that there appear the longitudinal modes in general.

B. Propagation of gravitational wave and scattering

Because

$$\begin{aligned}
\nabla^2 h_{\mu\nu} &= g^{\rho\sigma}\nabla_\rho \nabla_\sigma h_{\mu\nu} = g^{\rho\sigma}(\partial_\rho \nabla_\sigma h_{\mu\nu} - \Gamma^\tau_{\rho\sigma}\nabla_\tau h_{\mu\nu} \\
&\quad - \Gamma^\tau_{\rho\mu}\nabla_\sigma h_{\tau\nu} - \Gamma^\tau_{\rho\nu}\nabla_\sigma h_{\mu\tau}) \\
&= (\eta^{\rho\sigma} - \kappa^2 B^2 \zeta^{\rho\sigma})\partial_\rho \partial_\sigma h_{\mu\nu} \\
&\quad + \eta^{\rho\sigma}(\partial_\rho(-\Gamma^\tau_{\sigma\mu}h_{\tau\nu} - \Gamma^\tau_{\sigma\nu}h_{\mu\tau}) - \Gamma^\tau_{\rho\sigma}\partial_\tau h_{\mu\nu} \\
&\quad - \Gamma^\tau_{\rho\mu}\partial_\sigma h_{\tau\nu} - \Gamma^\tau_{\rho\nu}\partial_\sigma h_{\mu\tau}) + \mathcal{O}((\kappa^2 B^2)^2),
\end{aligned} \tag{62}$$

we find the following explicit expressions, which correspond to the equation $M^2\kappa^2 I^{(2)}_R = M^2\kappa^2 I^{(1,1)}_T$ in Eq. (6):

$$\begin{aligned}
\nabla^2 h_{xx} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xx} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{xx} \\
&\quad + \kappa^2 B^2\{h_{xx} + 2y\partial_y h_{xx} + (-y\partial_x + x\partial_y)(h_{xy} + h_{yx}) \\
&\quad + x\partial_t(h_{xt} + h_{tx})\},
\end{aligned} \tag{63}$$

$$\begin{aligned}
\nabla^2 h_{yy} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yy} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{yy} \\
&\quad + \kappa^2 B^2\{h_{yy} + 2x\partial_x h_{yy} + (-x\partial_y + y\partial_x)(h_{xy} + h_{yx}) \\
&\quad - y\partial_z(h_{yz} + h_{zy})\},
\end{aligned} \tag{64}$$

$$\begin{aligned}
\nabla^2 h_{zz} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{zz} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{zz} \\
&\quad + \kappa^2 B^2\{-h_{zz} - 2y\partial_y h_{zz} + y\partial_z(h_{zy} + h_{yz})\},
\end{aligned} \tag{65}$$

$$\begin{aligned}
\nabla^2 h_{xy} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xy} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{xy} \\
&\quad + \kappa^2 B^2\{h_{xy} + (x\partial_x + y\partial_y)h_{xy} \\
&\quad + (-x\partial_y + y\partial_x)(h_{xx} - h_{yy}) + x\partial_t h_{ty} \\
&\quad - y\partial_z h_{xz}\},
\end{aligned} \tag{66}$$

$$\begin{aligned}
\nabla^2 h_{xz} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xz} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{xz} \\
&\quad + \kappa^2 B^2\{(-y\partial_x + x\partial_y)h_{yz} + x\partial_t h_{tz} + y\partial_z h_{xy}\},
\end{aligned} \tag{67}$$

$$\begin{aligned}
\nabla^2 h_{yz} &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yz} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{yz} \\
&\quad + \kappa^2 B^2\{(x\partial_x - y\partial_y)h_{yz} + (-x\partial_y + y\partial_x)h_{xz} \\
&\quad + y\partial_z(h_{yy} - h_{zz})\}.
\end{aligned} \tag{68}$$

On the other hand,

$$\delta T_{\mu\nu} \equiv \frac{\partial T_{\mu\nu}}{\partial g_{\rho\sigma}} h_{\rho\sigma}, \qquad \delta T_{xx} = -\frac{1}{2} B^2 h_{yy},$$
$$\delta T_{yy} = -\frac{1}{2} B^2 h_{xx}, \qquad \delta T_{zz} = B^2 \left(-\frac{1}{2} h_{zz} + \frac{1}{2}(h_{xx} + h_{yy}) \right),$$
$$\delta T_{xy} = \frac{1}{2} B^2 h_{xy}, \qquad \delta T_{xz} = -\frac{1}{2} B^2 h_{xz},$$
$$\delta T_{yz} = -\frac{1}{2} B^2 h_{yz}. \tag{69}$$

By combining (63)–(69), we find Eq. (4) gives

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xx} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{xx} + \kappa^2 B^2\{2y\partial_y h_{xx} + (-y\partial_x + x\partial_y)(h_{xy} + h_{yx}) + x\partial_t(h_{xt} + h_{tx})\} + \frac{1}{2}\kappa^2 B^2(-h_{xx} + h_{yy} + h_{tt} + h_{zz}), \tag{70}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yy} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{yy} + \kappa^2 B^2\{2x\partial_x h_{yy} + (-x\partial_y + y\partial_x)(h_{xy} + h_{yx}) - y\partial_z(h_{yz} + h_{zy})\} + \frac{1}{2}\kappa^2 B^2(h_{xx} - h_{yy} - h_{zz} - h_{tt}), \tag{71}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{zz} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{zz} + \kappa^2 B^2\{-2y\partial_y h_{zz} + y\partial_z(h_{zy} + h_{yz})\} + \frac{1}{2}\kappa^2 B^2(h_{xx} - h_{yy} - h_{zz} - h_{tt}), \tag{72}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xy} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{xy} + \kappa^2 B^2\{(x\partial_x + y\partial_y)h_{xy} + (-x\partial_y + y\partial_x)(h_{xx} - h_{yy}) + x\partial_t h_{ty} - y\partial_z h_{xz}\} - \kappa^2 B^2 h_{xy}, \tag{73}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xz} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{xz} + \kappa^2 B^2\{(-y\partial_x + x\partial_y)h_{yz} + x\partial_t h_{tz} + y\partial_z h_{xy}\} - \kappa^2 B^2 h_{xz}, \tag{74}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yz} - \frac{\kappa^2 B^2}{2}(y^2(-\partial_x^2 + \partial_z^2) - x^2(\partial_y^2 + \partial_t^2))h_{yz} + \kappa^2 B^2\{(x\partial_x - y\partial_y)h_{yz} + (-x\partial_y + y\partial_x)h_{xz} + y\partial_z(h_{yy} - h_{zz})\}. \tag{75}$$

We investigate the propagation of the gravitational wave based on the above equations. In order to see the effect of the magnetic field, we assume

$$h_{xy} = h_\times^{(0)} \sin k(z-t) + \mathcal{O}(\kappa^2 B^2),$$
$$\text{other components} = \mathcal{O}(\kappa^2 B^2), \tag{76}$$

which corresponds to $\times$ mode propagating in parallel with the magnetic field. Then we obtain

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xx} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yy} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{zz} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yz}, \tag{77}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xy} + \frac{\kappa^2 B^2 k^2}{2}(y^2 - x^2)h_\times^{(0)} \sin k(z-t) - \kappa^2 B^2 h_\times^{(0)} \sin k(z-t), \tag{78}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xz} + \kappa^2 B^2 y k h_\times^{(0)} \cos k(z-t). \tag{79}$$

Therefore if we define $\Box \equiv \eta^{\rho\sigma}\partial_\rho\partial_\sigma$, we find

$$h_{xx} = h_{yy} = h_{zz} = h_{yz} = 0,$$
$$h_{xz} = -\kappa^2 B^2 k h_\times^{(0)} \Box^{-1}(y\cos k(z-t)),$$
$$h_{xy} = h_\times^{(0)}\left\{ \sin k(z-t) - \kappa^2 B^2 \Box^{-1}\left(\left(\frac{k^2(y^2 - x^2)}{2} - 1 \right) \sin k(z-t) \right) \right\}. \tag{80}$$

The $\mathcal{O}(\kappa^2 B^2)$ is given by the scattering of the gravitational wave by the magnetic field. It could be interesting that there appears a nontrivial h_{xz} component.

Next we explore the $+$ mode propagating in parallel with the magnetic field,

$$h_{xx} = -h_{yy} = h_+^{(0)} \sin k(z-t) + \mathcal{O}(\kappa^2 B^2),$$
$$\text{other components} = \mathcal{O}(\kappa^2 B^2). \tag{81}$$

Then we find

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xx} + \frac{\kappa^2 B^2 k^2}{2}(y^2 - x^2)h_+^{(0)} \sin k(z-t) + \kappa^2 B^2 h_+^{(0)} \sin k(z-t), \tag{82}$$

$$0 = -\eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yy} + \frac{\kappa^2 B^2 k^2}{2}(y^2 - x^2)h_+^{(0)} \sin k(z-t) + \kappa^2 B^2 h_+^{(0)} \sin k(z-t), \tag{83}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{zz} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xy} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xz}, \tag{84}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yz} - \kappa^2 B^2 y k h_+^{(0)} \cos k(z-t), \tag{85}$$

and the solution is given by

$$\begin{aligned} h_{zz} &= h_{xy} = h_{xz} = 0, \\ h_{yz} &= +\kappa^2 B^2 k h_+^{(0)} \Box^{-1}(y\cos k(z-t)), \\ h_{xx} &= -h_{yy} = h_+^{(0)}\Bigg\{\sin k(z-t) \\ &\quad -\kappa^2 B^2 \Box^{-1}\left(\left(\frac{k^2(y^2-x^2)}{2}+1\right)\sin k(z-t)\right)\Bigg\}. \end{aligned} \tag{86}$$

Then there appears a nontrivial h_{yz} component. The physical behavior of the $+$ mode (86) does not change from that of the $\times$ mode in (80).

We examine the $\times$ mode propagating perpendicular to the magnetic field,

$$\begin{aligned} h_{xz} &= \bar{h}_\times^{(0)} \sin k(y-t) + \mathcal{O}(\kappa^2 B^2), \\ &\text{other components} = \mathcal{O}(\kappa^2 B^2). \end{aligned} \tag{87}$$

Then we find

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xx} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yy} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{zz} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xy}, \tag{88}$$

$$\begin{aligned} 0 &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xz} - \kappa^2 B^2 k^2 x^2 \bar{h}_\times^{(0)} \sin k(y-t) \\ &\quad - \kappa^2 B^2 \bar{h}_+^{(0)} \sin k(y-t), \end{aligned} \tag{89}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yz} - \kappa^2 B^2 k \bar{h}_\times^{(0)} \cos k(y-t), \tag{90}$$

whose solution is given by

$$\begin{aligned} h_{xx} &= h_{yy} = h_{zz} = h_{xy} = 0, \\ h_{xz} &= \bar{h}_\times^{(0)}\{\sin k(y-t) + \kappa^2 B^2 \Box^{-1}((k^2x^2+1)\sin k(y-t))\}, \\ h_{yz} &= \kappa^2 B^2 k \bar{h}_\times^{(0)} \Box^{-1}(x\cos k(y-t)). \end{aligned} \tag{91}$$

In the case of the $+$ mode propagating perpendicular to the magnetic field,

$$\begin{aligned} h_{xx} &= -h_{zz} = \bar{h}_+^{(0)} \sin k(y-t) + \mathcal{O}(\kappa^2 B^2), \\ &\text{other components} = \mathcal{O}(\kappa^2 B^2), \end{aligned} \tag{92}$$

we obtain

$$\begin{aligned} 0 &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xx} - \kappa^2 B^2 k^2 x^2 \bar{h}_+^{(0)} \sin k(y-t) \\ &\quad + 2\kappa^2 B^2 k y \bar{h}_+^{(0)} \cos k(y-t) - \kappa^2 B^2 \bar{h}_+^{(0)} \sin k(y-t), \end{aligned} \tag{93}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yy} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xz} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{yz}, \tag{94}$$

$$\begin{aligned} 0 &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{zz} + \kappa^2 B^2 k^2 x^2 \bar{h}_+^{(0)} \sin k(y-t) \\ &\quad + 2\kappa^2 B^2 k y \bar{h}_+^{(0)} \cos k(y-t) \\ &\quad + \kappa^2 B^2 \bar{h}_+^{(0)} \sin k(y-t), \end{aligned} \tag{95}$$

$$0 = \eta^{\rho\sigma}\partial_\rho\partial_\sigma h_{xy} - \kappa^2 B^2 k x \bar{h}_+^{(0)} \cos k(y-t). \tag{96}$$

Then we find

$$\begin{aligned} h_{xx} &= \bar{h}_+^{(0)}\{\sin k(y-t) + \kappa^2 B^2 \Box^{-1}((k^2x^2+1)\sin k(y-t) \\ &\quad - 2ky\cos k(y-t))\}, \qquad h_{yy} = h_{xz} = h_{yz} = 0, \\ h_{zz} &= \bar{h}_+^{(0)}\{-\sin k(y-t) \\ &\quad + \kappa^2 B^2 \Box^{-1}(-(k^2x^2+1)\sin k(y-t) \\ &\quad - 2ky\cos k(y-t))\}, \\ h_{xy} &= \kappa^2 B^2 k \bar{h}_+^{(0)} \Box^{-1}(x\cos k(y-t)). \end{aligned} \tag{97}$$

The behavior of the $+$ mode in (97) seems to be rather different from that of the $\times$ mode (91).

In the above expressions, in order that the perturbation should be consistent, in addition to (60), we need to require

$$\frac{\kappa^2 B^2}{k^2} \ll 1. \tag{98}$$

We should note that $\Box^{-1}$ is the retarded propagator, which is nonlocal and satisfies the equation

$$\Box G(x^\mu, x'^\mu) = \delta^4(x^\mu - x'^\mu), \qquad G(x^\mu, x'^\mu) \equiv \Box^{-1}. \tag{99}$$

Therefore for any function $f(x^\mu)$ of the space-time coordinate x^μ, we have

$$\Box^{-1} f(x^\mu) \equiv \int_V d^4x' G(x^\mu, x'^\mu) f(x'^\mu). \tag{100}$$

The region V of the integration is given by the region of the space-time, where the magnetic field exists. Therefore the gravitational wave carries the information of the distribution of the magnetic field in the universe. In the above analysis, we have assumed that $\kappa^2 B^2$ should be small enough. Even if $\kappa^2 B^2$ is small, the integration over the space-time in (100) enhances the amplitude of the gravitational wave.

As an example, we consider NGC 6946, which is a spiral galaxy. The size of NGC 6946 is $\sim$100 k light years $\sim 10^{28}/\mathrm{eV}$, and its distance from the earth is 20 M light year $\sim 10^{30}/\mathrm{eV}$. We may estimate

$$\begin{aligned} \int_V d^4x' G(x^\mu, x'^\mu) f(x'^\mu) &\sim (10^{30}/\mathrm{eV})^{-1}(10^{28}/\mathrm{eV})^3 f \\ &= 10^{54}/\mathrm{eV}^2 f. \end{aligned} \tag{101}$$

The exponents −1 and 3 come because we are considering the static magnetic field. Because NGC 6946 has a magnetic field with μG, we may evaluate B^2 as

$$B^2 \sim 10^{-16}\ \mathrm{eV}^4, \tag{102}$$

and therefore

$$\kappa^2 B^2 \sim 10^{-72}\ \mathrm{eV}^2. \tag{103}$$

We may also estimate x and y from the size of the galaxy as

$$x \sim y \sim 10^{28}/\mathrm{eV}. \tag{104}$$

Therefore we obtain

$$\kappa^2 B^2 x^2 \sim \kappa^2 B^2 y^2 \sim 10^{-16} \ll 1, \tag{105}$$

and as a result the condition (60) is satisfied.

In the case of the gravitational wave GW150914, the typical frequency is 100–500 Hz, which corresponds to the wave number

$$k \sim 10^{-6}\ \mathrm{m}^{-1} \sim 10^{-13}\ \mathrm{eV}, \tag{106}$$

and hence

$$kx \sim ky \sim 10^{22}. \tag{107}$$

Because

$$\frac{\kappa^2 B^2}{k^2} \sim 10^{-46} \ll 1, \tag{108}$$

the condition (98) is also satisfied.

For example, in (80), we find

$$\begin{aligned} h_{xz} &= -\kappa^2 B^2 k h_\times^{(0)} \Box^{-1}(y \cos k(z-t)) \\ &\sim 10^{-72+54+22} h_\times^{(0)} \cos k(z-t) = 10^4 h_\times^{(0)} \cos k(z-t), \\ h_{xy} &= h_\times^{(0)} \left\{ \sin k(z-t) \right. \\ &\quad \left. - \kappa^2 B^2 \Box^{-1} \left(\left(\frac{k^2(y^2-x^2)}{2} - 1 \right) \sin k(z-t) \right) \right\} \\ &\sim (1+10^{26}) h_\times^{(0)} \sin k(z-t), \end{aligned} \tag{109}$$

which shows that the corrections are rather large. The large correction comes from the large size of the galaxy. Of course, we have neglected the numerical factors and we have assumed that the magnetic field is uniform at the large scale of the galaxy, which indicates that we should have overestimated the value. Furthermore the correction to the wave number is enhanced because it is multiplied by the distance of the propagation, which gives a non-negligible correction in the phase. However, we expand the expression with respect to $\kappa^2 B^2$, and there appears the large correction, for example, $\sin((k+\alpha\kappa^2 B^2)z) \sim \sin(kz) + \cos(kz)\alpha\kappa^2 B^2 z$, where we express the correction by $\alpha\kappa^2 B^2$, which might be small but $\alpha\kappa^2 B^2 z$ (multiplied with z) is not small in general. Anyway the magnetic field may give a contribution that cannot be neglected.

We may study magnetar [39], which has a very strong magnetic field of 10^{11} T. Then we find $B^2 \sim 10^{26}$ eV and therefore $\kappa^2 B^2 \sim 10^{-30}\ \mathrm{eV}^2$. If we consider the gravitational wave in (106), we find

$$\frac{\kappa^2 B^2}{k^2} \sim 10^{-4} \ll 1, \tag{110}$$

which is still small and the condition (98) is also satisfied. Thus the perturbation is still valid. The typical size of the magnetic field in magnetar is $2 \times 10^8\ \mathrm{km} \sim 10^{15}/\mathrm{eV}$, and therefore

$$kx \sim ky \sim 10^3 \tag{111}$$

and

$$\kappa^2 B^2 x^2 \sim \kappa^2 B^2 y^2 \sim 1, \tag{112}$$

which tells us that the condition (60) is not always satisfied. In the case of SGR 1806-20, which is the first magnetar found, the distance from the Earth is 5×10^4 light years $\sim 10^{27}/\mathrm{eV}$, instead of (101), and we may estimate

$$\begin{aligned} \int_V d^4x' G(x^\mu, x'^\mu) f(x'^\mu) &\sim (10^{27}/\mathrm{eV})^{-1} (10^{15}/\mathrm{eV})^3 f \\ &= 10^{18}/\mathrm{eV}^2 f. \end{aligned} \tag{113}$$

The expressions corresponding to (109) have the following form:

$$\begin{aligned} h_{xz} &= -\kappa^2 B^2 k h_\times^{(0)} \Box^{-1}(y \cos k(z-t)) \\ &\sim 10^{-30+3+18} h_\times^{(0)} \cos k(z-t) = 10^{-9} h_\times^{(0)} \cos k(z-t), \\ h_{xy} &= h_\times^{(0)} \left\{ \sin k(z-t) \right. \\ &\quad \left. - \kappa^2 B^2 \Box^{-1} \left(\left(\frac{k^2(y^2-x^2)}{2} - 1 \right) \sin k(z-t) \right) \right\} \\ &\sim (1+10^{-6}) h_\times^{(0)} \sin k(z-t), \end{aligned} \tag{114}$$

where the corrections are reasonably small. Hence, contrary to the case of the large magnetic field, it could be difficult to detect the effect of the magnetic field.

C. Propagation of gravitational wave by adiabatic approximation

In the last subsection, we have investigated the scattering of the gravitational wave by the magnetic field. As mentioned after Eq. (51), there are two kinds of sources for the scattering. One is given by the change of the geometry and the other is the fluctuation of the distribution of the magnetic field given by the gravitational wave. The obtained results seem to say that the effects in the change of the geometry are much larger than those of the fluctuation of the magnetic field.

The expressions (80), (86), and (97) [except of (92)] could not be valid for the magnetic field of the galactic size although they may be valid for the magnetic field with a smaller size. This could be mainly because the variation of the phase is not small although the correction of the wave number is small. Then we now try to solve Eqs. (70)–(75) by using the adiabatic approximation, where we neglect the derivative with respect to the background.

First we analyze the gravitational wave along the z axis and assume

$$h_{ij}=h_{ij}^{(0)}(x,y)\mathrm{e}^{i(k(x,y)z-\omega(x,u)t)}, \quad h_{ti}=h_{it}=h_{tt}=0. \tag{115}$$

Finally we choose the real part of the above expression. We take $h_{ij}^{(0)}(x,y)$, $k(x,y)$, and $\omega(x,u)$ can depend on the coordinates (x,y) although we neglect the derivative with respect to x and y for $h_{ij}^{(0)}(x,y)$, $k(x,y)$, and $\omega(x,u)$. Then Eqs. (70)–(75) give the following algebraic equations:

$$0=\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)\right)h_{xx}^{(0)} + \frac{1}{2}\kappa^2B^2(-h_{xx}^{(0)}+h_{yy}^{(0)}+h_{zz}^{(0)}), \tag{116}$$

$$0=\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)\right)h_{yy}^{(0)} - 2iky\kappa^2B^2h_{yz}^{(0)}+\frac{1}{2}\kappa^2B^2(h_{xx}^{(0)}-h_{yy}^{(0)}-h_{zz}^{(0)}), \tag{117}$$

$$0=\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)\right)h_{zz}^{(0)} + 2ikyh_{yz}^{(0)}+\frac{1}{2}\kappa^2B^2(h_{xx}^{(0)}-h_{yy}^{(0)}-h_{zz}^{(0)}), \tag{118}$$

$$0=\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)\right)h_{xy}^{(0)} - iky\kappa^2B^2h_{xz}^{(0)}-\kappa^2B^2h_{xy}^{(0)}, \tag{119}$$

$$0=\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)\right)h_{xz}^{(0)} + iky\kappa^2B^2h_{xy}^{(0)}-\kappa^2B^2h_{xz}^{(0)}, \tag{120}$$

$$0=\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)\right)h_{yz}^{(0)} + iky\kappa^2B^2(h_{yy}^{(0)}-h_{zz}^{(0)}). \tag{121}$$

Then the solution corresponding to the + mode is given by

$$h_{xx}^{(0)}=-h_{yy}^{(0)}, \qquad \text{other components}=0, \tag{122}$$

with the dispersion relation,

$$0=-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)-\kappa^2B^2. \tag{123}$$

As is clear from the dispersion relation, there appears the square of an effective mass $\sim\kappa^2B^2$ in addition to the x, y dependent shift of the phase. Equation (123) leads to

$$\left(1+\frac{\kappa^2B^2y^2}{2}\right)=\left(1+\frac{\kappa^2B^2x^2}{2}\right)\omega^2-\kappa^2B^2 \tag{124}$$

or

$$k^2=\left(1+\frac{\kappa^2B^2(x^2-y^2)}{2}\right)\omega^2-\kappa^2B^2+\mathcal{O}((\kappa^2B^2)^2). \tag{125}$$

The factor $1+\frac{\kappa^2B^2(x^2-y^2)}{2}$ in front of ω^2 comes from the change of the geometry, and therefore there appears the same correction even for the propagation of light but the term κ^2B^2 gives the square of the effective mass, which is absent in the photon.

The solution corresponding to the × mode is given by

$$h_{xz}^{(0)}=\frac{1}{iky\kappa^2B^2} \times\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)-\kappa^2B^2\right)h_{xy}^{(0)},$$

$$\text{other components}=0, \tag{126}$$

with a little bit of a complex dispersion relation,

$$\left(-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)-\kappa^2B^2\right)^2=k^2y^2\kappa^4B^4, \tag{127}$$

that is,

$$0=-k^2+\omega^2-\frac{\kappa^2B^2}{2}(-k^2y^2+\omega^2x^2)-\kappa^2B^2\pm ky\kappa^2B^2. \tag{128}$$

We should note that there should appear a h_{xz} component whose phase is different from that of the h_{xy} component by $\frac{\pi}{2}$.

Next we consider the gravitational wave along the y axis and assume

$$h_{ij} = h_{ij}^{(0)}(x,y)\mathrm{e}^{i(k(x,y)y-\omega(x,u)t)}, \quad h_{ti} = h_{it} = h_{tt} = 0. \tag{129}$$

Again we neglect the derivative with respect to x and y for $h_{ij}^{(0)}(x,y)$, $k(x,y)$, and $\omega(x,u)$ as an adiabatic approximation. Then we find the following equations:

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2)\right)h_{xx}^{(0)} + 2ik\kappa^2 B^2 (y h_{xx}^{(0)} + x h_{xy}^{(0)}) + \frac{1}{2}\kappa^2 B^2 (-h_{xx}^{(0)} + h_{yy}^{(0)} + h_{zz}^{(0)}), \tag{130}$$

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2)\right)h_{yy}^{(0)} - 2ik\kappa^2 B^2 x h_{xy}^{(0)} + \frac{1}{2}\kappa^2 B^2 (h_{xx}^{(0)} - h_{yy}^{(0)} - h_{zz}^{(0)}), \tag{131}$$

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2)\right)h_{zz}^{(0)} - 2ik\kappa^2 B^2 y h_{zz}^{(0)} + \frac{1}{2}\kappa^2 B^2 (h_{xx}^{(0)} - h_{yy}^{(0)} - h_{zz}^{(0)}), \tag{132}$$

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2)\right)h_{xy}^{(0)} + ik\kappa^2 B^2 \{y h_{xy}^{(0)} - x(h_{xx}^{(0)} - h_{yy}^{(0)})\} - \kappa^2 B^2 h_{xy}^{(0)}, \tag{133}$$

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2)\right)h_{xz}^{(0)} + ik\kappa^2 B^2 x h_{yz}^{(0)} - \kappa^2 B^2 h_{xz}^{(0)}, \tag{134}$$

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2)\right)h_{yz}^{(0)} + ik\kappa^2 B^2 (-y h_{yz}^{(0)} - x h_{xz}^{(0)}). \tag{135}$$

Note as in Eqs. (116)–(121), it is difficult to solve Eqs. (130)–(135). First we assume $h_{xz}^{(0)} = h_{yz}^{(0)} = 0$ and the following dispersion relation:

$$0 = -k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2+\omega^2) + 2ik\kappa^2 B^2 y - \kappa^2 B^2 + \lambda\kappa^2 B^2. \tag{136}$$

By substituting (136) into Eqs. (130)–(133), we obtain

$$0 = 2ikx h_{xy}^{(0)} + \frac{1}{2}(h_{xx}^{(0)} + h_{yy}^{(0)} + h_{zz}^{(0)}) + \lambda h_{xx}^{(0)}, \tag{137}$$

$$0 = -2ik(y h_{yy}^{(0)} + x h_{xy}^{(0)}) + \frac{1}{2}(h_{xx}^{(0)} + h_{yy}^{(0)} - h_{zz}^{(0)}) + \lambda h_{yy}^{(0)}, \tag{138}$$

$$0 = -4iky h_{zz}^{(0)} + \frac{1}{2}(h_{xx}^{(0)} - h_{yy}^{(0)} + h_{zz}^{(0)}) + \lambda h_{zz}^{(0)}, \tag{139}$$

$$0 = ik\{-y h_{xy}^{(0)} - x(h_{xx}^{(0)} - h_{yy}^{(0)})\} + \lambda h_{xy}^{(0)}, \tag{140}$$

which can be rewritten by using a matrix,

$$0 = \begin{pmatrix} \lambda+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2ikx \\ \frac{1}{2} & \lambda+\frac{1}{2}-2iky & -\frac{1}{2} & -2ikx \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}-4iky+\lambda & 0 \\ -ikx & ikx & 0 & -iky+\lambda \end{pmatrix} \begin{pmatrix} h_{xx}^{(0)} \\ h_{yy}^{(0)} \\ h_{zz}^{(0)} \\ h_{xy}^{(0)} \end{pmatrix}. \tag{141}$$

Then the dispersion relation (136) can be determined by solving the following equation:

$$0 = \begin{vmatrix} \lambda+\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2ikx \\ \frac{1}{2} & \lambda+\frac{1}{2}-2iky & -\frac{1}{2} & -2ikx \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}-4iky+\lambda & 0 \\ -ikx & ikx & 0 & -iky+\lambda \end{vmatrix}. \tag{142}$$

In the case that $|kx|$, $|ky| \ll 1$, we may approximate Eq. (141) as follows:

$$0 = \lambda\left\{\left(\lambda+\frac{1}{2}\right)^3 - \frac{3}{4}\left(\lambda+\frac{1}{2}\right) - \frac{1}{4}\right\} = \lambda\left(\lambda-\frac{1}{2}\right)(\lambda+1)^2, \tag{143}$$

whose solution is given by $\lambda = 0$, $\frac{1}{2}$, and two $\lambda = -1$. Because there appears the imaginary part in the dispersion relation (136), the amplitude of the gravitational wave is decaying. The mode corresponding to $\lambda = 0$ gives $h_{xx}^{(0)} = h_{yy}^{(0)} = h_{zz}^{(0)} = 0$ and $h_{xy}^{(0)} \neq 0$, and therefore the mode could be unphysical. The mode corresponding to $\lambda = \frac{1}{2}$ gives $h_{yy}^{(0)} = h_{zz}^{(0)} = -h_{xx}^{(0)}$ and the mode to $\lambda = -1$ gives $h_{xx}^{(0)} = h_{yy}^{(0)} + h_{zz}^{(0)}$. These are not connected with the modes of the gravitational wave in the vacuum. In the case that $|kx|$, $|ky| \gg 1$, by using Eq. (141), we acquire

$$0 = (\lambda - 4iky)(\lambda - iky)(\lambda^2 - 2iky\lambda - 2k^2x^2), \tag{144}$$

whose solution is given by

$$\lambda = iky, \qquad 4iky, \qquad iky \pm ik\sqrt{y^2 - 2x^2}. \tag{145}$$

For $\lambda = iky$, we obtain $h_{yy}^{(0)} = h_{yy}^{(0)}$, $h_{zz}^{(0)} = 0$, and $h_{xy}^{(0)} = -\frac{y}{2x}h_{xx}^{(0)}$. When $\lambda = 4iky$, $h_{xx}^{(0)} = h_{yy}^{(0)} = h_{xy}^{(0)} = 0$, and if

$\lambda = iky \pm ik\sqrt{y^2 - 2x^2}$, we find $h_{zz}^{(0)} = 0$ and $h_{xy}^{(0)} = \frac{i\lambda}{2kx} h_{xx}^{(0)}$, $h_{yy}^{(0)} = -\frac{\lambda}{\lambda - 2iky} h_{xx}^{(0)}$.

We may also investigate the mode where $h_{xx}^{(0)} = h_{yy}^{(0)} = h_{zz}^{(0)} = h_{xy}^{(0)} = 0$. Then by using (134) and (135), the dispersion relation is given by

$$0 = \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2 + \omega^2) - \kappa^2 B^2\right) \times \left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2 + \omega^2) - iky\kappa^2 B^2\right) - k^2 \kappa^4 B^4 x^2, \tag{146}$$

and we obtain

$$h_{yz}^{(0)} = -\frac{i}{k\kappa^2 B^2 x}\left(-k^2 + \omega^2 - \frac{\kappa^2 B^2 x^2}{2}(k^2 + \omega^2) - \kappa^2 B^2\right) h_{xz}^{(0)}. \tag{147}$$

Then we find that the adiabatic approximation gives reasonable results for the large magnetic field compared with simple perturbation with respect to $\kappa^2 B^2$ even if $\kappa^2 B^2$ is small. As mentioned after Eq. (109), the overestimated amplitude in the simple perturbation can be absorbed into the phase, and therefore the real amplitude does not become so large.

V. $F(R)$ GRAVITY CASE

Let us briefly discuss $F(R)$ gravity in a similar context. Its action is given by

$$S_{F(R)} = \int d^4x \sqrt{-g}\left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}}(g_{\mu\nu}, \Psi_i)\right), \tag{148}$$

where Ψ_i expresses the field corresponding to matters. It is well known that by using the scale transformation,

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}, \qquad \sigma = -\ln F'(A), \tag{149}$$

action (148) can be rewritten in the scalar-tensor form

$$S_E = \int d^4x \sqrt{-g}\left\{\frac{1}{2\kappa^2}\left(R - \frac{3}{2} g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right) + \mathcal{L}_{\mathrm{matter}}(\mathrm{e}^{\sigma} g_{\mu\nu}, \Psi_i)\right\},$$

$$V(\sigma) = \mathrm{e}^{\sigma} g(\mathrm{e}^{-\sigma}) - \mathrm{e}^{2\sigma} f(g(\mathrm{e}^{-\sigma})). \tag{150}$$

Here $g(\mathrm{e}^{-\sigma})$ is given by solving the equation $\sigma = -\ln(1 + f'(A)) = -\ln F'(A)$ as $A = g(\mathrm{e}^{-\sigma})$. In the case of the Einstein gravity in this paper, we have neglected the cosmic expansion. Hence we may assume the scalar field σ is a constant $\sigma = \sigma_0$ and $V(\sigma_0) = 0$. Furthermore, as long as we consider the gravitational wave, we do not consider the fluctuation of the scalar field σ. This also shows that the metric $\mathrm{e}^{\sigma} g_{\mu\nu}$ that appears in the Lagrangian density of matter $\mathcal{L}_{\mathrm{matter}}$ is different from the metric $g_{\mu\nu}$ by only a constant scale transformation, which effectively gives the change of the gravitational constant $\kappa^2 \to \kappa^2 \mathrm{e}^{\sigma_0}$. Thus as long as we neglect the cosmic expansion, the propagation of the gravitational wave in the $F(R)$ gravity is not qualitatively changed from that in the Einstein gravity.

By the variation of the action in (150) with respect to the metric $g_{\mu\nu}$, we obtain the Einstein equation in the Einstein frame,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 3\partial_\mu\sigma\partial_\nu\sigma + g_{\mu\nu}\left(-\frac{3}{2} g^{\rho\sigma}\partial_\sigma\phi\partial_\sigma\sigma - V(\sigma)\right) + \kappa^2 \mathrm{e}^{\sigma} T_{\mu\nu}. \tag{151}$$

Naively if the first two terms are dominant compared with the last term $\kappa^2 \mathrm{e}^{\sigma} T_{\mu\nu}$ as in the vacuum, we can neglect the contribution from the matter. On the other hand, if the last term $\kappa^2 \mathrm{e}^{\sigma} T_{\mu\nu}$ is dominant as in the dense matter, the expansion of the universe, which could be generated by the first two terms, could be negligible.

In the case of the $F(R)$ gravity, there appears a scalar mode corresponding to σ. The mass of the scalar mode should be very small in the bulk but the mass can become large inside the matter by the chameleon mechanism [40]. Therefore the propagation of the scalar mode is a little bit complicated.

The propagation of the graviton in other kinds of the modified gravity theories has also been actively investigated; see, for instance, [41–47].

VI. CONCLUSIONS

In the present paper, we have analyzed the propagation of gravitational waves in the medium in detail. We have shown how the propagation of gravitational waves could be changed by the medium.

In general, the radiation is made of the quanta or massless particles at high temperature. Usually the radiation consists of photons, which are quanta of the electromagnetic field. We should note that the radiation-dominated stage of the universe can be realized not only by the real radiation but also by the scalar-tensor theory. Then we have shown how to distinguish the radiation dominated universe generated by the real radiation with that generated by the scalar-tensor theory.

Motivated with the above observations, we have investigated the propagation of gravitational waves in the medium. Especially it has been found that the propagation of gravitational waves in the thermal radiation in general relativity is different from that in the scalar-tensor theory. Furthermore, we have explored the propagation of gravitational waves in the uniform magnetic field, and it has been found that the effects from the magnetic field to the propagation could not be negligible. For a small object such as magnetar, the perturbation with respect to $\kappa^2 B^2$ could be valid, but for the large object of galaxy size such as the large magnetic field, the perturbation breaks down. For the large object, the adiabatic approximation gives more reasonable results. Note that we limited to a flat space background where there is no qualitative difference between general relativity and, say, $F(R)$ gravity.

At the next stage, it would be extremely interesting to extend our study for evolving cosmological background. It is known that the evolution of gravitons in accelerating cosmologies for the case of extended gravity which has been considered in Ref. [48] is qualitatively different from that of general relativity. Then, the account of the cosmic magnetic field may even increase this qualitative difference. Then, as the first proposal for future possible extensions of the present work one can study the gravitational waves propagation in the anisotropic (Bianchi) universe with magnetic fields. Moreover, it has been examined that the stochastic background of gravitational waves can be tuned by the effect of $F(R)$ gravity [49]. This again may be generalized for the presence of the cosmic magnetic field.

Finally, it is very interesting to mention that there is a possibility of the existence of some relations between primordial magnetic fields and primordial gravitational fields [50]. This may be a clue to find a fundamental connection between electromagnetism and gravitation, which would be similar to that between thermodynamics and gravity. From the other side, such a study may give further bounds to gravitational wave propagation at the early universe with primordial magnetic fields.

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